CDS 140A – Solutions to Problems 1,9,10 from Homework 2

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1. **Problem 1.** First, verify that the given orbit satisfies the differential equation of the system.

$$\phi(t) = \pm\, 2\ \tan^{-1}(\sinh t)$$
$$\dot\phi(t) = \pm\, 2\ \frac{1}{1+\sinh^2 t}\cosh t$$
$$= \pm\, 2\ (\cosh t)^{-1}$$
$$\ddot\phi(t) = \mp\, 2\ \frac{1}{\cosh^2 t}\sinh t$$

Now, given that $\tan(\frac{\phi(t)}{2}) = \pm\ \sinh t$, we get,

$$\sinh t = \pm\ \tan\frac{\phi(t)}{2}$$
$$\cosh^2 t = 1 + \sinh^2 t$$
$$= 1 + \tan^2\frac{\phi(t)}{2}$$
$$= \sec^2\frac{\phi(t)}{2}$$

Therefore, we have,

$$\ddot\phi(t) = \mp\ \pm\ 2\ \frac{\tan\frac{\phi(t)}{2}}{\sec^2\frac{\phi(t)}{2}}$$
$$= -\sin\phi(t)$$

This finally gives us $\ddot\phi(t) + \sin\phi(t) = 0$. Therefore, the given orbits are solutions to the system. To verify that they are homoclinic, note that the equilibrium points of this system (in the range $-\pi \le 0 \le +\pi$) are at $-\pi$, 0 and $+\pi$. And, clearly, $\lim_{t\to+\infty} \pm 2\tan^{-1}(\sinh t) = \pm\pi$, and $\lim_{t\to-\infty} \pm 2\tan^{-1}(\sinh t) = \mp\pi$, which are equilibrium points. By definition, therefore, these orbits are homoclinic.

2. **Problem 9.** The given matrix is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

This can be decomposed as $A = S + N$, where

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2\, I$$

and

$$N = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since S is a constant multiple of the identity matrix, S and N commute. Therefore, $e^A = e^{S+N} = e^S e^N$. Clearly, $e^S = e^{2I} = e^2 I$. To compute e^N, notice that N is nilpotent:

$$N^0 = I,\ N^1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ N^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } N^k = 0\ \ \forall\, k \geq 3$$

Applying the definition of e^N, and using then nilpotency of N, we get

$$\begin{aligned} e^N &= I + N + \frac{N^2}{2} + 0 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} \end{aligned}$$

Finally, we get $e^A = e^2 I \cdot e^N = e^2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix}$

3. **Problem 10.** Given the system,

$$\begin{aligned} \dot{x} &= -2x - y \\ \dot{y} &= x - 2y \\ \dot{z} &= -z \end{aligned}$$

Clearly, the only equilibrium point is the origin, $(0, 0, 0)$. The system can be written in the standard notation as follows.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This is of the form $\dot{\mathbf{X}} = A \cdot \mathbf{X}$, where $\mathbf{X} = \begin{bmatrix} x & y & z \end{bmatrix}^T$. We note that the eigenvalues of A can be computed, and are given by -1, $-2 \pm i$. Because all the three eigenvalues have negative real parts, the stable subspace E^S, defined as the span of the generalized eigenvectors of A corresponding to eigenvalues with negative real part, is $\mathbb{R}^3$ itself. Now, invoking the *Stability Theorem* (Theorem 1.5 of the notes), we conclude that all the trajectories approach the origin as $t \to \infty$. This concludes the proof.