CDS140a - Introduction to Dynamics
Homework 1
Exercises 1, 2, and 5

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1. Consider the following planar system for $(x, v) \in \mathbb{R}^2$:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -x^3 \end{cases} \tag{1}$$

(a) Find the equilibrium points for the system.

The equilibrium points of this system are obtained by setting the right hand side to zero. Thus, the equilibria occurs in the xv-plane when $\dot{x} = v = 0$ and when x satisfies

$$\dot{v} = -x^3 = 0 \rightarrow x = 0.$$

$$\therefore (0,0) \text{ is the equilibrium point of the system.}$$

(b) Find a conserved energy for the system.

This system corresponds to a conservative mechanical system and can be rewritten as:

$$\begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m}\left(-\nabla V(x)\right). \end{cases}$$

Replacing the Equation 1 in this form, we can compute the potential energy $V(x)$:

$$\frac{1}{m}\left(-\nabla V(x)\right) = -x^3 \rightarrow V(x) = m\frac{x^4}{4}.$$

Finally, we find the conserved energy:

$$\therefore E(x, v) = m\frac{v^2}{2} + m\frac{x^4}{4}.$$

(c) Draw the phase portrait.

Figure 1.

(d) Argue informally that all the trajectories outside the origin are periodic.

Since the system preserves energy, its trajectories correspond to level-sets of the energy equation. Fixed an initial condition $(x_0, v_0) \neq (0,0)$, we verify that the level-sets compose ellipses. Therefore, the trajectories outside the origin are periodic.

2. Draw the phase portrait for the system:

$$\ddot{x} = -x^3 - \dot{x} \tag{2}$$

and comment on its structure.

First we rewrite the equation as a first order system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x^3 - y \end{cases}$$

The portrait is in Figure 2. The system has one equilibrium point, $(0,0)$. Such point is asymptotically stable, since all trajectories converge to it. The trajectories also have a spiral structure.

5. Discuss symmetry and reversibility properties (if any) of the equations in problems 1 and 2.

Given a system with solution $(x(t), v(t))$, first let's claim the conditions for symmetry and reversibility properties.

Reversibility A system has time reversibility if $(\tilde{x}(t), \tilde{v}(t)) = (x(-t), -v(-t))$ is also solution.

Symmetry A system has symmetry if $(\tilde{x}(t), \tilde{v}(t)) = (-x(-t), v(-t))$ is also solution.

Now, let's verify if equations of problems 1 and 2 are still valid for $(\tilde{x}(t), \tilde{v}(t))$.

Problem 1

Reversibility:

$$\dot{\tilde{x}}(t) = \dot{x(-t)} = -\dot{x}(-t) = -v(-t) = \tilde{v}(t) \qquad \checkmark$$

$$\dot{\tilde{v}}(t) = -\dot{v(-t)} = \dot{v}(-t) = (-x(-t))^3 = -\tilde{x}(t)^3 \quad \checkmark$$

$\therefore$ Problem 1 has time reversibility.

Symmetry:

$$\dot{\tilde{x}}(t) = -\overset{\cdot}{x(-t)} = \dot{x}(-t) = v(-t) = \tilde{v}(t) \qquad \checkmark$$

$$\dot{\tilde{v}}(t) = \overset{\cdot}{v(-t)} = -\dot{v}(-t) = -\left(-x(-t)\right)^3 = -\tilde{x}(t)^3 \qquad \checkmark$$

$\therefore$ Problem 1 has symmetry.

Problem 2

Reversibility:

$$\begin{aligned} \dot{\tilde{x}}(t) = &\quad \overset{\cdot}{x(-t)} = -\dot{x}(-t) = -v(-t) = \tilde{v}(t) && \checkmark \\ \dot{\tilde{v}}(t) = &\quad -\overset{\cdot}{v(-t)} = \dot{v}(-t) = -x(-t)^3 - v(-t) = \\ = &\quad -\tilde{x}(t)^3 + \tilde{v}(t) \neq -\tilde{x}(t)^3 - \tilde{v}(t) && FAILED \end{aligned}$$

$\therefore$ Problem 2 does not have time reversibility.

Symmetry:

$$\begin{aligned} \dot{\tilde{x}}(t) = &\quad -\overset{\cdot}{x(-t)} = \dot{x}(-t) = v(-t) = \tilde{v}(t) && \checkmark \\ \dot{\tilde{v}}(t) = &\quad \overset{\cdot}{v(-t)} = -\dot{v}(-t) = -\big(-x(-t)^3 - v(-t)\big) = \\ = &\quad -\tilde{x}(t)^3 + \tilde{v}(t) \neq -\tilde{x}(t)^3 - \tilde{v}(t) && FAILED \end{aligned}$$

$\therefore$ Problem 2 does not have symmetry.

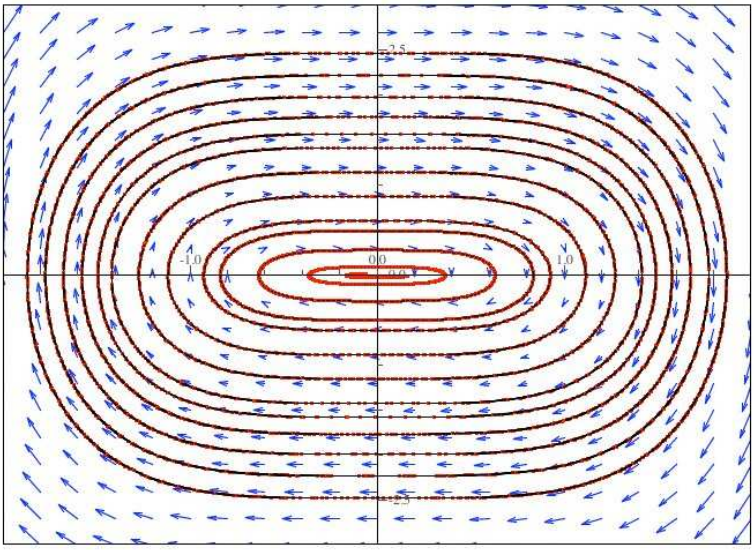

Figure 1: Phase Portrait for the system 1.

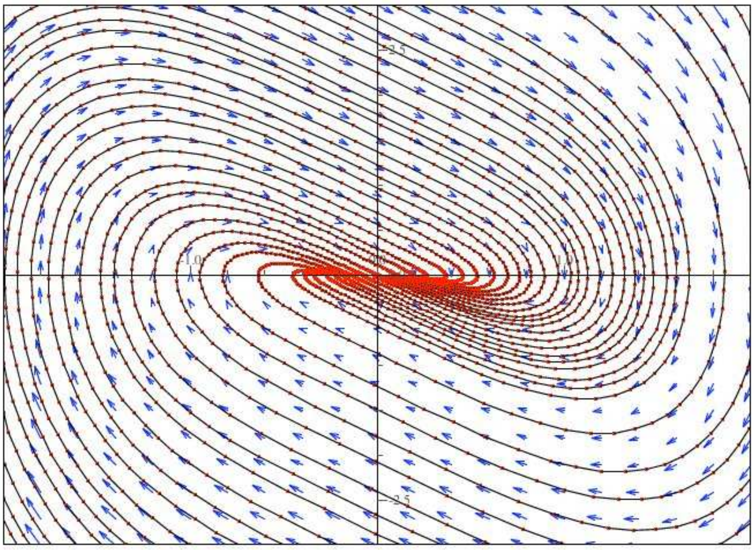

Figure 2: Phase Portrait for the system 2.