The Averaged Fluid and EPDiff Equations: (A Deja Vu Lecture)

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Early History

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$$\frac{d}{dt}\eta(x,t) = u(\eta(x,t),t)$$

  - **For each fixed t, letting $\eta_t(x) = \eta(x,t)$, $\eta_t \in \mathrm{Diff}_{\mathrm{vol}}(M)$, the group of volume preserving diffeomorphisms of M (Sobolev class H^s, $s > (n/2)+1$ to be precise).**

Early History

- Theorem: *the following two statements are equivalent:*

  (i) *The veclocity field* $u(x,t)$ *satisfies the* ***Euler equations for ideal (homogeneous, incompressible, inviscid) flow****; i.e.,*

$$\frac{\partial u}{\partial t} + \nabla_u u = -grad\ p$$

  (∇ *is the Levi-Civita connection; in* $\mathbb{R}^3$, $\nabla_u u = u \cdot \nabla u$*)*

  (ii) *The curve* $\eta_t \in \mathrm{Diff}_{\mathrm{vol}}(M)$ *is an* L_2 *geodesic.*

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  (ii) *The curve* $\eta_t \in \mathrm{Diff}_{\mathrm{vol}}(M)$ *is an* L_2 *geodesic.*

□ Ebin and JEM (1970) showed that while the Euler equations for u are a PDE, the equations for η are an ODE in the sense that it satisfies an equation ***with no derivative loss*** on $\mathrm{Diff}_{\mathrm{vol}}$. Lets call this the ***smoothness property***.

Early History

- In particular, this makes local existence and uniqueness "trivial"; many other consequences too: e.g., particle paths are C^∞ even if the initial data is only H^s, any two nearby diffeomorphisms can be uniquely joined by a solution, etc. Many of these properties were *rediscovered* in the fluids literature between 1970 and present.

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- Interestingly, these same properties hold for other systems; eg, the ***integrable shallow water equation***—the CH (Camassa-Holm) equation has both the geodesic and the smoothness property and the KdV equation has the geodesic property. But there is much more to the story!!

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- Here are the ***LAE-α equations***:

$$\frac{\partial}{\partial t} m + (u \cdot \nabla) m - \alpha^2 (\nabla u)^T \cdot \Delta u = -\operatorname{grad} p,$$

where $\alpha > 0$ is a small parameter, $m = (1 - \alpha^2 \Delta) u$, $\operatorname{div} u = 0$, and p is the fluid pressure.

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  where $\alpha > 0$ is a small parameter, $m = (1 - \alpha^2 \Delta)u$, $\operatorname{div} u = 0$, and p is the fluid pressure.
- The original formal derivation was improved by JEM and Shkoller in 2001 and then improved again by Bhat, Fetecau, JEM, Mohseni and West (2003).

LANS-α Equation

- The idea of the derivation is to average ***Hamilton's principle*** of ideal fluid mechanics. Recall that the action function is defined on curves in $\mathrm{Diff}_{\mathrm{vol}}$ and may be expressed in terms of either η or u. It is

$$S(\eta(\cdot)) = \frac{1}{2}\int_a^b \|u(x,t)\|^2 \, d^3x \, dt$$

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- Given a test curve $\eta_t \in \mathrm{Diff}_{\mathrm{vol}}$, one now considers a bundle of curves forming a tube surrounding η_t in $\mathrm{Diff}_{\mathrm{vol}}$; the quantity α ***measures the size of the fluctuations*** in this tube.

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- Choosing an averaging operation with natural/reasonable properties, one averages the action principle.
- Imposing a Lie advection flow rule on the fluctuations—a type of ***Taylor Hypothesis***, the system closes and one gets (after some ***slow and careful calculations***), the ***averaged action function***

$$S^{\alpha}(\eta(\cdot)) = \int_M \left\{ \frac{1}{2} \|u\|^2 + \frac{\alpha^2}{2} \|\nabla u\|^2 \right\} d^3x.$$

LANS-α Equation

- Finally, one computes the Euler–Lagrange equations associated to this averaged action function and one gets the LAE-α equations. In doing so, it is helpful to have ***Euler-Poincaré reduction theory*** handy. (see any modern textbook on geometric mechanics).

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- **Theorem.** (Shkoller, 1999) ***The LAE-α equations have, just as with the Euler equations, the smoothness property.***

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- Shkoller noted that one can get the same viscosity term by modifying (as in work of Peskin) the action principle to include ***stochastic variations***.
- Numerical simulations were begun by the Los Alamos group for forced turbulent flows in a periodic box, already in 1999 and continue to the present time.

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- The ***theoretical structure*** of LANS-α is very attractive: global existence of smooth solutions in three dimensions (even if α is the size of a molecule), a Kelvin theorem, a nice H^1 energy bound, ...
- Research in this area continues....

Compressible Flows

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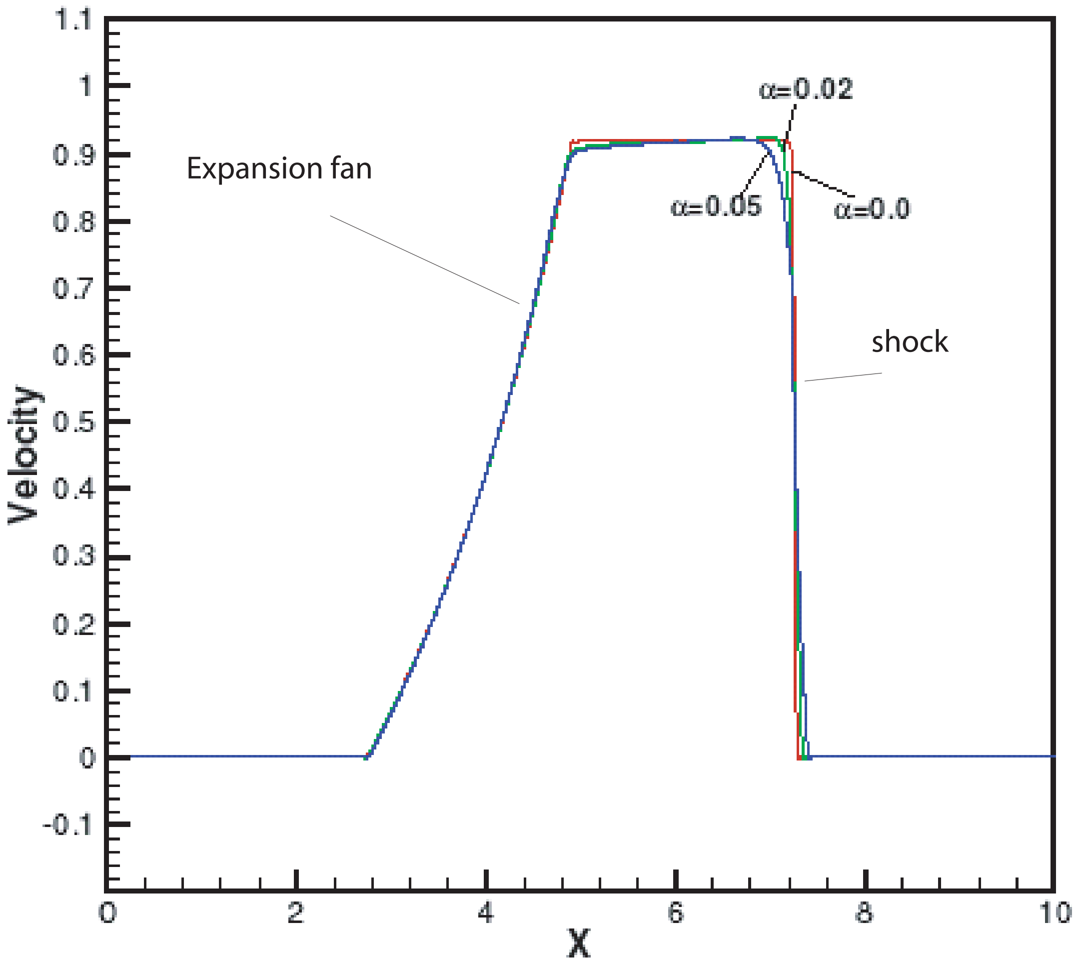
- Building on work of Holm, Bhat, Fetecau, JEM, Mohseni & West derived the ***CLANS-α equations***.
- This is really the subject of another talk, but suffice it to say that initial simulations, such as the ones shown here make us hopeful that these will provide good ***shock capturing methods*** as an alternative to ***artificial*** viscosity methods.

Compressible Flows

Compressible Flows



A shock capturing WENO scheme is used to solve a set of averaged compressible equations for a shock tube. The quantity α controls the thickness of the shock.

The EPDiff Equations

- We now turn to another set of equations, the ***EPDiff equations*** (Euler-Poincaré for the Diff group). These equations are an n-dimensional generalization of the ***CH equation for shallow water***:

$$u_t + 3uu_x = u_{xxt} + 2u_x u_{xx} + uu_{xxx}.$$

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$$u_t + 3uu_x = u_{xxt} + 2u_x u_{xx} + uu_{xxx}.$$

- Equivalently, the CH equation reads

$$m_t + um_x + 2u_x m = 0,$$

where $m = u - \alpha^2 u_{xx}$ and α^2 is a positive constant.

The EPDiff Equations

- In this m-form, the CH equation is the ***Lie-Poisson equation*** associated with the Lie algebra of one dimensional vector fields and with the Hamiltonian

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$$h(m) = \frac{1}{2} \int u\, m\, dx.$$

- The CH equation is also in ***Euler-Poincaré form*** with the Lagrangian corresponding to the H^1 metric corresponding to the Lagrangian,

$$l(u) = \frac{1}{2} \int (u^2 + \alpha^2 u_x^2)\, dx.$$

The EPDiff Equations

- Euler-Poincaré theory tells us that the one parameter curve of diffeomorphisms $\eta(x,t)$ defined, as in the case of the Euler equations, by

$$\frac{\partial}{\partial t}\eta(x,t) = u(\eta(x,t),t)$$

is a ***geodesic*** in the group of diffeomorphisms of $\mathbb{R}$ (or, in the spatially periodic case, the circle S^1) equipped with the right invariant metric equal to the H^1 metric at the identity.

Hey, what are the Equations?

- Here are the n-dimensional ***EPDiff equations***:

$$\frac{\partial}{\partial t}m + \underbrace{u \cdot \nabla m}_{\text{convection}} + \underbrace{\nabla u^T \cdot m}_{\text{stretching}} + \underbrace{m\,(\operatorname{div} u)}_{\text{expansion}} = 0,$$

where, as before, $m = (1 - \alpha^2 \Delta)u$.

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- In the 1d case, these reduce to the CH equations.
- These equations also come up in ***computer vision***! There they are known as the ***averaged template matching equations*** and are obtained (Mumford and company, Hirani and JEM) by an ***optimization process on deformations of one image to another***.

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- As a hint of the geometry to come, and as with much of fluid mechanics, there are other suggestive and more geometric ways of writing the equations.
- For the EPDiff equations, they can be written as

$$\frac{\partial m}{\partial t} + \pounds_u m = 0,$$

where $\pounds_u$ denotes the Lie derivative of m regarded as a one form density.

EPDiff Properties

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- It follows that the flows of solutions of the EPDiff equations are geodesics on Diff and conversely, geodesics give rise to solutions to the EPDiff equations.
- One more miracle: the same ***smoothness property*** as was shared by the Euler equations is also true for the EPDiff equations. Thus, the initial value problem is well posed (and many other things can be read off as well).

Singular Solutions

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- More or less the same thing is true for the EPDiff equations! More deja vu!.

Peakons for CH

- Now the CH equation is completely integrable and its soliton solutions are called ***peakons***. Unlike the usual solitons, they are not smooth. Otherwise they have properties similar to solitons.
- The CH equation has singular solutions whose momentum is supported at points on the real line:

$$m(x,t) = \sum_{i=1}^{N} p_i(t)\,\delta\big(x - q_i(t)\big)\,.$$

Peakons for CH

- The corresponding velocity is obtained by convolution with the Green's function,
$$G(|x-y|) = \tfrac{1}{2}e^{-|x-y|/\alpha}\,,$$
for the one-dimensional Helmholtz operator, $Q_{\rm op} = (1-\alpha^2\partial_x^2)$, appearing in the CH momentum velocity relationship, $m = Q_{\rm op}u$.
- Thus, the CH velocity corresponding to this momentum is given by a superposition of ***peaked traveling wave pulses***,
$$u(x,t) = \frac{1}{2}\sum_{i=1}^{N} p_i(t)\, e^{-|x-q_i(t)|/\alpha}\,.$$

Peakons for CH

- The resulting equations for $p_i(t)$ and $q_i(t)$, $i = 1, \ldots, N$, is an ***integrable system*** for any N. This system has a lot of fascinating algebraic geometry associated with it (Alber, Camassa, Fedorov, Holm and JEM).

Singular EPDiff Solutions

- Holm and Staley introduced the following measure-valued ansatz for $n-$dimensional solutions of the EPDiff equation:

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- These solutions are supported on N surfaces (or curves) of codimension $(n-k)$ for $s \in \mathbb{R}^k$ with $k < n$.
- For example, they may be supported on sets of points (***vector peakons***, $k=0$), one-dimensional filaments (***strings***, $k=1$), or two-dimensional surfaces (***sheets***, $k=2$) in three dimensions.

Singular EPDiff Solutions

- Substitution of the solution ansatz into the EPDiff equations gives the following integro-partial-differential equations:

$$\frac{\partial}{\partial t}Q^a(s,t) = \sum_{b=1}^{N} \int P^b(s',t)\, G(Q^a(s,t) - Q^b(s',t)\,)\, ds' \,,$$

$$\frac{\partial}{\partial t}P^a(s,t) = -\sum_{b=1}^{N} \int \big(P^a(s,t)\cdot P^b(s',t)\big)\, \frac{\partial}{\partial Q^a(s,t)} G\big(Q^a(s,t) - Q^b(s',t)\big)\, d$$

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- Again, one can check that the P's and Q's themselves evolve according to an interesting Hamiltonian system (integrability unknown!).

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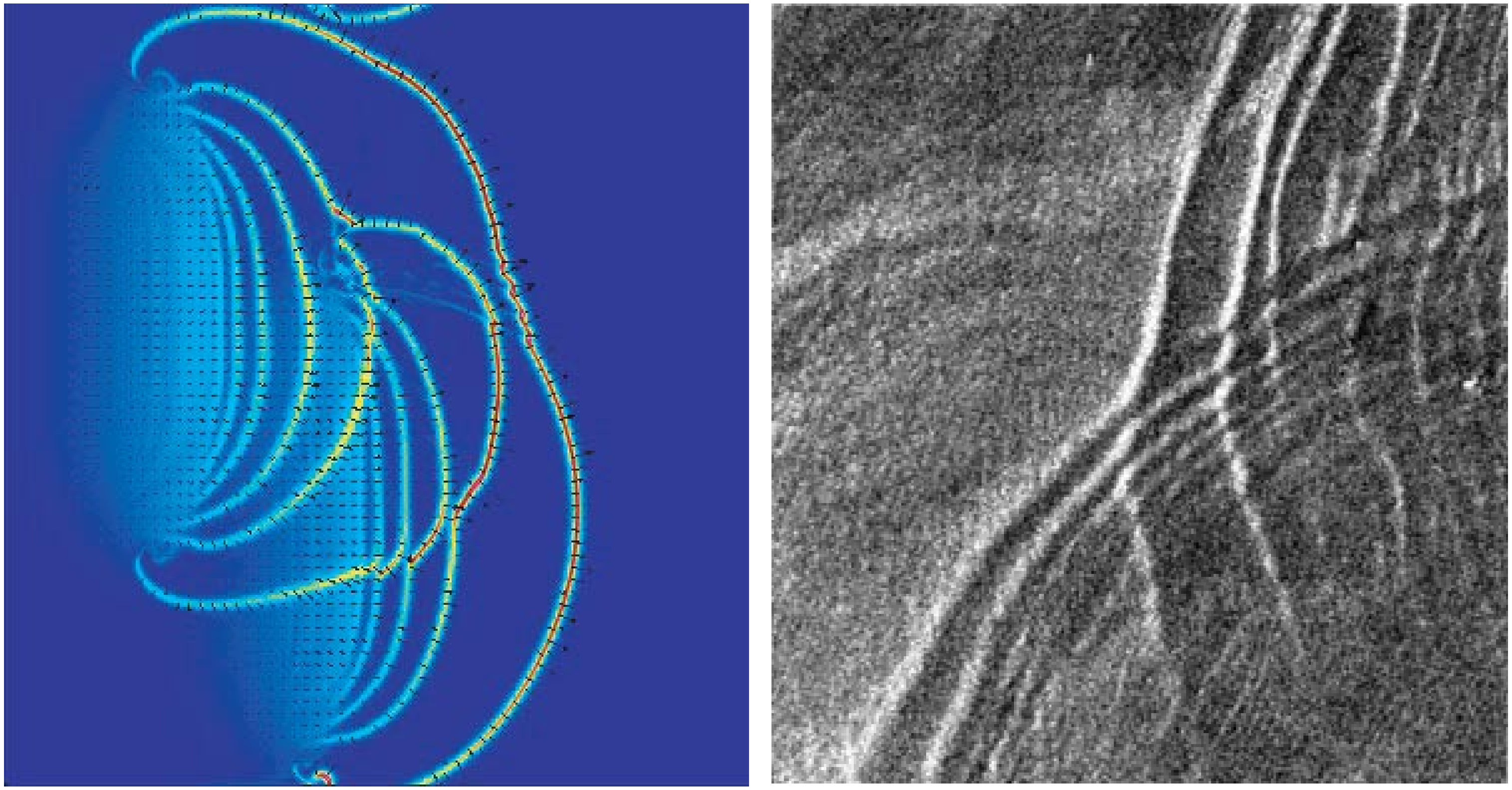
- Here, S is a manifold that is the support set of the P's and Q's and the momentum map is with respect to the natural ***left action*** of the group of diffeomorphisms of the ***target*** space. The whole system was, recall, ***right*** invariant, hence dual pairs, ...

Hey, Where's the Physics?



(a) (b)

Comparison of evolutionary EPDIFF solutions in two dimensions (a) and Synthetic Aperture Radar observations by the Space Shuttle of internal waves in the South China Sea (b). Both Figures show nonlinear reconnection occurring in the wave train interaction.

Hey, Where's the Physics?

Staley1

Hey, Where's the Physics?

$\boxed{Staley2}$

Hey, Where's the Physics?

Staley3

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- Integrable structures?
- Well posedness of the singular solutions?
- The geometric setting briefly indicated in this lecture suggests just a ***TON*** of interesting research questions.

Selected References

- Ebin, D.G. & J.E. Marsden [1970], *Groups of diffeomorphisms and the motion of an incompressible fluid*, *Ann. of Math.* 92, 102–163.
- Holm, D.D., J.E. Marsden, and T.S. Ratiu [1998], *Euler–Poincaré models of ideal fluids with nonlinear dispersion*, *Phys. Rev. Lett.* 349, 4173–4177 and *Adv. in Math.* 137, 1–81.
- Mohseni, K., B. Kosović, S. Shkoller, & J.E. Marsden [2003], *Numerical simulations of the Lagrangian averaged NavierStokes equations for homogeneous isotropic turbulence*, *Phys. of Fluids* 15, 524–544.
- Marsden, J.E. and S. Shkoller [2003], *The anisotropic averaged Euler equations*, *Arch. Rat. Mech. An.* 166, 27–46.

Selected References

- Bhat, H.S., R.C. Fetecau, J.E. Marsden, K. Mohseni, and M. West [2003], *Lagrangian averaging for compressible fluids*, (preprint).
- Holm, D.D. and J. E. Marsden [2003], *Momentum maps and measure valued solutions (peakons, filaments, and sheets) of the Euler-Poincaré equations for the diffeomorphism group*. In *The Breadth of Symplectic and Poisson Geometry*, (Marsden, J. E. and T. S. Ratiu, eds), ***Festshrift for Alan Weinstein (to appear)***. Birkhäuser Boston.
- Holm, D. D., and M. F. Staley [2003] *Wave structures and nonlinear balances in a family of evolutionary PDEs.* *SIAM J. Dyn. Syst.* To appear.

For papers, visit **http://www.cds.caltech.edu/∼marsden**

The End

TYPESETTING SOFTWARE: TEX, *Textures*, LATEX, `hyperref`, `texpower`, Adobe Acrobat 4.05
GRAPHICS SOFTWARE: Adobe Illustrator 10.0.1
LATEX SLIDE MACRO PACKAGES: Wendy McKay, Ross Moore