CDS 140a: Homework Set 6

Due: Friday, November 20, 2009.

1. Let a be a real parameter with $0 \leq a \leq 4$. The logistic map is the map of the unit interval $[0, 1]$ to itself that is defined by $f(x) = ax(1-x)$. Find the fixed points of f and determine their stability.

2. A *two cycle* of a map f is a point p together with its image $q = f(p)$ with the property that $f(q) = p$. Show that the logistic map has a two cycle if $a > 3$.

3. The standard map is the map of the plane $\mathbb{R}^2$ to itself that is given by

$$x_{n+1} = x_n + y_{n+1}$$
$$y_{n+1} = y_n + k \sin x_n,$$

   where k is a constant. Compute the Jacobian determinant of the associated map f and conclude that the standard map is area preserving.

4. Perform a stability analysis of the fixed point at the origin for the standard map.

5. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an area preserving map and that the origin is a fixed point; that is, $f(0, 0) = (0, 0)$. Is there a sense in which the eigenvalues of the linearization are symmetric in the unit circle? Verify this assertion for the standard map from the preceding problem.

6. Consider Duffing's equation

$$\ddot{x} - \beta x + \alpha x^3 = 0,$$

   where α and β are positive constants.

   (a) Show that the equations can be written as Euler–Lagrange equations for a suitable Lagrangian.

   (b) Transform the equations to Hamiltonian form

   (c) Determine the equilibrium points and study their stability from the point of view of Dirichlet's theorem as well as from the viewpoint of Lyapunov's theorem.

7. Consider a magnetic field B in $\mathbb{R}^3$ and suppose that $B = \nabla \times A$ (one calls A the magnetic potential). The Lagrangian for a particle with mass m and charge e moving in the field B is given by

$$L(q, \dot{q}) = \frac{1}{2} m \|\dot{q}\|^2 + \frac{e}{c} A(q) \cdot \dot{q}$$

   where q and $\dot{q}$ are vectors in $\mathbb{R}^3$.

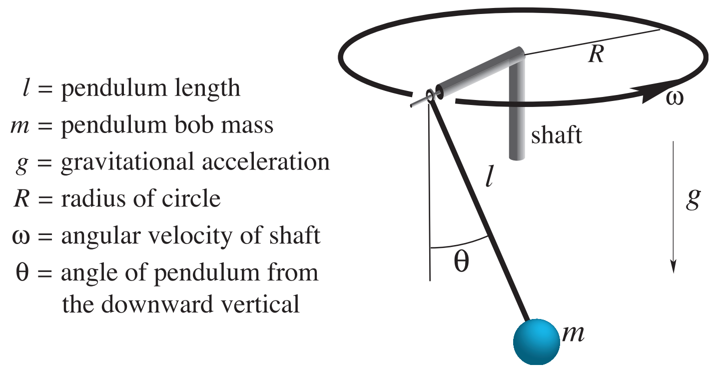
(a) Show that the Euler–Lagrange equations give Newton's equations with a ***Lorentz force law***:

$$m\ddot{q} = \frac{e}{c}\,\dot{q} \times B(q)$$

(b) What is the conserved energy?

8. Transform the system in the preceding problem to Hamiltonian form.

9. Consider the ***whirling pendulum*** shown in the figure.



It is a planar pendulum whose suspension point is being whirled in a circle with constant angular velocity ω by means of a vertical shaft, as shown. The plane of the pendulum is orthogonal to the radial arm of length R. Ignoring frictional effects and using the notation in the figure, find the equations of motion of the pendulum.

10. Continuing with the whirling pendulum from the preceding question, and regarding ω as a parameter, examine the bifurcation of equilibria that occurs as the angular velocity ω of the shaft is increased.