1. Show that if A is diagonalizable, then $\det(e^A) = e^{tr(A)}$.

$$A = P^{-1}\Lambda P$$

$$e^A = P^{-1}e^{\Lambda}P$$

$$e^{\Lambda} = \begin{bmatrix} e^{\lambda 1} & & \\ & \ddots & \\ & & e^{\lambda n} \end{bmatrix}$$

$$\det(e^{\Lambda}) = \det(P^{-1})\det(e^{\Lambda})\det(P) = \det(e^{\Lambda})$$

$$\det(e^{\Lambda}) = e^{\lambda 1} \ldots e^{\lambda n} = e^{\lambda 1 + \cdots + \lambda n}$$

Theorem: $tr(A) = \sum_{i=1}^{n} \lambda i$

So, $\det(e^A) = e^{tr(A)}$.

2. Use polar coordinates to solve

$$\dot{x} = ax - by$$
$$\dot{y} = ay + bx$$

$$x = rcos(\theta)$$
$$y = rsin(\theta)$$
$$\dot{x} = -rsin(\theta)\dot{\theta} + \dot{r}cos(\theta)$$
$$\dot{y} = rcos(\theta)\dot{\theta} + \dot{r}sin(\theta)$$

Plugging the last four expressions into the first two gives:

$$-rsin(\theta)\dot{\theta} + \dot{r}cos(\theta) = arcos(\theta) - brsin(\theta) \quad (1)$$
$$rcos(\theta)\dot{\theta} + \dot{r}sin(\theta) = arsin(\theta) + arcos(\theta) \quad (2)$$

Multiply (2) by $cos(\theta)$ and subtract (1) by $sin(\theta)$

$$r\dot{\theta} = br$$
$$\dot{\theta} = b$$
$$\theta = bt + \theta_0$$

Multiply (1) by $cos(\theta)$ and add (2) by $sin(\theta)$

$$\dot{r} = ar$$
$$r = r_0 e^{at}$$

Thus, we have

$$x = r_0 e^{at} cos(bt + \theta_0)$$
$$y = r_0 e^{at} sin(bt + \theta_0)$$

3. For $\dot{x} = x^2, x(0) = 1$, use local existence and uniqueness to estimate the time of existence. Solve the equation direction and find the actual time of existence.

We first show that $X(x) = \dot{x} = x^2$ is Lipschitz on an open ball U about the initial condition

$$U := (1 - (z + \varepsilon), 1 + (z + \varepsilon))$$

$$\|X(x) - X(y)\| = \|x^2 - y^2\| = |x + y||x - y|$$

$$\|X(x) - X(y)\| \leq 2(z + 2\varepsilon)|x - y|$$

$K = 2(z + 2\varepsilon)$, Lipschitz satisfied.

Now, take $B_z(1) = (1 - z, 1 + z) \subseteq U$ and find a bount, M such that

$$\|X(x)\| < M \; for \; all \; x \in B_z$$

$$M = \max\|x^2\|, x \in B_z$$

$$M = (z + 1)^2$$

To estimate the time of existence, we set, $\alpha = z/M$ with $t \in (-\alpha, \alpha)$. The best estimate will be the maximal value of alpha for $z > 0$.

$$t \in (-z/(z + 1)^2, -z/(z + 1)^2)$$

$$\max_{z>0} \frac{z}{(z + 1)^2} \rightarrow$$

$$\frac{d}{dz}\frac{z}{(z + 1)^2} = 0$$

$$(1 - z)(1 + z)^2 = 0$$

$$z = 1 \; maximizes \; \frac{z}{(z + 1)^2}$$

$$\alpha = \frac{1}{(1 + 1)^2} = 1/4$$

Solving the ODE directly we find

$$\int_1^x \frac{dx}{x} = \int_0^t dt$$

$$x = \frac{-1}{t - 1}$$

This blows up at $t = 1$, so the actual time of existence is 1.