Vector Calculus Solutions to Sample Final Examination #1

1. Let $f(x, y) = e^{xy} \sin(x + y)$.

   (a) In what direction, starting at $(0, \pi/2)$, is f changing the fastest?

   (b) In what directions starting at $(0, \pi/2)$ is f changing at 50% of its maximum rate?

   (c) Let $\mathbf{c}(t)$ be a flow line of $\mathbf{F} = \nabla f$ with $\mathbf{c}(0) = (0, \pi/2)$. Calculate

$$\frac{d}{dt}[f(c(t))]\bigg|_{t=0}.$$

Solution

(a) f is changing fastest in the direction of $\nabla f(0, \pi/2)$. But

$$\begin{aligned}\nabla f(x, y) &= [ye^{xy} \sin(x + y) + e^{xy} \cos(x + y)]\mathbf{i} \\ &\quad + [xe^{xy} \sin(x + y) + e^{xy} \cos(x + y)]\mathbf{j},\end{aligned}$$

and so

$$\nabla f(0, \pi/2) = \frac{\pi}{2}\mathbf{i}.$$

Thus f is increasing fastest in the direction $\mathbf{i}$ (and decreasing fastest in the direction -$\mathbf{i}$).

(b) If $\mathbf{n}$ is a unit vector, f is changing at the rate

$$\nabla f(0, \pi/2) \cdot \mathbf{n} = \frac{\pi}{2}\mathbf{n} \cdot \mathbf{i}$$

in the direction $\mathbf{n}$. The maximum value is $\pi/2$, so the rate is 50% of its maximum when

$$\frac{\pi}{2}\mathbf{n} \cdot \mathbf{i} = \frac{\pi}{2} \cdot \frac{1}{2}$$

i.e.,

$$\mathbf{n} \cdot \mathbf{i} = \frac{1}{2}$$

This means $\mathbf{n}$ makes an angle θ with $\mathbf{i}$ where $\cos\theta = 1/2$, or $\theta = \pm\pi/3$ or ± 60 degrees. Note that this defines two directions (if this were in space and not the plane, we would get a cone).

(c) By the chain rule, and since $\mathbf{c}'(t) = \nabla f(\mathbf{c}(t))$,

$$\begin{aligned} \left.\frac{d}{dt} f(\mathbf{c}(t))\right|_{t=0} &= \nabla f(\mathbf{c}(0)) \cdot \mathbf{c}'(0) \\ &= \nabla f\left(0, \frac{\pi}{2}\right) \cdot \nabla f\left(0, \frac{\pi}{2}\right) \\ &= \left\|\nabla f\left(0, \frac{\pi}{2}\right)\right\|^2 = \frac{\pi^2}{4}. \end{aligned}$$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a given mapping and write $f(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $g(u, v, w) = (u - v, u + w, w + v)$ and let $h = g \circ f$.

   (a) Write a formula for the derivative matrix $\mathbf{D}h$.

   (b) Show that $\mathbf{D}h$ cannot have rank 3 at any point (x, y, z).

   (c) Show that $\mathbf{D}h$ has an eigenvalue zero at every (x, y, z).

   Solution

   (a) By the chain rule,

$$\begin{aligned} Dh(x, y, z) &= Dg(u, v, w) \cdot Df(x, y, z) \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial x} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} & \frac{\partial u}{\partial z} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} & \frac{\partial u}{\partial y} - \frac{\partial w}{\partial y} & \frac{\partial u}{\partial z} - \frac{\partial w}{\partial z} \\ \frac{\partial v}{\partial x} - \frac{\partial w}{\partial x} & \frac{\partial v}{\partial y} - \frac{\partial w}{\partial y} & \frac{\partial \theta}{\partial z} - \frac{\partial w}{\partial z} \end{bmatrix} \end{aligned}$$

   (b) We claim that this matrix has determinant zero. Its determinant is the product of the determinants of the two factors. But

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

   Thus, $Dh(x, y, z)$ is not invertible, so it must have nullity $\neq 0$, so rank $Dh(x, y, z) \neq 3$.

   (c) Since nullity $\neq 0$ some non-zero vector must get sent to zero; this is an eigenvector with eigenvalue zero.

3. Extremize $f(x, y, z) = x$ subject to the constraints

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x + y + z = 1.$$

Solution We are to extremize $f(x, y, z) = x$ subject to

$$g_1 = x^2 + y^2 + z^2 - 1 = 0 \quad \text{and} \quad x + y + z - 1 = 0.$$

Using the method of Lagrange multipliers, this means

$$\begin{aligned} \nabla f &= \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 &= 0 \\ g_2 &= 0 \end{aligned}$$

i.e.,

$$\begin{aligned} 1 &= \lambda_1 \cdot \partial x + \lambda_2 && (1) \\ 0 &= \lambda_1 \cdot 2y + \lambda_2 && (2) \\ 0 &= \lambda_1 \cdot 2z + \lambda_2 && (3) \\ x^2 + y^2 + z^2 &= 1 && (4) \\ x + y + z &= 1. && (5) \\ & && (6) \end{aligned}$$

Subtracting (2) and (3) gives

$$2\lambda_1(y - z) = 0$$

and so either $\lambda_1 = 0$ or $y = z$. But $\lambda_1 = 0$ is not consistent with (1) and (2). Hence, $\lambda_1 \neq 0$ and so $y = z$. Thus we have

$$\begin{aligned} 1 &= \lambda_1 \cdot 2x + \lambda_2 && (7) \\ 0 &= \lambda_1 \cdot 2y + \lambda_2 && (8) \\ x^2 + 2y^2 &= 1 && (9) \\ x + 2y &= 1 && (10) \\ & && (11) \end{aligned}$$

Substituting (9) in (8) gives

$$(1 - 2y)^2 + 2y^2 = 1$$

i.e.,

$$1 - 4y + 4y^2 + 2y = 1$$

i.e.,

$$-2y + 3y^2 = 0.$$

Thus, either $y = 0$, or $y = 2/3$. If $y = 0$ then $x = 1$ (and $\lambda_2 = 0, \lambda_1 = 1/2$) and if $y = 2/3$ then $x = -1/3$.

Therefore, the solutions are

$$(1, 0, 0) \text{ and } (-1/3, 2/3, 2/3)$$

The former maximizes f while the latter minimizes it.

4. (a) Evaluate

$$\iiint_D \exp[(x^2 + y^2 + z^2)^{3/2}]\, dx\, dy\, dz$$

where D is the region defined by $1 \leq x^2 + y^2 + z^2 \leq 2$ and $z \geq 0$.

(b) Sketch or describe the region of integration for

$$\int_0^1 \int_0^x \int_0^y f(x, y, z) dz\, dy\, dx,$$

and interchange the order to $dy\, dx\, dz$.

Solution

(a) We use spherical coordinates

$$\begin{aligned} \int\int\int_D \exp[(x^2 + y^2 + z^2)^{3/2}] dxzdydz &= \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=1}^{\sqrt{2}} \exp(\rho^3) \times \rho^2 \sin\varphi d\rho d\theta d\varphi \\ &= 2\pi \int_{\varphi=0}^{\pi/2} \left(\int_{\rho=1}^{2} \exp(\rho^3)\rho^2 d\rho \right) \sin\varphi d\rho \\ &= 2\pi \times \frac{1}{3} \exp(\rho^3) \Big|_1^{\sqrt{2}} \times (-\cos\varphi) \Big|_0^{\pi/2} \\ &= \frac{2\pi}{3} (e^{\sqrt{8}} - e). \end{aligned}$$

(b) The region for

$$\int_0^1 \int_0^x \int_0^y f(x, y, z) dzdydx$$

is that under the plane $z = y$ and over the region in the plane bounded by the x-axis, the line $x = y$ and the line $x = 1$. (The student should draw a figure here).

(c) In the order $dydxdz$, the integral is easiest to write down by consulting the figure drawn in the previous part; one gets

$$\int_0^1 \int_z^1 \int_x^1 f(x,y,z)dydxdz.$$

5. Let $\mathbf{G}(x,y) = (xe^{x^2+y^2} + 2xy)\mathbf{i} + (ye^{x^2+y^2} + x^2)\mathbf{j}$.

   (a) Show that $\mathbf{G} = \nabla f$ for some f; find such an f.

   (b) Use (a) to show that the line integral of $\mathbf{G}$ around the edge of the triangle with vertices $(0,0),(0,1),(1,0)$ is zero.

   (c) State Green's theorem for the triangle in (b) and a vector field $\mathbf{F}$ and verify it for the vector field $\mathbf{G}$ above.

Solution

(a) If $\mathbf{G}(x,y) = P\mathbf{i} + Q\mathbf{j}, P = xe^{x^2+y^2} + 2xy, Q = ye^{x^2+y^2} + x^2$, note that

$$\frac{\partial P}{\partial y} = 2xye^{x^2+y^2} + 2x = \frac{\partial Q}{\partial x},$$

and so $\mathbf{G}$ is a gradient. Writing

$$P = \frac{\partial f}{\partial x}, \quad \text{and} \quad Q = \frac{\partial f}{\partial y}$$

we see that

$$f(x,y) = e^{x^2+y^2} + x^2 y.$$

(b) Let C be the boundary of the triangle T (the student should draw a figure of T.) Since the integral of a gradient around any closed curve is zero in general, it is zero in this particular case.

(c) Green's Theorem states, in this case that

$$\int_C Pdx + Qdy = \int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

We computed that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ above, so the right hand side is zero as well.

6. Let W be the three dimensional region under the graph of $f(x,y) = \exp(x^2+y^2)$ and over the region in the plane defined by $1 \le x^2 + y^2 \le 2$.

   (a) Find the volume of W.

   (b) Find the flux of the vector field $\mathbf{F} = (2x - xy)\mathbf{i} - y\mathbf{j} + yz\mathbf{k}$ out of the region W.

Solution

(a) The volume of W is

$$\begin{aligned}\iint_{1\le x^2+y^2\le 2} \exp(x^2+y^2)dxdy &= \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}} e^{r^2} r dr d\theta \\ &= 2\pi \times \frac{1}{2}(e^2-e) \\ &= \pi e(e-1).\end{aligned}$$

(b) The flux of $\mathbf{F}$ is, by Gauss' theorem

$$\begin{aligned}\iint_{\partial W} \mathbf{F}\cdot \mathbf{d}S &= \iiint_W div\mathbf{F} dxdydz \\ &= \iiint_W (2-y-1+y)dxdydz \\ &= \iiint_W dx\, dy\, dz = \pi(e^4-e).\end{aligned}$$

7. Let C be the curve $x^2+y^2=1$ lying in the plane $z=1$. Let $\mathbf{F}=(z-y)\mathbf{i}+y\mathbf{k}$.

(a) Calculate $\nabla \times \mathbf{F}$.

(b) Calculate $\int_C \mathbf{F}\cdot d\mathbf{s}$ using a parametrization of C and a chosen orientation for C.

(c) Write $C=\partial S$ for a suitably chosen surface S and, applying Stokes' theorem, verify your answer in (b) .

(d) Consider the sphere with radius $\sqrt{2}$ and center the origin. Let S' be the part of the sphere that is above the curve (*i.e.,* lies in the region $z \ge 1$), and has C as boundary. Evaluate the surface integral of $\nabla \times \mathbf{F}$ over S'. Specify the orientation you are using for S'.

Solution

(a) We evaluate the curl by writing out the expression for the curl as a cross product of ∇ and F:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & 0 & y \end{vmatrix} = \mathbf{i}+\mathbf{j}+\mathbf{k}$$

(b) Parameterize C by $x=\cos\theta, y=sin\theta, z=1, 0\le\theta\le 2\pi$, where the orientation is counter-clockwise as viewed from above. Then

$$\int_C \mathbf{F}\cdot \mathbf{d}s = \int_0^{2\pi}(1-\sin\theta)(-\sin\theta)d\theta = \int_0^{2\pi}\sin^2\theta d\theta = \pi$$

(since the average of $\sin^2\theta$ is $1/2$).

(c) Let us choose S to be the disk $x^2 + y^2 \leq 1, z = 1$. Then

$$\int_C \mathbf{F} \cdot \mathbf{d}s = \int\int_S \nabla \cdot \mathbf{F} \times dS = \int\int_S \mathbf{k} \cdot \mathbf{k} dx\, dy = \pi.$$

(d) Let the orientation of S'' be given by the outward normal. The student should draw a figure that shows $\partial S' = C$. Then,

$$\int\int_S^{''} (\nabla \cdot \mathbf{F}) \times dS = \int_C \mathbf{F} \times \mathbf{d}s = \pi.$$