CDS 140a: Homework Set 4

Due: Friday, October 30st, 2009.

1. Show that if A is diagonalizable, then
$$\det e^A = e^{\operatorname{trace} A}$$
Try this out on a few nondiagonalizble matrices and make a conjecture as to its general validity.

2. Solve the system
$$\dot{x} = ax - by$$
$$\dot{y} = bx + ay$$
by using polar coordinates.

3. Find the stable, unstable and center subspaces for the system
$$\dot{x} = y$$
$$\dot{y} = 0$$
$$\dot{z} = 2x - z$$
and comment on the phase portrait.

4. Find the stable, unstable and center subspaces for the system
$$\dot{x} = y$$
$$\dot{y} = -x$$
$$\dot{z} = 2x + z$$
and comment on the phase portrait.

5. Does the one dimensional equation $\dot{x} = x^{1/3}$, $x(0) = 0$ have a unique solution $x(t)$ defined for t in some interval $(-\epsilon, \epsilon)$?

6. What happens when you apply Picard iteration to the linear system $\dot{x} = Ax$?

7. Use the local existence and uniqueness theorem to estimate the time of existence of the solution of the one dimensional equation $\dot{x} = x^2$, where $x(0) = 1$. What is the actual positive lifetime of the solution?

8. Consider the solution $(x(t, \omega), y(t, \omega))$ of the problem
$$\dot{x} = x - \omega y$$
$$\dot{y} = \omega x + y$$
with the initial condition $x(0, \omega) = 1$, $y(0, \omega) = 0$. Is the solution a smooth function of ω? Let $X = \partial x/\partial\omega$ and $Y = \partial y/\partial\omega$. What equation, with what initial condition, does the pair (X, Y) satisfy?

9. Consider the solution $(x(t,x_0), y(t,y_0))$ of the problem

$$\begin{aligned}\dot{x} &= x - y \\ \dot{y} &= x + y\end{aligned}$$

with the initial condition $x(0,x_0) = x_0$, $y(0,y_0) = y_0$. Is the solution a smooth function of (x_0, y_0)? Let $X = \partial x/\partial x_0$ and $Y = \partial y/\partial y_0$. What equation, with what initial condition, does the pair (X, Y) satisfy?

10. Let $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field on an open set containing the origin. Suppose that $X(0) = 0$. Let $T > 0$ be a given positive real number. Show that there is a ball B about the origin such that any initial condition $x_0 \in B$ has a positive lifetime that is at least T.