CDS 140A — HOMEWORK 3 SCRIBE FILE

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Exercise 2. Do all solutions of the system

$$\begin{aligned}\dot{x} &= -x + y + z\\ \dot{y} &= -y + 2z\\ \dot{z} &= -2z\end{aligned}$$

converge to the origin as $t \to \infty$?

Proof. This system can be written in matrix form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \implies \frac{d}{dt}\begin{vmatrix} x\\ y\\ z\end{vmatrix} = \begin{vmatrix} -1 & 1 & 1\\ 0 & -1 & 2\\ 0 & 0 & -2\end{vmatrix}\begin{vmatrix} x\\ y\\ z\end{vmatrix}.$$

A is upper triangular, and therefore the eigenvalues are $-1, -1$ and -2. Each of these eigenvalues has a negative real part, so $E^S = \mathbb{R}^3$ and Theorem 1.5 then states that all trajectories approach the origin. □

Exercise 4. Find the Jordan canonical form, the $S + N$ decomposition and the matrix exponential for the matrix

$$A = \begin{vmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & -1\end{vmatrix}$$

Proof. First, notice that this matrix is triangular, so we can read the eigenvalues off the diagonal as 1, 1 and -1. The eigenvectors corresponding to 1 can be found by computing

$$\begin{vmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & -1\end{vmatrix}\begin{vmatrix} v_1\\ v_2\\ v_3\end{vmatrix} = \begin{vmatrix} v_1\\ v_2\\ v_3\end{vmatrix} \implies \begin{aligned} v_1 &= v_1\\ v_2 + v_3 &= v_2\\ -v_3 &= v_3\end{aligned}$$

Therefore, two linearly independent eigenvectors for the eigenvalue of 1 are $(1, 0, 0)$ and $(0, 1, 0)$. The eigenvector of $(0, 1, -2)$ can be computed analogously for the eigenvalue of -1. These eigenvectors form a basis for $\mathbb{R}^3$ so an aside in the notes yields that A is diagonalizable. In particular, consider

$$T = \begin{vmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & -2\end{vmatrix}, \text{ then } T^{-1}AT = D = \begin{vmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{vmatrix}$$

The Jordan canonical form is unique up to rearranging Jordan blocks, so it is equal to D. Moreover, the matrix A can be diagonalized, so it is semi-simple. We can take $S = A$, $N = 0$. The matrix exponential can be computed with the diagonalization

$$e^A = T^{-1}e^D T = \begin{vmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & -2\end{vmatrix}\begin{vmatrix} e & 0 & 0\\ 0 & e & 0\\ 0 & 0 & e^{-1}\end{vmatrix}\begin{vmatrix} 1 & 0 & 0\\ 0 & 1 & 1/2\\ 0 & 0 & -1/2\end{vmatrix} = \begin{vmatrix} e & 0 & 0\\ 0 & e & \frac{1}{2}(e - e^{-1})\\ 0 & 0 & e^{-1}\end{vmatrix}$$

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Exercise 5. Find the generalized eigenspaces of the matrix in the preceding problem and show directly that these subspaces are invariant under the equation $\dot{x} = Ax$ and span all of $\mathbb{R}^3$.

Proof. I computed the eigenvalues to be $1, 1$ and -1 in the preceeding problem. Furthermore, I found the eigenvectors to be

$$\underline{v}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}; \quad \underline{v}_1' = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}; \quad \underline{v}_{-1} = \begin{vmatrix} 0 \\ 1 \\ -2 \end{vmatrix}.$$

We can now obtain the generalized eigenspaces:

$$E^S = \text{span}\left(\begin{vmatrix} 0 \\ 1 \\ -2 \end{vmatrix} \right); \quad E^C = \text{span}\,(0)\,; \quad E^U = \text{span}\left(\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \right).$$

These subspaces span $\mathbb{R}^3$ because for any $\underline{x} \in \mathbb{R}^3$ we can write

$$\underline{x} = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = x_1 \underline{v}_1 + \left(x_2 + \frac{x_3}{2} \right) \underline{v}_1' - \frac{x_3}{2} \underline{v}_{-1}.$$

I will now show that these subspaces are invariant under the equation $\dot{x} = Ax$. Let $\underline{x} \in E^U$, this implies that $\underline{x}$ is of the form

$$\underline{x} = \begin{vmatrix} x_1 \\ x_2 \\ 0 \end{vmatrix}.$$

We can use this to show invariance of E^U:

a set S is invariant in the dynamical system $\dot{x} = Ax$ if and only if the flow maps points in the set to points in the set, i.e. $\phi_t(x) \in S, \forall x \in S, \forall t$. In this case the flow is $\phi_t = e^{At}$. We find the exponentail of At to be

$$e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & \sinh(t) \\ 0 & 0 & e^{-t} \end{bmatrix}$$

So that the point $(x_1, x_2, 0) \in E^U$ flows to $(e^t x_1, e^t x_2, 0) \in E^U$, so that E^U is invariant. The same is done for E^C and E^S. That is we choose an arbitrary point in each space, and show that point remains confined to that space under the map e^{At}

□