CDS 140a: Homework Set 3

Due: Friday, October 23th, 2009.

1. Solve the system

$$\dot{x} = x - y$$
$$\dot{y} = x + 3y$$

for given initial conditions (x_0, y_0).

2. Do all solutions of the system

$$\dot{x} = -x + y + z$$
$$\dot{y} = -y + 2z$$
$$\dot{z} = -2z$$

converge to the origin as $t \to \infty$?

3. Do all solutions of the system

$$\dot{x} = -x + y + z$$
$$\dot{y} = -y + 2z$$
$$\dot{z} = 2z$$

converge to the origin as $t \to \infty$?

4. Find the Jordan canonical form, the $S + N$ decomposition and the matrix exponential of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

5. Find the generalized eigenspaces of the matrix in the preceding problem and show directly that these subspaces are invariant under the equation $\dot{x} = Ax$ and span all of $\mathbb{R}^3$.

6. Find the Jordan canonical form, the $S + N$ decomposition and the matrix exponential of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

7. Find the generalized eigenspaces of the matrix in the preceding problem and show directly that these subspaces are invariant under the equation $\dot{x} = Ax$ and span all of $\mathbb{R}^4$.

8. Let A be a 3×3 (real) matrix.

   (a) If all the eigenvalues of A are zero, is it true that

   $$\exp(tA) = \mathrm{I} + tA + \frac{1}{2}t^2A^2 \ ?$$

   (b) Without giving details, explain how you could use (a) to solve the equations

   $$\begin{aligned} \frac{dx}{dt} &= -3x + y + 2z \\ \frac{dy}{dt} &= -4x + y + 3z \\ \frac{dz}{dt} &= -3x + y + 2z \end{aligned}$$

   with initial conditions $x(0) = 1, y(0) = 0, z(0) = 1$.

9. Consider the ordinary differential equation $\dot{x} = Ax$ where

   $$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -4 & 5 \\ 0 & 0 & -4 \end{bmatrix}.$$

   (a) Find the Jordan canonical form of the matrix A.

   (b) Compute the exponential e^{tA} in terms of the matrix P that brings A into Jordan canonical form (you don't need to compute P explicitly, but explain how you *would* compute it).

   (c) The positive orbit of x_0 under the flow generated by this differential equation is given by

   $$O(x_0) = \{x \mid x = e^{At}x_0, t \geq 0\}.$$

   For which $x_0 \in \mathbb{R}^3$ are the positive orbits bounded?

10. Let A be an $n \times n$ matrix, all of whose eigenvalues have positive real parts. Making use of appropriate results stated in class, show that for any initial condition $x_0 \in \mathbb{R}^n$, the solution of $\dot{x} = Ax$ tends to zero as $t \to -\infty$.