Lectures on Geometric Methods in Mathematical Physics

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Preface

The topics selected for these lectures aim to illustrate some of the ways geometry and analysis can be used in mathematical problems of physical interest. A recurring theme is the role of symmetry, bifurcation and Hamiltonian systems in diverse applications. Despite the old age of these topics and the current explosion of interest and research, the state of knowledge is in my opinion still very primitive. For example, very little is known about dynamical systems that are close to a completely integrable Hamiltonian system. The simplest classical examples, such as the harmonic oscillator, Duffing's equation, the spherical pendulum, the rigid body and the equations of a perfect fluid, show that this program is both interesting and complex.

Symmetry is relevant because many examples are, or are close to, a Hamiltonian system which is invariant under some Lie group. Such dynamical systems are very sensitive to perturbations in their equations; or, if you wish, these models are qualitatively unstable. This means that they are bifurcation points in the set of all dynamical systems. The following paragraphs briefly describe how this theme occurs in the various lectures.

The first lecture introduces some basic ideas about Hamiltonian systems, concentrating on the infinite dimensional case. Besides some background notation and examples, we give two recent results of a technical nature that can be skipped without affecting subsequent lectures; these are a new version of Darboux's theorem and the issue of differentiability of the evolution operators.

Lectures 2, 6 and 7 deal with elasticity. Lecture 2 discusses the general theory, focusing on its Hamiltonian structure and the role of symmetry and covariance in the basic equations. While the set–up may seem new, the ideas are old and well known. The context, however, may be more mathematically appealing than previous treatments. Lecture 6 describes a bifurcation that occurs when a beam is subject to small forcing and damping. The unperturbed beam dynamics contains a homoclinic orbit, a signal that symmetry is present. The perturbed system is shown to contain the complex dynamics of a Smale horseshoe. Lecture 7 studies the bifurcations that occur when a stress–free body is subjected to small surface loads. When these loads have a certain symmetry (called an axis of equilibrium), then the solutions can bifurcate; *i.e.,* singularities occur in the space of all solutions.

Lectures 3, 4, 5 and 8 deal with Hamiltonian systems per se. Lecture 3 gives background on symmetry and conserved quantities in Hamiltonian systems. Lectures 4 and 5 give some illustrations of how this is used: symplectic splittings, action angle variables, simple mechanical systems, magnetic fields, particles in Yang–Mills fields are touched on in Lecture 4, and the Calogero system, the Kostant–Symes theorem and the Toda lattice are treated in Lecture 5. Lecture 8 studies in some detail the bifurcations that occur in level sets of the Noether conserved quantity associated with a symmetry group. Here an explicit connection between symmetry and bifurcation is clear: a bifurcation occurs precisely at points in phase space that themselves have symmetry, *i.e.,* are fixed under a nontrivial subgroup of the given symmetry group.

Finally, Lectures 9 and 10 describe applications to general relativity. Here bifurcations in the space of solutions of Einstein's equations are found to exist precisely at solutions with symmetry. This is done by writing Einstein's equations as a coupled system of Hamiltonian and constraint equations. The constraints are equivalent to setting the Noether conserved quantity associated with the group of diffeomorphisms of spacetime equal to zero. This Hamiltonian and symmetry (or covariance) structure enables one to exploit the ideas of Lecture 8.

All of the lectures are related to the conventional theory of Hamiltonian systems, *i.e.,* triples (P, ω, H) where P is a manifold called the *phase space*, ω is a closed nondegenerate two–form called the *symplectic form* and H is a real valued function on P called the *energy* or *Hamiltonian.* This situation can be generalized and modified in several ways, some of which we list below with selected recent references:

i the theory of singular Hamiltonian systems wherein H is a distribution; see Marsden (1968a) and Parker (1979).

ii Lagrangian submanifolds; see Weinstein (1971), Tulczyjew (1976) and Weinstein (1977).

iii Degenerate symplectic forms and the Dirac theory of constraints; see Künzle (1969), Tulczyjew (1974), Sniatycki (1974), Menzio and

Tulczyjew (1978) and Gotay et al. (1978).

iv Forms of higher degree and field theory; see Goldschmidt and Sternberg (1973), Kijowski (1974), García (1974), Szczyryba (1976), Kijowski and Tulczyjew (1979) and Kuperschmidt (1980).

v Superhamiltonian systems on graded manifolds; see Kostant (1978).

vi Systems for which one has a map of Hamiltonians to vector fields $H \mapsto X_H$ but no symplectic structure; see Kupershmidt and Manin (1977).

vii Deformations of symplectic structures and Poisson algebras; see Lichnerowicz (1980).

viii Connections with partial differential equations, Fourier integral operators and quantizations; see Souriau (1970), Hörmander (1971), Duistermaat (1973), Guillemin and Sternberg (1977) and Sniatycki (1980).

ix Connections with group representations; see Kostant (1970) and Kirillov (1976).

x Results on periodic orbits, celestial mechanics and bifurcation of Hamiltonian systems; see Siegal and Moser (1971), Abraham and Marsden (1978), Weinstein (1978a) and Kummer (1979).

In addition to these topics in Hamiltonian systems, there are large numbers of topics and a vast literature on bifurcation theory that we shall not treat (see Cohen and Neu (1979); some references are given in Marsden (1978)), on elasticity (see Truesdell and Noll (1965) and Marsden and Hughes (1978)) and general relativity (see Misner et al. (1973) and Fischer and Marsden (1979b),Fischer and Marsden (1979a)). In fact, we shall not even attempt to list all of the topics of current interest that are related to these lectures.

We require some acquaintance with functional analysis, manifolds, Lie groups and finite dimensional Hamiltonian systems. This background can be obtained from many standard sources such as Yosida (1971), Lang (1972), Abraham and Marsden (1978) and Arnold (1978). Some of my previous lectures (Ebin et al. (1972) and Marsden (1974)) on related subjects may still also be of interest to some readers, although various topics treated there are now out of date.

These lectures were originally given at the CBMS–NSF Regional Conference held in Lowell, Massachusetts, March 19–23, 1979. The second edition maintains the spirit and topics of the original lectures, focusing on updating the material to take into account selected recent developments in the area.

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1

Infinite Dimensional Hamiltonian Systems

Definition 1.1. *A **symplectic manifold** is a pair (P, ω), where P is a C^∞ Banach manifold and ω is a C^∞ two–form on P, such that*

i $d\omega = 0$*; and*

ii ω *is (weakly) nondegenerate: for all* $x \in P$ *and* $v_x \in T_x P$ *(the tangent space to* P *at* x*),*

$$\omega_x(v_x, w_x) = 0$$

for all $w_x \in T_x P$ *implies* $v_x = 0$.

We now make a series of remarks concerning this definition.

1. Define the bundle map $\omega^\flat : TP \to T^*P$ by

$$\omega^\flat_x(v_x) \cdot w_x = \omega_x(v_x, w_x).$$

   Condition ii in the definition is equivalent to $\omega^\flat$ being one–to–one. If $\omega^\flat$ is onto, we call (P, ω) a *strong* symplectic manifold. In finite dimensions the notions of symplectic manifold and strong symplectic manifold coincide, but in infinite dimensions many interesting examples are *not* strong symplectic manifolds, as we shall see.

2. Let (P, ω) be a symplectic manifold.

   (a) If $T_x P$ is a reflexive linear space, *i.e.,* the natural injection $i : T_x P \to T_x^{**} P, i(v_x) \cdot \alpha_x = \alpha_x(v_x)$ is onto, then $\omega^\flat_x$ has closed range if and only if $\omega_x^\flat$ is onto. [Indeed, let $Y \subset T_x^*P$ be the range of $\omega_x^\flat$, and assume Y is closed and $Y \neq T_x^*P$. By the Hahn–Banach theorem there is a $\phi \in T_x^{**}P$ such that $\phi \neq 0$ and $\phi(Y) = \{0\}$. Let $\phi = i(v_x)$. Then, for any $w_x \in T_xP$,

$$\omega_x(v_x, w_x) = -\omega_x^\flat(w_x) \cdot v_x = -\phi(\omega_x^\flat(\omega_x)) = 0.$$

Thus, $v_x = 0$ and so $\phi = 0$, a contradiction.]

(b) The argument in (a) shows that if T_xP is reflexive, then the range of $\omega_x^\flat$ is dense. Thus, a common situation will be that $\omega_x^\flat$ has a range that is dense in, but unequal to, T_x^*P.

3. (a) If Q is a Banach manifold, then T^*Q carries a canonical symplectic structure ω. The form ω can be given in three equivalent ways:

   i $\omega = -d\theta$, where the canonical one–form θ is defined by $\theta(\alpha_q) \cdot w = \alpha_q \cdot (T\tau_Q^* \cdot w)$; here $\alpha_q \in T_q^*Q, w \in T_{a_q}(T^*Q), \tau_Q^* : T^*Q \to Q$ is the canonical projection and $T\tau_Q^*$ is its tangent.

   ii In local representation,

$$\omega_{\alpha_q}((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_2 \cdot v_1 - \alpha_1 \cdot v_2.$$

   iii If β is a one–form on Q, so that $\beta : Q \to T^*Q$, then $\beta^*\theta = \beta$ defines the canonical one–form θ and $\omega = -d\theta$.

Consult Abraham and Marsden (1978), pp. 178–9 for the proof of the equivalence of i, ii and iii. In finite dimensions, $\theta = \sum_{i=1}^n p_i dq^i$ and $\omega = \sum_{i=1}^n dq^i \wedge dp_i$ relative to the natural coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$.

(b) It is readily checked that Q is reflexive if and only if the canonical symplectic structure is a strong symplectic structure (Marsden (1968b)).

(c) By a *weak Riemannian metric* on a manifold Q we mean a smooth symmetric 2–covariant tensor $\langle \cdot, \cdot \rangle$ on Q such that $\langle \cdot, \cdot \rangle_q$ is a (not necessarily complete) inner product at each $q \in Q$. A weak Riemannian metric determines a smooth bundle map $\psi : TQ \to T^*Q$ by $\psi_q(v_q) \cdot w_q = \langle v_q, w_q \rangle_q$. Then $\Omega = \psi^*\omega$ is a symplectic structure on TQ, where ω is the canonical symplectic structure on T^*Q.

4. Reduction provides an important means of constructing symplectic forms. We shall use this procedure later. The general context is as follows. Let M be a Banach manifold and ω_M a closed two–form. Define its *characteristic bundle* by

$$E_x = \{v_x \in T_xM | \omega_M^\flat(v_x) = 0\}.$$

Assume that E is a smooth (split) subbundle of TM.[1] Then E is integrable; *i.e.,* if X and Y are two sections of E, so is $[X, Y]$. To see this, use the identity

$$i_{[X,Y]}\omega_M = L_X i_Y \omega_M - i_Y L_X \omega_M$$

to get

$$i_{[X,Y]}\omega_M = -i_Y(i_X d\omega_M + di_X\omega_M) = 0.$$

Frobenius' theorem implies that locally there is a foliation $\mathcal{F}$ whose tangent bundle is E. Locally, $M/\mathcal{F}$ is a manifold which has a unique symplectic form $\omega_{\mathcal{F}}$ such that $\omega_M = \pi^*_{\mathcal{F}}\omega_{\mathcal{F}}$, where $\pi_{\mathcal{F}} : M \to M/\mathcal{F}$ is the projection.

Often one starts with a symplectic manifold (P, ω) and constructs a submanifold $M \subset P$ and restricts ω to M; *i.e.,* we set $\omega_M = i^*\omega$, where $i : M \to P$ is inclusion. If f and g are functions on P, we can restrict them to M. If f and g are constant on leaves, they induce functions $\hat{f}, \hat{g}$ on $M/\mathcal{F}$. Thus $\{\hat{f}, \hat{g}\}$ is defined. However, $\{\hat{f}, \hat{g}\}$ need not be $\{f, g\}\hat{}$. There is, however, an important case when the brackets are related, namely when M is coisotropic, *i.e.,* when $(TM)^{\perp} \subset TM$, where $(TM)^{\perp}$ is the ω–orthogonal complement. In this case $\mathcal{F}$ is the foliation of the distribution $(TM)^{\perp}$, and if f and g are defined on P and are constant on leaves of $\mathcal{F}$, then so is $\{f, g\}$, and $\{f, g\}\hat{} = \{\hat{f}, \hat{g}\}$. This is readily verified. In Lectures 3 and 4 we shall develop a particular case of this procedure.

5. For functional analytic reasons, one often needs to restrict a strong symplectic form to a dense subspace with a different topology. This normally results in a (weak) symplectic structure. We formalize this idea as follows.[2] Let M and N be Banach manifolds with $N \subset M$ (as sets), and let $i : N \to M$ denote the inclusion map. We say that $N \subset M$ is a *manifold domain* if:

   i $i : N \to M$ is a C^∞ map.

   ii For each $x \in N, T_x i : T_x N \to T_x M$ is an injection of $T_x N$ onto a dense subspace of $T_x M$.

   iii Let M and N be modeled on the Banach spaces X and Y and let $Y \subset X$ be included as a dense subspace. For each point $x_0 \in N$

[1] Roughly speaking, this means that ω_x has locally constant rank. Some techniques useful in the infinite dimensional case are found in Lang (1972) and Ebin and Marsden (1970), Appendix A.

[2] This discussion is adapted from Marsden (1968b) and Chernoff and Marsden (1974). It is related to the idea of a "varieté bimodelé" of Chevallier (1975). The linear example $H^s \subset L^2$ may help fix ideas.

there are charts $\phi : U \subset M \to X$ and $\psi : V \subset N \to Y$ about x such that $V = i^{-1}(U)$ and $(\phi \circ i)|V = \psi$. (*i.e.,* M and N can be "simultaneously flattened").

For example, if A and B are compact manifolds (and B has no boundary), then the manifold of maps $H^{s+k}(A, B) = \{f : A \to B | f$ is of Sobolev class $H^{s+k}\}$ is a manifold domain in $H^s(A, B)$; the usual exponential charts that produce their manifold structure simultaneously flatten these two manifolds (see Palais (1968) for the necessary machinery; some may find the summary in Ebin and Marsden (1970) helpful).

If (P_0, ω_0) is a symplectic manifold and $P \subset P_0$ is a manifold domain, then $i^*\omega_0 = \omega$ is a symplectic form on P. Even if ω_0 is strong, ω will in general only be weak.

6. Now we discuss canonical forms (see [Abraham and Marsden (1978), p. 173], Chernoff and Marsden (1974) and Weinstein (1977) for further information).

   (a) If (P, ω) is a symplectic manifold and P is a Hilbert manifold, then P carries an almost complex structure, *i.e.,* a smooth bundle map $\mathbb{J} : TP \to TP$ such that $\mathbb{J}^2 = -I$. Moreover, $\mathbb{J}$ is symplectic and $\langle\langle v_x, w_x \rangle\rangle = \omega_x(v_x, \mathbb{J}w_x)$ is a weak Riemannian structure. The construction is as follows: write $\omega_x(v_x, w_x) = \langle A_x v_x, w_x \rangle_x$, where $\langle \cdot, \cdot \rangle_x$ is the Hilbert inner product, and let $\mathbb{J}_x = A_x(-A_x^2)^{-1/2}$.

   (b) *Example* Let $P_0 = L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with the symplectic structure ω_0 induced from the canonical symplectic structure on $L^2(\mathbb{R}) \times [L^2(\mathbb{R})]^*$ via the L^2–metric:

$$\omega_0((f_1, g_1), (f_2, g_2)) = \langle g_2, f_1 \rangle - \langle g_1, f_2 \rangle.$$

   Here $\mathbb{J}$ is the standard complex structure associated with the identification of P_0 with the complex Hilbert space $L^2(\mathbb{R}, \mathbb{C})$: $\mathbb{J}(f, g) = (-g, f)$. Here $\langle\langle (f_1, g_1), (f_2, g_2) \rangle\rangle = \langle f_1, f_2 \rangle_{L^2} \langle g_1, g_2 \rangle_{L_2}$. Next, let $P = H^1(\mathbb{R}) \times L^2(\mathbb{R})$, the phase space associated with the linear wave equation [3] with ω on P given by the same formula; *i.e.,* regard $P \subset P_0$ as a manifold domain and set $\omega = i^*\omega_0$. Associated with the Hilbert manifold structure of P, a short computation gives

$$\mathbb{J}(f, g) = (-(1 - \Delta)^{-1/2} g, (1 - \Delta)^{1/2} f)$$

[3] This example is discussed in Chernoff and Marsden (1974).

and

$$\langle\langle (f_1, g_1), (f_2, g_2) \rangle\rangle = \langle f_1, f_2 \rangle_{L^2} + \langle g_1, g_2 \rangle_{L^2}.$$

where Δ is the Laplace operator and $H^1(\mathbb{R})$ has the inner product

$$\langle f, g \rangle_{H^1} = \langle (1-\Delta)^{1/2} f, (1-\Delta)^{1/2} g \rangle_{L^2} = \langle (1-\Delta) f, g \rangle.$$

(c) In this example note that the weak Riemannian structure restricts naturally from P_0 to P but the complex structure does not. The same argument shows that *if* (P_0, ω_0) *is a symplectic manifold and* $i : P \subset P_0$ *is a manifold domain, if* $\omega = i^*\omega_0$, *and if* P_0 *and* P *are both Hilbert manifolds, then the weak Riemannian structure constructed above on* P *is the restriction of that constructed on* P_0. In this sense of manifold domains, then, the weak Riemannian structure is canonically associated with the symplectic structure and is independent of the Hilbert space structure. If this weak Riemannian structure happens to be complete on P_0, then a special complex structure $\mathbb{J}_0 : TP_0 \to TP_0$ is picked out, but note that $\mathbb{J}_0$ will not, in general, map TP to itself.

(d) Suppose Q is a weak Riemannian manifold with an associated smooth connection. Then TQ becomes a symplectic manifold and also carries a weak Riemannian structure $\langle\langle \cdot, \cdot \rangle\rangle$ (by declaring the horizontal and vertical projections to be orthogonal). By completing each tangent space in the weak tangent bundles $\widetilde{T}Q$ and $\widetilde{T}(TQ)$. Then there is complex structure $\mathbb{J}$ on the bundle $\widetilde{T}(TQ)$ such that $\omega, \mathbb{J}$ and $\langle\langle \cdot, \cdot \rangle\rangle$ stand in the correct relationship (in finite dimensions this complex structure is $\dot{q}^k + i\ddot{q}^k$).

Now we turn our attention to Darboux's theorem: "One can always find coordinates in which a symplectic form is constant and therefore after a further *linear* change of coordinates may be put in canonical form." The original proof of Darboux was by induction on the dimension; see, for example, Sternberg (1963). However, we now have available the well–known proof of Moser and Weinstein that also works for *strong* symplectic forms. This proof may be found in Abraham and Marsden (1978), p. 175, and further information is found in Weinstein (1977). Roughly speaking, Darboux's theorem puts the q's and p's into canonical form suitable for quantization; for some systems defined by reduction, this is a nontrivial and useful fact.

It is therefore of interest to investigate the validity of Darboux's theorem for (weak) symplectic forms. Without further hypotheses, the result is not true (Marsden (1972)). There is a version due to Tromba (1976), valid for a certain class of weak symplectic forms, but it is not appropriate for all of those arising in examples. We present the following version of Darboux's theorem.

Theorem 1.2. *Let (P, ω) be a (weak) symplectic manifold. Assume that there is a weak Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ on P and a complex structure $\mathbb{J} : TP \to TP$ (see Remark 6 above) satisfying the following conditions:*

i $\omega_x(v_x, w_x) = \langle\langle \mathbb{J}_x \cdot v_x, w_x \rangle\rangle$.

ii $\langle\langle \cdot, \cdot \rangle\rangle$ *has a smooth Riemannian connection.*

iii the Christoffel map (explained below) $\Gamma_x : T_xP \times T_xP \to T_xP$ *has a smooth* $\langle\langle \cdot, \cdot \rangle\rangle$*–adjoint; i.e., there is a smooth vector bundle map* $A : TP \times TP \to TP$ *such that*

$$\langle\langle \Gamma_x(v_x, w_x), t_x \rangle\rangle = \langle\langle v_x, A_x(w_x, t_x) \rangle\rangle_x.$$

Then there is a chart about each point $x_0 \in P$ in which ω is constant.

Remarks 1. In ii we require a torsion–free affine connection ∇ such that, for smooth vector fields U, V, W, on P,

(a) $U(\langle\langle V, W \rangle\rangle) = \langle\langle \nabla_U V, W \rangle\rangle + \langle\langle V, \nabla_U W \rangle\rangle$ and
(b) $\nabla_U V - \nabla_V U = [U, V]$.

The *connector* of ∇ is the bundle map $K : T^2P \to TP$ such that $\nabla_V W = K \circ TW \circ V$; cf. Dombrowski (1962). The *Christoffel map* $\Gamma : TP \times TP \to TP$ is defined by $\Gamma_x(v_x, w_x) = K([w_x]^h_{v_x}) = K([v_x]^h_{w_x})$, where $[w_x]^h_{v_x} \in T^2P$ is the horizontal lift of w_x to an element of $T_{v_x}(TP)$. (In finite dimensions, $\Gamma(v_x, w_x) = x^i, \Gamma^i_{jk} w^j v^k)$, where Γ^i_{jk} are the usual Christoffel symbols.) Since ∇ is torsion free, Γ is symmetric. Notice that if T_xP is completed relative to $\langle\langle \cdot, \cdot \rangle\rangle_x$ then Γ_x will have an adjoint automatically; condition iii requires that the adjoint maps $T_xP \times T_xP$ to T_xP.

2. For Hilbert manifolds with strong symplectic forms, $\langle\langle \cdot, \cdot \rangle\rangle$ is a strong metric, so the hypotheses regarding it are automatic. Thus in this case the theorem reduces to the Moser–Weinstein theorem.

   It is rather remarkable that most weak Riemannian metrics arising in examples *do* have smooth connections or, equivalently, smooth geodesic flows. The main examples we have in mind are Ebin (1970), Ebin and Marsden (1970) and Tromba (1977). (Of course, many examples occur for which P is already linear and $\langle\langle \cdot, \cdot \rangle\rangle$ is an inner product—it obviously has a smooth connection in that case.)

3. There is a related Morse lemma inspired by Palais (1968) and Tromba (1976). Let $f : M \to \mathbb{R}$ be C^2 and suppose $f(x_0) = 0, df(x_0) = 0$, and $d^2f(x_0)$ is weakly nondegenerate. Assume that M has a weak

Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ that has a smooth connection. Assume f has a C^1 gradient ∇f relative to $\langle\langle\cdot,\cdot\rangle\rangle$; *i.e.,* $\langle\langle\nabla f(x), v_x\rangle\rangle_x = df(x)\cdot v_x$ such that $D\nabla f(x_0) : T_{x_0}M \to T_{x_0}M$ is an isomorphism. Then there is a local chart in which f is quadratic.

We refer to Tromba (1977) and Choquet-Bruhat et al. (1979) for some nontrivial examples. This Morse lemma is useful in discussions of elastic stability and bifurcations as well; see Ball et al. (1978), Knops and Payne (1978) and Buchner and Schecter (1980).

4. Generalizations of the Darboux theorem (found in Weinstein (1977) and Abraham and Marsden (1978)) may be proved by similar methods.

♦

Proof of Darboux's theorem. Since the result is local, we can assume P is a Banach space X and that $x_0 = 0 \in X$. Let ω_1 be the constant symplectic form equaling $\omega(0)$. Let $\tilde{\omega} = \omega_1 - \omega$ and $\omega_t = \omega + t\tilde{\omega}, 0 \le t \le 1$. By the (proof of the) Poincaré lemma,

$$\tilde{\omega} = d\alpha \,\text{where}\, \alpha(x)\cdot y = \int_0^1 s\tilde{\omega}(sx)\cdot(x,y)ds.$$

We seek a smooth vector field Y_t such that $i_{Y_t}\omega_t = -\alpha$. Note that $\alpha(0) = 0$, so $Y_t(0) = 0$ as well. If this is done, the usual Weinstein–Moser proof will complete the argument; *i.e.,* if F_t denotes the evolution operator for Y_t with $F_0 =$ identity, then

$$\frac{d}{dt}F_t^*\omega_t = F_t^*(L_{Y_t}\omega_t) + F_t^*\frac{d}{dt}\omega_t = F_t^*(-d\alpha + \tilde{\omega}) = 0,$$

so $F_1^*\omega_1 = \omega$, and F_1^{-1} is the desired coordinate change. To show that Y_t exists, we write its defining equation as

$$(1-t)\omega(x)(Y_t(x), y) + t\omega(0)(Y_t(x), y) = -\alpha(x)\cdot y.$$

By condition i of the theorem, this is equivalent to

$$\begin{aligned}&(1-t)\langle\langle \mathbb{J}_x\cdot Y_t(x), y\rangle\rangle_x + t\langle\langle \mathbb{J}_0\cdot Y_t(x), y\rangle\rangle_0\\ &= \int_0^1 s\langle\langle \mathbb{J}_{sx}\cdot x, y\rangle\rangle_{sx}dx - \int_0^1 s\langle\langle \mathbb{J}_0\cdot x, y\rangle\rangle_0 ds.\end{aligned}$$ ■

Lemma 1.3. *There is an operator $B(x) : X \to X$, depending smoothly on x, such that, for each $u, v \in X$,*

$$\langle\langle u, v\rangle\rangle_x = \langle\langle B(x)u, v\rangle\rangle_0.$$

Proof. We derive a differential equation for $B(sx)$. We have

$$\begin{aligned}\frac{d}{ds}\langle\langle u,v\rangle\rangle_{sx} =&\langle\langle\Gamma_{sx}(x,u),v\rangle\rangle_{sx}+\langle\langle u,\Gamma_{sx}(x,v)\rangle\rangle_{sx}\\ =&\langle\langle\Gamma_{sx}(x,u)+A_{sx}(x,u),v\rangle\rangle_{sx}\\ =&\langle\langle B(sx)(\Gamma_{sx}(x,u)+A_{sx}(x,u)),v\rangle\rangle_0.\end{aligned}$$

Thus we should have

$$\frac{d}{ds}B(sx)=B(sx)\circ\Gamma_{sx}(x,\cdot)+A_{sx}(x,\cdot).$$

This equation has a unique solution with $B(0)=\mathrm{Id}$, and serves to construct $B(x)$. ■

Proof. Continuing with the proof of Darboux's theorem, we apply the lemma to the equation preceding it:

$$\begin{aligned}&(1-t)\langle\langle B(x)\cdot\mathbb{J}_x\cdot Y_t(x),y\rangle\rangle_0+t\langle\langle\mathbb{J}_0\cdot Y_t(x),y\rangle\rangle_0\\ &\quad=\int_0^1 s[\langle\langle B(sx)\cdot\mathbb{J}_{sx}\cdot x,y\rangle\rangle_0-\langle\langle\mathbb{J}_0\cdot x,y\rangle\rangle_0]\,ds.\end{aligned}$$

Thus

$$[(1-t)B(x)\circ\mathbb{J}_x+t\mathbb{J}_0]\cdot Y_t(x)=\int_0^1 s[B(sx)\cdot\mathbb{J}_{sx}x-\mathbb{J}_0\cdot x]\,ds.$$

For $x=0,(1-t)B(x)\cdot\mathbb{J}_x+t\mathbb{J}_0$ is the identity, so in a neighborhood of 0 it is invertible. Thus Y_t may be defined by

$$Y_t(x)=[(1-t)B(x)\circ\mathbb{J}_x+t\mathbb{J}_0]^{-1}\int_0^1 s[B(sx)\cdot\mathbb{J}_{sx}\cdot x-\mathbb{J}_0\cdot x]\,ds.$$

This therefore has the desired properties. ■

Now we turn to Hamiltonian systems.

Definition 1.4. *Let (P,ω) be a (weak) symplectic manifold and $H:D_H\to\mathbb{R}$ a C^1 function, where D_H is a manifold domain in P. We call the triple (P,ω,H) a Hamiltonian system. Set*

$$D_{X_H}=\{x\in D_H|dH(x)\in\mathrm{range}(i_x^*\circ\omega_x^\flat)\},$$

where $\omega_x^\flat:T_xP\to T_x^*P$ is defined as earlier and $i_x^*:T_x^*P\to T_x^*D_H$ is restriction, the dual of the inclusion map $i_x:T_xD_H\hookrightarrow T_xP$. Define

$$X_H:D_{X_H}\to TP|D_{X_H},\quad X_H(x)=(i_x^*\circ\omega_x^\flat)^{-1}dH,$$

and call X_H the Hamiltonian generator, or Hamiltonian vector field of H.

Notice that

$$\begin{aligned} D_{X_H} = &\{x \in D_H \mid \text{there exists } v_x \in T_xP \text{ such that} \\ &dH(x) \cdot w_x = \omega_x(v_x, w_x) \text{ for all } w_x \in T_xD_H\}, \end{aligned}$$

and $X_H(x) = v_x$. Also note that for $x \in D_{X_H}$, $dH(x)$ extends to T_x^*P.

Remarks and Examples 1. In finite dimensions we usually choose $D_H = P$, so that $D_{X_H} = P$ as well. In canonical (Darboux) coordinates, for which $\omega = \sum_{i=1}^n dq^i \wedge dp_i$, we have

$$X_H(q,p) = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i} \right).$$

2. (a) *Klein–Gordon equation.* Let $Q = L^2(\mathbb{R}^3)$ and $P_0 = TQ$, with symplectic structure induced from the L^2–metric (see Remark 6(b) following the definition of symplectic manifold). The energy associated with the Klein–Gordon equation $\partial^2\phi/\partial t^2 = \Delta\phi - m^2\phi$ is

$$H(\phi, \dot{\phi}) = \frac{1}{2}\left\langle \dot{\phi}, \dot{\phi} \right\rangle + \frac{1}{2}\langle \nabla\phi, \nabla\phi\rangle + \frac{m^2}{2}\langle \phi, \phi\rangle,$$

whose natural domain of definition is $H^1 \times L^2 = D_H$. In this example we choose $P = D_H$ with the symplectic structure pulled back from P_0. A straightforward calculation shows that $D_{X_H} = H^2 \times H^1$ and that $X_H(\phi, \dot{\phi}) \times (\dot{\phi}, \Delta\phi - m^2\phi)$.

(b) *Schrödinger equation.* Let $P = \mathcal{H}$ be a complex Hilbert space, $\omega(\phi, \psi) = -\operatorname{Im}\langle \phi, \psi\rangle, H_{OP}$ a given self–adjoint operator in $\mathcal{H}$ with domain $D_{H_{OP}}$, and $H(x) = 1/2\langle H_{OP}x, x\rangle$, the associated quadratic form (with domain $D_{\sqrt{H_{OP}}}$ if $H_{OP} \geq 0$). A straightforward calculation shows that $D_{X_H} = D_{H_{OP}}$ and $X_H(\phi) = iH_{OP}\phi$ corresponding to Schrödinger's equation $\partial\phi/\partial t = iH_{OP}\phi$. In a classical, *i.e.,* nonquantum, situation like (a), the choice $P = D_H$ is in agreement with the requirement that each "observable" state ought to have finite energy, and also is in agreement with the mathematics: in P the equations are well–posed, whereas in P_0 they are not. On the other hand, in (b) states of finite energy are not directly observable; rather it is the transition probabilities $|\langle \psi, \phi\rangle|^2$ that are observable, corresponding to the choice $P = \mathcal{H}$. By Stone's theorem, this choice is compatible with the dynamics.

(c) For a fairly complete discussion of the linear theory, including an abstract existence theorem, see Chernoff and Marsden (1974).

3. Some simple nonlinear examples are discussed in Chernoff and Marsden (1974). In these lectures we shall consider two fairly substantial examples, elasticity in Lecture 2 and general relativity in Lectures 9 and 10. See Abraham and Marsden (1978) for fluids and the KdV equation.

4. Given a symplectic manifold (P, ω) and two functions $f : D_f \subset P \to \mathbb{R}, g : D_g \subset P \to \mathbb{R}$, their *Poisson bracket* is defined on $D_{X_f} \cap D_{X_g}$ by the usual formula

$$\{f, g\}(x) = \omega_x(X_f(x), X_g(x)).$$

♦

Now we turn to the dynamics.

Definition 1.5. *Let P be a Banach manifold and $D \subset P$ be a manifold domain. Let $G : D \to TP$ be a vector field with domain D. By a semiflow for G we mean a map $F : R \subset D \times [0, \infty) \to D$ where $R \subset D \times [0, \infty)$ is open, with the following properties:*

i *F is continuous.*

ii *$D \times \{0\} \subset R$ and $F(x, 0) = x$ for all $x \in D$.*

iii *Let $t, s \geq 0$ and $x \in D$. Then*

$$(x, t + s) \in R \Leftrightarrow (x, s) \in R \,\text{and}(F(x, s), t) \in R.$$

In this case, $F(x, t + s) = F(F(x, s), t)$.

iv *For $t \geq 0$,*

$$\frac{d}{dt}(i \circ F(x, t)) = G(F(xt))$$

where $i : D \to P$ is the inclusion (the derivative is from the right at $t = 0$). We shall write $F_t(x)$ for $F(x, t)$.

Remarks. 1. The association of a unique F with a given G is an existence and uniqueness theorem. For example, in the context of elasticity or general relativity, this is proved in Hughes et al. (1977).

2. Separate continuity in x and t sometimes implies joint continuity; see Chernoff and Marsden (1974), Ball (1974) and Chernoff (1975).

3. If the definition is modified to allow $t \leq 0$ as well as $t \geq 0$, one speaks of a *flow*. This corresponds to reversibility, and may depend on the choice of spaces.[4]

♦

The next theorem is due to Chernoff and Marsden (1974).

Theorem 1.6. *Let (P, ω, H) be a Hamiltonian system and let $K : D_K \to \mathbb{R}$ be a C^1 function. Assume:*

i X_H has a semiflow F_t.

ii $D_{X_H} \subset P$ is a manifold domain.

iii $D_{X_K} \supset D_{X_H}$ and $X_K : D_{X_H} \to TP$ is continuous. Then, for each $x_0 \in D_{X_H}$ and $t > 0, (x_0, t) \in R$ (the domain of the flow),

$$\frac{d}{dt} K(F_t(x_0)) = \{K, H\}(F_t(x_0)).$$

Proof. We can work in a simultaneous chart for $D = D_{X_H}$ and P, reducing ourselves to the Banach space case, and can assume that $F_t(x_0) = 0$. The restriction $K|D$ is C^1,

$$dK(x) \cdot v = \omega_x(X_K(x), v), \quad x, v \in D,$$

and so

$$K(x) = K(0) + \int_0^1 \omega_{sx}(X_K(sx), x)\, ds.$$

Thus,

$$\frac{K(F_{t+h}(x_0)) - K(F_t(x_0))}{h} = \int_0^1 \omega_{sF_{t+h}}(\alpha_0)\left(X_K(sF_{t+h}(x_0)), \frac{F_{t+h}(x_0) - F_t(x_0)}{h}\right) ds.$$

As $h \to 0$, the integrand above converges uniformly in s to

$$\omega_{sF_t}(x_0)(X_K(sF_t(x_0)), X_H(F_t(x_0)) = \omega_0(X_K(0), X_H(0)) = \{K, H\}(0),$$

and so

$$\lim_{h \to 0} \frac{K(F_{t+h}(x_0)) - K(F_t(x_0))}{h} = \{K, H\}(F_t(x_0)),$$

as required. ■

[4]For instance, in the KdV equation one has a flow in $H^s, s \geq 3$ but only a semiflow in the spaces of Kato (1979).

Remarks. 1. One cannot simply use the chain rule to differentiate $K(F_t(x_0))$, since $F_t(x_0)$ need not be t–differentiable in the topology of D_K.

2. Taking $K = H$, we see that energy is conserved under rather weak hypotheses. This may be relevant in discussing the Hamiltonian structure's compatibility with the development of shocks in elasticity.

Results which are infinite dimensional analogues of "the flow consists of canonical transformations" require more care. Indeed, if $F_t(x)$ is only continuous, the assertion $F_t^*\omega = \omega$ does not make sense. For linear systems or smooth perturbations of linear systems (semilinear systems such as $\partial^2\phi/\partial t^2 = \Delta\phi + m^2\phi + F(\phi)), F_t : D \to D$ will be a smooth map and the justification of $F_t^*\omega = \omega$ is not difficult (see Segal (1962), Segal (1965) and Chernoff and Marsden (1974).) However, for quasilinear systems occurring in elasticity and general relativity, the situation is more delicate. We shall just sketch a few ideas, referring to Dorroh and Graff (????) and Hughes and Marsden (1978) for details and refinements. ♦

Definition 1.7. *Let X and Y be Banach spaces with $Y \hookrightarrow X$ continuously and densely included. A map $G : Y \to X$ is called generator differentiable if it is Fréchet differentiable and its derivative satisfies*

i

$$\lim_{||h||_Y \to 0} \frac{||G(x+h) - G(x) - DG(x) \cdot h||_x}{||h||_x} = 0$$

and

ii $||G(x+h) - G(x) - DG(x) \cdot h||_X / ||h||_X$ *is locally bounded in $x \in Y$ and $h \in Y$.*

A map $F : Y \to X$ is called *flow–differentiable* if it is generator differentiable and, moreover, $DF(x)$ extends to a bounded linear operator of X to X for each $x \in Y$.

Discussion

1. The concept of generator differentiable is useful because one can check that, on appropriate function spaces, nonlinear differential operators satisfy it.

2. The main result states that if G is generator differentiable and has a semiflow F_t and if the variational equations $DG(F_t(x))$ for x fixed have associated evolution operators (satisfying reasonable technical conditions) then F_t is flow–differentiable for each t.

3. For equations of the types in Hughes et al. (1977), these hypotheses can be checked.

4. The results make sense in the context of general manifold domains $N \hookrightarrow M$.

5. For flow–differentiable flows F_t, one can verify that the usual Lie derivative formalism
$$\frac{d}{dt}F_t^*\alpha = F_t^* L_G \alpha$$
makes sense and is true (combine the differential calculus in Dorroh and Graff (????) with the calculations in Chernoff and Marsden (1974)). In particular, for Hamiltonian systems, $F_t^*\omega = \omega$ holds.

6. A satisfactory infinite dimensional version of Liouville's theorem is not known (to the author's knowledge). The analogue of phase volume $\mu = \omega^n$ in finite dimensions is undoubtedly a Wiener–type measure. The work of Segal (1967) and Eells and Elworthy (see Eells (1972)) is relevant here.

7. One drawback to the concept of flow–differentiable maps is that it probably is not strong enough to yield the existence of invariant manifolds that are useful (perhaps necessary) in qualitative theory. To obtain this one may need $F_t : Y \to Y$ to be differentiable from Y to Y. See Marsden and Mccracken (1976) for conditions under which this is true (the easiest case is that of Segal (1962) mentioned earlier; see also Holmes and Marsden (1978)). For the KdV equation, see Ratiu (1979).

2

Elasticity as a Hamiltonian System

This lecture[1] describes *some* of the mathematics of nonlinear elastodynamics. For information on the mathematical and physical motivation, consult Truesdell and Noll (1965), Rivlin (1966), Malvern (1969) or Marsden and Hughes (1978).

Let $\mathcal{B}$ be a compact oriented smooth n–manifold, possibly with a boundary, and $\mathcal{S}$ a smooth oriented boundaryless m–manifold. We call $\mathcal{B}$ the *body* and $\mathcal{S}$ the (ambient) *space*. The *configuration space* consists of all *deformations* of $\mathcal{B}$ in $\mathcal{S}$; that is, all embeddings[2] $\phi : \mathcal{B} \to \mathcal{S}$. Write $\mathcal{C}^{s,p}$ for the embeddings of class $W^{s,p}$. (See Figure 2.1) Although consideration of nonsmooth configurations is important in elasticity (cf. Ball (1977)), we shall only consider ones that are at least C^1.

Remarks on $\mathcal{C}^{s,p}$

1. $\mathcal{C}^{s,p}$ is a smooth manifold modeled on $W^{s,p}(\mathbb{R}^n, \mathbb{R}^s)$ if $s > (n/p) + 1$ (see, e.g., Palais (1968)). The C^∞ deformations shall be denoted $\mathcal{C}$.

---

[1] This section is an outgrowth of joint work done with T.J.R. Hughes. Comments from a number of people, especially S. Antman, J. Ball and M. Gurtin, have been most helpful.

[2] Physically we should only ask for immersions that need not be one–to–one on $\partial\mathcal{B}$, to allow, for example, contact due to folding, but this is technically difficult.

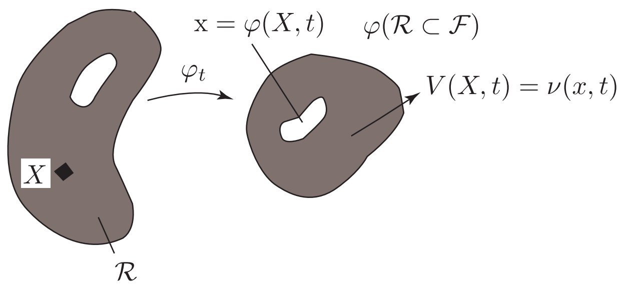

FIGURE 2.1.

2. The tangent space to $\mathcal{C}^{s,p}$ at ϕ is given by

$$\begin{aligned} T_\phi \mathcal{C}^{s,p} = &\{V : \mathcal{B} \to T\mathcal{S} | V \text{ is of class } W^{s,p} \text{ and for all} \\ &X \in \mathcal{B}, V(X) \in T_x\mathcal{S} \,\text{where}\, x = \phi(X)\}. \end{aligned}$$

3. A *motion* of $\mathcal{B}$ in $\mathcal{S}$ is a curve ϕ_t in $\mathcal{C}^{s,p}$. The *material velocity* is defined by

$$V(X,t) = \frac{\partial}{\partial t}\phi(X,t), \text{where}\, \phi_t(X) = \phi(X,t).$$

We can identify V_t, defined by $V_t(X) = V(X,t)$, with an element of $T_{\phi_t}\mathcal{C}^{s,p}$. The corresponding *spatial velocity* is defined by $v_t = V_t \circ \phi_t^{-1}$. It is a vector field on $\mathcal{S}$ with domain $\phi_t(\mathcal{B})$.

4. Different physical circumstances warrant variations in the definition. Three examples follow.

   (a) If one wishes to impose a *boundary condition of place*, then $\phi|\partial\mathcal{B}$ should be prescribed; *i.e.*, one considers a given $\phi_{\partial\mathcal{B}} : \partial\mathcal{B} \to \mathcal{S}$ and lets

$$\mathcal{C}^{s,p}_{\partial\mathcal{B}} = \{\phi \in \mathcal{C}^{s,p} | \phi|\partial\mathcal{B} = \phi_{\partial\mathcal{B}}\}.$$

   One can show that $\mathcal{C}^{s,p}_{\partial\mathcal{B}}$ is a smooth submanifold of $\mathcal{C}^{s,p}$ with

$$T_\phi\mathcal{C}^{s,p}_{\partial\mathcal{B}} = \{V \in T_\phi\mathcal{C}^{s,p} | V|\partial\mathcal{B} = 0\}$$

   (see Ebin and Marsden (1970) for the techniques needed to prove this).

   (b) If the deformations are to be confined to $\mathcal{B}$ (e.g., a fluid filling $\mathcal{B}$), one considers only maps such that $\phi(\mathcal{B}) = \mathcal{B}$, *i.e.*, $\mathcal{D}^{s,p}$, the $W^{s,p}$ diffeomorphisms of $\mathcal{B}$ to $\mathcal{B}$. Again, $\mathcal{D}^{s,p}$ is a smooth manifold.

(c) Let $\phi_0 : \mathcal{B} \to \mathcal{S}$ be a reference embedding, let $n = m$, and let $d\mu$ be a volume element on $\mathcal{S}$ (*i.e.,* $d\mu$ is a nowhere zero m–form on $\mathcal{S}$). A configuration ϕ is called *volume preserving* (relative to ϕ_0 and $d\mu$) if $(\phi \circ \phi_0^{-1})^* d\mu = d\mu$. For $s > (n/p) + 1$, the volume preserving configurations form a smooth submanifold.

♦

While $\mathcal{C}^{s,p}$ is to be the configuration space, $T\mathcal{C}^{s,p}$ is to be the phase space for elastodynamics. We make some remarks on this.

1. Let dm_0 be a given volume element on $\mathcal{B}$ called the *mass density.* The mass density on $\phi(\mathcal{B})$ in a configuration ϕ will be taken to be $dm = \phi_* dm_0$. The mass density on $\phi(\mathcal{B})$ is completely characterized by the condition $\int_{\phi(U)} dm = \int_U dm_0$ for any open set $U \subset \mathcal{B}$ (with smooth boundary, say) *i.e.,* by the law of conservation of mass. For a motion ϕ_t we have $\phi_t^* dm_t = dm_0$, and so if $n = m, \mathcal{L}_{v_t} dm_t = L_{v_t} dm_t + \partial dm_t/\partial t = 0$, where $\mathcal{L}_{v_t}$ is the Lie derivative and $\mathcal{L}_{v_t}$ is the "time frozen" or autonomous Lie derivative. This is the *equation of continuity.* (If $n \neq m$, the mean curvature of $\phi(\mathcal{B})$ in $\mathcal{S}$ is involved in the equation.)

2. Now assume that $\mathcal{S}$ has a Riemannian metric g. Define an inner product on $T_\phi \mathcal{C}^{s,p}$ by

$$\langle V, W \rangle = \int_{\mathcal{B}} \langle V(X), W(X) \rangle_X \, dm_0(X),$$

and let $\mathcal{K} : T\mathcal{C}^{s,p} \to \mathbb{R}, \mathcal{K}(V) = 1/2\langle V, V \rangle$ be the associated kinetic energy function. As in Lecture 1, this inner product induces a smooth map of $T\mathcal{C}^{s,p}$ to $T^*\mathcal{C}^{s,p}$, and hence the strong symplectic form on $T^*\mathcal{C}^{s,p}$ pulls back to a (weak) symplectic form on $T\mathcal{C}^{s,p}$. The results of Ebin and Marsden (1970) show that $X_\mathcal{K}$ is a smooth vector field on $T\mathcal{C}^{s,p}$. In particular, it is everywhere defined. Base integral curves of $X_\mathcal{K}$ are the geodesics of $\mathcal{C}^{s,p}$ and are given explicitly as follows: ϕ_t is a geodesic in $\mathcal{C}^{s,p}$ if and only if $\phi_t(X)$ is a geodesic in $\mathcal{S}$ for each fixed $X \in \mathcal{B}$; *i.e.,* $0 = A_t \equiv (D/Dt)V_t$ (acceleration = covariant derivative of velocity).

   For $n = m$ (so the spatial velocity v_t is a vector field on the open submanifold $\phi_t(\mathcal{B}) \subset \mathcal{S}$), the *spatial acceleration* is defined by $a_t = A_t \circ \phi_t^{-1}$. From the chain rule one gets $a_t = \partial v_t/\partial t + \nabla_{v_t} v_t$.

3. A special case of 2 is the fact that the solution of $\partial u/\partial t + u \cdot \nabla u = 0$ in $\mathbb{R}^n$ is the *spatial* velocity field of the motion $\phi_t(X) = X + tu_0(X)$, where u_0 is the initial value of u. In general ϕ_t ceases to be an embedding after a short time, and this solution is no longer valid.

4. The equations of an ideal fluid result if one replaces $\mathcal{C}^{s,p}$ by the volume–preserving diffeomorphisms of $\mathcal{B}$. Again it turns out that $X_{\mathcal{K}}$ is a smooth vector field, although this is by no means obvious (see Ebin and Marsden (1970) and Marsden (1976)).

Next we discuss some useful notation.

1. Given a configuration $\phi : \mathcal{B} \to \mathcal{S}$, let $F = T\phi$ be the derivative of ϕ, and call F the *deformation gradient.*

2. Let $C = \phi^* g$ be the pullback of the metric on $\mathcal{S}$, called the *right Cauchy Green tensor.*

3. If ϕ_t is a motion, $D = 1/2(\partial C/\partial t)$ is the *material rate of deformation tensor.* If $n = m, D = \phi_t^* d$, where $d = L_v g$ is the *spatial rate of deformation tensor.*

4. Let G be a Riemannian metric on $\mathcal{B}$ and let its associated volume element be denoted dv. The *Jacobian* of a configuration $\phi : \mathcal{B} \to \mathcal{S}$ is denoted $J : \mathcal{B} \to \mathbb{R}$ and is defined by $\phi_* d\mu = J dv$. The *mass density* ρ is defined by $\rho_0 dv = dm_0$ and $\rho d\mu = dm$.

5. Let w be a vector field on $\phi(\mathcal{B})$ and $n = m$. Its *Piola transform* is the vector field W on $\mathcal{B}$ defined by $W = J\phi^* w$, or, equivalently, $i_w dv = \phi^*(i_w d\mu)$, where i_w is the interior product. Let DIV W be the divergence of W with respect to dv and div w the divergence of w with respect to $d\mu$. The *Piola identity* states that DIV $W =$ (div $w)\circ\phi$. (*Proof*. $\mathrm{DIV}\, W dv = L_w dv = d i_w dv = d(\phi^* i_w d\mu) = \phi^* d i_w d\mu = \phi^* L_w d\mu = \phi^* \operatorname{div} w$.)

Particular elastic materials are characterized by certain potential energy functions. We will produce a map $\mathcal{V} : \mathcal{C}^{s,p} \to \mathbb{R}$ that will serve as the potential energy; our Hamiltonian will be $\mathcal{H} : T\mathcal{C}^{s,p} \to \mathbb{R}, \mathcal{H}(V_\phi) = \mathcal{K}(V_\phi) + \mathcal{V}(\phi)$, where $V_\phi \in T_\phi \mathcal{C}^{s,p}$.

Let $\mathcal{O}_{\mathcal{S}}$ denote an orbit in $\mathcal{M}_{\mathcal{S}}$, the space of C^∞ Riemannian metrics on $\mathcal{S}$ under the action by pullback of $\mathcal{D}_{\mathcal{S}}$, the orientation preserving C^∞

diffeomorphisms of $\mathcal{S}$. Thus

$$\mathcal{O}_{\mathcal{S}} = \{\eta^* g_0 | g_0 \text{ is a given metric and } \eta \in \mathcal{D}_{\mathcal{S}}\}.$$

The orbit consists of all metrics geometrically equivalent to a given one, such as the Euclidean metric on $\mathbb{R}^3$. Likewise, let $\mathcal{O}_{\mathcal{B}}$ denote an orbit of metrics on $\mathcal{B}$. Let $\Lambda_{\mathcal{B}}$ denote the smooth densities on $\mathcal{B}$.

Definition 2.1. *An elastic stored energy function $\mathcal{W}$ is a smooth map*[3]

$$\mathcal{W} : \mathcal{C} \times \mathcal{O}_{\mathcal{S}} \times \mathcal{O}_{\mathcal{B}} \to \Lambda_{\mathcal{B}}$$

satisfying:

1. *Material frame indifference. For $\phi \in \mathcal{B}, g \in \mathcal{O}_{\mathcal{S}}, G \in \mathcal{O}_{\mathcal{B}}$ and $\xi \in \mathcal{D}_{\mathcal{S}}$, we have*
$$\mathcal{W}(\phi, g, G) = \mathcal{W}(\xi \circ \phi, \xi_* g, G).$$

2. *Locality If (ϕ_1, g_1, G_1) and (ϕ_2, g_2, G_2) agree on an open set $U \subset \mathcal{B}$ (i.e., $\phi_1 = \phi_2$ on $U, g_1 = g_2$ on $\phi_1(U) = \phi_2(U)$ and $G_1 = G_2$ on U), then $\mathcal{W}(\phi_1, g_1, G_1) = \mathcal{W}(\phi_2, g_2, G_2)$ on U.*

   If we have the identity

$$\mathcal{W}(\phi \circ \eta, g, \eta^* G) = \eta^*(\mathcal{W}(\phi, g, G)), \quad \phi \in \mathcal{C}, \eta \in \mathcal{D}_{\mathcal{B}},$$

   we call $\mathcal{W}$ materially covariant.

Remarks. 1. (a) In condition 1, one can view ξ either actively or passively; *i.e.,* $\xi \circ \phi$ can be viewed as a superposed displacement or as the same displacement ϕ in a different representation.

(b) For $\mathcal{S} = \mathbb{R}^3$ and ξ an isometry, *i.e.,* a rigid motion, condition 1 expresses the invariance of $\mathcal{W}$ as a function of ϕ under $\phi \to \xi \circ \phi$ (interpreted either as a superposed rigid motion or as an observer transformation). Hence the name "material frame indifference" (although "spatial frame indifference" seems just as appropriate).

(c) Condition 1 is equivalent to $\mathcal{W}$ depending on ϕ and g only through $C = \phi^* g$. Then material covariance reads

$$\eta^*(\mathcal{W}(C, G)) = \mathcal{W}(\eta^* C, \eta^* G).$$

[3]In C^∞ topologies there is no canonically agreed–on differential calculus. Any one in which the elementary rules of calculus (especially the chain rule) hold will do here, for example, the one in Lang (1972).

2. Condition 2 is a natural physical requirement that the elastic stored energy in a piece of material is independent of what the material elsewhere is doing. (For global constraints such as incompressibility this need not hold.)

3. (a) Material covariance refers to the transformation property of $\mathcal{W}$ under a change of *reference* configuration. It is closely related to the notion of *material symmetry* at $X \in \mathcal{B}$, *i.e.,* and $\eta \in \mathcal{D}_{\mathcal{B}}$ such that $\eta(X) = X, (\eta^* G)(X) = G(X)$ and

$$\mathcal{W}(\phi \circ \eta, g, G)(X) = \mathcal{W}(\phi, g, G)(X).$$

   If, for every $A \in SO(X, G)$ (the special orthogonal group of $T_X \mathcal{B}$), there is an $\eta \in \mathcal{D}_{\mathcal{B}}$ that is a material symmetry at X and $T_X \eta = A$, then the material is called *isotropic.* One can show that *material covariance is equivalent to isotropy.*

   (b) Notice that $\mathcal{D}_{\mathcal{B}}$ acts on $\mathcal{C}$ by composition on the *right* while $\mathcal{D}_{\mathcal{S}}$ acts on $\mathcal{C}$ by composition on the *left.*

4. (a) Let $S_2(\mathcal{B})$ denote the bundle of symmetric two–tensors on $\mathcal{B}$, and $S_2^+(\mathcal{B})$ the positive definite ones. Thus a section of $S_2^+(\mathcal{B})$ is just a Riemannian metric on $\mathcal{B}$. Let

$$\widehat{\mathcal{W}} : S_2^+(\mathcal{B}) \times S_2^+(\mathcal{B}) \to \mathbb{R}$$

   be a given smooth map. Define $\mathcal{W} : \mathcal{C} \times \mathcal{O}_{\mathcal{S}} \times \mathcal{O}_{\mathcal{B}} \to \Lambda_{\mathcal{B}}$ by

$$\mathcal{W}(\phi, g, G) = \widehat{\mathcal{W}} \circ (C \times G) dv,$$

   where $C = \phi^* g$ and dv is the volume element of G. Following standard abuse of notation we shall write

$$\widehat{\mathcal{W}} \circ (C \times G) dv = \widehat{\mathcal{W}}(C, G) dv,$$

   and say that $\widehat{\mathcal{W}}$ "depends only on the point values of C and G". We say that $\widehat{\mathcal{W}}$ is *natural* if for any diffeomorphism η of $\mathcal{B}$, and points $C_X \in S_2^+(\mathcal{B}), G_X \in S_2^+(\mathcal{B})$,

$$\widehat{\mathcal{W}}(\eta_* C_X, \eta_* G_X) = \widehat{\mathcal{W}}(C_X, G_X).$$

   (b) It is readily verified that a $\mathcal{W}$ *defined by a natural map* $\widehat{\mathcal{W}}$ *is a materially covariant stored energy function.* Under some additional hypotheses discussed below, the converse is also true.

(c) (Neo–Hookean material). Here is an example of a natural $\mathcal{W}$. Let $\lambda_1, \ldots \lambda_n$ be the eigenvalues of $\sqrt{C_X}$ (taken on $T_X\mathcal{B}$ with the inner product G_X), and let $\widehat{\mathcal{W}}(C_X, G_X) = \lambda_1^2 + \ldots + \lambda_n^2 - n$. (Generalizations of this due to Mooney–Rivlin and Ogden are also natural.)

♦

Given a stored energy function $\mathcal{W}$ and keeping g, G fixed, define

$$\mathcal{V} : \mathcal{C} \to \mathbb{R}, \mathcal{V}(\phi) = \int_{\mathcal{B}} \mathcal{W}(\phi, g, G).$$

We shall remain in C^∞ until we have given more functional form to $\mathcal{W}$; at the moment there is no restriction on the number of derivatives of ϕ or g on which $\mathcal{W}$ may depend.[4]

Now let

$$\mathcal{H} : T\mathcal{C} \to \mathbb{R}, \mathcal{H}\left(V_\phi\right) = K\left(V_\phi\right) + \mathcal{V}(\phi).$$

(If there are body forces or surface tractions, additional terms must be added to $\mathcal{H}$). Let us formally compute the Hamiltonian vector field $X_{\mathcal{H}}$ for $\mathcal{H}$. Indeed, we have already described $X_{\mathcal{H}}$. It is well known (e.g., Abraham and Marsden (1978), p. 227) that $X_{\mathcal{V}}$ is the vertical lift of $-\mathrm{grad}\, \mathcal{V}$. By definition, grad $\mathcal{V}$ is the vector field on $\mathcal{C}$ such that

$$\langle \mathrm{grad}\, \mathcal{V}(\phi), W\rangle = d\mathcal{V}(\phi) \cdot W$$

for $W \in T_\phi\mathcal{C}$. But we can write

$$d\mathcal{V}(\phi) \cdot W = \int_{\mathcal{B}} D_\phi \mathcal{W}(\phi, g, G) \cdot W = \int_{\mathcal{B}} \langle W, D_\phi \mathcal{W}(\phi, g, G)^* \cdot 1\rangle \, dv,$$

which defines the adjoint of $D_\phi\mathcal{W}$. Thus grad $\mathcal{V}(\phi) = \rho_0^{-1} D_\phi \mathcal{W}(\phi, g, G)^* \cdot 1$. At this stage, $D_\phi\mathcal{W}$ may be a high order differential operator. The equations defining an integral curve of $X_{\mathcal{H}}$ are therefore

$$\rho_0 A_t = -D_\phi \mathcal{W}(\phi_t, g, G)^* \cdot 1 \quad \text{on}\, mathcalB,$$

where $A_t = DV_t/Dt$ is the acceleration of the curve ϕ_t.

---

[4] If the theory is to couple to gravity, one could demand "minimal coupling," *i.e.,* dependence only on the point values of g and hence on first derivatives of ϕ alone; cf. Hawking and Ellis (1973).

Next let us compute the rate of change of total energy of a material volume $U \subset \mathcal{B}$ (that is assumed to have smooth boundary) along a solution:

$$\begin{aligned}\frac{d}{dt}&\left\{\int_U \frac{\rho_0}{2}||V_t(X)||^2\, dv(X) + \int_U \mathcal{W}(\phi_t, g, G)\right\}\\ &= \int_U \{\rho_0(A_t, (X), V_t(X)dv(X) + D_\phi \mathcal{W}(\phi_t, g, G)\cdot V_t\}\\ &= \int_U \{\langle -D_\phi \mathcal{W}(\phi_t, g, G)^* 1, V_t(X)\rangle dv(X) + D_\phi \mathcal{W}(\phi_t, g, G)\cdot V_t\}.\end{aligned}$$

If $U = \mathcal{B}$, then this vanishes as it should. Now we invoke a basic axiom of continuum physics.

Cauchy's Axiom of Power *Given a solution curve ϕ_t as above, there exists a smooth function T of the arguments X, t and $N \in T_X\mathcal{B}$ with values in $T_x^*\mathcal{S}$, where $x = \phi_t(X)$, such that*

$$\frac{d}{dt}\left\{\int_U \frac{\rho_0}{2}||V_t||^2 dv + \mathcal{W}(\phi_t, g, G)\right\} = \int_{\partial U} T(X, t, N_X)\cdot V_t da(X);$$

in this formula, N_X is the unit outward normal to ∂U and da is the Riemannian area element on ∂U.

Remarks. 1. Cauchy's axiom says that if a piece U of material undergoes a motion, then the rate of change of the kinetic plus elastic potential energies equals the power expended by some force field on the surface ∂U. This force field is the *first Piola–Kirchhoff traction field.*

A theorem of Cauchy asserts that T is necessarily linear in N_X; we shall omit the proof.[5] Thus T may be regarded as a (time–dependent) section of the bundle

$$T^*\mathcal{B} \otimes \phi_t^*(T^*\mathcal{S}).$$

In the continuum mechanics literature such sections are examples of *two–point tensors.* We shall write

$$T(X, t)(N_X, V_t(X))$$

for this tensor, and call it the *first Piola–Kirchhoff stress tensor.* Under the metrics g and G there is an associated tensor S which goes by the same name. Thus S is a section of $T\mathcal{B} \otimes \phi_t^*(T\mathcal{S})$. The Piola transform (defined above) applied to the first slot of S produces

[5] See, for example, Truesdell and Toupin (1960), Malvern (1969) and Gurtin and Martins (1976). The proof depends only on Cauchy's axiom of power.

a section of $\phi_t^*(T\mathcal{S}\otimes T\mathcal{S})$. We can regard this as a section of $T\mathcal{S}\otimes T\mathcal{S}$ over $\phi_t(\mathcal{B})$ if $\phi_t(\mathcal{B})\subset\mathcal{S}$ is open. The resulting two–tensor is τ, the *Cauchy (or spatial) stress tensor.* On the other hand, the two–tensor P on $\mathcal{B}$ defined by pulling S back to $\mathcal{B}$, *i.e.,*

$$P(X,t)(\alpha_1,\alpha_2) = S(X,t)\cdot(\alpha_1,\phi_t * \alpha_2)$$

so that P is a section of $T\mathcal{B}\otimes T\mathcal{B}$, is called the *second Piola–Kirchhoff stress.*

2. Cauchy's axiom may be rewritten using the divergence theorem as

$$\frac{d}{dt}\left\{\int_U \frac{\rho_0}{2}||V_t||^2 dv + \mathcal{W}(\phi_t,g,G)(X)\right\} = \int_U \mathrm{DIV}(S(X,t)\cdot V_t(X))dv(X),$$

where $\cdot$ denotes contraction in the second slot of S and DIV is the divergence on $\mathcal{B}$ of the resulting vector field. Combining this with our previous expression for the rate of change of the total energy of U and using the arbitrariness of U yields the identity

$$D_\phi\mathcal{W}(\phi_t,g,G)\cdot V_t = \langle D_\phi\mathcal{W}(\phi_t,g,G)^*1,V_t\rangle + \langle \mathrm{DIV}\, S, V_t\rangle dv + S\cdot\nabla V_t dv.$$

We assume that an identity of this type holds for all possible solutions, in particular, at $t=0$; we can choose ϕ and V arbitrarily. Choose $X_0\in\mathcal{B}$ and the fact that V and ∇V can be varied independently at X_0, to conclude that

$$(a) \qquad D_\phi\mathcal{W}(\phi,g,G)^*\cdot 1 = -\,\mathrm{DIV}\, S$$

and

$$(b) \qquad D_\phi\mathcal{W}(\phi,g,G)\cdot V = S\cdot\nabla dv.$$

Locality and (b) show that $\mathcal{W}$ is a function only of the point values of the derivative of ϕ. (See Gurtin (1972) for a similar result.) However, we know that $\mathcal{W}$ depends on ϕ and g only through C by material frame indifference. Thus, $\mathcal{W}$ is *induced by a map* $\widehat{\mathcal{W}}$. Moreover, $\mathcal{W}$ is materially covariant if and only if $\widehat{\mathcal{W}}$ is natural.

3. By abuse of notation, let us write $\widehat{\mathcal{W}}(F,g,G)$, $\widehat{\mathcal{W}}(C,G)$ and $\widehat{\mathcal{W}}(\phi,g,G)$ to indicate the intended variables. The three basic stress tensors can be related to $\mathcal{W}$ via the identity (b) and the chain rule; one gets three

equivalent formulae:

$$\text{First Piola–Kirchhoff stress,} \qquad S = \rho_0 \frac{\partial \hat{\xi}}{\partial F};$$
$$\text{Second Piola–Kirchhoff stress} \qquad P = 2\rho_0 \frac{\partial \hat{\xi}}{\partial C};$$
$$\text{Cauchy stress} \qquad \tau = 2\rho \frac{\partial \hat{e}}{\partial g};$$

where[6] $\hat{\xi} dm_0 = \widehat{\mathcal{W}}$; *i.e.,* $\hat{\xi}$ is the stored energy per unit mass and $\hat{e}$ denotes $\hat{\xi} \circ \phi^{-1}$.

4. In terms of S, the basic equations of motion defining the Hamiltonian system are
$$\rho_0 A_t = \mathrm{DIV}\, S,$$
where S is regarded as a function of ϕ through the constitutive relation $S = \rho_0 \partial\hat{\xi}/\partial F$. Boundary conditions appropriate to the problem at hand must be added as well.

5. In local coordinates, on $\mathcal{S} = \mathbb{R}^3$ and $\rho_0 =$ constant, the equations of motion read
$$\frac{\partial^2 \phi^\alpha}{\partial t^2} = \sum_j \frac{\partial}{\partial x^j}[S^{j\alpha}(\phi^\beta_{,i})] = \sum_{i,j,\beta} \frac{\partial S^{j\alpha}}{\partial \phi^\beta_{,i}} \frac{\partial^2 \partial\beta}{\partial x^j \partial x^i}.$$
The *elasticity tensor* is $A^{j\alpha i}{}_{\beta} = \partial S^{j\alpha}/\partial \phi^\beta_{,i}$. This is a quasilinear second order system. It is generally assumed to be hyperbolic; *i.e.,* the strong ellipticity (or Legendre–Hadamard condition) holds:
$$A^{j}_{\alpha\beta}{}^{i} \xi^\alpha \xi^\beta \eta_j \eta_i \geq \varepsilon |\xi|^2 |\eta|^2$$
for some $\varepsilon > 0$. Under this condition one can show that $X_{\mathcal{H}} : T\mathcal{C}^s \to T^2\mathcal{C}^{s,2}(\mathcal{C}^s = \mathcal{C}^{s,2})$ defines a local flow on $T\mathcal{C}^s$ if $s > n/2 + 1$, and the set up described in Lecture 1 holds. See Hughes et al. (1977) and Kato (1977).

6. It is generally believed that the flow defined by elasticity is valid only for a short time, and that shocks will develop. It is not known how to deal properly with this situation. We refer to Marsden and Hughes

[6] This formula for the Cauchy stress is similar to one due to Doyle and Erickson (1956). For its relationship to the Green–Naghdi–Rivlin balance of energy arguments, see Marsden and Hughes (1978).

(1978) for references, and to the interesting recent work of Klainerman (1979).

7. The static problem consists of describing the critical points of $\mathcal{V}$. This is of special interest because the usual methods in the calculus of variations (such as convexity or the Palais–Smale condition) do not apply. Rather one must use weak methods developed by Morrey; this approach has been developed by Ball (1977). It would be interesting to see whether Ball's results fit the context developed by Graff (1978).

8. There are many interesting open problems in elasticity related to Remarks 6 and 7. One of these is how to describe the formation of holes or ruptures in bodies under large stress. In this direction, recent work of Ball (1982) is especially interesting. Another problem is that of least stability: is a minimum of $\mathcal{V}$ dynamically stable? This problem is subtle because of the *weak* nondegeneracy of the second derivative of $\mathcal{V}$ at minima. See Knops and Wilkes (1973), Ball et al. (1978) and Marsden and Hughes (1978) for further information.

9. Ideas similar to those presented here for elasticity seem to be useful for continuum physics in general. For example, Arnold (1966) and Ebin and Marsden (1970) use these methods to describe the Euler equations of a fluid. More recently, the Vlasov–Maxwell equations of plasma physics have been understood as an infinite dimensional Hamiltonian system using the symplectic diffeomorphism group of $T^*\mathbb{R}^3$ by Morrison, Marsden and Weinstein; see Marsden and Weinstein (1982)).

♦

3

Symmetry and Reduction

Several of the remaining lectures are related to the interaction between Hamiltonian systems and symmetries. This lecture reviews some of the material we shall need. (Consult Abraham and Marsden (1978), Chapter 4 for more information.) We shall deal with the finite dimensional case first; the generalization to infinite dimensions can be carried out in the context of Lecture 1.

Definition 3.1. *Let G be a Lie group and M a manifold. An action of G on M is a smooth mapping $\Phi : G \times M \to M$ satisfying $\Phi(e, x) = x$ and $\Phi(g_1, g_2, x) = \Phi(g_1, \Phi(g_2, x))$. Let $\Phi_g : M \to M$ be defined by $\Phi_g(x) = \Phi(g, x) = \Phi(g, x)$. (Thus $\Phi_e = Id$ and $\Phi_{g_1,g_2} = \Phi_{g_1} \circ \Phi_{g_2}$; so Φ may be regarded as a homomorphism from G to $\mathcal{D}(M)$, the diffeomorphism group of M.)*

Let $\mathfrak{g}$ be the Lie algebra of G. For $\xi \in \mathfrak{g}$ define the corresponding *infinitesimal generator* ξ_M, a vector field on M, by requiring it to be the generator of the flow $F_t = \Phi_{\exp i\xi}$; thus

$$\xi_M(x) = \frac{d}{dt}\Phi_{\exp i\xi}(x)|_{t=0} = T\Phi^x \cdot \xi|_{g=e},$$

where $\Phi^x(g) = \Phi(g, x)$.

Remark. 1. The orbit of a point $x \in M$ is denoted by $\mathcal{O}_x = G \cdot x = \{\Phi(g, x) | g \in G\}$. This is an immersed submanifold with

$$T_x(G \cdot x) = \{\xi_M(x) | \xi \in \mathfrak{g}\}.$$

2. The *adjoint action* of G on $\mathfrak{g}$ is defined by

$$(g,\xi) \mapsto \mathrm{Ad}_g\,\xi = T_e(R_{g^{-1}} \circ L_g) \cdot \xi,$$

where R_g and L_g denote left and right translation by g respectively. The corresponding infinitesimal generators are

$$\xi_{\mathfrak{g}} = \mathrm{ad}\,\xi : \eta \mapsto [\xi,\eta].$$

3. Two useful general identities are

$$\text{(a)} \qquad (\mathrm{Ad}_g\,\xi)_M = (\Phi_g)_*\xi_M$$

and

$$\text{(b)} \qquad [\xi_M,\eta_M] = -[\xi,\eta]_M.$$

4. The *coadjoint action* of G on $\mathfrak{g}^*$ is defined by

$$(g,\mu) \mapsto \mathrm{Ad}^*_{g^{-1}} \cdot \mu.$$

(Note that for linear transformations, the pullback map coincides with the dual.) The corresponding infinitesimal generators are

$$\xi_{g^*} = -\,\mathrm{ad}^*_\xi\,.$$

5. A theorem of Kirillov states that for any Lie group G, if $\mu \in \mathfrak{g}^*$ then the orbit of μ under the coadjoint action is a symplectic manifold. The symplectic form ω_μ is determined by

$$\omega_\mu(\xi_{g^*}(\mu),\eta_{g^*}(\mu)) = -\mu([\xi,\eta]).$$

We shall see shortly that this is a special case of a general theorem on reduction.

♦

Now we turn to the conserved quantities associated with the symmetries of a symplectic manifold, due to Souriau and Kostant.

Definition 3.2. *Let (P,ω) be a symplectic manifold and Φ an action of a Lie group G on P. Assume the action is symplectic : $\Phi_g^*\omega = \omega$ for all $g \in G$. A momentum mapping is a smooth mapping $J : P \to \mathfrak{g}^*$ such that*

$$\langle dJ(x)\cdot v_x,\xi\rangle = \omega_x(\xi_P(x),v_x)$$

for all $\xi \in \mathfrak{g}, v_x \in T_xP$, where $dJ(x)$ is the derivative of J at x, regarded as a linear map of T_xP to $\mathfrak{g}^*$, and $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

A momentum map is Ad^*–equivariant when the following diagram commutes for each $g \in G$:

$$\begin{tikzcd} P \arrow[r, "\Phi_g"] \arrow[d, "J"'] & P \arrow[d, "J"] \\ \mathfrak{g}^* \arrow[r, "\mathrm{Ad}^*_{g^{-1}}"'] & \mathfrak{g}^* \end{tikzcd}$$

If J is Ad^*–equivariant, we call (P, ω, G, J) a *Hamiltonian G–space.*

Remarks 1. Given $J : P \to \mathfrak{g}^*$, let $\hat{J} : \mathfrak{g} \times P \to \mathbb{R}$, $\hat{J}(\xi, x) = \langle \hat{J}(x), \xi \rangle$ and $\hat{J}_\xi(x) = \hat{J}(\xi, x)$. The condition for a momentum map is equivalent to

$$\xi_P = X_{\hat{J}_\xi}.$$

2. (Commutation relations for Ad^*–equivariant momentum maps). By differentiating $J \circ \Phi_g = \mathrm{Ad}^*_{g^{-1}} \circ J$ in g, we find that

$$\{\hat{J}_\xi, \hat{J}_\eta\} = \hat{J}_{[\xi,\eta]}.$$

3. (a) If $H : P \to \mathbb{R}$ is invariant under the symplectic action Φ and J is a momentum map for the action, then J is constant on the orbits of X_H. Indeed, $H \circ \Phi_g = H$ implies that $\{H, \hat{j}_\xi\} = 0$. Moreover, the flow F_t of X_H and Φ_g commute.

   (b) The momentum map for the action Φ of $G \times \mathbb{R}$ on P, namely $\Psi_{(g,t)} = \Phi_g \circ F_t$, is $J \times H$, the *energy–momentum map.*

4. (Construction of J).

   (a) *Suppose that $\omega = -d\theta$ and the action Φ preserves θ. Then we can choose*

$$\hat{J}(\xi, x) = i_{\xi_P}\theta \textit{ (interior product)} .$$

Proof. Since $\Phi_g^*\theta = \theta, \mathcal{L}_{\xi_p}\theta = 0$ (Lie derivative). Thus $i_{\xi_P} d\theta + di_{\xi_P}\theta = 0$, or $di_{\xi_P} - i_{\xi_P}\omega$. Thus ξ_P has $i_{\xi_P}\theta$ as a Hamiltonian. One can check that this momentum map is Ad^*–equivariant.

(b) Suppose that G acts on Q and hence on T^*Q. Then, as a special case of (a), one finds that

$$\hat{J}(\xi, \alpha_q) = \langle \alpha_q, \xi_Q(q) \rangle.$$

(This reproduces the usual linear and angular momentum for n particles in $\mathbb{R}^3$ as a special case.) ■

5. (Adjoint formalism). Let the symplectic form ω on P be related to a Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ on P and a complex structure $\mathbb{J}$ on P by the formula discussed in Lecture 1 : $\omega(v_x, w_x) = \langle\langle \mathbb{J} v_x, w_x \rangle\rangle$. By definition of the momentum mapping,

$$\langle dJ(x) \cdot v_x \xi \rangle = \omega_x(\xi_P(x), v_x) = \langle\langle \mathbb{J}\xi_P(x), v_x \rangle\rangle.$$

Define the adjoint of $dJ(x) : \mathfrak{g} \to T_x P$ by

$$\langle dJ(x) \cdot v_x, \xi \rangle = \langle\langle v_x, dJ(x)^* \xi \rangle\rangle.$$

Thus $\mathbb{J}\xi_P(x) = dJ(x)^* \xi$. Since $\mathbb{J}^2 = -I$, we get

$$\xi_P(x) = -\mathbb{J} \circ [dJ(x)]^* \cdot \xi.$$

This way of relating the "Hamiltonian" J to its family of generators $\{\xi_P\}$ is convenient in a variety of situations.

6. General relativity provides motivation for considering situations more general than the preceding. One possibility is to replace the group G by a manifold E and the action Φ by an equivalence relation on P. (For a group the equivalence classes are the orbits.) We assume that there is a generator, *i.e.,* a section of the bundle $L(TE, TP)$ of linear maps of $T_e E$ to $T_x P$ over $E \times P$. Thus, for each $e \in E, x \in P$ and $\xi \in T_e E$, we have a vector $\xi_P(x) \in T_x P$, depending linearly on ξ. These vectors are assumed to span, as ξ varies, the tangent space to the equivalence class of x. A *momentum map* is then a map

$$\hat{J} : TE \times P \to \mathbb{R},$$

such that, for each $\xi \in T_e E, \xi_P = X_{\hat{J}_\zeta}$ as before. The adjoint formalism is unaltered in this formulation. (In general relativity there is, secretly, a group lurking in the background, but it is masked when dynamics is considered; see Lectures 9 and 10.)

7. Associated to a momentum map $\hat{J} : \mathfrak{g} \times P \to \mathbb{R}$ is a Lagrangian submanifold of T^*P when J is thought of as a Morse family (see Weinstein (1977), Lecture 6). Another one of interest is described in Abraham and Marsden (1978), Exercise 5.31.

8. Sometimes G represents a "gauge" group and, simultaneously, J incorporates the dynamics. In this case the formula $\xi_P(x) = -\mathbb{J} \circ DJ(x)^*\xi$ shows how the dynamics changes with different choices of gauge ξ. We will see this explicitly in general relativity later. There is often a physical constraint $J = \text{const}$ associated with this situation. To appreciate this requires lengthy excursions into classical relativistic field theory and the Dirac theory of constraints. In fact, these topics are a subject of current research.

♦

We now review the reduction process in the context of momentum mappings. The formulation we use is due to Marsden and Weinstein (1974). The ideas will be utilized in subsequent lectures.

First we have the notation. Let $J : P \to \mathfrak{g}^*$ be an Ad^*–equivariant momentum mapping for a symplectic group of G on P. Let $\mu \in \mathfrak{g}^*$, and suppose $J^{-1}(\mu)$ is a submanifold of P with tangent space at x given by ker $dJ(x)$ (for example, suppose μ is a regular value for J). Let

$$G_\mu = \{g \in G | \, \mathrm{Ad}^*_{g^{-1}} \, \mu - \mu\},$$

the isotropy group of μ. Note that if G is abelian or if $\mu = 0$, then $G_\mu = G$. By Ad^*–equivariance, G_μ acts on $J^{-1}(\mu)$. Suppose $J^{-1}(\mu)/G_\mu$ is a C^∞ manifold for which the canonical projection $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ is a smooth submersion; *i.e.,* the action is free and proper. Set

$$P_\mu = \frac{J^{-1}(\mu)}{G_\mu},$$

called the *reduced phase space.* (If P_μ is not a manifold for global reasons, the constructions may still be done locally.) Let $i_\mu : J^{-1}(\mu) \to P$ be inclusion.

Theorem 3.3. *Under the hypotheses described, there is a unique symplectic structure ω_μ on P_μ such that*

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega.$$

Comments 1. The proof follows from the general process of Cartan described in Lecture 1, together with these key identities for each $x \in J^{-1}(\mu)$:

(a) $T_x(G_\mu \cdot x) = T_x(G \cdot x) \cap T_x(J^{-1}(\mu))$.

(b) $T_x(G \cdot x)$ is the ω_x–orthogonal complement of $T_x(J^{-1}(\mu))$.

See Abraham and Marsden (1978), §4.2 for details.

2. If f and g are G–invariant, then $\{f, g\}_\mu = \{f_\mu, g_\mu\}$, where f_μ is the induced function on P_μ.

3. If $H : P \to \mathbb{R}$ is G–invariant, then X_H projects to a vector field on P_μ, namely X_{H_μ}. Fixed points of the latter are called *relative equilibria* of X_H. The stability of relative equilibria can thus be studied using reduction (see the next lecture for an example).

4. If G is abelian and dim $G = k$, then the momentum mapping represents k integrals in involution; *i.e.,* $f_1, \ldots, f_k$ are functionally independent and $\{f_i, f_j\} = 0$. Then P has dimension dim $P - 2k$ and represents the classical reduction due to integrals in involution. If dim $P_\mu = 0$ *i.e.,* dim $P = 2k$, the system is *completely integrable. One can use the same definition for nonabelian or infinite dimensional groups.*[1] For $G = SO(3)$, and $\mu \neq 0, G_\mu = S^1$ and reduction now reduces to Jacobi's "elimination of the node."

5. If $P = T^*G$ and G acts on itself (and hence on P) by left translations, the above theorem reproduces the Kirillov theorem.

6. (Due to A. Weinstein). Reduction at a general μ can, in some sense, be replaced by reduction at $\mu = 0$ as follows. Let $\mathcal{O}_\mu$ be the coadjoint orbit through μ in $\mathfrak{g}^*$, and consider the product $P \times \mathcal{O}_\mu$ be the coadjoint orbit through μ in $\mathfrak{g}^*$, and consider the product $P \times \mathcal{O}_\mu$ with symplectic structure $\omega \ominus \omega_\mu$. Its momentum map is $J =$ (inclusion). Then $(P \times \mathcal{O}_\mu)_0$ and P_μ are symplectically diffeomorphic. (Related results will occur in Example (1a) in the following lecture.)

♦

[1] In the infinite dimensional case, this definition appears to us to be the natural one; for example, the KdV equation is completely integrable in our sense (and everybody else's too); see also the recent work of Berger and Church (1979).

4
Applications of Reduction

This lecture gives three related examples of reduction. Additional examples are given in the next lecture. The reader may wish to "warm up" by trying to work out the passage to center of mass coordinates for n particles as an example of reduction.

(*Moncrief's splitting*) Let $J : P \to \mathfrak{g}^*$ be an Ad *–equivariant momentum mapping for a symplectic action of G on (P, ω). We shall split the tangent space to P at a point $x \in P$ into three summands which reflect the reduction process. The method turns out to be useful in a number of contexts, but its original purpose was to unify various decompositions that occur in geometry and relativity (see Moncrief (1975a) and Arms et al. (1975) for more information).

If V and W are finite dimensional inner product spaces and $T : V \to W$ is a linear transformation, then we have two orthogonal decompositions,

$$V = \ker T \oplus \operatorname{range} T^*$$

and

$$W = \operatorname{range} T \oplus \ker T^*.$$

♦

These basic decompositions extend to the infinite dimensional case when T or T^* is an elliptic operator (the splittings are called the Fredholm alternative).

Assume that the symplectic form ω on P arises from an inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on P and a complex structure $\mathbb{J}$, as explained in Lecture 1; thus $\omega(v_x, w_x) = \langle\langle \mathbb{J} v_x, w_x\rangle\rangle$, and the adjoint formalism gives

$$\xi_P(x) = -\mathbb{J} \circ [dJ(x)]^* \cdot \xi.$$

We shall not immediately use an inner product on $\mathfrak{g}$; the adjoint $dJ(x)^* : \mathfrak{g} \to T_x P$ is defined by

$$\langle\langle v_x, dJ(x)^* \cdot \xi\rangle\rangle = \langle dJ(x) \cdot v_x, \xi\rangle,$$

where the pairing on the right–hand side is the natural one between $\mathfrak{g}$ and $\mathfrak{g}^*$.

We have

$$T_x P = \ker dJ(x) \oplus \operatorname{range} dJ(x)^*.$$

(If $J(x) = \mu$, this represents an orthogonal decomposition of $T_x P$ along and normal to the level set $J^{-1}(\mu)$. However, no condition on the manifold structure of $J^{-1}(\mu)$ is required for the validity of the decomposition).

There is another decomposition associated with the linear operator $\alpha_x : \mathfrak{g}_\mu \to T_x P, \xi \mapsto \xi_p(x)$, namely,

$$T_x P = \operatorname{range} \alpha_x \oplus \ker \alpha_x^*,$$

where $\alpha_x^* : T_x P \to \mathfrak{g}_\mu^*$ is given by $\langle \alpha_x^* \cdot v_x, \xi\rangle = \langle\langle v_x, \alpha_x \cdot \xi\rangle\rangle$.

The fact that G_μ preserves the set $J^{-1}(\mu)$, *i.e.,* Ad^*–equivariance, implies the inclusion range $\alpha_x \subset$ ker $dJ(x)$. Thus the two decompositions may be intersected to give Moncrief's splitting:

$$T_x P = \operatorname{range} \alpha_x \oplus (\ker dJ(x) \cap \ker \alpha_x^*) \oplus \operatorname{range} dJ(x)^*.$$

The summands represent, respectively, the tangent space to the orbit $G_\mu \cdot x$ (gauge directions), the tangent space to P_μ and the orthogonal space to $J^{-1}(\mu)$. Using the adjoint formalism, and assuming $G = G_\mu$ for simplicity, we get

$$T_x P = \operatorname{range} \mathbb{J} \circ dJ(x)^* \oplus [\ker dJ(x) \oplus \ker (dJ(x) \circ \mathbb{J})] \oplus \operatorname{range} dJ(x)^*.$$

(*Action angle decomposition*). [1] We shall now rearrange the summands in Moncrief's decomposition to produce a *symplectic decomposition* of $T_x P$. In the case when G is abelian and the system is completely integrable, this reduces to standard action angle variables.

We shall produce a symplectic isomorphism,

$$T_x P \approx T_\mu(G \cdot \mu) \oplus [\mathfrak{g}_\mu / \mathfrak{g}_x \oplus (\mathfrak{g}_\mu/\mathfrak{g}_x)^*] \oplus T_{[x]} P_\mu,$$

[1] See Abraham and Marsden (1978), 5.21 and Mishchenko and Fomenko (1978b).

where $\mathfrak{g}_\mu$ is the Lie algebra of G_μ, the isotropy subgroup of $\mu, T_{[x]}P_\mu = [\ker dJ(x) \cap \ker \alpha_x^*]$ and $\mathfrak{g}_x \subset \mathfrak{g}_\mu$ is the Lie algebra of G_x, the isotropy group of x for the action of G_μ.[2]

From the fact that $T_x(G \cdot x)$ is the ω_x–orthogonal complement of ker $dJ(x)$, we see that $T_x(G \cdot x) \approx$ range $dJ(x)^*$. On the other hand, range $\alpha_x \approx T_x(G_\mu \cdot x)$. But $T_\mu(G \cdot \mu) \approx \mathfrak{g}/\mathfrak{g}_\mu$ and $\mathfrak{g}_\mu/\mathfrak{g}_x \approx T_x(G_\mu \cdot x)$. Now Ad^*–equivariance implies that G_x is also the isotropy group at x for the action of G, so $\mathfrak{g}/\mathfrak{g}_x \approx T_x(G \cdot x)$, Thus,

$$\begin{aligned} T_\mu(G \cdot \mu) \oplus \mathfrak{g}_\mu/\mathfrak{g}_x \oplus (\mathfrak{g}_\mu/\mathfrak{g}_x)^* \approx & T_\mu(G \cdot \mu) \oplus T_x(G_\mu \cdot x) \oplus (\mathfrak{g}_\mu/\mathfrak{g}_x)^* \\ \approx & \mathfrak{g}/\mathfrak{g}_\mu \oplus \mathfrak{g}_\mu/\mathfrak{g}_x \oplus T_x(G_\mu \cdot x) \\ \approx & \mathfrak{g}/\mathfrak{g}_x \oplus T_x(G_\mu \cdot x) \\ \approx & T_x(G \cdot x) \oplus T_x(G_\mu \cdot x) \\ \approx & \operatorname{range} dJ(x)^* \oplus \operatorname{range} \alpha_x. \end{aligned}$$

This produces the stated isomorphism, and one can check that it is symplectic. Thus, *locally*, one has a decomposition of P into symplectic manifolds: if μ is a regular value of J (and hence $\mathfrak{g}_x = \{0\}$),

$$P \approx G \cdot \mu \times T^* G_\mu \times P_\mu.$$

For this decomposition of manifolds, it is interesting to start over and take another, global point of view suggested by Alan Weinstein and Jedrzej Sniatycki. We proceed by a series of remarks. We recall first the following reduction lemma mentioned in Lecture 3 (throughout, μ will be a regular value of J). ♦

Lemma 4.1. *(a)* $T_x(G_\mu \cdot x) = T_x(G \cdot x) \cap T_x(J^{-1}(\mu))$.

(b) $T_x(J^{-1}(\mu))$ *and* $T_x(G \cdot x)$ *are* ω*–orthogonal complements.*

As usual, $G \cdot x$ denotes the orbit of $x \in P$ under the action of G. A corollary of this is that *the orbit* $G_\mu \cdot x$ is an *isotropic submanifold of* P (*i.e.,* $T(G_\mu \cdot x) \subset T(G_\mu \cdot x)^\perp$, where $^\perp$ denotes the ω–orthogonal complement).

2. Now consider the manifold $J^{-1}(\mathcal{O}_\mu)$, where $\mathcal{O}_\mu \subset \mathfrak{g}^*$ is the orbit of μ under the coadjoint action. There are two natural projections (cf. Marle (1976)):

$$J_\mu = J | J^{-1}(\mathcal{O}_\mu) : J^{-1}(\mathcal{O}_\mu) \to \mathcal{O}_\mu$$

and

$$\pi_\mu : J^{-1}(\mathcal{O}_\mu) \to P_\mu;$$

[2] In Lecture 8 we will see that x is a regular value of J if and only if $\mathfrak{g}_x = \{0\}$. We will denote $\mathfrak{g}_x = s_x$ in that lecture, and call it the *symmetry group* of x.

π_μ is the composition of the projection $J^{-1}(\mathcal{O}_\mu) \to J^{-1}(\mathcal{O}_\mu)/G$ and the natural identification of $J^{-1}(\mathcal{O}_\mu)/G$ with $J^{-1}(\mu)/G_\mu$.

Let $i_\mu : J^{-1}(\mathcal{O}_\mu) \to P$ be the natural inclusion, and Ω_μ the canonical symplectic structure, on $\mathcal{O}_\mu$; $\Omega_\mu(\xi_{\mathfrak{g}^*}(\bar{\mu}), \eta_{\mathfrak{g}^*}(\bar{\mu})) = \langle \bar{\mu}, [\xi, \eta] \rangle$.

Proposition 4.2. *2(a)* $i_\mu^* \omega = \pi_\mu^* \omega_\mu + J_\mu^* \Omega_\mu$.

To prove this, we use the following:

2(b) $$T_x(J^{-1}(\mathcal{O}_\mu)) = T_x(G{\cdot}x) + \ker dJ(x).$$

These are ω–orthogonal by Remark 1. (Note that $G \cdot x \subset J^{-1}(\mathcal{O}_\mu)$ by equivariance.)

Proof of 2(b) $T_x(J^{-1}(\mathcal{O}_\mu)) = dJ(x)^{-1}(T_{\bar{\mu}}(G \cdot \mu))$, where $\bar{\mu} = J(x)$. By Ad*–equivariance,

$$dJ \circ \xi_P = \xi_{\mathfrak{g}^*} \circ J \tag{4.1}$$

(see [Abraham and Marsden (1978), p. 270]). Thus,

$$\begin{aligned} T_{\bar{\mu}}(G \cdot \mu) = \{\xi_{\mathfrak{g}^*}(\bar{\mu}) | \xi \in \mathfrak{g}\} &= \{dJ(x) \cdot \xi_P(x) | \xi \in \mathfrak{g}\} \\ &= \{dJ(x) \cdot v | v \in T_x(G \cdot x)\} \end{aligned}$$

Applying $dJ(x)^{-1}$ gives the result. To prove 2(a) we need another remark.

3.

Claim 4.3. *$J_\mu^* \Omega_\mu$ restricted to $T_x(G \cdot x) \times T_x(G \cdot x)$ coincides with ω restricted to the same place.*[3]

Proof. Let $v, w \in T_x(J^{-1}(\mathcal{O}_\mu))$ be of the form $v = \xi_P(n) + v'$ and $w = \eta_P(x) + w'$ where $v', w' \in \ker\, dJ(x)$. We prove that

$$J_\mu^* \Omega_\mu(v, w) = \omega(\xi_P(x), \eta_P(x)), \tag{4.2}$$

[3]Remark 3 reduces to Kostant's theorem on homogeneous Hamiltonian G–spaces if G acts transitively; cf. Guillemin and Sternberg (1977).

which implies the claim. To see 4.2, write

$$\begin{aligned}
J_\mu^*(\Omega_\mu)(v,w) =& \Omega_\mu(dJ(x)\cdot v, dJ(x)\cdot w)\\
=&\Omega_\mu(dJ(x)\cdot \xi_P(x), dJ(x)\cdot \eta_P(x))\\
=&\Omega_\mu(\xi_{\mathfrak{g}^*}(\bar{\mu}), \eta_{\mathfrak{g}^*}(\bar{\mu}))\,\text{by}(1))\\
=&\langle \bar{\mu}, [\xi,\eta]\rangle(\text{ definition of } \Omega_\mu)\\
=&\langle J(x), [\xi,\eta]\rangle\\
=&\{\langle J,\xi\rangle, \langle J,\eta\rangle\}(x) \text{ (by equivariance)}\\
=&\omega(\xi_P(x), \eta_P(x))\,\text{since}\, X_{\langle J,\xi\rangle} = \xi_P \text{ by definition of } J).
\end{aligned}$$

By definition of the reduced form ω_μ and π_μ,

$$\pi_\mu^*\omega_\mu(\xi_P(x)+u, \eta_P(x)+v) = \omega(u,v) \tag{4.3}$$

for $u, v \in \ker\, dJ(x)$. The above proposition therefore follows from 4.2, 4.3 and the ω–orthogonality of $T_x(G\cdot x)$ and $\ker\, dJ(x)$. ■

4. Now we are ready to discuss our local symplectic decomposition. Fix a point $x_0 \in P$. The group action defines an isotropic embedding

$$i : G_{\mu_0} \to G_{\mu_0}\cdot x_0, \text{where}\, \mu_0 = J(x_0).$$

The symplectic normal bundle has fiber at $g \in G_{\mu_0}$ given by

$$\begin{aligned}
Ti(T_gH_{\mu_0})^\perp/T_i(T_gG_{\mu_0}) =& T_x(G_{\mu_0}\cdot x_0)^\perp/T_x(G_{\mu_0}\cdot x_0), \quad x = g\cdot x_0\\
=&[T_x(G\cdot x_0)\cap \ker dJ(x)]^\perp/T_x(G_{\mu_0}\cdot x_0)(\text{Remark 1(a)})\\
=&[T_x(G\cdot x_0)+ \ker dJ(x)]/T_x(G_{\mu_0}\cdot x_0)(\text{Remark 1(b)})\\
=&T_x(J^{-1}(\mathcal{O}_{\mu_0}))/T_x(G_\mu\cdot x_0).(\text{Remark 2(b)})
\end{aligned}$$

Next, embed G_{μ_0} as an isotropic submanifold of $T^*G_{\mu_0}\times P_{\mu_0}\times \mathcal{O}_{\mu_0}$ by $g \mapsto 0_g \times [g\cdot x_0]\times g\cdot\mu_0$, where $0_g \in T^*G_{\mu_0}$ is the zero element at g.

5. By Proposition 2(a), these two embeddings have naturally isomorphic symplectic normal bundles. Thus, Weinstein (1977) isotropic embedding theorem gives:

Theorem 4.4. *There is a neighborhood U of $G_{\mu_0}\cdot x_0$ in P and V of G_{μ_0} in $T^*G_{\mu_0}\times P_{\mu_0}\times \mathcal{O}_{\mu_0}$ and a symplectic diffeomorphism $F : U \to V$. The construction of F shows also that it is natural with respect to $J^{-1}(\mathcal{O}_{\mu_0})$:*

$$\begin{array}{ccc} U \cap J^{-1}(\mathcal{O}_{\mu_0}) & \xrightarrow{\quad F \quad} & V \cap (G_{\mu_0} \times P_{\mu_0} \times \mathcal{O}_{\mu_0}) \\ \Big\downarrow {\scriptstyle \pi_{\mu_0} \times J_{\mu_0}} & & \Big\downarrow {\scriptstyle \text{Project}} \\ P_{\mu_0} \times \mathcal{O}_{\mu_0} & \xrightarrow[\text{Identity}]{} & P_{\mu_0} \times \mathcal{O}_{\mu_0} \end{array}$$

It would be of interest to investigate when this can be done globally; for such a discussion in the abelian case, see Duistermaat (1980). In particular, this reference isolates a global obstruction in this case and shows that it actually occurs in the spherical pendulum.

6. As Weinstein (1977) notes, the diffeomorphism F above still has some ambiguity. We now show how to remove this on the infinitesimal level, and how to relate it to our infinitesimal decomposition above. Let $\langle\langle \cdot, \cdot \rangle\rangle$ be a Riemannian metric on P that is G–invariant, and $\mathbb{J}$ a complex structure, also G–invariant, such that

$$\omega(u, v) = \langle\langle \mathbb{J}u, v \rangle\rangle.$$

Fix $x \in P$, and let $W_x =$ the $\langle\langle \cdot, \cdot \rangle\rangle$–orthogonal complement of $T_x(J^{-1}(\mathcal{O}_\mu))$. Then W_x is isomorphic to $T_x(G_\mu \cdot x)$ by the map $\mathbb{J}$; see Lemma 4.1 (b). Identify $T_x(G_\mu \cdot x)$ with $T_x^*(G_\mu \cdot x)$, and hence with $\mathfrak{g}_\mu^*$ by $\langle\langle \cdot, \cdot \rangle\rangle$.

Proposition 4.5 (6a). *Let (V, ω) be a symplectic vector space, $E \subset V$ a subspace and suppose*

$$(E + E^\perp) \oplus W = V.$$

Then V has the following ω–orthogonal decomposition into symplectic subspaces:

$$V = [W \oplus (E \cap E^\perp)] \oplus [E \cap W^\perp] \oplus [E^\perp \cap W^\perp].$$

The proof is a direct verification.

Applying Proposition 6(a) to $T_xP = V, E = T_x(G \cdot x), E^\perp = \ker dJ(x)$ and $W = W_x \approx \mathfrak{g}_\mu^*$ gives

Proposition 4.6 (6b). *There is a symplectic decomposition*

$$T_xP \approx (\mathfrak{g}_\mu \times \mathfrak{g}_\mu^*) \oplus T_{[x]}P_\mu \oplus T_\mu\mathcal{O}_\mu.$$

If μ is regular, this is the same as the decomposition with which we began this example.

(*Example 2 Simple mechanical systems*) Sometimes one can identify reduced Hamiltonian systems a little more explicitly than just as $(P_\mu, \omega_\mu H_\mu)$. This is the case, for instance, with the simple mechanical systems studied by Smale (1970); see Satzer (1977) and Abraham and Marsden (1978).

Let Q be a Riemannian manifold and $K : T^*Q \to \mathbb{R}$ the kinetic energy (we work on the cotangent bundle for convenience only). Let G act on Q and hence on T^*Q; assume that the action is by isometries (leaves K invariant), and let $J : T^*Q \to \mathfrak{g}, \hat{J}(\xi, \alpha_q) = \alpha_{\mathfrak{g}}(\xi_Q(q))$ be the associated momentum mapping. Assume that μ is a regular value for J and that the action of G_μ on Q is free and proper (so Q/G_μ is a manifold).

Define a one–form α_μ on Q at q by minimizing K over the α_q such that $J(\alpha_q) = \mu$. (This set of α_q is an affine subspace of $T^*_x Q$.) One can check that α_μ is G_μ–equivariant.

Let $Q_\mu = Q/G_\mu$, let $\hat{\alpha}_\mu$ be the one–form on Q_μ induced from α_μ, and then regard (by pullback) $\hat{\alpha}_\mu$ as a one–from on T^*Q_μ. Let

$$\Omega_\mu = \omega_0 + d\hat{\alpha}_\mu,$$

where ω_0 is the canonical symplectic form on T^*Q_μ.

Now $T^*Q_\mu \approx \{\alpha_q \in T^*Q | \alpha_q \cdot \xi_Q(q) = 0 \text{ for all } \xi \in \mathfrak{g}_\mu\}/G_\mu$, and we can embed $J^{-1}(\mu)/G_\mu$ into T^*Q_μ by $\alpha_q \mapsto \alpha_q - \alpha_\mu(q)$ and passing to the quotient. One verifies that this produces a symplectic embedding

$$\phi_\mu : P_\mu \hookrightarrow T^*Q_\mu.$$

If $\mathfrak{g} = \mathfrak{g}_\mu$ (e.g., G is abelian) then ϕ_μ is onto.

By a *simple mechanical system*, we mean a Q as above together with a G–invariant potential $V : Q \to \mathbb{R}$. Thus we get a G–invariant Hamiltonian $H = K + V$ on T^*Q, and hence an induced Hamiltonian on P_μ. The corresponding Hamiltonian system on T^*Q_μ is also a simple mechanical system: there is a canonically induced metric but we use the *amended potential:*

$$V_\mu(q) = V(q) + K(\alpha_\mu(q)).$$

The verification is not difficult. The presence of the α_μ in the symplectic structure Ω_μ on T^*Q_μ means that the system on T^*Q_μ may be regarded as a particle moving under the combined influence of a potential V_μ and a *magnetic potential* α_μ. (This phenomenon was first pointed out in a special case by Whittaker (1959), as far as we know.) ◆

Example 2a A seemingly simple but interesting special case was pointed out by Alan Weinstein. Consider n harmonic oscillators in the plane. Choose $Q = (\mathbb{R}^2)^n \backslash \{0\}$ with coordinates $((x_1, y_1), \ldots, (x_n, y_n))$, and let $V((x_1, y_1), \ldots, (x_n, y_n)) = 1/2\{\alpha_1(x_1^2 + y_1^2) + \ldots + \alpha_n(x_n^2 + y_n^2)\}, \alpha_i > 0$. Let $G = S^1$ act on Q by rotation in each $\mathbb{R}^2$, and let $\mu \neq 0$ be given. One checks that, with the Euclidean metric on Q,

$$\alpha_\mu(x_1, \ldots, y_n) = \frac{(-y_1 dx_1 + x_1 dy_1) + \ldots + (-y_n dx_n + x_n dy_n)}{x_1^2 + y_1^2 + \ldots + x_n^2 + y_n^2}\mu.$$

Regard $Q = \mathbb{C}^n \backslash \{0\} \approx S^{2n-1} \times \mathbb{R}^+$, so that S^1 acting on Q gives an induced action on the complex sphere S^{2n-1}; its quotient is the complex projective space $\mathbb{C}P^{n-1}$; the S^1 bundle S^{2n-1} over $\mathbb{C}P^{n-1}$ is the *Hopf fibration.* Thus

$$Q_\mu \approx \mathbb{C}P^{n-1} \times \mathbb{R}^+.$$

The metric on Q_μ is $dr^2 + r^2 d\Omega^2$, where $d\Omega^2$ is the standard metric on $\mathbb{C}P^{n-1}$. The amended potential is given by $V_\mu((x_1, y_1), \ldots, (x_n, y_n)) = V((x_1, y_1), \ldots, (x_n, y_n)) + K(\alpha_\mu((x_1, y_1), \ldots, (x_n, y_n)))$ as before.

The original Hamiltonian system contains stable periodic orbits with angular momentum μ; these are stable fixed points of the reduced system on T^*Q_μ. However, V_μ need not have minima there. Their stability may be viewed as an instance of *magnetic stabilization.* (The special case $n = 2$ results in the motion of a particle in $\mathbb{C}P^1 \times \mathbb{R}^+ \approx \mathbb{R}^3 - \{0\}$ under the influence of a *magnetic monopole.*)[4] ♦

(*A particle moving in a Yang–Mills field*). See Kerner (1968), Torrence and Tulczyjew (1973), Sternberg (1977), Menzio and Tulczyjew (1978), Weinstein (1978b) and Sniatycki (1979). Sternberg proceeds as follows.

Let $\pi : B \to Q$ be a principal G–bundle and (P, ω) a symplectic manifold. Let G act symplectically on P and J be an Ad^*–equivariant momentum mapping. Let $\tau : T^*Q \to Q$, and let $\tilde{\pi} : \widetilde{B} \to T^*Q$ be the pullback bundle by τ. Let $\widetilde{B} \times_G P$ be the associated bundle over T^*Q, *i.e.,* $(\widetilde{B} \times P)/G$. A *connection* γ on B (A Yang–Mills field) is a splitting $\gamma_b : T_q Q \to \mathfrak{g} \hookrightarrow T_b B$ for the sequence

$$0 \to \mathfrak{g} \to T_b B \to T_q Q \to 0;$$

i.e., γ is a $\mathfrak{g}$–valued one–form on B. From this we can construct a symplectic structure ω_γ on $\widetilde{B} \times_G P$ as follows. Let $\tilde{\gamma}$ be the induced connection on $\widetilde{B}$. Then the pairing $\langle \gamma, J \rangle$ may be regarded as a one–form on $\widetilde{B} \times P$. Then $\tilde{\omega}$ be the pullback on $\widetilde{B} \times P$ of ω on P via projection on the second factor. Then

$$d\langle \gamma, J \rangle + \tilde{\omega}$$

is a two–form on $\widetilde{B} \times P$, and one checks that it is G–invariant (see the momentum lemma in Abraham and Marsden (1978)), and so is defined on $\widetilde{B} \times_G P$. Let $\tilde{\omega}_0$ be the pullback of the canonical symplectic structure on T^*Q to $\widetilde{B} \times_G P$. Then let

$$\Omega = \tilde{\omega}_0 + d\langle \gamma, J \rangle + \tilde{\omega}.$$

One can check that Ω is nondegenerate, and so is a symplectic form.

Given a Hamiltonian $H : T^*Q \to \mathbb{R}$, we can let $\widetilde{H}$ be H composed with the projection $\widetilde{B} \times_G P \to T^*Q$, and then *the Hamiltonian system describing*

[4]Related examples and remarks have been given by Miller (1976) and Kummer (1981).

the motion of the particle in the Yang–Mills field is $(\widetilde{B} \times_G P, \omega_\gamma, \widetilde{H})$. (Usually one chooses P to be a coadjoint orbit in $\mathfrak{g}^* \ldots$; for electromagnetism P is a point $\{e\}$, the charge).

Weinstein proceeds as follows. The *right* action of G on B lifts to a symplectic action of G on T^*B with momentum map $J_B : T^*B \to \mathfrak{g}^*$; on the fiber over $b \in B, J_B$ is the dual of the natural inclusion $\mathfrak{g} \to T_bB$. The momentum map of the associated *left* action is $-J_B$. The momentum map for the action of G on $T^*B \times P$ is thus $-J_B + J$. From the reduced symplectic manifold at the regular value $\mu = 0$:[5]

Now let $H : T^*Q \to \mathbb{R}$ and γ be given. The dual of the connection defines a map $\gamma^* : T^*B \to T^*Q$, which is constant on G–orbits and so induces a projection $\pi_\gamma : (T^*B \times P)_0 \to T^*Q$. Let $H_\gamma = H \circ \pi_\gamma$. The Hamiltonian system is now $((T^*B \times P)_0, \omega_0, H_\gamma)$.

These two Hamiltonian systems are isomorphic by a diffeomorphism constructed as follows: the map $\gamma^* : T^*B \to T^*Q$ naturally factors through a map $\tilde{\gamma}^* : T^*B \to \widetilde{B}$, so we get a G- equivariant map $\tilde{\gamma}^*_P : T^*B \times P \to \widetilde{B} \times P$. The restriction of $\tilde{\gamma}^*_P$ to $(-J_B + J)^{-1}(0)$ is a diffeomorphism, and hence we get an induced diffeomorphism γ_G of the G–quotients, *i.e.,* of $(T^*B \times P)_0$ with $\widetilde{B} \times_G P$. This diffeomorphism is verified to be symplectic, and to map $\widetilde{H}$ to H_γ. ♦

The situation is summarized as follows:

$$\left\{ \begin{array}{c} \text{Modify } H \text{ via} \\ \text{connection } \gamma\text{, but} \\ \text{use a universal} \\ \text{symplectic} \\ \text{structure} \end{array} \right\} \overset{\text{isomorphic}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{“Universal” } H \\ \text{but symplectic} \\ \text{structure depends} \\ \text{on } \gamma \end{array} \right\}$$

[5]This construction is an abstraction of that in Torrence and Tulczyjew (1973); the level set $(-J_B + J)^{-1}(0)$ plays the role of a constraint in the sense of the Dirac theory of constraints. We remark that level sets of momentum maps seem to play this role quite generally; we note that $J^{-1}(\mu)$ is first class (coisotropic) if and only if $\mathfrak{g} = \mathfrak{g}_\mu$. In particular, the constraint here is first class. (See the references in the Introduction.)

$$
\begin{tikzcd}
(T^*B \times P)_0 \arrow[rr, "\gamma_G"] \arrow[d, "\pi_\gamma"'] & & \widetilde{B} \times_G P \arrow[d] \\
T^*Q \arrow[dr, "H"'] & & T^*Q \arrow[dl, "H"] \\
& \mathbb{R} &
\end{tikzcd}
$$

Examples 2 and 3 are related as follows. For the special case of the motion of a charged nonrelativistic particle in an electric potential and a magnetic field, we consider a principal circle bundle $\pi : B \to Q$ over a Riemannian manifold Q. The momentum map for the natural S^1 action on T^*B is $J(\alpha_b) \cdot \xi = \langle \alpha_b, \xi_B(b) \rangle$. The level set $J^{-1}(e)$ corresponds to the *charge constraint*. Given a metric on B we can reduce a simple mechanical system on B at $\mu = e$ to obtain the motion of a particle in $B_e \approx Q$ in the presence of a magnetic field (electromagnetic if Q is space–time[6]).

This discussion is in fact a reformulation of the examples in Menzio and Tulczyjew (1978). Passing to Example 3 corresponds to shifting the momentum from e to 0 by enlarging T^*B to $T^*B \times \{e\}$ and reducing at 0 instead. For a discussion of the relationship between the methods described here and the original one of Kerner (1968), see Sniatycki (1979).

As Guillemin and Sternberg (1978) point out, the method of Sternberg above is obtained if one derives the motion of a particle by the Einstein–Infeld–Hoffman limiting procedure done on the four–dimensional coupled Einstein–Yang–Mills system. (The limit is done as the Yang–Mills source concentrates on a time–like line.) This hints that the Sternberg method is "more basic". However, one can also treat the Einstein–Yang–Mills equations as a Hamiltonian system *dynamically*, as in Arms (1979). It seems that if the limit is taken here, one recovers Weinstein's picture. Thus our view is that both are equally basic.

There are many more applications of reduction, some of which are given in the next lecture. For an application to the traction problem is elasticity (and the "significance" of Korn's second inequality), see Marsden and Hughes (1978).

For an application of reduction to plasma physics, see Marsden and Weinstein (1982).

[6] In the relativistic case one also has a mass constraint corresponding to a level set of the momentum map for the action of $\mathbb{R}$ in the homogeneous formulation ([Abraham and Marsden (1978), p. 235]).

5

Two Completely Integrable Systems

A Hamiltonian system with symmetry is called *completely integrable* when its reduced phase space is a point. Under rather general hypotheses, Mishchenko and Fomenko (1978b) proved that this implies complete integrability via an abelian group; *i.e.,* one has integrals in involution. We shall now describe two classical examples of such systems, the Calogero system and the Toda lattice. Both are closely related to reduction, but each in a different, somewhat unexpected way. An excellent general reference is Moser (1980).

As indicated in the previous lecture, reduction provides a method of producing complicated Hamiltonian systems out of simple ones. This principle applies to the Calogero system, a result due to Kazhdan et al. (1978).

Definition 5.1. *The Calogero system is the system of n particles on the line, i.e., $P = T^*\mathbb{R}^n$, with*

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \frac{1}{2}\sum p_i^2 + \frac{1}{2}\sum_{i \neq j} \frac{1}{(q_i - q_j)^2}.$$

Remarks 1. The system is translation invariant, so that $p_1 + \cdots + p_n$ is conserved. Passing to the center of mass coordinates (reduction), we shall assume $p_1 + \ldots + p_n = 0, q_1 + \cdots + q_n = 0$. As we shall see, this system is completely integral. The $n-1$ independent integrals in involution, $H = f_1, \ldots, f_{n-1}$, will be exhibited as traces of powers of a certain matrix.

2. The Calogero system is related to the KdV equation $u_t - 3uu_x + (1/2)u_{xxx} = 0$ in the following way, called a *pole expansion*. Consider the locus

$$L = \{(q,p) \mid \operatorname{grad} H(q,p) = 0\} = \left\{ (q,0) \;\middle|\; \sum_{\substack{j,k=1 \\ k \neq j}}^{n} (q_j - q_k)^{-3} = 0 \right\}.$$

The restriction to L of the f_2–flow $\ddot{q}_i = 6\sum_{k\neq j}(q_j - q_k)^{-2}$ is the same as the time–reversed KdV flow of $u(x,t) = i\sum_{j=1}^{n}(x - q_j(t))^{-2}$. $L = \varnothing$ if all q_j are real. If q_j are complex and $n = d(d+1)/2$ for some $d \in \mathbb{N}, L \neq \varnothing$ and its closure $\bar{L}$ is diffeomorphic to $\mathbb{C}^d$; see Airault et al. (1977) and Chudnovsky and Chudnovsky (1977).

3. Another interesting pole expansion whose complete integrability is not known is one associated to the two–dimensional Euler equations by a vortex approximation. One ends up with n one–dimensional particles moving via the Hamiltonian

$$H(\vec{x}_1, \ldots, \vec{x}_n) = -\frac{1}{4\pi}\sum_{i\neq j}\Gamma_i\Gamma_j \log \|\, \vec{x}_i - \vec{x}_j \,\|,$$

where $\vec{x}_i \in \mathbb{R}^2$ and Γ_i are constants. (See Chorin and Marsden (1979), p. 85.) The vector $(\Gamma_1, \ldots, \Gamma_n)$ is a discretization of the vorticity and plays the role of $\mu \in g^*$ in reduction.

4. A bridge between the equations one is studying and Lie group methods is the Lax equation. We shall bypass this for economy and pass directly to the Lie group structure associated with the Calogero system. The next remark will be the general context.

5. Let G be a semisimple Lie group with Killing form $\langle\cdot,\cdot\rangle$, and let G act on $\mathfrak{g} \times \mathfrak{g}$ by two copies of the adjoint representation

$$\Phi_g(\xi, \eta) = (\operatorname{Ad}_g \xi, \operatorname{Ad}_g \eta).$$

The infinitesimal generator of $\xi \in \mathfrak{g}$ is

$$\xi_{\mathfrak{g}\times\mathfrak{g}} = (\operatorname{ad}\xi, \operatorname{ad}\xi).$$

Put on $\mathfrak{g}\times\mathfrak{g} = T_{\mathfrak{g}}$ the symplectic structure associated with the Killing form

$$\omega((\xi_1,\eta_1),(\xi_2,\eta_2)) = \langle \eta_2, \xi_1\rangle - \langle \xi_2, \eta_1\rangle.$$

The above action is clearly symplectic, and the associated momentum map is given by

$$J : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}^*, \quad J(\xi, \eta) = \langle [\eta, \xi], \cdot \rangle.$$

Thus, if $\varepsilon \in \mathfrak{g}^* \approx \mathfrak{g}$, the isotropy group of ε is

$$G_\varepsilon = \{g \in G \mid \mathrm{Ad}_g \, \varepsilon = \varepsilon\},$$

and the reduced space

$$(\mathfrak{g} \times \mathfrak{g})_\varepsilon = \frac{J^{-1}(\varepsilon)}{G_\varepsilon}$$

is a symplectic manifold.

If $f_1, \ldots, f_n$ are functions in involution on $\mathfrak{g} \times \mathfrak{g}$ and are G–invariant, then the corresponding induced functions $(f_i)_\varepsilon$ on $(\mathfrak{g} \times \mathfrak{g})_\varepsilon$ are also in involution by our general remarks on reduction. For the Calogero system one can find $f_1, \ldots, f_n$ by inspection, although the direct discovery of $(f_1)_\varepsilon, \ldots, (f_n)_\varepsilon$ is by no means transparent.

6. Now consider a specific case of Remark 5. Make the following choices: $G = SU(n)$, so that $\mathfrak{g} = su(n) = n \times n$ traceless skew Hermitian matrices, and

$$\varepsilon = - \begin{pmatrix} 0 & i & \ldots & i \\ i & \ldots & i & \vdots \\ \vdots & i & \ldots & i \\ i & \vdots & i & 0 \end{pmatrix}.$$

One computes that $J^{-1}(\varepsilon) = \{(\eta, \xi) |$ there is a $g \in G$ such that $\eta = \mathrm{Ad}_g \, \delta, \xi = \mathrm{Ad}_g \, \lambda$, where δ is some diagonal matrix

$$\delta = \begin{pmatrix} q_1 & 0 \\ \ldots & \ldots \\ 0 & q_n \end{pmatrix}$$

and $\lambda = (\lambda_{ij})$ is of the form $\lambda_{ij} = 1/(q_i - q_j), i \neq j$ with λ_{ij} arbitrary$\}$. Also,

$$G_\varepsilon = \left\{ g \in G | g \, \text{has} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \text{ as an eigenvector} \right\}.$$

It follows that $(\mathfrak{g} \times \mathfrak{g})_\varepsilon$ is identifiable with the set of pairs (δ, λ) of the stated form and hence with $T^*(\mathbb{R}^n)$, where $p_j = i\lambda_{ij}$, minus collision points where $q_i = q_j$ for some $i \neq j$. The reduced symplectic form is exactly $\omega_\varepsilon = \sum_k dp_k \wedge dp_k$. Define $H : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by $H(\eta, \xi) = 1/2$ Re trace (ξ^2). This is G–invariant, and a computation shows that the reduced Hamiltonian H_ε is minus the Calogero Hamiltonian. On $\mathfrak{g} \times \mathfrak{g}$, Hamilton's equations are

$$\dot{\lambda} = -[\zeta, \lambda], \text{ (a Lax–type equation) },$$

$$\dot{\delta} = -(\operatorname{grad} H)(\lambda) - [\zeta, \delta],$$

where

$$\zeta_{jk} = i\delta_{jk} \sum_{k \neq l} \frac{1}{(q_k - q_l)^2} - i(1 - \delta_{jk}) \frac{1}{(q_j - q_k)^2}.$$

Let $f_k(\eta, \xi) = 1/(k+1)$ Re trace $(\xi^{k+1}), k = 1, \ldots, n$, so that $f_1 = H$ and $(f_k)_\varepsilon = 1/(k+1)$ trace (λ^{k+1}). Since the f_k are functions only of ξ, it is obvious that $\{f_k, f_j\} = 0$ and hence $\{(f_k)_\varepsilon, (f_j)_\varepsilon\} = 0$. Chevalley's theorem[1] implies independence of the f_k (see Moser (1980) for a direct proof). This proves complete integrability of the Calogero system.

7. Similar things hold for the Moser–Sutherland system in which one uses the potential $1/\sin^2(q_i - q_j)$ and $G = SU(n)$. This time it is convenient to use G acting on TG rather than on $\mathfrak{g} \times \mathfrak{g}$. For the $\sinh^{-2}$ potential or $\mathfrak{p}$–potential ($\mathfrak{p}$ = Weierstrass $\mathfrak{p}$–function), one replaces G by "Kac–Moody groups."

◆

Next we turn to the Toda lattice. This time we shall get the integrals in involution not by inspection but by the application of the Kostant–Symes theorem (see Kostant (1979) and Symes (1979), Symes (1980)). We shall formulate it in a way that clarifies the role of reduction.[2] The setup is as follows.

Let (P, ω, G, J) be a Hamiltonian G–space. Let $i : R \to P$ be a symplectic submanifold that is invariant under a subgroup $H \subset G$. Thus, $(R, i^*\omega, H, j)$ is a Hamiltonian H–space, where the momentum mapping is $j = \pi_{\mathfrak{h}^*} \circ J|R, \pi_{\mathfrak{h}^*} : \mathfrak{g}^* \to \mathfrak{h}^*$ being the canonical projection and

[1] Let $\mathfrak{g}$ be a Lie subalgebra of $sl(n)$. Then the functions $g_k(A) = tr(A^k), k = 2, \ldots, n$, are independent. This may be regarded as a theorem of algebraic geometry.

[2] What follows represents joint work with T. Ratiu. See Ratiu (1980a) for more information. Another approach which directly uses Weinstein's co–normal reduction theorem (Weinstein (1977), pp. 25–26) and Remark 4 in Lecture 1 was recently obtained by Symes (1980).

Assumptions 1. The Lie algebra splits $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$, where the sum is a vector space sum.

2. $\mu \in \mathfrak{h}^*$ is a regular value for j and H_μ acts freely and properly on $j^{-1}(\mu)$, so the reduced space R_μ is a manifold. Assume each $x \in j^{-1}(\mu)$ is a regular point for J.

3. For any $\xi, \eta \in \mathfrak{t}$, one has the identity

$$\{\hat{J}_\xi \mid R, \hat{J}_\eta \mid R\} = \{\hat{J}_\xi, \hat{J}_\eta\} \mid R.$$

♦

Remark on Assumption 3. An example of sufficient conditions to check the validity of 3 is as follows. Suppose $P = T^*Q_1$ and $R = T^*Q_2$, where $Q_2 \hookrightarrow Q_1$ and where J is induced by a cotangent action; *i.e.,* the lift of an action on Q_1 and H leaves Q_2 invariant; assume that the infinitesimal generators $\{\xi_{Q_1} \mid \xi \in \mathfrak{t}\}$ are integrable, producing a foliation $\mathcal{F}$ of manifolds whose tangent space meets T_xQ_2 only in $\{0\}$ for each $x \in Q_2$. (Roughly speaking, there is a K–action for a subgroup $K \subset G$ whose orbits are transverse to the H–invariant subspace Q_2). Then an easy computation in local coordinates shows that each side of the required identity is zero. ♦

Theorem 5.2. *Let Assumptions 1, 2 and 3 hold and let f and g be G–invariant functions on P that are constant on each surface $J^{-1}(\hat{\mu})$, where $\hat{\mu} \in \{\mu\} \oplus \mathfrak{t}$.*[3] *Then*

$$\{f_\mu, g_\mu\} = 0 \,\text{on}\, R_\mu,$$

where f_μ and g_μ are the functions induced on R_μ by f and g.

Proof. The conclusion is equivalent to $\{f \mid R, g \mid R\}(x) = 0$ for $x \in j^{-1}(\mu)$. Since f and g are G–invariant, $X_f(x)$ and $X_g(x)$ belong to $\ker T_xJ$. Since f and g are constant on $J^{-1}(\hat{\mu})$, where $\hat{\mu} = J(x) \in \{\mu\} \oplus \mathfrak{t}^*$, $X_f(x)$ and $X_g(x)$ are ω–orthogonal to $\ker T_xJ$. Therefore $X_f(x)$ and $X_g(x)$ belong to $T_x(G_{\hat{\mu}} \cdot x)$ (see Comment 1 following the main theorem on reduction in Lecture 3). Thus there exist $\xi\eta \in \mathfrak{g}_{\hat{\mu}}$ such that

$$X_f(x) = \xi_P(x) \,\text{and}\, X_g(x) = \eta_P(x).$$

Write, according to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$,

$$\xi = \xi' + \xi'' \,\text{and}\, \eta = \eta' + \eta'',$$

[3]Secretly, this means that f and g are constant when reduced by $\hat{\mu}$.

so that

$$X_f(x) = \xi'_P(x) + \xi''_P(x) = \xi'_R(x) + \xi''_P(x)$$

(since H leaves R invariant and $\xi' \in \mathfrak{h}$), with a similar formula for $X_g(x)$.

The following Lie algebra identity holds:

$$\langle \hat{\mu}, [\xi'', \eta'] \rangle = -\langle \hat{\mu}, [\xi'', \eta''] \rangle.$$

Indeed, $\eta \in \mathfrak{g}_{\hat{\mu}}$, so $\langle \hat{\mu}, [\xi'', \eta] \rangle = 0$.

Now $T_xP = T_xR \oplus (T_xR)^\perp$, a symplectic orthogonal decomposition. Let $\pi_x : T_xP \to T_xR$ be the projection onto the first factor. One has this general fact for symplectic submanifolds:

$$X_{f|R}(x) = \pi_x \cdot X_f(x).$$

Thus

$$\begin{aligned}
\{f \mid R, g \mid R\}(x) =& \omega_x(\pi_x \cdot X_f(x), X_{g|R}(x)) \\
=& \omega_x(X_f(x), X_{g|R}(x)) \\
=& \omega_x(\xi'_R(x) + \xi''_P(x), X_{g|R}(x)) \\
=& \omega_x(\xi'_R(x), X_{g|R}(x)) + \omega(\xi''_P(x), X_{g|R}(x)).
\end{aligned}$$

The first term vanishes by conservation of j on R. Thus

$$\begin{aligned}
\{f \mid R, g \mid R\}(x) =& \omega_x(\xi''_P(x), X_{g|R}(x)) \\
=& \omega_x(\xi''_P(x), \pi_x \cdot X_g(x)) \\
=& \omega_x(\xi''_P(x), \eta'_R(x) + \pi_x \cdot \eta''_P(x)) \\
=& \omega_x(\xi''_P(x), \eta'_P(x)) + \omega_x(\xi''_P(x), \pi_x \cdot \eta''_P(x)) \\
=& \{\hat{J}_{\xi''}, \hat{J}_{\eta'}\}(x) + \omega_x(\pi_x \cdot \xi''_P(x), \pi_x \cdot \eta''_P(x)) \\
=& \hat{J}_{[\xi'', \eta']}(x) + \{\hat{J}_{\xi''} \mid R, \hat{J}_{\eta''} \mid R\}(x),
\end{aligned}$$

by Ad*–equivariance of J. The first term is $\langle \hat{\mu}, [\xi'', \eta'] \rangle = -\langle \hat{\mu}, [\xi'', \eta''] \rangle$ by our earlier remark. By Assumption 3, the second term is $\{\hat{J}_{\xi''}, \hat{J}_{\eta''}\}(x) = \langle \hat{\mu}, [\xi'', \eta''] \rangle$, so they cancel and $\{f \mid R, g \mid R\}(x) = 0$ as required. ■

Corollary 5.3. *(Kostant–Symes theorem). Let G be a Lie group with $H \subset G$ a Lie subgroup. Assume $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$ a vector space sum, where $[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t}$; i.e., $\mathfrak{t}$ is a Lie subalgebra.*

Let $f, g : \mathfrak{g}^* \to \mathbb{R}$ be two functions that are constant on coadjoint orbits in $\mathfrak{g}^*$. Then, for $\mu \in \mathfrak{h}^*$,

$$\{f_\mu, g_\mu\} = 0,$$

where f_μ and g_μ are the restrictions of f and g to the coadjoint orbit of μ in $\mathfrak{h}^*$.

Proof. Apply the theorem with $P = T^*G$ and $R = T^*H$, recalling that passing to the reduced manifolds is equivalent to restricting to coadjoint orbits (see Lecture 3). The condition $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ implies the identity in Assumption 3 by the remark on Assumption 3. ■

One can vary the hypotheses and assume, instead of $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, that $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$. This is useful in the n–dimensional rigid body; see Manakov (1976), Mishchenko and Fomenko (1978b),Mishchenko and Fomenko (1978a), Adler and Van Moerbeke (1980), Adler and Moerbeke (1980) and Ratiu (1980b), Ratiu (1980c).

Our second completely integrable system, the Toda lattice, is defined as follows.

Definition 5.4. *The nonperiodic Toda lattice is the Hamiltonian system of n particles on the line with*

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}.$$

Since Toda's original paper in 1967, there has been a great deal of work on this system and its relation to the KdV equation. See for example Flaschka (1974), Moser (1975),Mckean (1979) and Kostant (1979). We shall confine ourselves to demonstrating its complete integrability using the Kostant–Symes theorem. We proceed to do this by a number of steps, and then make some additional remarks.

1. The system is translation–invariant, so that total momentum $p_1 + \cdots + p_n$ is conserved. Passing to the center of mass coordinates (reduction!), we can assume $p_1 + \cdots + p_n = 0$ and $\sum q_i = 0$.

2. Let $b_i = -p_i$ and $a_i = e^{q_i - q_{i+1}}$, and

$$L = \begin{bmatrix} b_1 & a_1 & & & 0 \\ a_1 & b_2 & a_2 & & \\ & a_2 & & & \\ & & & & a_{n-1} \\ & & & b_n & \\ 0 & & & a_{n-1} & \end{bmatrix}, B = \begin{bmatrix} 0 & a_1 & & & 0 \\ -a_1 & 0 & a_2 & & \\ & -a_2 & 0 & & \\ & & & & a_{n-1} \\ & & & 0 & \\ 0 & & & -a_{n-1} & \end{bmatrix}.$$

The equations of motion can now be written in Lax form:

$$\dot{L} = [B, L], \quad H = \frac{1}{2}\operatorname{trace} L^2.$$

Since B is skew, this implies that the eigenvalues of L are constant in time, the *isospectral* property.

3. Now let $\mathfrak{g} = sl(n, \mathbb{R})$, $\mathfrak{t} = so(n, \mathbb{R})$ and $\mathfrak{h}$ = lower triangular matrices. Let us identify $\mathfrak{h}^*$ with the upper triangular matrices (the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form (A, B) = trace (AB^*)). Let $A \to A^+$ denote taking the upper triangular part of the matrix A. Let

$$\mu = \begin{bmatrix} 0 & 1 & & & & 0 \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \cdots & & \\ & & & & & 1 \\ 0 & & & & & 0 \end{bmatrix} \in \mathfrak{h}^*,$$

and compute that the coadjoint orbit of μ in $\mathfrak{h}^*$ consists of the matrices

$$L^+ = \begin{bmatrix} b_1 & a_1 & & 0 \\ & b_2 & & \\ & & & a_{n-1} \\ 0 & & & b_n \end{bmatrix},$$

and that the canonical symplectic structure on this coadjoint orbit is

$$\omega = \sum_{i=1}^{n-1} db_i \wedge \sum_{j=1}^{n-1} \frac{d\alpha_j}{a_j} = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

The Lax equations are just the Hamilton equations for L^+ on this coadjoint orbit with Hamiltonian $H(A) = 1/2$ trace A^2.

4. Let $f_k : \mathfrak{g}^* \to \mathbb{R}$, $f_k(A) = (1/(k+1))$ trace (A^{k+1}), so $f_1 = H$. Clearly, f_k is constant on coadjoint orbits in $\mathfrak{g}^*$ by invariance of the trace under conjugation. Thus, by the Kostant–Symes theorem, the f_k are in involution on coadjoint orbits in $\mathfrak{h}^*$. They are independent by the Chevalley theorem,[4] so *the Toda lattice is completely integrable.*

5. Kostant (1979) generalizes this procedure to any simple Lie algebra.

6. The periodic Toda lattice is a Hamiltonian system on orbits of Kac–Moody Lie algebras; see Adler and Van Moerbeke (1980), Adler and Moerbeke (1980).

[4]See footnote 1, p. ?.

7. There is a great deal of algebraic geometry behind the Toda lattice and the KdV equation. See the lectures of Mckean (1979) for details and references.

8. The group underlying the KdV equation is not yet completely understood. The work of Adler (1979) and Ebin and Marsden (1970) and comments of Duistermaat suggest that there is good reason both for technical purposes (see Ratiu (1979)) and aesthetic interest, to *regard the KdV equation as a right invariant Hamiltonian system on the group of invertible Fourier integral operators.* (See Ratiu and Schmid (1981).)

6

Bifurcations of a Forced Beam

Symmetries[1] in physical systems are often associated with bifurcations. The remaining lectures all illustrate various facets of this philosophy. This lecture illustrates a particular case of a completely integrable Hamiltonian system undergoing a dynamic symmetry breaking bifurcation as an external parameter is varies.

A physical model will help motivate the analysis. One considers a beam that is buckled by an external load Γ, so that there are two stable equilibrium states and one unstable (see Fig. 6.1). The whole structure is then shaken with a transverse periodic displacement $f \cos \omega t$. The beam moves due to its inertia. In a (related) experiment (see Moon and Holmes (1979) and Tseng and Dugundji (1971)), one observes periodic motion about the two stable equilibria for small f, but as f increases, the motion becomes aperiodic or "chaotic." The mathematical problem is to develop theorems to explain this bifurcation.

Chaotic motion in dynamical systems is now a burgeoning industry; the results given here represent one mechanism among many. See, for example, Cohen and Neu (1979) for a survey of some of the current papers, and Guckenheimer (1979) for a related mechanism in reaction–diffusion equations.

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[1] This lecture is based on joint work with Philip Holmes. For further background and related work, see Holmes (1979a), Holmes (1979b), Chow et al. (1980) and Holmes and Marsden (1978), Holmes and Marsden (1980).

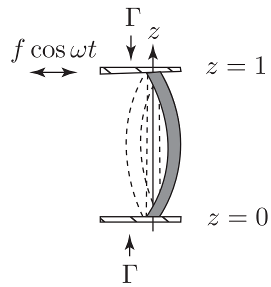

FIGURE 6.1. A buckled beam undergoing periodic forcing.

There are a number of specific models that can be used to describe the beam in Figure 6.1 One such model is the following p.d.e. for the deflection $w(z,t)$ of the center line of the beam:

$$\ddot{w} + w'''' + \Gamma w'' = \kappa \left(\int_0^1 [w']^2 d\zeta \right) w'' = \varepsilon(f \cos\omega t - \delta \dot{w}),$$

where $\cdot = \partial/\partial t, ' = \partial/\partial z, \Gamma =$ external load, $\kappa =$ stiffness, $\delta =$ damping, and ε is a parameter used to measure the relative size of f and δ. We use the "hinged" boundary conditions $w = w'' = 0$ at $z = 0, 1$. We also assume the beam is in its first buckled state, $\pi^2 < \Gamma < 4\pi^2$.

A simpler model is obtained by looking for "lowest mode" solutions of the form $w(z,t) = x(t)\sin(\pi z)$. Substituting into the p.d.e. and taking the inner product with $\sin(\pi z)$, one finds the following Duffing type equation for x:

$$\ddot{x} - \beta x + \alpha x^3 = \varepsilon(\gamma \cos\omega t - \delta x),$$

where

$$\beta = \pi^2(\Gamma - \pi^2) > 0, \quad \alpha = \frac{\kappa\pi^4}{2}, \quad \gamma = \frac{4f}{\pi}.$$

The methods used are inspired by Melnikov (1963); see also Arnold (1964) and Holmes (1979a). We shall set it up in an abstract fashion that applies to the above p.d.e.

It is known that the time t–maps of the Euler and Navier–Stokes equations written in Lagrangian coordinates are smooth. Thus the methods of this paper apply to these equations, in principle. On regions with no boundary, one can regard the Navier–Stokes equations with forcing as a perturbation of a Hamiltonian system (the Euler equations; see Ebin and Marsden (1970)). Thus, if one knew a homoclinic orbit for the Euler equations, then the methods of this paper would produce infinitely many periodic orbits with arbitrarily high period, indicative of turbulence. No specific examples of this are known to us (one could begin by looking at T^2 and studying Arnold (1966)).

Similar situations probably arise in traveling waves and the current–driven Josephson junction. For example, an unforced sine–Gordon equation with damping studied by M. Lévi seems to possess a homoclinic orbit (cf. M. Levi and Miranker (1978)). Presumably the ideas will be useful for the KdV equation as well.

Abstract hypotheses and technical lemmas. We consider an evolution equation in a Banach space X of the form

$$\dot{x} = f_0(x) + \varepsilon f_1(x, t),$$

where f_1 is periodic of period T in t.

Assumption 1

(a) *Assume $f_0(x) = Ax + F(x)$, where A generates a C^0 one–parameter group of transformations on X and where $F : X \to X$ is $C^\infty, F(0) = 0, DF(0) = 0$.*

(b) *Assume $f_1 : X \times S^1 \to X$ is C^∞, where $S^1 = \mathbb{R}/(T)$, the circle of length T.*

(c) *Assume F_t^ε is globally defined, for $\varepsilon \geq 0$ sufficiently small.*

Assumption 1 implies that the associated suspended autonomous system on $X \times S^1$,

$$\dot{x} = f_0(x) + \varepsilon f_1(x, \theta), \quad \dot{\theta} = 1,$$

has a smooth local flow, F_t^ε (by Segal (1962)).

In examples, Assumption 1(c) may sometimes be proved using energy estimates. This is related to the next assumptions (see Lecture 1 for the terminology).

Assumption 2

Assume that the system $\dot{x} = f_0(x)$ (the unperturbed system) is Hamiltonian with energy $H_0 : X \to \mathbb{R}$. Assume that there is a symplectic two–manifold $\Sigma \subset X$ invariant under the flow F_t^0, and that the origin $p_0 = 0 \in \Sigma$ is a saddle point and has a homoclinic orbit$x_0(t)$; i.e.,

$$p_0 = \lim_{t \to +\infty} x_0(t) = \lim_{t \to \infty} x_0(t).$$

See Figure 6.2 (a). In specific applications one must be able to calculate with the orbit $x_0(t)$, either explicitly or numerically.

In the p.d.e. example above, $x_0(t)$ is known analytically (see below). Let $x_0 = x_0(0)$, a conveniently chosen point on the orbit.

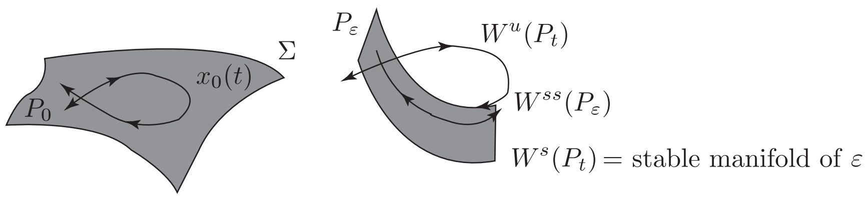

FIGURE 6.2. (a) Phase portrait on Σ for $\varepsilon = 0$. (b) Perturbation of invariant manifolds, $\varepsilon > 0$.

Next we introduce a nonresonance hypothesis.
Assumption 3

(a) *Assume that the forcing term $f_1(x,t)$ in (3) has the form*

$$f_1(x,t) = A_1 x + f(t) + g(x,t),$$

where $A_1 : X \to X$ is a bounded linear operator, f is periodic with period $T, g(x,t)$ is t–periodic with period T and satisfies $g(0,t) = 0, D_x g(0,t) = 0$, so g admits the estimate

$$||g(x,t)|| \leq \text{const}\, ||x||^2,$$

for x in the neighborhood of 0.

(b) Suppose that the "linearized" system

$$\dot{x}_L = Ax_L + \varepsilon A_1 x_L + \varepsilon f(t)$$

has a T–periodic solution $x_L(t,\varepsilon)$ such that $x_L(t,\varepsilon) = O(\varepsilon)$.

For finite dimensional systems, this can be replaced by the assumption that 1 does not lie in the spectrum of e^{TA}; *i.e.*, none of the eigenvalues of A resonate with the forcing frequency.

For the beam problem, with $f(t) = f(z)\cos\omega t$, Assumption 3(b) means that $\omega \neq \pm\lambda_n, n = 1, 2, \ldots,$ where $i\lambda_n$ are the purely imaginary eigenvalues of A. This is seen by solving the component forced linear oscillator equations. As we shall see, more delicate nonresonance requirements would be necessary for general (smooth) T–periodic perturbations.

Next, we need an assumption that A_1 contributes positive damping and that $p_0 = 0$ is a saddle.

Assumption 4

(a) For $\varepsilon = 0, e^{TA}$ has a spectrum consisting of two simple real eigenvalues $e^{\pm\lambda T}, \lambda \neq 0$, with the rest of the spectrum on the unit circle.

(b) For $\varepsilon > 0, e^{T(A+\varepsilon A_1)}$ has a spectrum consisting of two simple real eigenvalues $e^{T\lambda_\varepsilon^\pm}$ (varying continuously in ε from perturbation theory; cf. Kato (1977)) with the rest of the spectrum, σ_R^ε, inside the unit circle $|z| = 1$ and obeying the estimates

$$C_2\varepsilon \leq \operatorname{dist}(\sigma_R^\varepsilon, |z| = 1) \leq C_1\varepsilon$$

for C_1, C_2 positive constants.

In general it can be awkward to estimate the spectrum of e^{TA} in terms of the spectrum of A. Some information is contained in Hille and Phillips (1957) and Carr (1981). For the beam problem with $\varepsilon = 0$, it is sufficient to use these two facts or a direct calculation:

(a) If A is skew adjoint, then $\sigma(e^{tA}) =$ closure of $e^{t\sigma(A)}$.

(b) If $X = X_1 \oplus X_2$, where X_2 is finite dimensional (the eigenspace of the real eigenvalues in the beam problem) and B_1 is skew adjoint on X_1 and $B_2 : X_2 \to X_2$ is a (bounded) linear operator, then

$$\sigma(e^{t(B_1 \oplus B_2)}) = \operatorname{closure}(e^{\sigma t B_1}) \cup e^{t\sigma(B_2)}.$$

The estimate dist $(\sigma_R^\varepsilon, |z| = 1) \geq C_2\varepsilon$ guarantees that

$$L_\varepsilon = \operatorname{Id} - e^{T(A+\varepsilon A_1)}$$

is invertible and

$$||L_\varepsilon^{-1}|| \leq \operatorname{const} / \varepsilon.$$

Finally, we need an extra hypothesis on the nonlinear term. We have already assumed that B vanishes at least quadratically, as does g. Now we assume that B vanishes cubically.

Assumption 5

$$B(0) = 0, \quad DB(0) = 0, \quad D^2B(0) = 0.$$

This means that in a neighborhood of 0,

$$||B(x)|| \leq \text{const}\, ||x||^3.$$

(Actually $B(x) = o(||x||^2)$ would do.)

The necessity of having B vanish cubically is due to the possibility of the spectrum of A accumulating at zero. If this can be excluded for other reasons, Assumption 5 can be dropped and Assumption 4 simplified. There is a similar phenomenon for ordinary differential equations noted by Jack Hale; namely, if the linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} \dot{x} \\ x \\ -\varepsilon y \end{pmatrix}$$

is perturbed by nonlinear terms plus forcing, to guarantee that the trivial solution $(0,0,0)$ perturbs to a periodic solution as in Lemma 6.1 below, one needs the nonlinear terms to be $O(|x| + |\dot{x}| + |y|)^3$.

Consider the suspended system with its flow $F_t^\varepsilon : X \times S^1 \to X \times S^1$. Let $P^\varepsilon : X \to X$ be defined by

$$P^\varepsilon(x) = \pi_1 \cdot (F_T^\varepsilon(x, 0)),$$

where $\pi_1 : X \times S^1 \to X$ is the projection onto the first factor. The map P^ε is just the Poincaré map for the flow F_t^ε. Note that $P^0(p_0) = p_0$, and that fixed points of P^ε correspond to periodic orbits of F_t^ε.

Lemma 6.1. *For $\varepsilon > 0$ small, there is a unique fixed point p_ε of P^ε near $p_0 = 0$; moreover, $p_\varepsilon - p_0 = O(\varepsilon)$; i.e., there is a constant K such that $||p_\varepsilon|| \leq K_\varepsilon$ (for all (small) ε).*

For ordinary differential equations, Lemma 6.1 is a standard fact about persistence of fixed points, assuming 1 does not lie in the spectrum of e^{TA} (*i.e.*, p_0 is hyperbolic). For general partial differential equations, the validity of Lemma 6.1 can be a delicate matter. However, the same *ideas* can be used to prove this and the following technical lemma (see Holmes and Marsden (1980) for details).

Lemma 6.2. *For $\varepsilon > 0$ sufficiently small, the spectrum of $DP^\varepsilon(p_\varepsilon)$ lies strictly inside the unit circle with the exception of the single real eigenvalue $e^{T\lambda_\varepsilon^+} > 1$.*

The method

1. From invariant manifold theory (Hirsch et al. (1977)), there are two invariant curves for P^ε emanating from the fixed point p_ε, say $W^u(p_\varepsilon)$ and $W^{ss}(p_\varepsilon)$, that correspond to the two eigenvalues $\pm\lambda$ of the unperturbed system. See Figure 6.2 (b).

   The Poincaré map P^ε was associated with the section $X \times \{0\}$ in $X \times S^1$. Equally well, we can take the section $X \times \{t_0\}$ to get Poincaré maps $P^\varepsilon_{t_0}$. By definition,

   $$P^\varepsilon_{t_0}(x) = \pi_1(F^\varepsilon_T(x, t_0)).$$

   There is an analogue of Lemmas 1 and 2 for $P^\varepsilon_{t_0}$. Let $P_\varepsilon(t_0)$ denote its unique fixed point and $W^{ss}(p_\varepsilon(t_0))$ and $w^u(p_\varepsilon t_0))$ be its strong stable and unstable manifolds. Assumption 4 implies that the stable manifold $W^s(p_\varepsilon)$ of P_ε has codimension 1 in X. The same is then true of $W^s(p_\varepsilon(t_0))$ as well.

2. Let $\gamma_\varepsilon(t)$ denote the periodic orbit of the (suspended system with $\gamma_\varepsilon(0) = (p_\varepsilon, 0)$. We have

   $$\gamma_\varepsilon(t) = (p_\varepsilon(t), t).$$

   The invariant manifolds for the periodic orbit γ_ε are denoted $W^{ss}(\gamma_\varepsilon), W^s(\gamma_\varepsilon)$ and $W^u(\gamma_\varepsilon)$. We have

   $$\begin{aligned} W^s(p_\varepsilon(t_0)) =& W^s(\gamma_\varepsilon) \cap (X \times \{t_0\}), \\ W^{ss}(p_\varepsilon(t_0)) =& W^{ss}(\gamma_\varepsilon) \cap (X \times \{t_0\}), \\ W^u(p_\varepsilon(t_0)) =& W^\mu(\gamma_\varepsilon) \cap (X \times \{t_0\}). \end{aligned}$$

   See Figure 6.3.

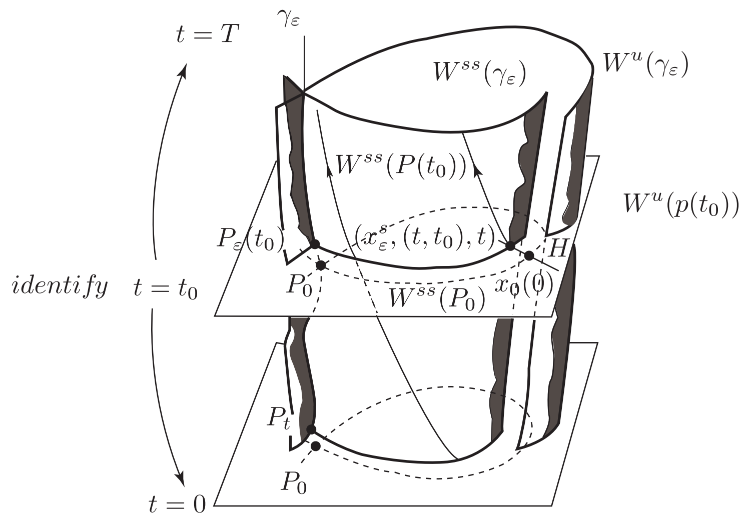

FIGURE 6.3. Invariant manifolds of the suspended system.

3. Our goal is the study of how $W^u(p_\varepsilon(t_0))$ and $W^s(p_\varepsilon(t_0))$ intersect. Preparatory to this goal, we study the perturbation of solution curves in $W^{ss}(\gamma_\varepsilon), W^s(\gamma_\varepsilon)$ and $W^u(\gamma_\varepsilon)$.

   Choose a point, say $x_0(0)$, on the homoclinic orbit for the unperturbed system. Choose a codimension–one hyperplane H transverse to the homoclinic orbit at $x_0(0)$. Since $W^{ss}(p_\varepsilon(t_0))$ is C^r close to $x_0(0)$, it intersects H in a unique point, say $x^s_\varepsilon(t_0,t_0)$. Define $(x^s_\varepsilon(t,t_0),t)$ to be the unique integral curve of the suspended system with initial condition $x^s_\varepsilon(t_0,t_0)$. Define $x^u_\varepsilon(t,t_0)$ in a similar way.

   The initial conditions $x^s_\varepsilon(t_0,t_0)$ and $x^u_\varepsilon(t_0,t_0)$ are not conveniently computable. This difficulty turns out not to be important; as we shall see, this problem is taken care of by the boundary conditions at $t\to\pm\infty$. We have

$$x^s_\varepsilon(t_0,t_0)=x_0(0)+\varepsilon v^s+O(\varepsilon^2),$$

   and

$$x^u_\varepsilon(t_0,t_0)=x_0(0)+\varepsilon v^u+O(\varepsilon^2),$$

   by construction, where $||O(\varepsilon^2)||\leq \mathrm{const}\cdot\varepsilon^2$, and v^s and v^u are fixed vectors. Notice that

$$(P^\varepsilon_{t_0})^n(y^s_\varepsilon(t_0))=x^s_\varepsilon(t_0+nT,t_0)\to p_\varepsilon(t_0)\,\mathrm{as}\,n\to\infty.$$

Since $x^s_\varepsilon(t,t_0)$ is an integral curve of a perturbation, we can write

$$x^s_\varepsilon(t,t_0) = x_0(t-t_0) + \varepsilon x^s_1(t,t_0) + O(\varepsilon^2),$$

where $x^s_1(t,t_0)$ is the solution of the first variation equation,

$$\frac{d}{dt}x^s_1(t,t_0) = Df_0(x_0(t-t_0)) \cdot x^s_1(t,t_0) + f_1(x_0(t-t_0),t),$$

with $x^s_1(t_0,t_0) = v^s$.

4. Let ω denote the (weak) symplectic form on X associated with the Hamiltonian generator f_0. Define the *Melnikov function* by

$$\Delta_\varepsilon(t,t_0) = \omega(f_0(x_0(t-t_0)), x^s_\varepsilon(t,t_0) - x^u_\varepsilon(t,t_0))$$

and set $\Delta_\varepsilon(t_0) = \Delta_\varepsilon(t_0,t_0)$.

5. Let $d = x^s_\varepsilon(t_0,t_0) - x^u_\varepsilon(t_0,t_0)$, and let d_Σ be the component of d along $T_{x_0}\Sigma$ by the symplectic decomposition $X = T_{x_0}\Sigma \oplus (T_{x_0}\Sigma)^\perp$. To leading order, d is parallel to Σ. Suppose that Δ_ε has a simple zero as a function of t_0. This means that d_Σ changes its orientation relative to $f_0(x_0)$ as t_0 changes. Therefore, $x^u_\varepsilon(t_0,t_0)$ must cross from one side of $W^s(p_\varepsilon)$ to the other near x_0. This is the main idea behind the proof of the following:

Lemma 6.3. *If ε is sufficiently small and $\Delta_\varepsilon(t_0)$ has a simple zero at some t_0 and maxima and minima that are at least $O(\varepsilon)$, then $W^u(p_\varepsilon(t_0))$ and $W^s(p_\varepsilon(t_0))$ intersect transversally near $x_0(0)$.*

6. The following formula is used to verify the condition that Δ_ε have a simple zero:

$$\Delta_\varepsilon(t_0) = -\varepsilon \int_{-\infty}^{\infty} \omega(f_0(x_0(t-t_0), f_1(x_0(t-t_0),t)dt + O(\varepsilon^2).$$

Proof. Write

$$\Delta_\varepsilon(t, t_0) = \Delta_\varepsilon^+(t, t_0) - \Delta_\varepsilon^-(t, t_0) + O(\varepsilon^2),$$

where

$$\Delta_\varepsilon^+(t, t_0) = \omega(f_0(x_0(t - t_0)), \varepsilon x_1^s(t, t_0))$$

and

$$\Delta_\varepsilon^-(t, t_0) = \omega(f_0(x_0(t - t_0)), \varepsilon x_1^u(t, t_0)).$$

Now

$$\begin{aligned}\frac{d}{dt}\Delta_\varepsilon^+(t, t_0) =& \omega(Df_0(x_0(t - t_0)) \cdot f_0(x_0(t - t_0)), \varepsilon x_1^s(t, t_0)) \\ &+ \omega(f_0(x_0(t - t_0)), \varepsilon\{Df_0(x_0(t - t_0)) \cdot x_1^s(t, t_0) \\ &+ f_1(x_0(t - t_0), t)\}).\end{aligned}$$

Since f_0 is Hamiltonian, Df_0 is ω–skew. Therefore

$$\frac{d}{dt}\Delta_\varepsilon^+(t, t_0) = \omega(f_0(x_0(t - t_0)), \varepsilon f_1(x_0(t - t_0), t));$$

i.e.,

$$-\Delta_\varepsilon^+(t_0, t_0) = \varepsilon \int_{t_0}^{\infty} \omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt.$$

Similarly,

$$\Delta_\varepsilon^-(t_0, t_0) = \varepsilon \int_{-\infty}^{t_0} \omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt.$$

so adding gives the stated formula. ■

The expression $\int_{-\infty}^{\infty} \omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t)\, dt$ is an "averaged bracket" over the orbit $x_0(t)$; if f_1 is Hamiltonian (time–dependent), this is just an integrated Poisson bracket over the orbit $x_0(t)$. The power of Melnikov's method rests in the fact that this formula renders the leading term of $\Delta_\varepsilon(t_0)$ computable. Before drawing consequences of transversal intersection, we discuss some examples.

Example 1 $\ddot{x} - \beta x + \alpha x^3 = \varepsilon(\gamma\cos\omega t - \delta\dot{x})$. Here the unperturbed system is $\ddot{x} - \beta x + \alpha x^3 = 0$; *i.e.*,

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \beta x - \alpha x^3 \end{pmatrix},$$

which is Hamiltonian in $X = \mathbb{R}^2 = \Sigma$ with

$$H(x, \dot{x}) = \frac{\dot{x}^2}{2} - \frac{\beta x^2}{2} + \frac{\alpha x^4}{4}.$$

The flow of this system is the familiar figure–eight pattern (Fig. 6.4) with a homoclinic orbit given by

$$x_0(t)\sqrt{\frac{2\beta}{\alpha}}\,\mathrm{sech}(\sqrt{\beta}t).$$

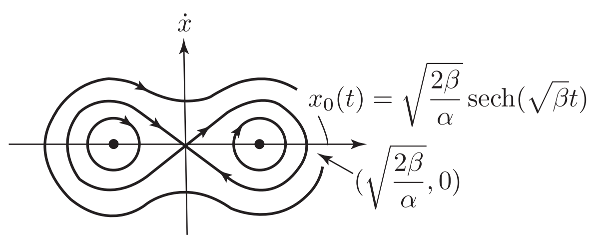

FIGURE 6.4. The homoclinic orbit in Duffing's equation.

The Melnikov function is

$$\begin{aligned}\Delta_\varepsilon(t_0) &= -\,\varepsilon\int_{-\infty}^{\infty}\omega\begin{pmatrix} \dot{x} \\ \beta x - \alpha x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma\cos\omega t - \delta\dot{x} \end{pmatrix} dt + O(\varepsilon^2) \\ &= -\,\varepsilon\int_{-\infty}^{\infty}\dot{x}(\gamma\cos\omega t - \delta x)\,dt + O(\varepsilon^2),\end{aligned}$$

where x stands for $x_0(t - t_0) = \sqrt{2\beta/\alpha}$ sech $\sqrt{\beta}(t - t_0)$. This integral may be computed by using residues:

$$\Delta_\varepsilon(t_0) = \varepsilon\left\{2\gamma\pi\omega\sqrt{\frac{2}{\alpha}}\left(\cosh\left(\frac{\pi\omega}{2\sqrt{\beta}}\right)\right)^{-1}\sin\omega t_0 + \frac{4\delta\beta^{3/2}}{3\alpha}\right\} + O(\varepsilon^2).$$

Thus, if

$$\gamma > \gamma_c \equiv \frac{2\delta\beta^{3/2}}{3\omega\sqrt{2\alpha}}\cosh\left(\frac{\pi\omega}{2\sqrt{\beta}}\right).$$

Then for ε small, Δ_ε has simple zeros and so the stable and unstable manifolds have transversal intersection. ♦

Example 2 The differential equation of the beam:

$$\ddot{w} + w'''' + \Gamma w'' - \kappa\left(\int_0^1 [w']^2 d\zeta\right) w'' = \varepsilon(f\cos\omega t - \delta\dot{w}),$$
$$w = w'' = 0 \,\text{at}\, z = 0, 1.$$

The basic space is $X = H_0^2 \times L^2$, where H_0^2 denotes the H^2 functions on $[0,1]$ satisfying the boundary conditions $w = 0$ at $z = 0, 1$. For $x = (w, \dot{w}) \in X$, write the equation as

$$\frac{dx}{dt} = f_0(x) + \varepsilon f_1(x),$$

where

$$f_0(x) = Ax + B(x),$$

$$A\begin{pmatrix} w \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \dot{w} \\ -w'''' \end{pmatrix}, D(A) = \left\{(w,\dot{w}) \in H^4 \times H^2 | w = w'' = 0 = \dot{w} \,\text{at}\, z = 0, 1\right\},$$

$$B\begin{pmatrix} w \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 \\ \kappa\left(\int_0^1 [w']^2 d\zeta\right) w'' - \Gamma w' \end{pmatrix},$$

and

$$f_1\begin{pmatrix} w \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 \\ f\cos\omega t - \delta\dot{\omega} \end{pmatrix}.$$

The methods of Holmes and Marsden (1978) show that the equation generates a global flow $F_t^\varepsilon X \to X$ consisting of C^∞ maps for each ε and t. If x_0 lies in domain of the (unbounded) operator A, then $F_t^\varepsilon(x_0)$ is t–differentiable and the equation is satisfied.

For $\varepsilon = 0$, the equation is Hamiltonian with

$$\omega((w_1,\dot{w}_1),(w_2,\dot{w}_2)) = \int_0^1 \{\dot{w}_2(z)w_1(z) - \dot{w}_1(z)w_2(z)\}\, dz$$

and

$$H(w,\dot{w}) = \frac{1}{2}||\dot{w}||_{L^2}^2 - \frac{\Gamma}{2}||w'||_{L^2}^2 + \frac{1}{2}||w''||_{L^2}^2 + \frac{\kappa}{4}||w'||_{L^2}^4.$$

The invariant symplectic two–manifold Σ is the plane in X spanned by the functions $(a \sin \pi z, b \sin \pi z)$, and the homoclinic loop is given by

$$\omega_0(z,t) = \frac{2}{\pi}\sqrt{\frac{\Gamma - \pi^2}{\kappa}}(\sin \pi z)\,\mathrm{sech}(t\pi\sqrt{\Gamma - \pi^2}).$$

By direct calculation one shows that the spectral and nonresonance conditions are met for $\delta > 0$ and $j^2\pi^2(j^2\pi^2 - \Gamma) \neq \omega^2, j = 2, 3, 4 \ldots$. The real eigenvalues for $\varepsilon = 0$ are $\lambda = \pm\pi(\Gamma - \pi^2)^{1/2}$, while the rest are on the imaginary axis at $\lambda_n = \pm n\pi(\Gamma - n^2\pi^2)^{1/2}, n = 2, 3 \ldots$.

The perturbation assumptions on the saddle point and its stable and unstable manifolds are valid because of the form of the equations: $\dot{x}$ = (linear generator) + (smooth map). The proof can be given in the context of Holmes and Marsden (1978). The calculation of the Melnikov function now reduces to that of the Duffing equation in Example 1. With α, β, γ related as given earlier, we find that the stable and unstable manifolds of $(0, 0)$ intersect transversally if $\gamma > \gamma_\varepsilon$. (For $\varepsilon > 0$ the unstable manifold leaves Σ, so one cannot directly deduce the results for Example 2 from Example 1.) ♦

Consequences of transversal intersection If the hypotheses above hold, we end up with a Poincaré map $P^\varepsilon : X \to X$ that has a hyperbolic saddle point p_ε which has a one–dimensional unstable manifold intersecting a codimension–one stable manifold transversally. For $X = \mathbb{R}^2$, this situation implies that the dynamics contains a horseshoe (see Smale (1967)). For instance, one can conclude the existence of infinitely many periodic points with arbitrarily high period. Since the flow is globally attracting, this also suggests the presence of a strange attractor (cf. Holmes (1979)). See Figure 6.5.

For $X = \mathbb{R}^2$ similar conclusions hold, as has been shown by an elegant argument of Conley and Moser; Moser (1973). The attractive feature of their method is that it basically reduces the proof to one of finding *explicit estimates* on what P^ε does to horizontal and vertical strips near the saddle point. This enables one to generalize the argument to Banach spaces X, and in particular to the beam example. One can conclude, for instance, that (some power of) the map P^ε on its nonwandering set Λ is conjugate to a shift on two symbols, that the periodic points are dense in Λ, and that there exist periodic solutions of arbitrarily high period. See Holmes and Marsden (1980) for details.

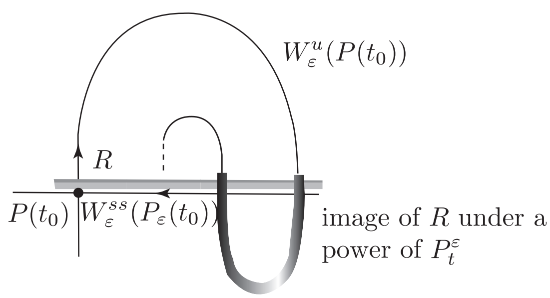

FIGURE 6.5. Horseshoes

The same methods can be used to study subharmonic bifurcations of the periodic orbits in Figure 6.4 Bifurcations near the homoclinic loop are especially interesting (see Chow et al. (1980) and recent work of Holmes and Greenspan). In addition, Feigenbaum (1978) subharmonic sequences, Newhouse sinks (Newhouse (1974)) and the Henon (1976) map all occur in this bifurcation.

Similar methods can be used to study horseshoes in Hamiltonian systems such as in the Sitnikov problem (Moser (1973)), the motion of vortices (Ziglin (1980)), the pendulum–oscillator and rigid bodies, as well as Arnold diffusion (Holmes and Marsden [in preparation]).

What year is the paper by Holmes and Marsden(inpreparation)?

7

The Traction Problem in Elastostatics

Some[1] interesting singularities in the solution manifold for the traction problem in elastostatics were discovered by Signorini in the 1930's. These investigations led to an extensive literature; especially noteworthy are Stoppelli (1958), Grioli (1962), Truesdell and Noll (1965), Van Buren (1968) and Capritz and Podio-Guidugli (1974). The methods are generally so analytic that their geometric beauty gets lost. The purpose of this lecture is to take the first few steps in a program using geometric methods; further details are available in Chillingworth et al. (1982).

The problem. Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with smooth boundary; assume $0 \in \Omega$. Let $\mathcal{C}$ be the space of $W^{s,p}$ maps $\phi : \bar{\Omega} \to \mathbb{R}^3, s > 3/p + 1$ such that $\phi(0) = 0$ and ϕ is a $W^{s,p}$ diffeomorphism onto its image.

As in Lecture 2, let $F = T\phi$ and let C be the right Cauchy–Green tensor. Let the stored energy function W be a function of the point values of C, and let $T = \rho_0 \partial W / \partial F$ be the first Piola–Kirchhoff stress tensor.

[1] Based on joint work with D.R.J. Chillingworth and Y.H. Wan.

Assume that the undeformed state is stress–free, *i.e.,* $T = 0$ when $\phi =$ identity, and assume that strong ellipticity holds.

Let $B : \Omega \to \mathbb{R}^3$ denote a given body force (per unit volume) and $\tau : \partial\Omega \to \mathbb{R}^3$ a given surface traction per unit area. These are dead loads; *i.e.,* the equilibrium equations we are studying are

$$\begin{gathered} \mathrm{DIV}\, T + B = 0 \,\mathrm{in}\, Omega, \\ T \cdot N = \tau \,\mathrm{on}\, \partial\Omega. \end{gathered}$$

Let $\mathcal{L}$ be the space of pairs $l = (B, \tau)$ of loads of class $W^{s-2,p}$ on Ω and $W^{s-3/2,p}$ on $\partial\Omega$, such that

$$\int_\Omega B(X) dV(X) + \int_{\partial\Omega} \tau(X) dA(X) = 0;$$

i.e., the total force is zero. By the divergence theorem, if l is a set of loads satisfying the equilibrium equations for some ϕ, then $l \in \mathcal{L}$.

If we were studying the displacement problem, it would follow directly from the implicit function theorem that for any B near zero, there would be a unique ϕ near the identity satisfying the equilibrium equations. For the traction problem the kernel of the linearized equations consists of infinitesimal rigid body motions and the implicit function theorem fails. In fact, the solution set bifurcates near the identity, and the geometry of the rotation group $SO(3)$ plays a crucial role.

Definitions and notation

1. Let $\Phi : \mathcal{C} \to \mathcal{L}$ be defined by

$$\Phi(\phi) = (-\,\mathrm{DIV}\, T, T \cdot N),$$

so that the equilibrium equations are $\Phi(\phi) = l$.

2. Let

$$\mathcal{C}_l \{ u \in T_{\mathrm{id}}\mathcal{C} \mid Du(0) \text{ is symmetric } \}$$

(we automatically have $u(0) = 0$ as well), and let the *equilibrated loads* be those whose torque in the reference configuration is zero; *i.e.,*

$$\mathcal{L}_e = \left\{ l \in \mathcal{L} \,\middle|\, \int_\Omega X \times B(X) dV(X) + \int_{\partial\Omega} X \times \tau(X) dA(X) = 0 \right\}.$$

From linear elasticity we know that

$$D\Phi(\mathrm{id})|_{\mathcal{C}_l} : \mathcal{C}_l \to \mathcal{L}_e$$

is an isomorphism. (See for instance Marsden and Hughes (1978) for this standard proof.)

3. Let $SO(3)$ act on $\mathcal{C}$ and $\mathcal{L}$ in the obvious way. For $Q \in SO(3), \phi \in \mathcal{C}$ and $l \in \mathcal{L}$, let

$$(Q, \phi) \mapsto Q \circ \phi \,\text{and}(Q, l) \mapsto (Q \circ B, Q \circ \tau) = Ql.$$

For $l \in \mathcal{L}$; let $\mathcal{O}_l$ denote the $SO(3)$ orbit of l,

$$\mathcal{O}_l = \{Ql \mid Q \in SO(3)\}.$$

4. Let $l \in \mathcal{L}_e$. Then l is said to have *no axis of equilibrium* if, for all $\xi \in so(3)$, the Lie algebra of $SO(3)\xi \neq 0$ we have

$$\xi l \notin \mathcal{L}_e;$$

i.e., any rotation of l destroys the equilibrium. If l has an axis of equilibrium, then there is a vector $e \in \mathbb{R}^3$ such that rotations of l about e map l into $\mathcal{L}_e$, as is readily checked.

Lemma 7.1. *(Da Silva's theorem). Let $l \in \mathcal{L}$. Then $\mathcal{O}_l \cap \mathcal{L}_e \neq \varnothing$.*

Proof. Define the *astatic load map* $k : \mathcal{L} \to M_3(3{\times}3 \text{ matrices})$ by

$$k(l) = k(B, \tau) = \int_\Omega B(X) \otimes X dV(X) + \int_{\partial\Omega} \tau(X) \otimes X dA(X),$$

so that $l \in \mathcal{L}_\varepsilon$ if and only if $k(l)$ is symmetric. Now k is $SO(3)$ equivariant,

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "k"] \arrow[d, "so(3)"'] & M_3 \arrow[d, "so(3)"] \\
\mathcal{L} \arrow[r, "k"'] & M_3
\end{tikzcd}$$

where the action on M_3 is $(Q, A) \mapsto QA$; *i.e.,*

$$k(Ql) = Qk(l).$$

The result is now obvious from the polar decomposition. ■

6. Notice that Φ is also equivariant from material frame indifference:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Phi"] \arrow[d, "so(3)"'] & \mathcal{L} \arrow[d, "so(3)"] \\
\mathcal{C} \arrow[r, "\Phi"'] & \mathcal{L}
\end{tikzcd}$$

Thus, to study the solutions of $\Phi(\phi) = l$ for a given l, we can assume that $l \in \mathcal{L}_e$.

No axis of equilibrium Suppose now that $l \in \mathcal{L}_e$ is given and has no axis of equilibrium. The main theorem in this case is due to Stoppelli (1958), which we prove in a series of remarks.

Lemma 7.2. *(a) dim* $\mathcal{O}_l = 3$ *and*

(b) $T_l\mathcal{O}_l \oplus \mathcal{L}_e = \mathcal{L}$.

Proof. If dim $\mathcal{O}_l < 3$, there would be a $\xi \neq 0, \xi \in so(3)$ such that $\xi l = 0$, which contradicts $\xi l \notin \mathcal{L}_e$. Thus (a) holds. Also, by the no–axis–of–equilibrium assumption, $T_l\mathcal{O}_l \cap \mathcal{L}_e = \{0\}$. Since $\mathcal{L}_e$ has codimension three in $\mathcal{L}$ and (a) holds, we get (b). ■

2. Let $\widetilde{\Phi}$ be the restriction of Φ to $\mathcal{C}_l$, regarded as an affine subspace of $\mathcal{C}$ centered at the identity. As remarked before,

$$D\tilde{\Phi}(\mathrm{id}) : \mathcal{C}_l \to \mathcal{L}_e$$

is an isomorphism. In particular, it is one–to–one, and so for $\tilde{\Phi}$ restricted to a neighborhood of the identity,

$$\operatorname{range}\tilde{\Phi} \equiv N$$

is a submanifold of $\mathcal{L}$ tangent to $\mathcal{L}_e$ at the origin. (See Figure 7.1) By the above lemma,

$$\{Ql \mid Q \in \text{ a neighborhood } U \text{ of } \mathrm{Id} \in SO(3)\}$$

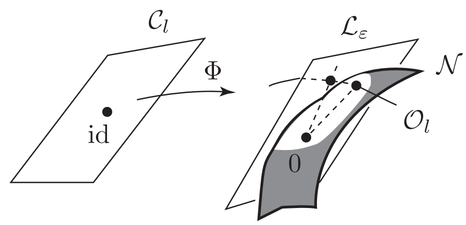

FIGURE 7.1. The geometry of Stoppelli's theorem.

is a neighborhood of l in the normal direction to $\mathcal{L}_e$. Thus

$$\{\lambda Ql \mid Q \in U, \lambda \in (-\varepsilon, \varepsilon)\}$$

is a cone in the normal bundle to $\mathcal{L}_e$.

Since N is tangent to $\mathcal{L}_e$ at 0, for λ sufficiently small $\mathcal{O}_{\lambda l}$ will intersect N.[2] Thus, for λ sufficiently small, there is a unique Q in a neighborhood of the identity such that

$$\Phi(\bar{\phi}) = \lambda Ql$$

has a unique solution $\bar{\phi} \in \mathcal{C}_l$. Thus $\phi = Q^{-1}\bar{\phi}$ solves $\Phi(\phi) = \lambda l$. Thus we have proved.

Theorem 7.3. *(Stoppelli (1958)). Suppose $l \in \mathcal{L}_e$ has no axis of equilibrium. Then, for λ sufficiently small, there is a unique $\bar{\phi} \in \mathcal{C}_l$ and Q in a neighborhood of the identity such that $\phi = Q^{-1}\bar{\phi}$ solves the traction problem*

$$\Phi(\phi) = \lambda l.$$

[2]This explains the somewhat mysterious estimates needed in Stoppelli's arguments; see Van Buren (1968) for a good account.

Axis of equilibrium (sketch). The full story of the geometry of the solution space when there is an axis of equilibrium is a complicated one. We shall confine ourselves in the following series of remarks to highlighting a few crucial points.

1. Let
$$\Psi = k \circ \tilde{\Phi} : \mathcal{C}_l \to M_3.$$
An application of Gauss' theorem shows that Ψ is the total stress,
$$\Psi(\phi) = \int_\Omega T dV,$$
where T is the first Piola–Kirchhoff stress tensor. The idea now is the following. To see how the range of $\tilde{\Phi}$ meets orbits in $\mathcal{L}$, we study the way Ψ meets in M_3 and then use information about the map k.

2. The kernel of k consists of those loads l for which every axis $e \in \mathbb{R}^3$ is an axis of equilibrium. This is a calculation.

3. The map k, the decomposition $M_3 = \text{Skew} \oplus \text{Sym}$ into skew symmetric and symmetric matrices and the L^2–inner product on $\mathcal{L}$ produce a decomposition into orthogonal subspaces:
$$\mathcal{L} \approx \mathcal{L}_e \oplus \text{Skew} \approx \ (\ker\, k \oplus \text{Sym})\ \oplus \text{Skew}\,.$$
We can express N uniquely as the graph of a smooth map $F : \mathcal{L}_e \to \text{Skew}$ satisfying $F(0) = 0$ and $DF(0) = 0$, since N is tangent to $\mathcal{L}_e$ at 0. (Remarkably enough, direct calculation using Ψ shows that $D^2F(0)$ can be computed using only data obtained from linearized elasticity.) Define a rescaled map $\widetilde{F} : \mathbb{R} \times \mathcal{L}_e \to \text{Skew}$ by $\bar{F}(\lambda, l) = F(\lambda l)/\lambda^2$. The problem of how $\mathcal{O}_l$ meets N now becomes that of solving the equation
$$\text{Skew}(QA) = \lambda \bar{F}(\lambda, l)$$
for Q, where $A = k(l)$ and Skew $(QA) = 1/2((QA) - (QA)^T)$ is the skew part of QA.

This equation is now to be solved by the Lyapunov–Schmidt procedure in bifurcation theory; this procedure is briefly reviewed below in a formulation relevant to this problem.

4. The first step in a bifurcation analysis is to study zeros of Skew (QA), *i.e.,* to study how the orbits of the $SO(3)$ action on M_3 by left multiplication, meet Sym, the symmetric matrices. This is done by a straightforward analysis and matrix computation. It turns out that there are exactly five types of orbits, called types $0, 1, 2, 3$ and 4. They are summarized in Table 7.1.

It is interesting that these are the only cases that can arise, and that some of them were not discussed in detail by Signorini and Stoppelli. Additional solutions were missed because not enough parameters where included to give a full neighborhood description. Indeed, cases $2, 3$ and 4 can be very complicated. Also notice that the global properties of the rotation group are involved in a crucial way; for example, if a load l has an axis of equilibrium that is simple (type 1, the case considered by Stoppelli) then there is a rotation (far from the identity) Q such that Ql has *no* axis of equilibrium.

In particular, by the previous theorem, there is always a solution of type 1, although it may not be near the identity. (This is consistent with the results of Ball (1977).)

Figure 7.2 illustrates a few simple examples of loads of the form $(0, \tau)$ of types $1, 2, 3$ and 4.

Lyapunov–Schmidt procedure. We develop the bifurcation analysis in a series of remarks.

1. First we recall the classical procedure. Let X, Λ and Y be Banach spaces and $f : X \times \Lambda \to Y$ a C^k map, $k \geq 1$. Let $D_x f(x, \lambda)$ be the (Fréchet) derivative of f with respect to x, a continuous linear map of X to Y. Let $f(x_0, \lambda_0) = 0$ and let

$$X_1 = \ker D_x f(x_0, \lambda_0).$$

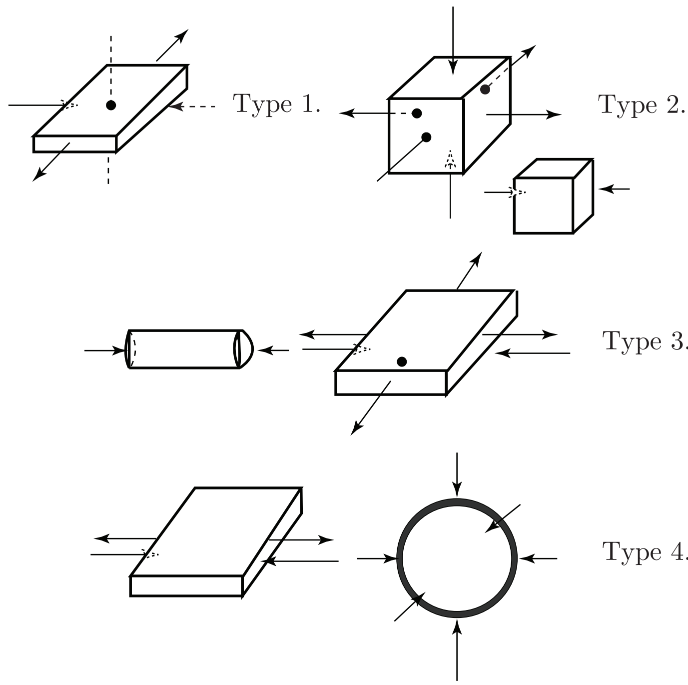

Figure 7.2. Load types.

Assume X_1 has a closed complement X_2, so that $X = X_1 \oplus X_2$. Also, assume

$$Y_1 = \operatorname{range} D_x f(x_0, \lambda_0)$$

is closed and has a closed complement Y_2. If X_1 and Y_2 are finite dimensional, then $D_x f(x_0, y_0)$ is a Fredholm operator. Write $Y = Y_1 \oplus Y_2$, and let $P : Y \to Y_1$ be the projection. By the implicit function theorem, the equation

$$Pf(x_1 + x_2, \lambda) = 0$$

has a unique solution $x_2 = u(x_1, \lambda)$ near x_0, λ_0, where $x = x_1 + x_2 \in X = X_1 \oplus X_2$. Thus, the equation $f(x, \lambda) = 0$ is equivalent to the *bifurcation equation*

$$(I - P)f(x_1 + u(x_1, \lambda), \lambda) = 0,$$

a "system of dim Y_2 equations in dim X_1 unknowns". This reduction of $f(x, \lambda) = 0$ to the bifurcation equation is the

Lyapunov–Schmidt procedure.

2. Next we recall a general topological context for bifurcation of vector fields that will be applied to our situation. Let M and Λ be manifolds and $X : M \times \Lambda \to TM$ a vector field on M depending on parameters $\lambda \in \Lambda$. We seek the zeros of X. For $\lambda = \lambda_0$ suppose the zero set S is a smooth compact submanifold. Assume that M carries a Riemannian metric and that the range of $D_x X(x, \lambda_0)$ is the orthogonal complement to $T_x S$. The normal bundle E to TS trivializes a neighborhood U of S; so, for each $x \in U$, let $P_x : T_x M \to T_x S_{\pi(x)}$ be the orthogonal projection to the fiber $S_{\pi(x)}$ over $\pi(x)$, where $\pi : E \to S$ is the projection. By the inverse function theorem, there is a unique section $\phi_\lambda : S \to E$ such that $P_x X(\varphi_\lambda(x), \lambda) = 0$ for $x \in S$ and λ in a neighborhood of λ_0 (assume, e.g., that M is compact.) Let $\widetilde{X}(x, \lambda)$ be the orthogonal projection of $X(x, \lambda)$ onto the tangent space to the graph of ϕ_λ at a point x on the graph. Thus, $\widetilde{X}(x, \lambda)$ is a vector field on the graph of ϕ_λ, and finding its zeros is clearly equivalent (for small λ) to finding zeros of X. We call the equation $\widetilde{X}(x, \lambda) = 0$ on the graph of ϕ_λ the *bifurcation equation.*

3. The scheme just sketched actually fits in with our problem. We let $M = SO(3), \Lambda = \mathbb{R} \times M_3, \lambda_0 = (0, A_0)$ and let X be Skew $(QA) - \lambda \bar{F}(\lambda, l)$ regarded as a left invariant vector field on $SO(3)$. (Recall that Skew is the Lie algebra of $SO(3)$.) For A_0 of type 1, the zero set of $X(Q, \lambda_0)$ is two points and a circle. The hypotheses in the above procedure can be checked by a calculation, so the bifurcation equation becomes the study of vector fields on S^1. One now has to do some calculations in singularity theory to analyze it: one finds under hypotheses on the linearized elasticities that as (λ, A_0) vary, one encounters up to four cusps. Likewise for type 2 one analyzes vector fields on $\mathbb{RP}^2$ and double cusps occur, so that up to nine solutions can occur locally

in $SO(3)$ (and twelve globally in $SO(3)$).

4. In carrying out the analysis, the variational (or Hamiltonian) nature of the problem can be exploited. Solutions of the equilibrium equations are critical points for the functional

$$V_l(\phi) = \int_\Omega W(\phi) + \int_\Omega \phi(X)\cdot B(X)dV(X) + \int_{\partial\Omega} \phi(X)\cdot\tau(X)dA(X)$$

(see Lecture 2). This function is needed to study the stability of solutions. One should note, however, that stability here is taken in the sense of minima of V_l; whether or not this implies dynamic stability is unknown (see Ball et al. (1978) and references therein).

5. It is hoped that methods like these will be useful in other problems as well. For example, Rivlin has found seven homogeneous solutions for incompressible deformations of a cube. His analysis, described in Gurtin (1981), bears some similarities to the problem here. One would like to know, for example, whether there are any other (nonhomogeneous) solutions. Recently, some progress on this problem has been made by J. Ball and D. Scheaffer.

6. The analysis of this problem shows that the equations of elastostatics are linearization unstable when there is an axis of equilibrium; *i.e.,* perturbation expansions are not always literally valid, but need some adjustment. This phenomenon is actually shared by a number of other problems, as the following lectures will demonstrate.

8
Bifurcations of Momentum Mappings

Bifurcation theory[1] is a vast subject dealing with qualitative changes in systems, usually as some parameter is varied. The previous two lectures gave specific examples of this, one dynamic and one static. This lecture is concerned with bifurcations of the level sets $J^{-1}(\mu)$ of the momentum mapping of a Hamiltonian system with symmetry, as the momentum $\mu \in \mathfrak{g}^*$ is varied. The crucial question in this study, and the one we concentrate on, is the structure of the set $J^{-1}(\mu_0)$ when μ_0 is a critical value of J. Figure 8.1 illustrates a very simple case of what is going on.

For general background on this topic of "topology and mechanics," see Smale (1970) and Abraham and Marsden (1978), §4.5. The results of this lecture are an extension and refinement of part of these previous results and are, as far as we know, new.

The main example that motivated this abstract study is the occurrence of singularities in the space of solutions of Einstein's equations, described in Lectures 9 and 10. However, the reader may wish to think about some simpler examples while proceeding. For example, the total angular momentum functions for 1

[1]This section is based on joint work of J.M. Arms, V. Moncrief, and J. Marsden (Arms et al. (1981)).

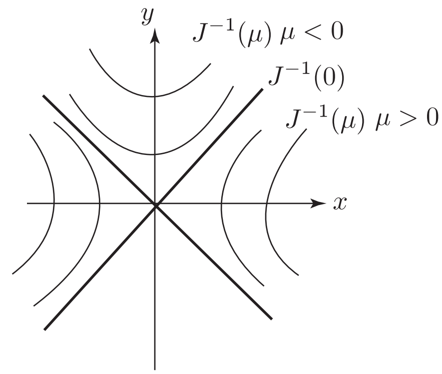

FIGURE 8.1. The level sets of $J^{-1}(\mu)$, for $J(x,y) = x^2 - y^2$, bifurcate at $\mu = 0$. (This is a "piece" of the angular momentum map $J(x,y,\dot{x},\dot{y}) = x\dot{y} - \dot{x}y$, which has a nondegenerate critical point at $(0,0,0,0) \in \mathbb{R}^4$.)

or 2 particles moving in $\mathbb{R}^3$ are already interesting at the critical case of zero total angular momentum.[2] Another beautiful example is the "north pole" and the Huygens solutions for the spherical pendulum; see Duistermaat (1980).

Let (P, ω, G, J) be a Hamiltonian G–space. If $x_0 \in P, \mu_0 = J(x_0)$, and if

$$dJ(x_0) : T_x P \to \mathfrak{g}^*$$

is surjective (with split kernel), then locally $J^{-1}(\mu_0)$ is a manifold and $\{J^{-1}(\mu) \mid \mu \in \mathfrak{g}^*\}$ forms a regular local foliation of a neighborhood of x_0. Thus, when $dJ(x_0)$ fails to be surjective, we look for bifurcations. We shall begin by relating this idea to symmetries.

Definition 8.1. *Let $\mathcal{S}_{x_0}$ = the component of the identity of $\{g \in G_{\mu_0} | gx_0 = x_0\}$, called the symmetry group of x_0. Its Lie algebra is denoted s_{x_0}, so*

$$s_{x_0} = \{\xi \in \mathfrak{g}_{\mu_0} | \xi_P(x_0) = 0\}.$$

1. Recall that G_{μ_0} is the isotropy group of $\mu_0 = J(x_0)$ relative to the coadjoint action of G on $\mathfrak{g}^*$, and $\mathfrak{g}_{\mu_0}$ is its Lie

[2]For one particle, the answer is as follows. $J^{-1}(0) = \{(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \times p = 0\} \approx$ a cone over M, where M is a compact three–manifold which is $S^2 \times S^1 \sim$, where $\sim$ is the equivalence relation $(x,z) \sim (-x,-z)$.

algebra; see Lecture 3.

2. Assume that P carries a (weak) metric $\langle\langle\cdot,\cdot\rangle\rangle$ and a complex structure $\mathbb{J}$, so that we can use the adjoint formalism described in Lecture 4. Below we shall also need the assumption that the action of G on TP commutes with $\mathbb{J}$.

3. We note that

   $$\mathcal{S}_{x_0} = \{g \in G \mid g \cdot x_0 = x_0\} \text{ and } s_{x_0} = \{\xi \in \mathfrak{g} \mid \xi_P(x_0) = 0\}.$$

   Indeed, let $g \in G$ and $g \cdot x_0 = x_0$. Thus $J(g \cdot x_0) = J(x_0) = \mu_0$. By Ad^*–equivariance, $J(g \cdot x_0) = \mathrm{Ad}^*_{g^{-1}} J(x_0)$, so $\mathrm{Ad}^*_{g^{-1}}\mu_0 = \mu_0$ and so $g \in G_{\mu_0}$.

Lemma 8.2. *$dJ(x)$ is surjective iff $s_x = \{0\}$; i.e., x has "no symmetries" (not counting discrete symmetries).*

Proof. From the adjoint formalism,

$$\operatorname{range} dJ(x) = \mathfrak{g}^* \Leftrightarrow \ker dJ(x)^* = \{0\}$$

and

$$-\mathbb{J} \circ dJ(x)^*\xi = \xi_P(x),$$

so the result is clear. ■

Thus, *the only way a (local) bifurcation can occur is near points with symmetries.* To examine the situation near such points we perform a reduction (as above, called the *Lyapunov–Schmidt procedure)* in the following remarks. *From now on, we shall assume that $\mu_0 = 0$ for simplicity, so that $\mathfrak{g}_{\mu_0} = \mathfrak{g}$.*

1. Fix $x_0 \in P$ with $s_{x_0} \neq \{0\}$, let $J(x_0) = 0$ and let $\mathcal{C} = J^{-1}(0)$, the solution space we are interested in. Let

   $$\mathbb{P} : \mathfrak{g}^* \to \operatorname{range} dJ(x_0),$$

   be the orthogonal projection associated with the splitting

   $$\mathfrak{g} = \operatorname{range} dJ(x_0) \oplus \ker dJ(x_0)^*.$$

This construction requires an inner product on $\mathfrak{g}^*$; assume $(\cdot,\cdot)_{x_0}$ is an inner product invariant under $\mathcal{S}_{x_0}$ relative to which $dJ(x_0)^*$ is taken, so $\mathbb{P}$ is defined.

2. The equation $J(x) = 0$ is equivalent to the equations

$$\mathbb{P}J(x) = 0, \quad (I - \mathbb{P})J(x) = 0.$$

The map $\mathbb{P}J : P \to \text{range } dJ(x_0)$ is a submersion at x_0, and so

$$\mathcal{C}_{\mathbb{P}} = \{x \mid \mathbb{P}J(x) = 0\}$$

is a smooth manifold in a neighborhood of x_0.

3. Note that

$$T_{x_0}\mathcal{C}_{\mathbb{P}} = \ker dJ(x_0),$$

since $d\mathbb{P}J(x_0) = dJ(x_0)$,

4. Thus $\mathcal{C} = f^{-1}(0)$, where $f : \mathcal{C}_{\mathbb{P}} \to \ker dJ(x_0)^*$ is given by

$$f(x) = (I - \mathbb{P})J(x).$$

5. Note that $df(x_0) = 0$.

These remarks reduce the problem of investigating the level sets of J to that of f. Since $df(x_0) = 0$, the reduced problem is essentially a problem in singularities of mappings. The most basic result to try in this context is the Morse lemma; it will yield a picture of the level sets like that in Figure 8.1.[3] *It is a remarkable fact about momentum mappings that under reasonable hypotheses, the Morse lemma is always applicable when* dim ker $dJ(x_0)^* = 1$. To see this, we shall pass to a slice S_{x_0} and show that within S_{x_0}, J has an identifiable nondegenerate critical manifold. We do this in the following series of remarks.

1. Assume that the action of G on P admits a slice at x_0, *i.e.,* a submanifold $S_{x_0} \subset P$ containing x_0 and satisfying:

[3]For the use of the Morse lemma in the context of more traditional bifurcation theory, see Nirenberg (1974).

i If $g \in \mathcal{S}_{x_0}$, then $g \cdot S_{x_0} = S_{x_0}$.

ii If $g \cdot S_{x_0} \cap S_{x_0} \neq \varnothing$ then $g \in \mathcal{S}_{x_0}$.

iii There is a local cross–section $\chi : G/\mathcal{S}_{x_0} \to G$ defined in a neighborhood of the identity coset such that the map $S_{x_0} \times G/\mathcal{S}_{x_0} \to P : (x, u) \to \chi(u) \cdot x$ gives a local diffeomorphism of $S_{x_0} \times G/\mathcal{S}_{x_0}$ with P.

For instance, compact groups acting on manifolds have slices (cf. Palais (1957)), and so does the diffeomorphism group acting on the space of metrics (cf. Ebin (1970)).

(a) More specifically, we shall assume for convenience that S_{x_0} has been chosen to be an "affine slice": a ball in the subspace of $T_{x_0}P$ orthogonal to the orbit of x_0, namely,

$$S_{x_0} = \text{ballinker}(dJ(x_0 \cdot \mathbb{J})),$$

where we identify ker $dJ(x_0) \circ \mathbb{J}$ with an affine subspace of P and assume P is open in a linear space or has been localized. The ball is taken in a Riemannian metric[4] on $T_{x_0}P$ invariant under S_{x_0}.

2. From the slice properties, it follows that there is a neighborhood V of x_0 such that if $x \in V$ then $\mathcal{S}_x$ is conjugate to a subgroup of $\mathcal{S}_{x_0}$. Denote by N_{x_0} those $x \in V$ such that $\mathcal{S}_x$ is conjugate to $\mathcal{S}_{x_0}$ itself; *i.e.*, N_{x_0} consists of elements of the same symmetry type as x_0 (or elements of the same "orbit type"). It is known (see, for instance, Hermann (1968)) that N_{x_0} is a smooth manifold.

   It is clear that the orbit $G \cdot x_0 \subset N_{x_0}$, but in interesting examples N_{x_0} will be strictly larger.

3. From properties of the slice,

[4] In infinite dimensions a strong metric is required; cf. Lectures 9 and 10. We acknowledge the helpful remarks of R. Palais concerning slices.

$$\begin{aligned}\mathcal{N}_{x_0} \equiv& N_{x_0} \cap S_{x_0}\\ =&\{x \in S_{x_0} \mid \mathcal{S}_x = \mathcal{S}_{x_0}\}\\ =&\{x \in S_{x_0} \mid \xi_P(x) = 0 \,\text{forall}\, \xi \in s_{x_0}\}.\end{aligned}$$

Now $\mathcal{N}_{x_0}$ is the set of points of fixed symmetry type for the action of $\mathcal{S}_{x_0}$ on S_{x_0}, so it too is a smooth manifold. Since $\xi_P(x) = -\mathbb{J} \circ dJ(x)^* \xi$,

$$\begin{aligned}T_{x_0}\mathcal{N}_{x_0} =&\{u \in \ker(dJ(x_0) \circ \mathbb{J}) \mid d_x(dJ(x_0)^* \cdot \xi) \cdot u = 0 \,\text{forall}\, \xi \in s_{x_0}\}\\ =&\{u \in \ker dJ(x_0) \circ \mathbb{J} | \langle \xi, d^2J(x_0)(u,v)\rangle = 0 \,\text{forall}\, v \in T_{x_0}P \,\text{and}\, \xi \in s_{x_0}\}.\end{aligned}$$

4. We have range $(-\mathbb{J} \circ dJ(x_0)^*) \subset$ ker $(dJ(x_0))$, and so range $(dJ(x_0)^*) \subset$ ker $(dJ(x_0) \circ \mathbb{J})$. Therefore, ker $dJ(x_0)+$ ker $(dJ(x_0) \circ \mathbb{J}) \supset$ ker $(dJ(x_0))+$ range $dJ(x_0)^* = T_{x_0}P$. Hence $\mathcal{C}_{\mathbb{P}}$ and S_{x_0} intersect transversally at x_0, and so $\mathcal{C}_{\mathbb{P}} \cap S_{x_0}$ is a manifold near x_0 with tangent space

$$T_{x_0}(\mathcal{C}_{\mathbb{P}} \cap S_{x_0}) = \ker dJ(x_0) \cap \ker(dJ(x_0) \circ \mathbb{J}),$$

which is one of the components ("true degrees") in Moncrief's decomposition (see Lecture 4).

5. Let

$$\begin{aligned}\mathfrak{g}^*_{x_0} =&\{\nu \in \mathfrak{g}^* \mid \mathrm{Ad}^*_{g-1}\, \nu = \nu \text{ for all } g \in \mathcal{S}_{x_0}\}\\ =&\{\nu \in \mathfrak{g}^* \mid \langle \nu, [\xi, \eta]\rangle = 0 \text{ for all } \xi, \eta \in s_{x_0}\},\end{aligned}$$

i.e., points of $\mathfrak{g}^*$ with the same symmetry as x_0. By Ad^*–equivariance, if $x \in \mathcal{N}_{x_0}$, then $J(x) \in \mathfrak{g}^*_{x_0}$. Thus $dJ(x_0) : T_{x_0}\mathcal{N}_{x_0} \to \mathfrak{g}^*_{x_0}$. One can show that

$$dJ(x_0)^* : \mathfrak{g}^*_{x_0} \to T_{x_0}\mathcal{N}_{x_0},$$

and so

$$dJ(x_0) \circ dJ(x_0)^* : \mathfrak{g}^*_{x_0} \to \mathfrak{g}^*_{x_0}.$$

We claim that the map $\mathbb{P}J : \mathcal{N}_{x_0} \to \mathbb{P}\mathfrak{g}^*_{x_0} \equiv (\text{range } dJ(x_0))^*_{x_0}$ is a submersion. Indeed, let $\nu \in \text{range } dJ(x_0)$ and have the same symmetry as x_0 (as above). Now $dJ(x_0) \circ dJ(x_0)^* : (\text{range } dJ(x_0))^*_{x_0} \to (\text{range } dJ(x_0))^*_{x_0}$ is an isomorphism, so we can write $\nu = dJ(x_0) \circ dJ(x_0)^* \eta$ for $\eta \in \mathfrak{g}^*_{x_0}$. Thus $dJ(x_0)^* \eta \in T_{x_0}\mathcal{N}_{x_0}$.

We have proved that $\mathcal{N}_{x_0} \cap \mathcal{C}_{\mathbb{P}}$ *is a smooth manifold.*

6. In fact, $\mathcal{N}_{x_0} \cap \mathcal{C}_{\mathbb{P}} = \mathcal{N}_{x_0} \cap \mathcal{C}$. In other words, elements of $\mathcal{N}_{x_0} \cap \mathcal{C}_{\mathbb{P}}$ are actually solutions. This follows from differentiating $\langle \xi, J(x) \rangle$ in x to get

$$d\langle \xi, J(x) \rangle \cdot u = \langle \xi, dJ(x) \cdot u \rangle = \langle dJ(x)^* \xi, u \rangle.$$

If $x \in \mathcal{N}_{x_0}$ and $\xi \in s_{x_0}$ this vanishes identically. Therefore $\langle \xi, J(x) \rangle$ vanishes on $\mathcal{N}_{x_0} \cap \mathcal{C}_{\mathbb{P}}$. But this means $(I - \mathbb{P})J(x) = 0$, so $J(x) = 0$.

Thus, *solutions of* $J(x) = 0$ *with the same symmetry type as* x_0 *and in the slice at* x_0 *form a smooth manifold.*

7. Consider $f : \mathcal{C}_{\mathbb{P}} \cap S_{x_0} \to \ker\, dJ(x_0)^*; x \mapsto (I - \mathbb{P})J(x)$, as introduced earlier. The calculation just done shows that $\mathcal{N}_{x_0} \cap \mathcal{C}_{\mathbb{P}}$ is a manifold of critical points for f; *i.e.,* $dJ(x) = 0$ if $x \in \mathcal{N}_{x_0} \cap \mathcal{C}_{\mathbb{P}}$. We want to determine to what extent $\mathcal{N}_{x_0} \cap \mathcal{C}_{\mathbb{P}}$ is a nondegenerate critical manifold for f. For $\dim \ker\, dJ(x_0)^* = 1$, this will literally be true.

8. We have

$$\langle \xi, d^2 f(x_0) \cdot (u, v) \rangle = \langle \xi, d^2 J(x_0) \cdot (u, v) \rangle.$$

If $\xi \in \ker\, dJ(x_0)^*$ and $u \in T_{x_0}\mathcal{N}_{x_0}$, then this vanishes for all $v \in T_{x_0}(\mathcal{C}_{\mathbb{P}} \cap S_{x_0})$ by the expression for $T_{x_0}\mathcal{N}_{x_0}$ in Remark 3 above. The *degeneracy space* for f at x_0 is, by definition, all $u \in T_{x_0}(\mathcal{C}_{\mathbb{P}} \cap S_{x_0})$ such that $d^2 f(x_0)(u, v) = 0$ for all $v \in T_{x_0}(\mathcal{C}_{\mathbb{P}} \cap S_{x_0})$. *We claim that the degeneracy space is exactly* $T_{x_0}\mathcal{N}_{x_0}$. To prove this, we proceed as follows:

(a) (*Gauge invariance of* d^2J). For $\xi \in \mathfrak{g}, u \in \ker\, dJ(x_0)$ and $v \in$ range $\mathbb{J} \circ dJ(x_0)^*$, we claim that

$$\langle \xi, d^2J(x_0)(u, v)\rangle = 0.$$

Indeed, (from [Abraham and Marsden (1978), Prop. 4.1.26**i**] and the fact that the action is symplectic), we have this identity for momentum mappings:

$$\langle \xi, dJ(g^{-1}x) \cdot u\rangle = \langle \mathrm{Ad}_g\, \xi, dJ(x) \cdot u\rangle.$$

If this is evaluated at $x = x_0$ and $u \in \ker\, dJ(x_0)$, we get the identity

$$\langle \xi, dJ(g^{-1}x_0) \cdot u\rangle = 0.$$

Differentiation in g then yields the result.

(b) Suppose now $u \in \ker\, dJ(x_0) \cap \ker\, dJ(x_0) \circ \mathbb{J}$, and $\langle \xi, d^2J(x_0)(u, v)\rangle = 0$ for all $v \in \ker dJ(x_0) \cap \ker (dJ(x_0) \circ \mathbb{J})$. By gauge invariance, we also have $\langle \xi, d^2J(x_0)(u, v)\rangle = 0$ for all $v \in \ker\, dJ(x_0)$. To complete the argument we need one more ingredient.

(c) ($\mathbb{J}$–invariance of d^2J). Assume that the original action commutes with $\mathbb{J}$ (*i.e.,* $\mathbb{J} \circ T\Phi_g = T\Phi_g \circ \mathbb{J}$). Then, since $\mathbb{J}$ is symplectic, differentiation of this relation in g and employing the definition of J gives

$$\langle \xi, d^2J(x_0)(u, v)\rangle = \langle \xi, d^2J(x_0) \cdot (\mathbb{J}u, \mathbb{J}v)\rangle$$

for $\xi \in \ker\, dJ(x_0)^*$ and $u, v \in T_{x_0}P$.

Returning to Remark 8(b), we have $\langle \xi, d^2J(x_0)(u, v)\rangle = 0$ for fixed $u \in \ker\, dJ(x_0) \cap \ker\, dJ(x_0) \circ \mathbb{J}$ and all $v \in \ker\, dJ(x_0)$. Now let $w \in$ range $dJ(x_0)^*$. Then

$$\langle \xi, d^2J(x_0)(u, w)\rangle = \langle \xi, J^2J(x_0)(\mathbb{J}u, \mathbb{J}\omega)\rangle = 0$$

by gauge invariance. Now $v + w$ is a general element of $T_{x_0}P$, so $\langle \xi, d^2J(x_0)(u, v)\rangle = 0$ for all $v \in T_xP$. Thus $u \in T_{x_0}\mathcal{N}_{x_0}$.

We summarize our finds as follows.

Theorem 8.3. *Let* $f : \mathcal{C}_{\mathbb{P}} \cap S_{x_0} \to$ *ker* $dJ(x_0)^*, x \mapsto (I - \mathbb{P})J(x)$. *Then* $\mathcal{N}_{x_0} \cap \mathcal{C}$ *is a smooth manifold of critical points for* f. *Moreover, the degeneracy space for* $d^2 f$ *at a point* $x \in \mathcal{N}_{x_0} \cap \mathcal{C}$ *is exactly* $T_x(\mathcal{N}_{x_0} \cap \mathcal{C})$.

In particular, if dim ker $dJ(x_0)^* = 1$, *then* $\mathcal{N}_{x_0} \cap \mathcal{C}$ *is a nondegenerate critical manifold for* f *and so*[5] $\mathcal{C} \cap S_{x_0}$ *is a cone* $\times(\mathcal{N}_{x_0} \cap \mathcal{S}_{x_0})$ *and hence* $\mathcal{C}$ *itself is a cone* $\times(\mathcal{N}_{x_0} \cap \mathcal{S}_{x_0}) \times (G/S_{x_0})$, *in a neighborhood of* x_0.

The surprising thing is that *the singularity* in $\mathcal{C}$ *is necessarily quadratic, even though no explicit nondegeneracy assumption is made on* J. A simple example may help here.

Example. Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a Hamiltonian with a critical point at the origin. Suppose H has all its orbits periodic of the same period. Then H has a nondegenerate critical point at $(0, 0)$. This follows from the above theorem by using $G = S^1$ acting by the flow of X_H.[6] ♦

In the case where dim ker $dJ(x_0)^* > 1$, we expect $\mathcal{C}$ to look like "cones on cones". To really prove this, we begin with the case of quadratic J. We shall prove directly that $\mathcal{C} \cap S_{x_0}$ *is diffeomorphic to the zero set of* $(I - \mathbb{P})d^2J(x_0)(u, u)$ [for $u \in T_{x_0}S_{x_0}$], without any appeal to degeneracy spaces. This case is already interesting and applies to three nontrivial examples: zero angular momentum for n particles in $\mathbb{R}^3$, the constraint equations in gauge theory (see below and Arms (1980)) and the supermomentum constraint in relativity (see the next two lectures).

We shall prove this claim in the next series of remarks.

1. Our assumptions are as above, except that now assume P is (open in) a linear space and J is quadratic; *i.e.,* in a neighborhood of $x_0 \in P$ where $J(x_0) = 0$, we have

$$J(x_0 + h) = dJ(x_0) \cdot h + Q(h),$$

[5] See Bott (1954), or use a parameterized Morse lemma.

[6] In fact, as Alan Weinstein pointed out, in a neighborhood of $(0, 0)$ the equivariant Darboux theorem shows that it is a harmonic oscillator.

where $Q(h) = (1/2)B(h,h)$ and $B(u,v) = d^2J(x_0)(u,v)$.

2. Now $dJ(x_0) \circ dJ(x_0)^*$ is an isomorphism of range $dJ(x_0)$ onto itself. Let this map be denoted Δ, so Δ: range $dJ(x_0) \to$ range $dJ(x_0)$. Let $G = \Delta^{-1} \circ \mathbb{P} : \mathfrak{g}^* \to$ range $dJ(x_0)$, the "Green's function" for Δ. The crucial map we deal with is

   $$F : P \to P, \quad F(x) = x + dJ(x_0)^* \circ G \circ Q(h), \quad h = x - x_0.$$

   This map is inspired by a similar one used in deformations of complex structures by Kuranishi (1965) and is used in Atiyah et al. (1978).[7]

3. Clearly $DF(x_0) = \mathrm{id}$, so that F is a local diffeomorphism in a neighborhood of x_0.

4.

   Claim 8.4. *F takes a neighborhood of x_0 in $\mathcal{C}_{\mathbb{P}}$ to a neighborhood of x_0 in $\{x_0\}+$ ker $dJ(x_0)$.*

   Proof. We need to show that

   $$x \in \mathcal{C}_{\mathbb{P}} \Leftrightarrow F(x) - x_0 \in \ker dJ(x_0).$$

   Now

   $$dJ(x_0)\cdot(F(x)-x_0) = dJ(x_0)\cdot h + dJ(x_0)\circ dJ(x_0)^* \circ G \circ Q(h).$$

   But from its definition we clearly have $dJ(x_0) \circ dJ(x_0)^* \circ G = \mathbb{P}$, and so

   $$\begin{aligned} dJ(x_0)(F(x) - x_0) =& dJ(x_0) \cdot h + \mathbb{P}Q(h) \\ =& \mathbb{P}(dJ(x_0) \cdot h + Q(h)) \\ =& \mathbb{P}(J(x_0 + h)). \end{aligned}$$

   The assertion is thus obvious. ■

[7] Suggestions on this point by L. Nirenberg and I. Singer are gratefully acknowledged.

5. *F maps S_{x_0}* to *S_{x_0}*.

Proof. Recall that S_{x_0} is (a ball in) $\{x_0\}+$ ker $(dJ(x_0)\circ\mathbb{J})$. Then

$$\begin{aligned} dJ(x_0)\circ\mathbb{J}(F(x)-x_0) =& dJ(x_0)\circ\mathbb{J}(h+dJ(x_0)^*\circ G\circ Q(h)) \\ =& dJ(x_0)\circ\mathbb{J}(h), \end{aligned}$$

since $dJ(x_0)\circ\mathbb{J}\circ dJ(x_0)^*\equiv 0$. The assertion is then clear. ■

Thus F is a local diffeomorphism:

$$F:\mathcal{C}_{\mathbb{P}}\cap S_{x_0}\to\{x_0\}+\ker dJ(x_0)\cap\ker(dJ(x_0)\circ\mathbb{J}).$$

(Recall from above that $\mathcal{C}_{\mathbb{P}}\cap S_{x_0}$ is a manifold; F actually is an explicit chart for it.) So far we have not really used the fact that J is quadratic. Now we will.

Theorem 8.5. *F maps $\mathcal{C}\cap S_{x_0}$ locally one–to–one onto the cone $C_{x_0}=\{x_0\}+[(\mathrm{Id}-\mathbb{P})Q]^{-1}(0)$. Thus, $\mathcal{C}\approx C_{x_0}\times G/S_{x_0}$.*

Proof. We need to show that, for $x\in\mathcal{C}_{\mathbb{P}}\cap S_{x_0}$,

$$(\mathrm{Id}-\mathbb{P})Q(F(x)-x_0)=0\Leftrightarrow(\mathrm{Id}-\mathbb{P})J(x)=0.$$

By Claim 4, $F(x)-x_0\in$ ker $dJ(x_0)$, so, letting $\widehat{\mathbb{P}}$ be the projection onto ker $dJ(x_0)$, we have

$$F(x)-x_0=\widehat{\mathbb{P}}(F(x)-x_0)=\widehat{\mathbb{P}}(h+dJ(x_0)^*\circ G\circ Q(h))=\widehat{\mathbb{P}}h.$$

Thus

$$\begin{aligned} (\mathrm{Id}-\mathbb{P})Q(F(x)-x_0) =& (\mathrm{Id}-\mathbb{P})Q(\widehat{\mathbb{P}}h) \\ =& (\mathrm{Id}-\mathbb{P})Q(h-(\mathrm{Id}-\widehat{\mathbb{P}})h) \\ =& (\mathrm{Id}-\mathbb{P})\{Q(h)-B(h,(\mathrm{Id}-\widehat{\mathbb{P}})h\} \\ &+\frac{1}{2}B((\mathrm{Id}-\widehat{\mathbb{P}})h,((\mathrm{Id}-\widehat{\mathbb{P}})h). \end{aligned}$$

Now, by $\mathbb{J}$–invariance of $d^2J(x_0)$,

$$(\mathrm{Id} - \mathbb{P}) \circ B(h, (\mathrm{Id} - \widehat{\mathbb{P}})h) = (\mathrm{Id} - \mathbb{P}) \circ B(\mathbb{J}h, \mathbb{J}(\mathrm{Id} - \widehat{\mathbb{P}})h).$$

But $\mathbb{J}h \in \ker\, dJ(x_0)$ since $h \in \ker\, dJ(x_0) \circ \mathbb{J}$, as $x \in S_{x_0}$. Thus, by gauge invariance of $d^2J(x_0), B(\mathbb{J}h, \mathbb{J}(\mathrm{Id} - \widehat{\mathbb{P}})h) = 0$ since $\mathbb{J}(\mathrm{Id} - \widehat{\mathbb{P}})h \in \text{range } \mathbb{J} \circ dJ(x_0)^*$. Thus,

$$(\mathrm{Id} - \mathbb{P})Q(F(x) - x_0) = (\mathrm{Id} - \mathbb{P})Q(h) = (\mathrm{Id} - \mathbb{P})J(x),$$

since $J(x) = dJ(x) \cdot h + Q(h)$. This proves the theorem. ■

In finite dimensions, there is a proof that does not require J to be quadratic. The idea is to work in $\mathcal{C}_{\mathbb{P}} \cap S_{x_0}$ which is a symplectic manifold such that $\mathcal{S}_{x_0}$ acts on it, with the fixed point x_0. (We still assume that our group action has a slice as above.) In a neighborhood of x_0 there is a symplectic chart in which the group action is linear, by the equivariant Darboux theorem (Weinstein (1977), p. 24). These coordinates make the momentum map $j = (I - \mathbb{P})J|(\mathcal{C}_{\mathbb{P}} \cap S_{x_0})$ for this action homogeneous quadratic yielding the desired conclusion. Details about this cone can *then* be obtained from our determination of the degeneracy spaces in the theorem on p. 72.

Gauge theory .[8] Existence and uniqueness theory for the Cauchy problem shows that the structure of singularities in the solution space of the four–dimensional Yang–Mills field equations on a fixed background space–time is the same as that for the constraint equations. These constraint equations are well known and may be described as follows: Let M be a fixed compact three–manifold (a Cauchy surface in the fixed background space–time). Let $\pi : B \to M$ be a principal G–bundle and let $\mathcal{U}$ denote the space of ($W^{s,p}, s > 3/p+1$) connections on this bundle. Elements $A \in \mathcal{U}$ represent vector potentials for gauge fields restricted to M. Let $P = T^*\mathcal{U}$ be the basic symplectic space, elements of which are pairs (A, η); η represents the generalized electric field density. Assume $\mathfrak{g}$, the Lie algebra of G, carries an adjoint–action invariant inner product $(\cdot, \cdot)$, so that $T_{(A,\eta)}(T^*\mathcal{U})$

[8]This example is due to Judy Arms.

(elements of which are denoted (b, θ)) carries a preferred L^2–inner product $\langle\langle \cdot, \cdot \rangle\rangle$. This, the canonical symplectic structure and the complex structure $\mathbb{J}(b, \theta) = (-\theta, b)$ (appropriately dualized by $(\cdot, \cdot)$) are in the correct relationship.

The constraint equations are $J(A, \eta) = 0$, where $J(A, \eta)$ is the gauge covariant divergence of η using the connection A. In fact, J is the momentum map for the action of the group G of bundle automorphisms of B on P. This is the group G in the general theory; its Lie algebra is g, the $\mathfrak{g}$–valued functions on M. The dual g^* is the $\mathfrak{g}^*$–valued densities; thus $J : P \to g^*$. The adjoint operator $dJ(A, \eta)^*$ is elliptic and so one can construct a slice using ker $(dJ(A, \eta) \circ \mathbb{J})$; the spaces here are infinite dimensional, but ellipticity of $dJ(A, \eta)^*$ validates the technical points.

Moreover, J is quadratic. The quadratic term Q is

$$Q(b, \theta) = [b, \theta],$$

where b and θ are perturbations of A and η, so $h = (b, \theta)$, and $[b, \theta]$ is the bracket in $\mathfrak{g}$. (From this simple form gauge and $\mathbb{J}$–invariance can be verified directly.) The preceding theorem therefore applies. For gauge fields, infinitesimal symmetries of (A, η) are A–covariant constant $\mathfrak{g}$–valued functions on B that commute with η. The existence of such symmetries implies that the gauge field is reducible to a field with a smaller gauge group $H \subset G, H \neq G$. The space $\mathcal{N}_{(A,\eta)} \cap \mathcal{C}$ consists of solutions of the constraint equations which are reducible to the gauge group H; the rest of the solution set containing the conical singularities consists of solutions with a gauge group K intermediate between H and G, the conical singularities of specified symmetry type K then fit together to produce the entire conical singularity in the constraint set $\mathcal{C} = \{(A, \eta) J(A, \eta) = 0\}$. Finally, we remark that the space of solutions modulo gauge transformations seems to be a stratified symplectic manifold. (The details of the needed slice theorem and related technical facts have recently been proved by J. Rogulski.)

9

The Space of Solutions of Einstein's Equations: Regular Points

This lecture[1] will be devoted to a discussion of the following main theorem.

Theorem 9.1. *Let V be a four–manifold, ${}^{(4)}g_0$ a Lorentz metric on V and $M \subset V$ a compact spacelike three–manifold. Assume $(V, {}^{(4)}g_0)$ is a globally hyperbolic[2] space–time satisfying the vacuum Einstein equations:*

$$\mathrm{Ein}({}^{(4)}g_0) = 0$$

where $\mathrm{Ein}({}^{(4)}g_0) = \mathrm{Ric}({}^{(4)}g_0) - (1/2)R({}^{(4)}g_0){}^{(4)}g_0$ *denotes the Einstein tensor of* ${}^{(4)}g_0$.

Let $\mathcal{E}$ denote the set of all (globally hyperbolic) solutions of Einstein's vacuum equations (in a suitable Sobolev topology). Then

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[1]This lecture is based on joint work with A. Fischer and V. Moncrief. See Fischer and Marsden (1973), Fischer and Marsden (1979b), Fischer and Marsden (1979a) and Moncrief (1975b), Moncrief (1975a).

[2]Basic definitions in general relativity may be found in Hawking and Ellis (1973) or Misner et al. (1973).

$$\left(\begin{array}{c}\mathcal{E} \text{ is a smooth manifold in a} \\ \text{neighborhood of } {}^{(4)}g_0 \text{ with} \\ \text{tangent space the space of solutions} \\ \text{of the linearized Einstein equations}\end{array}\right) \Leftrightarrow \left(\begin{array}{c}{}^{(4)}g_0 \text{ has no Killing fields,} i.e., \\ \text{infinitesimal symmetries.}\end{array}\right)$$

Before embarking on the strategy of proof, we make a few preliminary remarks.

1. If ${}^{(4)}g$ is near ${}^{(4)}g_0$, then M will still be spacelike for ${}^{(4)}g$. In fact, M will be a Cauchy surface. (See Budic et al. [1978].)

2. The topology on $\mathcal{E}$ is actually that of H^s convergence on compact sets of V. Since $V \approx M \times \mathbb{R}$ is noncompact, this is a Fréchet topology. However, using the initial value formulation of the Einstein equations, explained below, we can restrict attention to a compact neighborhood of M and hence to a Banach space setting.

3. The compactness of M is important. In the noncompact case, symmetries of the background metric ${}^{(4)}g_0$ do not preclude $\mathcal{E}$ being a manifold. For example, if ${}^{(4)}g_0$ is the Minkowski metric on $\mathbb{R}^4$ (with symmetry group the Poincaré group), then $\mathcal{E}$ is a manifold near ${}^{(4)}g_0$, if appropriate asymptotic conditions are built in. This is related to the fact that asymptotically flat space–times have a well–defined energy and angular momentum, whereas cosmological space–times (*i.e.,* those with compact Cauchy surfaces) do not. For details of this case, see Choquet-Bruhat et al. (1979) and Cantor (1979).

4. In some sense the results of this and the next lecture are special cases of the theory in Lecture 8, with the symmetry group being the group of diffeomorphisms of V. There are quite a few steps required to nail this down precisely, which are outlined below.

5. For analogous results in geometry for the scalar curvature equation, see Fischer and Marsden (1975).

6. For the situation in gauge theory, see Atiyah et al. (1978) and Arms (1979), Arms (1980). The main ideas carry over for pure gauge theory and for gauge theory coupled to gravity. (For the coupling of gravity to matter fields the answer seems to depend on how the problem is formulated, *i.e.,* what physical quantities are taken to be the basic variables.)

As we shall see shortly, the breakdown of the manifold picture for $\mathcal{E}$ near a vacuum metric with symmetries is closely related to the breakdown of linear perturbation theory about a symmetrical background (with compact Cauchy surfaces). We first give a brief history of this linearization stability problem as it occurs in gravity.

1. Brill and Deser (1973) considered perturbations of the flat metric on $T^3 \times \mathbb{R}$ and discovered the first example of this trouble in perturbation theory. They found in going to a second order perturbation analysis that they had to readjust the first order perturbations in order to avoid inconsistencies at second order. This was the first hint of a conical structure for $\mathcal{E}$ near solutions with symmetry.

   Similarly Signorini (1930) had found a difficulty in the traction problem in elasticity by this perturbative approach; see Lecture 7.

2. Fischer and Marsden (1973) found general sufficient conditions for $\mathcal{E}$ to be a manifold in terms of the Cauchy data for vacuum spacetimes.

3. Choquet-Bruhat and Deser (1973) proved a version of the theorem that $\mathcal{E}$ is a manifold near Minkowski space, which was later improved by Choquet-Bruhat et al. (1979).

4. Moncrief (1975b) showed that the sufficient conditions derived by Fischer and Marsden for the compact case were equivalent to the requirement that $(V, {}^{(4)}g_0)$ have no Killing fields. This then led to the link between symmetries and bifurcations explained in Lecture 8, and completed the implication "$\Leftarrow$" in the theorem.

5. Moncrief (1975a) discovered the general splitting of gravitational perturbations; the generalization to momentum maps was found by Arms et al. (1975) and was described in Lecture 4.

6. Moncrief (1976) discovered the space–time significance of the second order conditions that arise when one has a Killing field and identified them with conserved quantities of Taub. This is described in Lecture 10.

7. Arms and Marsden (1979) used the second order conditions of Brill and Deser, as generalized by Fischer, Marsden and Moncrief, to complete the implication "$\Rightarrow$".

8. The description of the conical singularity in $\mathcal{E}$ near a spacetime with symmetries is due to Fischer et al. (1980) for one Killing field and to Arms, Fischer, Marsden and Moncrief in the general case. This situation is discussed in the next lecture.

In Lecture 8 we saw that linearization, bifurcation and symmetry phenomena are all related. Let us begin here with a general treatment of linear perturbations.

Let f be a map between two Banach spaces X, Y,

$$f : X \to Y$$

(the argument works just as well if X, Y are Banach manifolds), and consider trying to solve

$$f(x) = 0$$

near x_0, where x_0 is a solution. The linearized equations are

$$Df(x_0) \cdot h = 0.$$

From the implicit function theorem, if

$$Df(x_0) : X \to Y$$

is surjective, then the solution set $f^{-1}(0)$ is a manifold near x_0 and ker $Df(x_0)$ is its tangent space.

Technically we need, in the infinite dimensional case, some splitting property. This follows from ellipticity of either $Df(x_0)$ or its adjoint.

We define linearization stability of the equations $f(x) = 0$ as follows.

Definition 9.2. *The equation $f(x) = 0$ is called linearization stable at a solution X_0 if all $h \in X$ satisfying the linearized equations are integrable, i.e., if there exists a C^1 curve $x(\lambda)$, where λ is the curve parameter, of solutions of the exact equations such that $x(0) = x_0, x'(0) = h$.*

Remarks 1. Think of the solution of the linearized equation as providing the first term in a perturbation series about x_0,

$$x(\lambda) = x_0 + \lambda h + \frac{\lambda^2}{2!} h^{(2)} + \cdots .$$

The idea of linearization stability is to decide whether $x_0 + \lambda h$ is a good first approximation to a curve of solutions. If $Df(x_0)$ is surjective, then $f^{-1}(0)$ is a manifold and all solutions of $Df(x_0) \cdot h = 0$ form its tangent space. In this case it is obvious that perturbations can be extended to curves of solutions.

2. What goes wrong with the above argument in interesting examples such as gravity is that $Df(x_0)$ fails to be surjective at some point, so that we do not know whether we have got a manifold, or whether, if so, its tangent space coincides with ker $Df(x_0)$.

3. A simple example is provided by

$$f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x^2 - y^2.$$

The solution set is clearly a cone with vertex at the origin. However, $Df(0, 0) = 0$, so that any tangent vector satisfies the linearized equations at this point. Thus linearizing this is misleading, and must be adjusted by considering quadratic corrections. In this problem the exact equation is purely quadratic, so that second order corrections would clearly suffice to decide integrability. One could, however, add higher order terms to the f in this example to bend the cone. Then it would be a nontrivial statement that exact solution curves would be directions of the perturbations adjusted to satisfy the second order conditions. We will derive the general second order conditions in the next lecture.

♦

In general relativity or Yang–Mills theory, one is dealing with *hyperbolic* equations and there are no direct invertibility or splitting properties to use in the four–dimensional context. Thus we want to pass to the Cauchy problem, and split the problem into one of the initial value equations and evolution equations. In this initial value form of the theory, we can use elliptic theory and proceed as outlined above. Therefore we need to consider the dynamics of general relativity from the initial value or ADM (Arnowitt et al. (1962)) point of view. We follow the formulation of Fischer and Marsden (1979b).

Let $(V, {}^{(4)}g)$ be a space–time satisfying Ein $({}^{(4)}g) = 0$. The equations follow from the well–known Hilbert variational principle with Lagrangian density,

$$\mathcal{L} = d\mu_{{}^{(4)}g} \mathcal{R}({}^{(4)}g),$$

where $\mathcal{R}({}^{(4)}g)$ is the scalar curvature of ${}^{(4)}g$ and $d\mu_{{}^{(4)}g}$ its associated volume element. We want to split the Einstein equations into initial value equations and evolution equations, after introducing suitable dynamical variables. The technique for doing this is well known, so we shall just summarize the results.

First we need some notation. Let M be a compact three–manifold, and define

$$\mathrm{Emb}(M, V, {}^{(4)}g) = \{ \text{ set of all spacelike embeddings of } M \text{ into } V_4 \}.$$

This space (which is not a group) will play a role analogous to that of a symmetry group for a Hamiltonian system. The natural occurrence of this space of embeddings is a reflection of the true symmetry group

$$\mathcal{D} = \text{ four–dimensional diffeomorphisms of } V,$$

which the space–time formulation of Einstein's equations has. Though Emb $(M, V, {}^{(4)}g)$ is not a group, it will act rather like a group on the *solution* set of the Einstein equations, and we shall be able to use techniques similar to those developed in Lectures 3, 4 and 8 for Hamiltonian systems with symmetry groups. The space Emb $(M, V_4, {}^{(4)}g)$ is in fact a C^∞ infinite dimensional manifold (see Palais (1968) and Ebin and Marsden (1970)), with tangent space at $i \in$ Emb $(M, V, {}^{(4)}g)$ given by

$$T_i \,\mathrm{Emb}(M, V, {}^{(4)}g) = \{ {}^{(4)}X : M \to TV \mid {}^{(4)} X(x) \in T_{i(x)}V \}.$$

If you like, you can think of such a manifold of maps as a space of sections of a fiber bundle $(M \times V \to {}_\pi M)$. Notice that the space of embeddings also plays a key role in elasticity; see Lecture 2.)

We can decompose any ${}^{(4)}X$ in T_i Emb into a piece normal to the embedded hypersurface in V and a piece tangent to this surface. If we let ${}^{(4)}Z$ designate the unit future pointing normal field to $i(M)$, then we can write

$$ {}^{(4)}X = N {}^{(4)}Z + X, $$

where X is tangent to $i(M)$. This decomposition defines the *lapse function* N and the *shift vector* field X of the ADM formalism. Using i to pull these objects back to M, we can think of N and X as a function and a vector field on the three–manifold M; see Figure 9.1.

The analogue of $\mathfrak{g}$ (the gauge Lie algebra) is

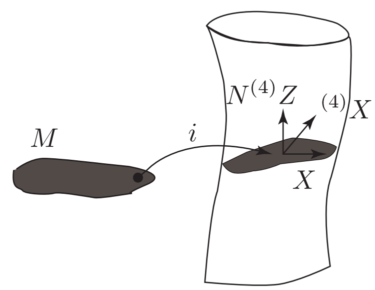

FIGURE 9.1. The lapse and shift.

$$
\begin{aligned}
&T_i \operatorname{Emb}(M, V_4, {}^{(4)}g)\\
&\qquad = \{ \text{ space of } N\text{'s and } X\text{'s } \}\\
&\qquad \approx C^\infty(M) \times \mathcal{H}^\infty(M) = C^\infty \text{ functions } \times C^\infty \text{ vector fields,}
\end{aligned}
$$

where we have used pullback by i to identify the space of N's and X's with the space of functions and vector fields on M, and have worked in C^∞ for simplicity. Note that there is no "identity" element of Emb $(M, V_4, {}^{(4)}g)$. Now define

$$
\begin{aligned}
g = i^{*(4)}g = {} & \text{the first fundamental form of } i(M)\\
& \text{regarded as a Riemannian metric on } M,
\end{aligned}
$$

and

$$
\begin{aligned}
k = {} & \text{the second fundamental form of } i(M) \text{ regarded}\\
& \text{as a symmetric two–tensor on } M.
\end{aligned}
$$

One can regard (g, k) as an element of $T\mathcal{M}$, where $\mathcal{M}$ is the space of Riemannian metrics of M. We now define

$$\pi = \mu(g)((\operatorname{trace} k)g - k)^{\#}$$

which is a contravariant two–tensor density on M. We now regard (g, π) as an element of $T^*\mathcal{M}$, where here is meant the L_2 dual space which is naturally provided by the ADM variational principle. Thus π is in the L_2 dual to $T_g\mathcal{M}$ (not the Sobolev dual defined by the topology on $\mathcal{M}$).

Now define the constraint maps

$$\Phi : T^*\mathcal{M} \to (\mathcal{C}^\infty(M)\times\mathcal{H}^\infty(M))^*, \quad \Phi(g,\pi) = (\mathcal{H}(g,\pi), \mathcal{J}(g,\pi)),$$

where

$$\mathcal{H}(g,\pi) = \mu(g)^{-1}\left(\pi\cdot\pi - \frac{1}{2}(\operatorname{trace}\pi)^2\right) - \mu(g)\mathcal{R}(g),$$

and

$$\mathcal{J}(g,\pi) = -2\pi^{ij}_{|j} = 2\delta\pi;$$

here $\mathcal{R}(g)$ is the curvature scalar of $g, \mu(g)$ its volume element and "|" represents covariant differentiation with respect to g. Yielding to the spiral of inflation of "super" geometric quantities in relativity, we define the super–momentum map

$$J(g,\pi)(N,X) = \int_M \langle (N,X), \Phi(g,\pi)\rangle = \int_M (N\mathcal{H} + H\cdot\mathcal{J}).$$

The following theorem (in a form first suggested by Fischer around 1973) geometrizes the content of the ADM formalism.

Theorem 9.3. *Let i_λ be any curve in* Emb $(M, V_4, {}^{(4)}g)$*, and let $(N(\lambda), X(\lambda))$ and $(g(\lambda), \pi(\lambda))$ be the corresponding curves induced by i_λ. Then the equations*

i
$$\Phi(g,\pi) = 0,$$

ii
$$\frac{\partial}{\partial\lambda}\begin{pmatrix} g(\pi) \\ \lambda(\pi)\end{pmatrix} = -\mathbb{J}\circ D\Phi(g(\lambda),\pi(\lambda))^*\cdot(N(\lambda),X(\lambda))$$

are equivalent to Ein $({}^{(4)}g) = 0$ *on the part of space–time swept out by $i_\lambda(M)$.*

It is well known that the evolution equations **ii** preserve the constraints **i**, so that the latter need only be imposed on an initial surface. In the above formula $D\Phi(g,\pi)^*$ is the L^2–adjoint of $D\Phi(g,\pi)$, and $\mathbb{J}$ is the complex structure associated with the L^2–metric $\langle\langle\cdot,\cdot\rangle\rangle$ on $T^*\mathcal{M}$ and the symplectic structure (see Lecture 1).

For comparison recall the situation for a Hamiltonian G–space (G, P, ω, J). There we found that the generators of the group associated with the moment map J were the Hamiltonian vector fields

$$\xi_P(x) = -\mathbb{J} \circ DJ(x)^* \cdot \xi.$$

Here (N, X) plays the role of $\xi =$ an element of the Lie algebra $\mathfrak{g}$. In the case of Hamiltonian G–spaces we required the property of Ad*–equivariance, which gave

$$\{\widehat{J}(\xi), \widehat{J}(\eta)\} = \widehat{J}([\xi, \eta]),$$

where $\{\cdot, \cdot\}$ was the Poisson bracket and $[\cdot, \cdot]$ was the Lie algebra bracket. In relativity, the analogous property of Poisson brackets of the moment map does not hold on all of phase space, but instead only on the subset defined by the constraints $J = 0$. This is one of the "penalties" of the fact that Emb $(M, V_4, {}^{(4)}g)$ is not a group, though it acts like a group on solutions of the field equations.

The equations $\Phi(g(\lambda)\pi(\lambda)) = 0$ are equivalent to the normal–normal and normal–tangential projections of the Einstein equations to the hypersurfaces $i_\lambda(M)$. The adjoint form of the evolution equations (**ii** above) provides a first order form for the remaining tangential–tangential projections of Ein $({}^{(4)}g)$ to the hypersurfaces $i_\lambda(M)$. The contracted Bianchi identities obeyed by Ein $({}^{(4)}g)$ ensure that if **i** holds on an initial surface then these constraints will be automatically propagated by **ii**.

The main theorem now is a formal consequence of our work in Lecture 8. There we showed that $J^{-1}(0)$ is a manifold near x_0 when x_0 has no symmetries; we did this by identifying ker $dJ(x_0)^*$ with symmetries, and concluded that if there were no symmetries, then J is a submersion.

There are a number of issues that must be overcome to really carry this out. First of all, Emb is not a group, so one must verify directly the steps where this is used. On a formal level this causes no difficulties. Secondly, there are the technical issues caused by the infinite dimensional nature of the problem.

Finally, there is the initial value theory which must be used to propagate our information on the constraint space $J^{-1}(0)$ to that on space–time itself. All of these issues have, in fact, been overcome (see the references cited earlier).

10

The Space of Solutions of Einstein's Equations: Singular Points

The main result[1] to be discussed in this lecture is as follows.

Theorem 10.1. *Let $(V, {}^{(4)}g_0)$ be a globally hyperbolic vacuum spacetime with a compact Cauchy hypersurface of constant mean curvature. Let $\mathcal{E}$ denote the space of solutions of the Einstein vacuum equations, as in Lecture 9.*

Suppose the space of Killings fields for ${}^{(4)}g_0$ has dimension $k \geq 1$. Then:

1. *$\mathcal{E}$ has a conical singularity at ${}^{(4)}g_0$ in the sense that $\mathcal{E} \approx C \times$ (manifold), where C is the zero set of a quadratic form with values in $\mathbb{R}^k$.*

2. *A solution ${}^{(4)}h$ of the linearized equations D Ebin $({}^{(4)}g_0){}^{(4)}h = 0$ is integrable if and only if the conserved quantities of Taub vanish; i.e., for every Killing field ${}^{(4)}X$ of ${}^{(4)}g_0$.*

$$\int_M {}^{(4)}X \cdot [D^2 \operatorname{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z d\mu = 0$$

where ${}^{(4)}Z$ is the forward point unit normal to M.

[1] Based on joint work with J. Arms, A. Fischer and V. Moncrief.

Discussion. 1. Notice that we have added the condition that $(V, {}^{(4)}g_0)$ admit a hypersurface of constant mean curvature. This is required for technical reasons in the proof. It is believed to be a relatively harmless assumption; indeed, any perturbation of ${}^{(4)}g_0$ has a hypersurface of constant mean curvature as well (Choquet-Bruhat et al. (1979) and they are believed to exist under rather general conditions (Marsden and Tipler (1980)).

2. One way of viewing the result is through perturbation series. Suppose one wishes a convergent expansion about the background ${}^{(4)}g_0$,

$$ {}^{(4)}g(\lambda) = {}^{(4)}g_0 + \lambda {}^{(4)}h + \frac{\lambda^2}{2} {}^{(4)}h_2 + \ldots; $$

such an expansion can be performed if and only if the Taub quantities formed from ${}^{(4)}h$ and the Killing fields of ${}^{(4)}g_0$ vanish. It is rather remarkable that the only obstruction to forming perturbation series occurs at second order. The complexity of the Einstein tensor might naively be expected to produce a more complicated solution set. *The "reason" it has only quadratic singularities is that secretly the Einstein tensor conceals a momentum mapping.*

3. The main idea for the proof of part 1 of the theorem is already contained in Lectures 8 and 9. There are a number of important points, however; one has to check that the slice theorem is valid and one has to be very careful about the fact that Emb $(M, V, {}^{(4)}g_0)$ is not a group and how to express, in terms of Cauchy data, the fact that ${}^{(4)}g_0$ and ${}^{(4)}g$ have the same symmetry type. The spacelike gauges are dealt with using the diffeomorphism group of M, while the timelike gauges require using the constraint of constant mean curvature. However, with this care in mind, the *outline* of the proof is the same as that in Lecture 8 using the momentum map of Lecture 9. The *details* are a rather long story: see Fischer et al. (1980) and Arms et al. (1982).

4. A model problem involving homogeneous cosmologies, where the conical structure can be seen explicitly, has been given by Jantzen (1979).

5. A good example of the theorem is the description of symmetry breaking in the flat universe $\mathbb{T}^3 \times \mathbb{R}$; if the time symmetry is broken, one passes to a Kasner universe (Bianchi type 1) and if a spacelike symmetry is then broken one gets a Gowdy universe. Each of these symmetry losses involves a conical singularity in the solution space.

6. The conical singularities in the solution space entail difficulties with other issues such as quantization (see Moncrief (1978)) and notions of "general solutions" (see Barrow and Tipler (1979)).

7. The solution manifold for asymptotically flat space–times is nonsingular, as was mentioned earlier. However, the difficulties in the cosmological case still arise if one studies perturbations of the total energy and angular momentum.

♦

The rest of the lecture will be devoted to explaining how the Taub quantities come in. This can be done by a complicated calculation relating D^2 Ein directly to $D^2 J$ and invoking the general results in Lecture 8. However, we can also proceed directly from the space–time point of view. Since this is simpler and more instructive, we shall do so.

We begin with the general procedure for discovering second order conditions. Again, let $f : X \to Y$ be a smooth map and let $x_0 \in X$, $f(x_0) = 0$. Suppose that $Df(x_0)$ is not surjective but has closed range, and let $h \in X$ satisfy the linearized equations

$$Df(x_0) \cdot h = 0.$$

Recall that h is *integrable* if there is a C^1 curve $x(\lambda)$ in X such that

$$f(x(\lambda)) = 0, \quad x(0) = x_0, \quad x'(0) = h.$$

Now differentiate $f(x(\lambda)) = 0$ successively with respect to λ to obtain

$$\begin{aligned}
&Df(x(\lambda)) \cdot x'(\lambda) = 0, \\
&D^2 f(x(\lambda)) \cdot (x'(\lambda), x'(\lambda)) + Df(x(\lambda)) \cdot x''(\lambda) = 0,
\end{aligned}$$

so that, setting $\lambda = 0$, we get

$$0 = D^2 f(x_0) \cdot (h, h) + Df(x_0) \cdot x''(\lambda).$$

Now let $l \in Y^*$ be such that $l \neq 0$ and $l = 0$ (as a linear functional) on the range of $Df(x_0)$. Such an l exists, since $Df(x_0)$ is not surjective, by the Hahn–Banach theorem. The above equation, paired with l, then gives

$$\langle l, D^2 f(x_0) \cdot (h, h) \rangle = 0,$$

since l paired with $Df(x_0) \cdot x''(0) \in$ range $(Df(x_0))$ gives zero. Thus we have the quadratic restrictions on h:

$$\langle l, D^2 f(x_0) \cdot (h, h) \rangle = 0 \text{ for all } l \text{ such that } l = 0 \text{ on the range of } Df(x_0).$$

Remarks 1. It is possible, of course, that these quadratic constraints are an identity on the solutions of $Df(x_0) \cdot h = 0$. In that case one should proceed as above and look for cubic or higher order constraints on h.

2. We have considered l to lie in the dual space Y^*. However, if we have an inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on Y we could use this to define the pairing in the above argument; this is what happens in practice.

3. The above method is well known in bifurcation theory as a technique for finding the directions of bifurcation.

4. In singularities of mappings, $\langle l, D^2 f(x_0) \cdot (h,k)\rangle$ is called the "intrinsic second derivative" of f at x_0.

♦

Let us now see how this idea works for the Einstein equations. To do this, we shall need a few preliminary calculations.

Lemma 10.2. *If* Ein $({}^{(4)}g) = 0$ *and* ${}^{(4)}h$ *is any symmetric two tensor, then*

$$\delta[D\,\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h] = 0,$$

where $\delta = \delta_{{}^{(4)}g}$ *is the divergence with respect to* ${}^{(4)}g$.

Proof. The contracted Bianchi identities assert that δ Ein $({}^{(4)}g) = 0$. Differentiation with respect to ${}^{(4)}g$ gives the identity

$$[D\delta({}^{(4)}g) \cdot {}^{(4)}h] \cdot \mathrm{Ein}({}^{(4)}g) + \delta[D\,\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h] = 0,$$

where $\delta({}^{(4)}g) = \delta_{{}^{(4)}g}$ indicates the functional dependence of δ on ${}^{(4)}g$, and $[D\delta({}^{(4)}g) \cdot {}^{(4)}h]\cdot$ Ein $({}^{(4)}g)$ is the linearized divergence operator acting on Ein $({}^{(4)}g)$. The lemma follows since Ein $({}^{(4)}g) = 0$. ■

Lemma 10.3. *Suppose* Ein $({}^{(4)}g) = 0$ *and* D Ein $({}^{(4)}g) \cdot {}^{(4)}h = 0$. *Then*

$$\delta[D^2\ \mathrm{Ein}({}^{(4)}h) \cdot ({}^{(4)}h, {}^{(4)}h)] = 0.$$

Proof. This follows from differentiating the contracted Bianchi identities twice to give

$$\begin{aligned}&[D^2\delta({}^{(4)}g) \cdot ({}^{(4)}h, {}^{(4)}h)] \cdot \mathrm{Ein}({}^{(4)}g) + [D\delta({}^{(4)}g) \cdot {}^{(4)}h] \cdot (D\ \mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h)\\ &+ D\delta({}^{(4)}g) \cdot {}^{(4)}h \cdot (D\ \mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h) + \delta D^2\,\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, {}^{(4)}h) = 0,\end{aligned}$$

and then using the hypotheses Ein $({}^{(4)}g) = 0$ and D Ein $({}^{(4)}g) = 0$. ■

Proposition 10.4 (Taub (1970))**.** *Suppose* Ein $({}^{(4)}g) = 0, D\,\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h = 0$ *and* ${}^{(4)}X$ *is a Killing field for* ${}^{(4)}g$. *Then the vector field*

$${}^{(4)}T = {}^{(4)}X \cdot [D^2\,\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, {}^{(4)}h)]$$

has zero divergence. (*Here the first "$\cdot$" denotes contraction.*)

Proof. From Lemma 10.2, the bracketed quantity has zero divergence. Thus ${}^{(4)}T$ is the contraction of a Killing field and a symmetric divergence–free two–tensor field, and hence has zero divergence. ■

As a consequence, if Σ_1 and Σ_2 are two compact spacelike hypersurfaces, then

$$\int_{\Sigma_1} {}^{(4)}T \cdot {}^{(4)}Z_{\Sigma_1} d^2\Sigma_1 = \int_{\Sigma_2} {}^{(4)}T \cdot {}^{(4)}Z_{\Sigma_2} d^3\Sigma_2,$$

where ${}^{(4)}Z_{\Sigma_i}, i = 1, 2$, is the unit forward pointing normal to Σ_i and $d^3\Sigma_i$ is its Riemannian volume element.

Lemma 10.5. *Suppose* Ein $({}^{(4)}g) = 0, {}^{(4)}X$ *is a Killing field of* ${}^{(4)}g, {}^{(4)}h$ *is a symmetric two–tensor field and* Σ *is a compact spacelike hypersurface. Then*

$$B(\Sigma, h) \equiv \int_{\Sigma} {}^{(4)}X \cdot [D\,\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h] \cdot {}^{(4)}Z_{\Sigma} d^3\Sigma = 0.$$

Proof. By Lemma 10.1.1, $D\,\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h$ is divergence free, and since ${}^{(4)}X$ is a Killing vector field, ${}^{(4)}X \cdot [D\,\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h]$ is a divergence–free vector field. Thus for two spacelike compact hypersurfaces, $B(\Sigma_1, {}^{(4)}h) = B(\Sigma_2, {}^{(4)}h)$. Choose Σ_1 and Σ_2 disjoint and replace ${}^{(4)}h$ by a symmetric two–tensor ${}^{(4)}k$ that equals ${}^{(4)}h$ on Σ and vanishes on a neighborhood of Σ_2. Then

$$B(\Sigma_1, {}^{(4)}h) = B(\Sigma_1, {}^{(4)}k) = B(\Sigma_2, {}^{(4)}k) = 0.$$ ■

5. Now we are ready to connect these ideas with linearization stability. If Ein $({}^{(4)}g) = 0$ and D Ein $({}^{(4)}g) \cdot {}^{(4)}h = 0$, we call ${}^{(4)}h$ an *infinitesimal deformation.* An *actual deformation* is a smooth curve ${}^{(4)}g(\lambda)$ of Lorentz metrics through ${}^{(4)}g_0$ satisfying Ein $({}^{(4)}g(\lambda)) = 0$. We say ${}^{(4)}h$ is *integrable* if, for every compact set $C \subset V_4$, there is an actual deformation ${}^{(4)}g(\lambda)$ defined on C such that ${}^{(4)}g(0) = {}^{(4)}g_0$ on D and

$$\frac{d}{d\lambda} {}^{(4)}g(\lambda)|_{\lambda=0} = {}^{(4)}h \text{ on } D.$$

By the chain rule, every integrable ${}^{(4)}h$ is an infinitesimal deformation. A spacetime is called *linearization stable* if every infinitesimal deformation is integrable.

In the presence of Killing fields, the necessary second order condition for integrability is as follows.

Proposition 10.6 (Second order conditions)**.** *Suppose Ein $({}^{(4)}g_0) = 0, {}^{(4)}X$ is a Killing field of ${}^{(4)}g_0$ and ${}^{(4)}h$ is integrable. Then the conserved quantity of Taub vanishes identically when integrated over any compact spacelike hypersurface Σ:*

$$\int_\Sigma {}^{(4)}X \cdot [D^2 \operatorname{Ein}({}^{(4)}g_0)({}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma = 0.$$

Proof. Differentiation of Ein $({}^{(4)}g(\lambda)) = 0$ twice with respect to λ at $\lambda = 0$ gives the identity

$$D^2 \operatorname{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h) + D \operatorname{Ein}({}^{(4)}g_0) \cdot {}^{(4)}k = 0,$$

where ${}^{(4)}k = (d^2/d\lambda^2){}^{(4)}g(\lambda)|_{\lambda=0}$. Contracting with ${}^{(4)}X$, integrating over Σ and using Lemma 4 gives the result. ■

Our main theorem says not only that are these conditions necessary for integrability but that they are sufficient as well.

We saw in Lecture 8 that gauge invariance of $d^2J(x_0)$ is a crucial part of the analysis. We conclude this lecture by showing how to obtain the required identities on the space–time (these identities can then be projected to the geometrodynamical data).

1. If $F : V_4 \to V_4$ is a diffeomorphism, then

$$\operatorname{Ein}(F^{*(4)}g) = F^*(\operatorname{Ein}({}^{(4)}g)),$$

where F^* denotes the pullback of tensors. This equation asserts the covariance of the Einstein operator. The infinitesimal version of covariance is the following:

Proposition 10.7. *Let ${}^{(4)}X$ be any vector field on V, and ${}^{(4)}h$ a symmetric two–tensor field. Then*

$$D \operatorname{Ein}({}^{(4)}g) \cdot (L_{{}^{(4)}X}{}^{(4)}g) = L_{{}^{(4)}X}(\operatorname{Ein}({}^{(4)}g))$$

and

$$D^2 \operatorname{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, L_{{}^{(4)}X}{}^{(4)}g) + D \operatorname{Ein}({}^{(4)}g) L_{{}^{(4)}X}{}^{(4)}h$$
$$= L_{{}^{(4)}X}(D \operatorname{Ein}({}^{(4)}g) \cdot {}^{(4)}h),$$

where $L_{{}^{(4)}X}$ denotes Lie differentiation.

Proof. Let F be the flow of ${}^{(4)}X$, and $F_0 = \mathrm{id}_V$, the identity diffeomorphism on V. (Of course, F_λ may be only locally defined.) Thus, locally,

$$\operatorname{Ein}(F_\lambda^{*(4)}g) = F_\lambda^* \operatorname{Ein}({}^{(4)}g).$$

Differentiating this relation in λ gives

$$D \operatorname{Ein}(F_\lambda^{*(4)}g) \cdot F_\lambda^*(L_{{}^{(4)}X}{}^{(4)}g) = F_\lambda^*(L_{{}^{(4)}X} \operatorname{Ein}({}^{(4)}g)).$$

Setting $\lambda = 0$ gives the first relation. Then, differentiating this result with respect to ${}^{(4)}g$ gives the second relation. ∎

2. If Ein $({}^{(4)}g) = 0$, it follows that

$$D \operatorname{Ein}({}^{(4)}g) \cdot L_{{}^{(4)}X}{}^{(4)}g = 0$$

   for any vector field ${}^{(4)}X$. Perturbations of the form $L_{{}^{(4)}X}{}^{(4)}g$ are *gauge* perturbations, so this equation shows that the linearized Einstein operator D Ein $({}^{(4)}g)$ is gauge invariant if ${}^{(4)}g$ is a solution to the empty space equations. Similarly, if ${}^{(4)}h$ solves the linearized equations

$$D \operatorname{Ein}({}^{(4)}g) \cdot {}^{(4)}h = 0,$$

   then we have the gauge invariance identity

$$D^2 \operatorname{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, L_{{}^{(4)}X}{}^{(4)}g) + D \operatorname{Ein}({}^{(4)}g) \cdot L_{{}^{(4)}X}{}^{(4)}h = 0$$

   for any ${}^{(4)}X$. We shall use this relationship to prove gauge invariance of Taub's conserved quantities ${}^{(4)}T$.

3. The next proposition establishes the *gauge invariance* of Taub's conserved quantities ${}^{(4)}T$ when integrated over a hypersurface.

Proposition 10.8. *Let* Ein $({}^{(4)}g) = 0, {}^{(4)}X$ *be a Killing field of* ${}^{(4)}g, D$ *Ein* $({}^{(4)}g) \cdot {}^{(4)}h = 0$ *and* ${}^{(4)}Y$ *an arbitrary vector field. Then, for any compact space–like hypersurface* Σ,

$$\int_\Sigma {}^{(4)}X \cdot [D^2 \operatorname{Ein}({}^{(4)}g)({}^{(4)}h + L_{{}^{(4)}Y}{}^{(4)}g, {}^{(4)}h + L_{{}^{(4)}Y}{}^{(4)}g)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma$$
$$= \int_\Sigma {}^{(4)}X \cdot [D^2 \operatorname{Ein}({}^{(4)}g)({}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma.$$

Proof. By the bilinearity of D^2 Ein $({}^{(4)}g)$, we need only show that

$$\int_\Sigma {}^{(4)}X \cdot [D^2 \operatorname{Ein}({}^{(4)}g)({}^{(4)}k, L_{{}^{(4)}Y}{}^{(4)}g)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma = 0,$$

where ${}^{(4)}k = {}^{(4)}h + L_{{}^{(4)}Y}{}^{(4)}g$ satisfies D Ein $({}^{(4)}g) \cdot {}^{(4)}k = 0$. But this follows by contracting the gauge invariance identity with ${}^{(4)}X$, integrating over Σ and using Lemma 4 above. ■

With these identities one can now plug into the machinery of Lecture 8.

As was the case with Yang–Mill fields, we expect that $\mathcal{E}/\mathcal{D}$ is a stratified set, each stratum being a symplectic manifold.

There are many other impressive applications of "global analysis" to general relativity which we have not mentioned. Perhaps the most spectacular of these is the solution of the positive mass conjecture by Schoen and Yau (1979c).

There are several papers by Schoen and Yau in 1979, which one is the corresponding paper? Schoen and Yau (1979c), Schoen and Yau (1979f), Schoen and Yau (1979d), Schoen and Yau (1979a), Schoen and Yau (1979e), Schoen and Yau (1979b)

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