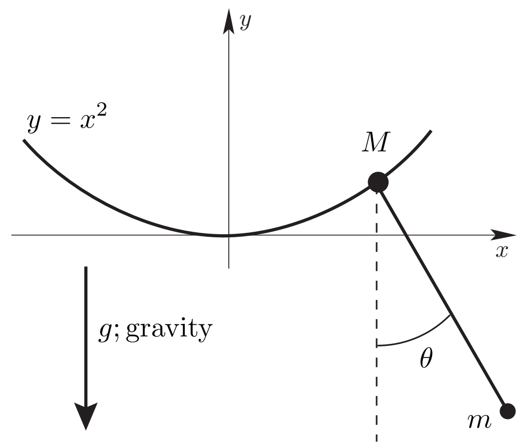
CDS 140a: Homework Set 7

Due: Wednesday, December 2, 2009.

For the first three questions, consider the mechanical system shown in the Figure below. The mass M slides (without friction) along the curve $y = x^2$. The mass m hangs by a light rod of length l (as a planar pendulum) from the mass M. Let θ denote the angle with the vertical, as shown. Both masses are subject to a downward gravitational force.



1. Write down a Lagrangian for the system and compute the Euler–Lagrange equations.

2. Does this system have a conserved energy? Explain how to get this expression in both the Lagrangian and the Hamiltonian formulations of the problem.

3. Note that when M is stationary at $x = 0$ and m is hanging vertically (with $\theta = 0$), one has an equilibrium. Is it stable? Asymptotically stable? What if one adds friction?

4. The support point of a simple planar pendulum of mass m and length l moves along the horizontal x-axis with position $x(t) = a \cos \omega t$.

   (a) Find the Lagrangian for the system in terms of θ (the angle that the pendulum makes with the vertical direction) and $\dot{\theta}$.

   (b) Derive the equation of motion for the pendulum and give the energy equation for the system.

5. Consider a Hamiltonian system on $\mathbb{R}^4$ with coordinates (x, q, p, r) of the form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{x} &= \frac{\partial H}{\partial r} \\ \dot{r} &= -\frac{\partial H}{\partial x} \end{aligned}$$

where

$$H = \frac{1}{2}\left(r^2 + p^2 - x^2 - q^2\right) + \Omega\left(xp - qr\right) + x^2q^2 + x^4 + q^4$$

and Ω is a constant. Compute the spectrum of the linearized system at the origin for various values of Ω and verify that it has the symmetry that you expect from general theory.

6. Consider the planar dynamical system

$$\begin{aligned} \dot{x} &= x^2 - x^3 \\ \dot{y} &= y \end{aligned}$$

Find all the equilibrium points and their linearizations.

7. Find the stable, unstable, and center manifolds of all these equilibrium points in the preceding problem. Verify with a computer simulation.

8. Consider the following dynamical system in $\mathbb{R}^2$:

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= 2y - 5x^3 \end{aligned}$$

Show that $y = x^3$ is an invariant manifold.

9. Determine the stable and unstable manifolds of the origin in the preceding problem. Verify with a computer simulation.

10. Consider the planar system

$$\begin{aligned} \dot{x} &= x^2 + xy \\ \dot{y} &= -y + x^2 \end{aligned}$$

Find the stable manifold of the origin explicitly and find an analytic approximation to the center manifold to leading order. Determine, approximately, the dynamics on the center manifold and verify with a computer simulation of the system.