CDS 140a: Homework 1 Solutions

1. Consider the planar system $(x, v) \in \mathbb{R}^2$ given by

$$\dot{x} = v \tag{0.1}$$
$$\dot{v} = -x^3 \tag{0.2}$$

(a) The equilibrium points for the system can be found by setting equations (0.1) and (0.2) equal to 0. from equation (0.1) we get $v = 0$ and from equation (0.2) we get $x^3 = 0$ and so $x = 0$. Therefore, there is only one equilibrium points given by $(x, v) = (0, 0)$.

(b) From equation (0.2), we can write

$$\dot{v} = v\frac{\partial v}{\partial x} = -x^3.$$

We assert that

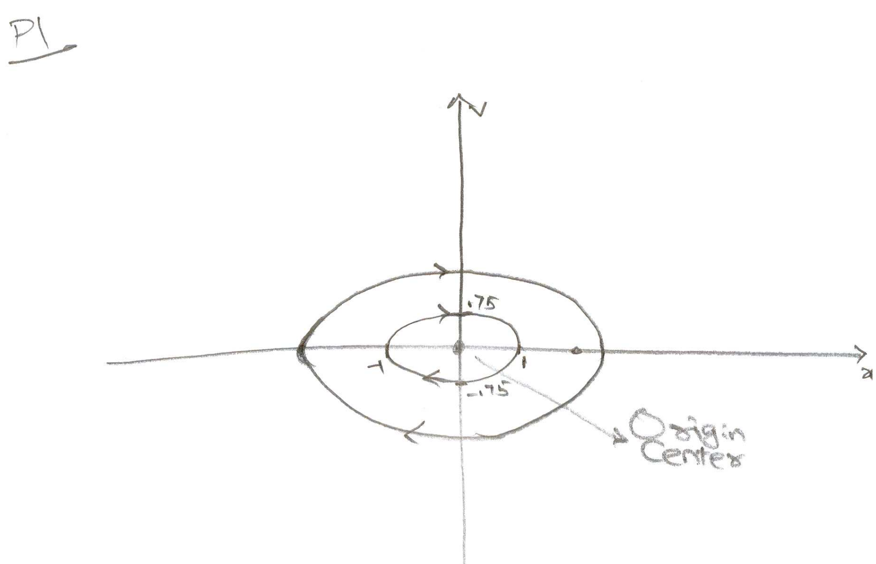
$$v^2/2 + x^4/4 = \text{constant} = \text{Energy}$$

This *conservation of energy equation* can be proved by $\frac{d}{dt}E = 0$; using equation (0.2) we get:

$$\frac{d}{dt}E = (v + x^3)\dot{v} = 0 \tag{0.3}$$

Hence energy is conserved.

(c) the phase portrait for the given problem is :



(d) In the energy equation, replacing

$$(x(t), v(t)) \text{ by } (x(t), -v(-t)), (-x(-t), v(t)), (-x(-t), -v(-t)),$$

doesn't change the energy value. Thus, the trajectory is symmetric about all axis. since, they crosses the axis, this implies that they are a closed trajectory. Also, the energy equation has the topological form of a circle with coordinates (x^2, v), which says its a closed trajectory.

To be more specific, lets check the Jacobian of the given equation.

$$J(x,v) = \begin{bmatrix} 0 & 1 \\ -3x^2 & 0 \end{bmatrix}$$

Therefore,

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The trace of this matrix $trace = p = 0$, and also $det = q = 0$, this tell us that both the eigen values are zero. Hence, this is a non-hyperbolic fixed point. therefore it act as a center and hence the orbit surrounding to the fixed point are periodic.

2. Given

$$\ddot{x} = -x^3 - \dot{x} \tag{0.4}$$

This equation (0.4) is same as that in problem 1, but with a dissipation term. Since, the dissipation term does not effect the equilibrium points, they are still the same: $(x, v) = (0, 0)$. Since we have a dissipation term, the system will lose its energy, it will move towards stable point. In this case, $(0, 0)$ is a stable point and hence all the trajectory will move towards $(0, 0)$

We can check this by writing the equation in the form

$$\dot{x} = v \tag{0.5}$$

$$\dot{v} = -x^3 - v \tag{0.6}$$

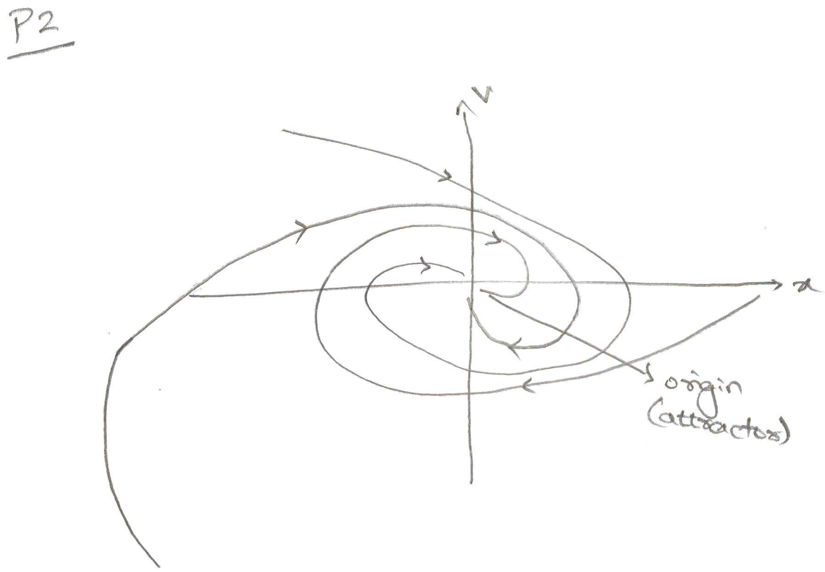
by solving this equation, we get the fixed point as $(x, v) \equiv (0, 0)$ The Jacobian of the system of equation is:

$$J(x,v) = \begin{bmatrix} 0 & 1 \\ -3x^2 & -1 \end{bmatrix}$$

Therefore,

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

The trace of this matrix $trace = p = -1$, and also $det = q = 0$, this tell us that one of the eigen value is zero. Hence, this is a non-hyperbolic fixed point and therefore it act as a attractor and hence the trajectory will sink on the fixed point.



3. Given: planar system $(x, v) \in \mathbb{R}^2$:

$$\ddot{x} = 2x + x^2 - x^3$$

This can be written in the following form.

$$\dot{x} = v \tag{0.7}$$
$$\dot{v} = 2x + x^2 - x^3 \tag{0.8}$$

(a) The equilibrium points for the system can be found by setting equation ((0.7) & (0.8)) equal to 0. from equation (0.7) we get:

$$\dot{x} = 0$$
$$\Rightarrow v = 0$$

and from equation (0.8) we get

$$\dot{v} = 0$$
$$\Rightarrow 2x + x^2 - x^3 = 0$$
$$\Rightarrow -x(x-2)(x+1) = 0$$

Thus, $x = 0, 2, -1$. Thus, the equilibrium points are $(x, v) = (0,0), (2,0), (-1,0)$

(b) From equation (0.8), we can write

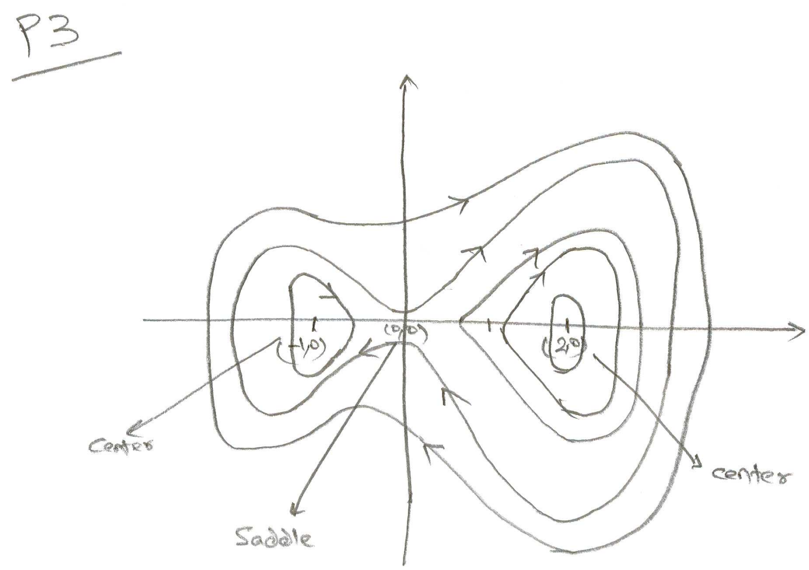
$$\dot{v} = v\frac{dv}{dx} = 2x + x^2 - x^3$$
$$\Rightarrow \quad v^2/2 + x^4/4 - x^3/3 - x^2 = constant = Energy \tag{0.9}$$

That energy is conserved can be proved by $\frac{d}{dt}E = 0$ from equation (0.9), using equation (0.8) we have:

$$\frac{d}{dt}E = (v + x^3 - x^2 - 2x)\dot{v} = 0 \tag{0.10}$$

Hence energy is conserved.

(c) The phase portrait of the given system is:



(d) Since, in the energy equation ((0.9)), replacing $(x(t), v(t))$ by $(x(t), -v(-t))$, doesn't change the energy value. Therefore, the trajectory is symmetric about x axis. since, they crosses the axis, this means, they are closed trajectory. Also, energy equation ((0.9)) is of the form of ellipse, this implies periodic trajectory. Also, from the phase portrait, we can see that the point $(2,0)\&(-1,0)$ are stable point, with the point $(2,0)$ being a strong attractor, compare to point $(-1,0)$.

To check the nature of fixed points formally,

The Jacobian of the system of equation is:

$$J(x, v) = \begin{bmatrix} 0 & 1 \\ 2 + 2x - 3x^2 & 0 \end{bmatrix}$$

Therefore, fixed point 1:

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

The trace of this matrix $trace = p = 0$, and also $det = q = -2$, this tell us that the eigen values are of opposite sign and equal in magnitude. therefore, origin act as a saddle point.

fixed point 2:

$$J(2,0) = \begin{bmatrix} 0 & 1 \\ -6 & 0 \end{bmatrix}$$

The trace of this matrix $trace = p = 0$, and also $det = q = 6$, Thus, $(2,0)$ acts as an center.

fixed point 3:

$$J(-1,0) = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$$

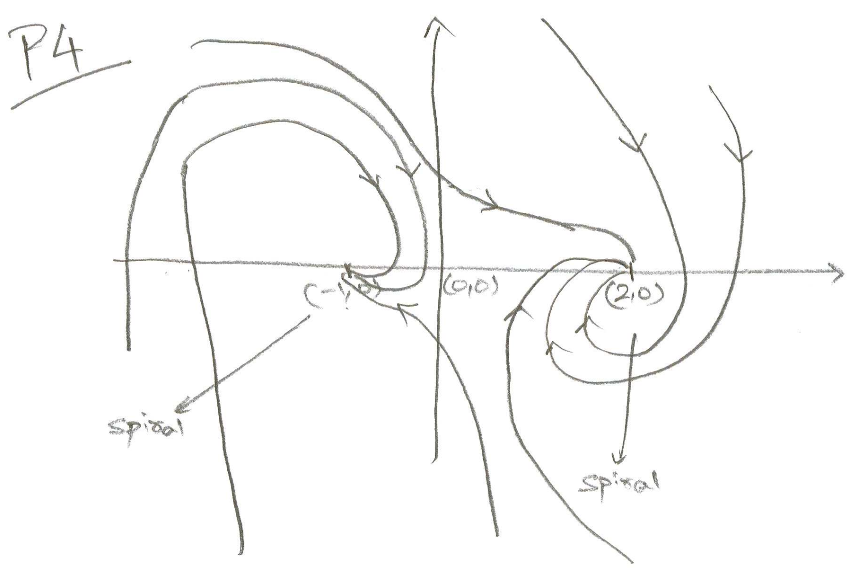
The trace of this matrix $trace = p = 0$, and also $det = q = 3$, Thus, $(-1,0)$ also acts as an center.

4. Given

$$\ddot{x} = 2x + x^2 - x^3 - 2\dot{x} \tag{0.11}$$

The equation ((0.11)) is same as problem 2, with a dissipative term. Since, the dissipation term does not effect the equilibrium points, therefore, the equilibrium points are still the same $(x, v) = (0,0), (2,0), (-1,0)$.

Since, we have a dissipation term in the equation ((0.11)) This means, as the system will lose its energy, it will move towards stable point. In this case, $(2,0)\&(-1,0)$ are stable points and $(0,0)$ is a saddle point, hence all the trajectory will move towards $(2,0)\&(-1,0)$ them in a spiral motion. Depending on the initial conditions of the system, which describes the initial energy of the system, the point will either move towards $(2,0)$ or $(-1,0)$. Also, $(-1,0)$ is a weak attractor as compare to $(2,0)$, therefore a system with enough energy will most likely will be attracted by $(2,0)$. (This can be verified by taking Jacobian matrix and checking the trace and determinant of the Jacobian matrix.)



5. In the energy equation, replacing

$$(x(t), v(t)) \text{ by } (x(t), -v(-t)), (-x(-t), v(t)), (-x(-t), -v(-t)),$$

doesn't change the energy value. Therefore, the trajectory is symmetric about all axis and also reversible. since, they crosses the axis, this implies that they are a closed trajectory. Also, the level set of the energy is of the form of circle with coordinates (x^2, v), which says its a closed trajectory with a constant period.

For the system in problem (2), because of the dissipation term, the energy equation is no longer stable and hence, the trajectory will fall on the origin, as origin acts as a stable attractor. Thus, the equation are no longer symmetric or reversible. Although, since the dissipative term is v, therefore, the trajectories are anti-symmetric about $x = v$ line, as can be seen in the phase portrait.

6. Since, in the energy equation ((0.9)), replacing $(x(t), v(t))$ by $(x(t), -v(-t))$, doesn't change the energy value. Therefore, the trajectory is symmetric about x axis. since, they crosses the axis, this means, they are closed trajectory. Also, energy equation ((0.9)) is of the form of ellipse, this implies periodic trajectory.

Also, from the phase portrait, we can see that the point $(2, 0)\&(-1, 0)$ are stable point, with the point $(2, 0)$ being a strong attractor, compare to point $(-1, 0)$. The point $(2, 0)$ being a strong attractor as compare to $(-1, 0)$ can be checked by calculating the eigenvalues or the checking the trace and determinant of the Jacobian matrix (as shown in problem 3).

In case of problem (4), the system will loses its energy, therefore, it will move towards stable points. In this case, $(2, 0)\&(-1, 0)$ are stable points and $(0, 0)$ is a saddle point, hence all the trajectory will move towards $(2, 0)\&(-1, 0)$ them in a spiral motion. Also, $(-1, 0)$ is a weak attractor as compare to $(2, 0)$, therefore a system with enough energy will most likely will be attracted by $(2, 0)$, hence, the system will no longer be symmetric and due to stable attractor, the system will not be reversible.

7. dynamical system provided:

$$\dot{x} = -x(x^2 + y^2 - \mu) - y(x^2 + y^2) \tag{0.12}$$

$$\dot{y} = -y(x^2 + y^2 - \mu) + x(x^2 + y^2) \tag{0.13}$$

To analyze the system, we can leave off the nonlinear terms, therefore, we get is:

$$\dot{x} = \mu x \tag{0.14}$$

$$\dot{y} = \mu y \tag{0.15}$$

therefore, the equilibrium point of the system is origin $(0, 0)$.

$$Df(0, \mu) = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$$

Thus, origin is a stable point for $\mu < 0$.

The Jacobian of the nonlinear equation system is:

$$J(x,v) = \begin{bmatrix} -(3x^2 + y^2 - \mu - 2xy) & -(x + x^2 + 3y^2) \\ -(2xy) + (3x^2 + y^2) & -(x^2 + 3y^2 - \mu - x) \end{bmatrix}$$

Therefore, fixed point 1:

$$J(0,0) = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$$

The trace of this matrix $trace = p = 2\mu$, and also $det = q = \mu^2 > 0$, this tell us that the origin will act as attractor for $\mu < 0$, repeller for $\mu > 0$, and as a center for $\mu = 0$

Also, from the energy point of view.

$$E(x,y) = \frac{\mu^2}{2}(x^2 + y^2) \tag{0.16}$$

Thus introduce polar coordinates $(r\theta)$ in the usual way:

$$x = r\cos(\theta), \qquad y = r\sin(\theta)$$

Differentiating the relation $r^2 = x^2 + y^2$ and using equation (0.12) & (0.13), we get: (as long as r is not zero)

$$\dot{r} = r(\mu - r^2) \tag{0.17}$$

Similarly, by differentiating $x = r\cos(\theta)$ and making use of the equations for $\dot{x}$ and $\dot{r}$we find that

$$\dot{\theta} = r^2 \tag{0.18}$$

Thus from equation ((0.17)), the system has a fixed point at origin and a limit cycle at $r = \sqrt{\mu}$. for $\mu > 0$, there is a unique and stable circular limit cycle that exists. This corresponds to a periodic orbit in the (x, y) plane. As μ increases past 0, the fixed point at the origin switches from attractor to repeller and a stable limit cycle (attractor) emerges for $\mu > 0$.

8. (part of this problem is already explained in previous problem) when $\mu < 0$, we have $\dot{r} < 0$, and therefore, r decreases and it the trajectory moves towards the origin. Also, from the Jacobian matrix, for $\mu < 0$, the origin act as stable attractor and no limit cycle is present. $\because \quad \theta = r^2$, this implies, as r decreases, θ changes too. and this trajectory goes to origin. when $\mu > 0$, a stable limit cycle appears due to the change of eigenvalue of the equation. and when $\mu > \mu_0$, the limit cycle act as an attractor and the limit cycle is asymptotic to the circle of radius $\sqrt{\mu_0}$.

9. Given $\ddot{x} = \alpha x - x^3$ we can write this equation as

$$\dot{x} = v \tag{0.19}$$
$$\dot{v} = \alpha x - x^3 \tag{0.20}$$

The equilibrium points for the system can be found by setting equation ((0.19) & (0.20)) equal to 0. from equation (0.19) we get:

$$\begin{aligned} \dot{x} &= 0 \\ \Rightarrow v &= 0 \end{aligned}$$

and from equation (0.20) we get

$$\begin{aligned} \dot{v} &= 0 \\ \Rightarrow \alpha x - x^3 &= 0 \\ \Rightarrow -x(x^2 - \alpha) &= 0 \\ \Rightarrow x &= 0, \sqrt{\alpha}, -\sqrt{\alpha} \end{aligned}$$

therefore, the equilibrium points are $(x, v) = (0, 0), (\sqrt{\alpha}, 0), (-\sqrt{\alpha}, 0)$

The Jacobian of the nonlinear equation system is:

$$J(x, v) = \begin{bmatrix} 0 & 1 \\ \alpha - 3x^2 & 0 \end{bmatrix}$$

Therefore, fixed point 1:

$$J(0, 0) = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}$$

The trace of this matrix $trace = p = 0$, and also $det = q = -\alpha$, this tell us that the origin will act as saddle for $\alpha > 0$, center for $\alpha \leq 0$.

fixed point 2 & 3:

$$J(\pm\sqrt{\alpha}, 0) = \begin{bmatrix} 0 & 1 \\ -2\alpha & 0 \end{bmatrix}$$

The trace of this matrix $trace = p = 0$, and also $det = q = 2\alpha$, this tell us that the points will act as center for $\alpha > 0$. For $\alpha < 0$, the fixed points will be complex value.

10. Given $\ddot{x} = \alpha x - x^3 - \dot{x}$ we can write this equation as

$$\dot{x} = v \tag{0.21}$$
$$\dot{v} = \alpha x - x^3 - v \tag{0.22}$$

The equilibrium points for the system can be found by setting equation ((0.21) & (0.22)) equal to 0. from equation (0.21) we get:

$$\begin{aligned} \dot{x} &= 0 \\ \Rightarrow v &= 0 \end{aligned}$$

and from equation (0.22) we get

$$\begin{aligned} \dot{v} &= 0 \\ \Rightarrow \alpha x - x^3 - v &= 0 \\ \Rightarrow -x(x^2 - \alpha) &= 0 \\ \Rightarrow x = 0, \sqrt{\alpha}, -\sqrt{\alpha} \end{aligned}$$

therefore, the equilibrium points are $(x, v) = (0, 0), (\sqrt{\alpha}, 0), (-\sqrt{\alpha}, 0)$

The Jacobian of the nonlinear equation system is:

$$J(x, v) = \begin{bmatrix} 0 & 1 \\ \alpha - 3x^2 & -1 \end{bmatrix}$$

Therefore, fixed point 1:

$$J(0, 0) = \begin{bmatrix} 0 & 1 \\ \alpha & -1 \end{bmatrix}$$

The trace of this matrix $trace = p = -1$, and also $det = q = -\alpha$, this tell us that the origin will act as saddle for $\alpha > 0$, stable spiral for $\alpha \leq 0$.

fixed point 2 & 3:

$$J(0, 0) = \begin{bmatrix} 0 & 1 \\ -2\alpha & 0 \end{bmatrix}$$

The trace of this matrix $trace = p = 0$, and also $det = q = 2\alpha$, this tell us that the points will act as center for $\alpha > 0$. For $\alpha < 0$, the fixed points will be complex value.

fixed point 2 & 3:

$$J(\pm\sqrt{\alpha}, 0) = \begin{bmatrix} 0 & 1 \\ -2\alpha & -1 \end{bmatrix}$$

The trace of this matrix $trace = p = -1$, and also $det = q = 2\alpha$, this tell us that the points will act as stable spiral for $\alpha > 0$. For $\alpha < 0$, the fixed points will be complex value.