CDS 140a: Homework Set 5

Due: Friday, November 6, 2009.

1. (Computer work). Plot the phase portrait of the van der Pol oscillator

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= -x + v(1 - x^2)\end{aligned}$$

   and conjecture about its global structure. Do you think that solutions exist for all time?

2. Linearize the system in the preceding problem about the origin and compute the eigenvalues of the linearization. Is this consistent with the phase portrait?

3. (Computer work). Plot the phase portrait of the system

$$\begin{aligned}\dot{x} &= 2xy \\ \dot{y} &= x^2 - y^2\end{aligned}$$

   and conjecture about its global structure. Do you think that solutions exist for all time?

4. Linearize the system in the preceding problem about the origin and compute the eigenvalues of the linearization. Is this consistent with the phase portrait?

5. Do solutions of the system

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= x + x^3 - x^4 - x^5 - 3v\end{aligned}$$

   exist for all time for any set of initial conditions?

6. Linearize the system in the preceding problem at the origin and compute the associated eigenvalues. Is the origin a source or a sink?

7. Show that the solutions of the system

$$\begin{aligned}\dot{x} &= 3y + \frac{x}{1 + y^2 + z^2} \\ \dot{y} &= x + z + \sin yz \\ \dot{z} &= x + y + z + \frac{z}{2 + \cos xz}\end{aligned}$$

   exist for all time for any set of initial conditions.

8. Consider the Hopf example

$$\begin{aligned}\dot{x} &= -y + x\left(\mu - x^2 - y^2\right) \\ \dot{y} &= x + y\left(\mu - x^2 - y^2\right)\end{aligned}$$

   where μ is a real parameter. Calculate the linearization at the origin and the associated eigenvalues and show that the result is consistent with the phase portrait.

9. Let $(X(t,\mu), Y(t,\mu))$ be the solution of the Hopf equation in the preceding exercise with initial condition $(X(0,\mu), Y(0,\mu)) = (1,1)$ Find a differential equation that determines $\frac{\partial X}{\partial \mu}$ and $\frac{\partial Y}{\partial \mu}$.

10. Using the results of Proposition 1.3.10 in the class notes or otherwise, show the following for solutions of a smooth vector field X. If $\gamma(t)$ is a periodic orbit of X with period say τ, then for any $T > 0$, there is an $\epsilon > 0$ such that a trajectory with initial conditions a distance no more than ϵ from the periodic orbit exists for time at least T.