CDS 140a: Homework Set 2

Due: Friday, October 16th, 2009.

1. Verify directly that the homoclinic orbits for the simple pendulum equation $\ddot{\phi} + \sin\phi = 0$ are given by $\phi(t) = \pm 2\tan^{-1}(\sinh t)$.

2. Derive the same formula as in Exercise 1 using conservation of energy, determining the value of the energy on the homoclinic orbit, solving for $\dot{\phi}$ and then integrating.

3. Ignoring friction, the equations of motion for a particle in a hoop spinning about a line a distance $\epsilon > 0$ off center are given by

$$\ddot{\theta} = \omega^2 \left(\sin\theta - \frac{\epsilon}{R}\right)\cos\theta - \frac{g}{R}\sin\theta \,.$$

Show graphically that If ω is large enough, there will be four equilibrium points and if ω is small enough there are only two equilibrium points.

4. Find the general solution of

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and sketch the phase portrait.

5. Find the general solution of

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and sketch the phase portrait.

6. Find the general solution of

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and sketch the phase portrait.

7. Let T be an invertible $n \times n$ matrix and $\|T\|$ be its operator norm. Show that

$$\|T^{-1}\| \geq \frac{1}{\|T\|}$$

8. Let T be an $n \times n$ matrix satisfying $\|I - T\| < 1$. Show that T is invertible and that the series

$$\sum_{k=0}^{\infty} (I - T)^k$$

converges to T^{-1}.

9. Write the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

as the sum of two commuting matrices and use this to compute e^A.

10. Show that all solutions of the linear system

$$\begin{aligned} \dot{x} &= -2x - y \\ \dot{y} &= x - 2y \\ \dot{z} &= -z \end{aligned}$$

converge to the origin as $t \to \infty$.