Variational Principles, Dirac Structures, and Reduction

Dedicated to Alan@60

Jerrold E. Marsden

Joint work with Hiroaki Yoshimura (and others)

Control and Dynamical Systems, Caltech

http://www.cds.caltech.edu/~marsden/

Alanfest, Vienna, August 7, 2003

What Dirac did; what we do

- Dirac (who knew mechanics!), studied Lagrangian systems with constraints, including those arising from degeneracies.

What Dirac did; what we do

- Dirac (who knew mechanics!), studied Lagrangian systems with constraints, including those arising from degeneracies.
- See his article on “Generalized Hamiltonian Mechanics” in the *Canadian J. of Math.*, circa 1950.

What Dirac did; what we do

- Dirac (who knew mechanics!), studied Lagrangian systems with constraints, including those arising from degeneracies.
- See his article on "Generalized Hamiltonian Mechanics" in the *Canadian J. of Math.*, circa 1950.
- He also worked on the program of going to the Hamiltonian side via the Legendre transformation and computing the associated Poisson brackets.

What Dirac did; what we do

- Dirac (who knew mechanics!), studied Lagrangian systems with constraints, including those arising from degeneracies.
- See his article on "Generalized Hamiltonian Mechanics" in the *Canadian J. of Math.*, circa 1950.
- He also worked on the program of going to the Hamiltonian side via the Legendre transformation and computing the associated Poisson brackets.
- Lesson learned from examples and applications:
  In many if not most cases, one does ***not start*** on the Hamiltonian side, but rather on the Lagrangian side with a variational principle.

What Dirac did; what we do

- Dirac understood this very clearly and it is how his papers are written; but this seems to have been a forgotten lesson!

What Dirac did; what we do

- Dirac understood this very clearly and it is how his papers are written; but this seems to have been a forgotten lesson!
- While it is quite appropriate that *Dirac structures* are named after him, it seems that workers in the field have so far left out Lagrangian mechanics from the story! *Our goal is to fill this gap (or canyon)*.

What Dirac did; what we do

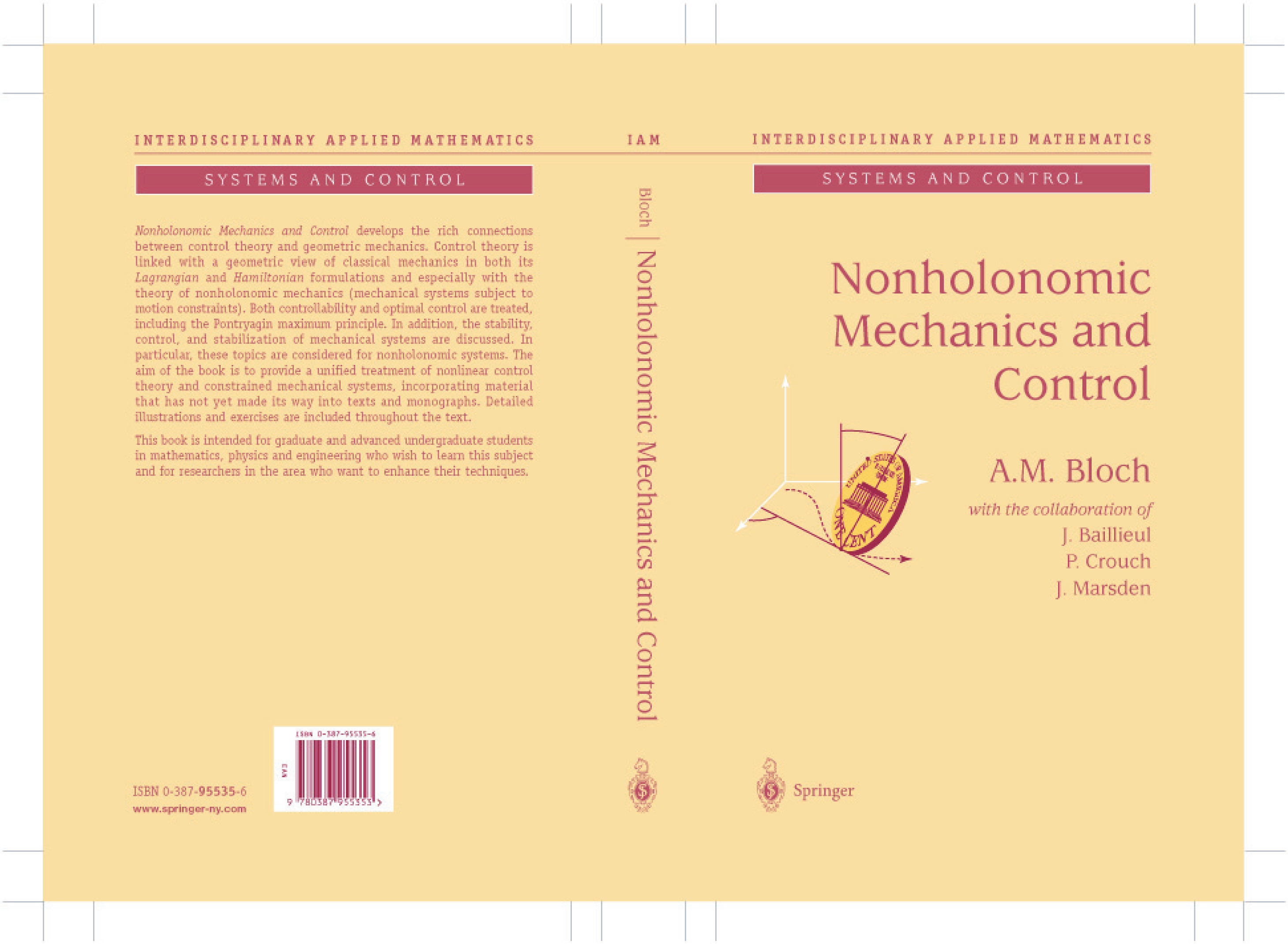
- Dirac understood this very clearly and it is how his papers are written; but this seems to have been a forgotten lesson!
- While it is quite appropriate that *Dirac structures* are named after him, it seems that workers in the field have so far left out Lagrangian mechanics from the story! *Our goal is to fill this gap (or canyon)*.
- You are wrong if you believe that this gap can be trivially filled by simply waving a Legendre transformation wand.

Examples

- Standard nondegenerate Lagrangian and Hamiltonian systems, possibly with symmetry, possibly reduced.
- Specific case: *the dynamics of asteroid pairs*, such as Ida and Dactyl:

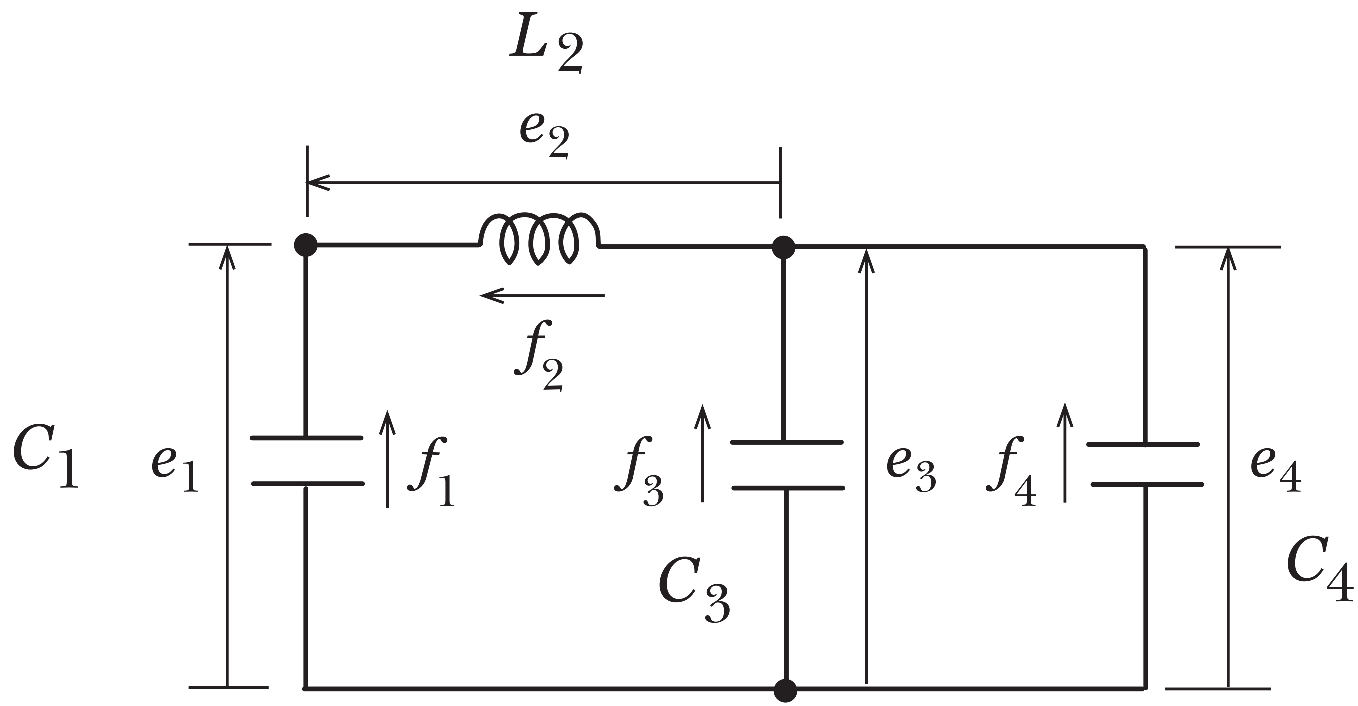
Examples

- **Nonholonomic mechanics.** Specific example is the rolling penny. Have a look at one of my favorite books:



Examples

- **Electrical Networks.** Specific example; how to analyze the dynamics associated with this network:



Theoretical Developments

- General development of Dirac structures (Courant, Weinstein, Dorfman) and reduction theory of Dirac structures (Van der Schaft, Blankenstein, Ratiu).
- Application to nonholonomic systems and circuits, but on the Hamiltonian side (horrors!) by Van der Schaft, Maschke, Bloch, Crouch and others.

Theoretical Developments

- General development of Dirac structures (Courant, Weinstein, Dorfman) and reduction theory of Dirac structures (Van der Schaft, Blankenstein, Ratiu).
- Application to nonholonomic systems and circuits, but on the Hamiltonian side (horrors!) by Van der Schaft, Maschke, Bloch, Crouch and others.
- Lagrangian and Hamiltonian reduction theory (see article of JM and Alan on the history of the subject).
- General symplectic and Poisson reduction are fine, but one wants more detail for the case of tangent and cotangent bundles. Why? Well, that is how one does examples!

Theoretical Developments

- **Hamiltonian reduction of cotangent bundles**[1] Start with $H : T^*Q \to \mathbb{R}$ and a Lie group G acting (free and proper for simplicity) on Q. Choose a principal connection A on the *shape space bundle* $Q \to Q/G$.

[1] Weinstein, Sternberg, Montgomery, Cendra, JEM, Ratiu,....

Theoretical Developments

- Hamiltonian reduction of cotangent bundles[1] Start with $H : T^*Q \to \mathbb{R}$ and a Lie group G acting (free and proper for simplicity) on Q. Choose a principal connection A on the *shape space bundle* $Q \to Q/G$.
- Mechanics and examples tell you that this global perspective is the right one–that is, one should really choose a connection at this point. Then,

$$T^*Q/G \cong_A T^*(Q/G) \times \tilde{\mathfrak{g}}^*$$

with its natural Poisson structure (containing curvature terms from A), etc.

[1] Weinstein, Sternberg, Montgomery, Cendra, JEM, Ratiu,....

Theoretical Developments

- Hamiltonian reduction of cotangent bundles[1] Start with $H : T^*Q \to \mathbb{R}$ and a Lie group G acting (free and proper for simplicity) on Q. Choose a principal connection A on the *shape space bundle* $Q \to Q/G$.
- Mechanics and examples tell you that this global perspective is the right one–that is, one should really choose a connection at this point. Then,

$$T^*Q/G \cong_A T^*(Q/G) \times \tilde{\mathfrak{g}}^*$$

  with its natural Poisson structure (containing curvature terms from A), etc.
- Reduced equations on this space are called the *Hamilton-Poincaré equations.* When $Q = G$, you get the *Lie-Poisson equations* on $\mathfrak{g}^*$.

[1] Weinstein, Sternberg, Montgomery, Cendra, JEM, Ratiu,....

Theoretical Developments

- If you do symplectic reduction in this context by imposing a momentum map constraint $\mathbf{J} = \mu$ then one gets an *associated coadjoint orbit bundle*

$$\mathbf{J}^{-1}(\mu)/G_\mu = T^*(Q/G) \times \tilde{\mathcal{O}}_\mu$$

  whose symplectic structure is a combination of the KKS orbit bracket with the cotangent bracket plus a curvature term.

Theoretical Developments

- If you do symplectic reduction in this context by imposing a momentum map constraint $\mathbf{J} = \mu$ then one gets an *associated coadjoint orbit bundle*

$$\mathbf{J}^{-1}(\mu)/G_\mu = T^*(Q/G) \times \tilde{\mathcal{O}}_\mu$$

  whose symplectic structure is a combination of the KKS orbit bracket with the cotangent bracket plus a curvature term.

- This is actually *useful* in examples; eg, the asteroid pair problem, in the dynamics of a fluid with a free surface. (The Hodge decomposition provides the connection).

Theoretical Developments

- **Lagrangian reduction of tangent bundles**[2]: Start with $L : TQ \to \mathbb{R}$ and a Lie group G acting (free and proper for simplicity) on Q. Choose a principal connection A on the *shape space bundle* $Q \to Q/G$, so that

$$TQ/G \cong_A T(Q/G) \times \tilde{\mathfrak{g}}$$

[2] JEM, Scheurle, Weinstein, Cendra, Ratiu, ...

Theoretical Developments

- **Lagrangian reduction of tangent bundles**[2]: Start with $L : TQ \to \mathbb{R}$ and a Lie group G acting (free and proper for simplicity) on Q. Choose a principal connection A on the *shape space bundle* $Q \to Q/G$, so that

$$TQ/G \cong_A T(Q/G) \times \tilde{\mathfrak{g}}$$

- Reduced equations on this space are called the *Lagrange–Poincaré equations.* When $Q = G$, you get the *Euler–Poincaré equations* on $\mathfrak{g}$.

[2] JEM, Scheurle, Weinstein, Cendra, Ratiu, ...

Theoretical Developments

- **Lagrangian reduction of tangent bundles**[2]: Start with $L : TQ \to \mathbb{R}$ and a Lie group G acting (free and proper for simplicity) on Q. Choose a principal connection A on the *shape space bundle* $Q \to Q/G$, so that

$$TQ/G \cong_A T(Q/G) \times \tilde{\mathfrak{g}}$$

- Reduced equations on this space are called the *Lagrange–Poincaré equations.* When $Q = G$, you get the *Euler–Poincaré equations* on $\mathfrak{g}$.
- Strategy: reduce variational principles, not symplectic or Poisson structures. One can close up this category so that reduction keeps you in it (reduction by stages).

[2] JEM, Scheurle, Weinstein, Cendra, Ratiu, ...

Theoretical Developments

- Lagrangian reduction of tangent bundles[2]: Start with $L : TQ \to \mathbb{R}$ and a Lie group G acting (free and proper for simplicity) on Q. Choose a principal connection A on the *shape space bundle* $Q \to Q/G$, so that

$$TQ/G \cong_A T(Q/G) \times \tilde{\mathfrak{g}}$$

- Reduced equations on this space are called the *Lagrange–Poincaré equations.* When $Q = G$, you get the *Euler–Poincaré equations* on $\mathfrak{g}$.
- Strategy: reduce variational principles, not symplectic or Poisson structures. One can close up this category so that reduction keeps you in it (reduction by stages).
- *Don't you dare* choose a metric and identify TQ and T^*Q or a Killing form and identify $\mathfrak{g}$ and $\mathfrak{g}^*$!!

[2] JEM, Scheurle, Weinstein, Cendra, Ratiu, ...

Theoretical Developments

- Many other variations on the theme: For example, *semi-direct product reduction theory*[3] and its recent generalization to group extensions (including Bott Virasoro, Camassa-Holm, Dym, etc).

[3] Guillemin-Sternberg, JEM, Weinstein, Ratiu, Leonard, Holm,...

Theoretical Developments

- Many other variations on the theme: For example, *semi-direct product reduction theory*[3] and its recent generalization to group extensions (including Bott Virasoro, Camassa-Holm, Dym, etc).
- Stability theory[4] for relative equilibria; energy-momentum method (Arnold method, energy-Casimir, block diagonalization, ...).

[3] Guillemin-Sternberg, JEM, Weinstein, Ratiu, Leonard, Holm,...
[4] Arnold, JEM, Simo, Lewis,...

Nonholonomic Mechanical Systems

- Constraints such as rolling constraints. Dynamics governed by the *Lagrange–d'Alembert principle*: start with a distribution $\Delta \subset TQ$ and ask that

$$\delta \int L \, dt = 0$$

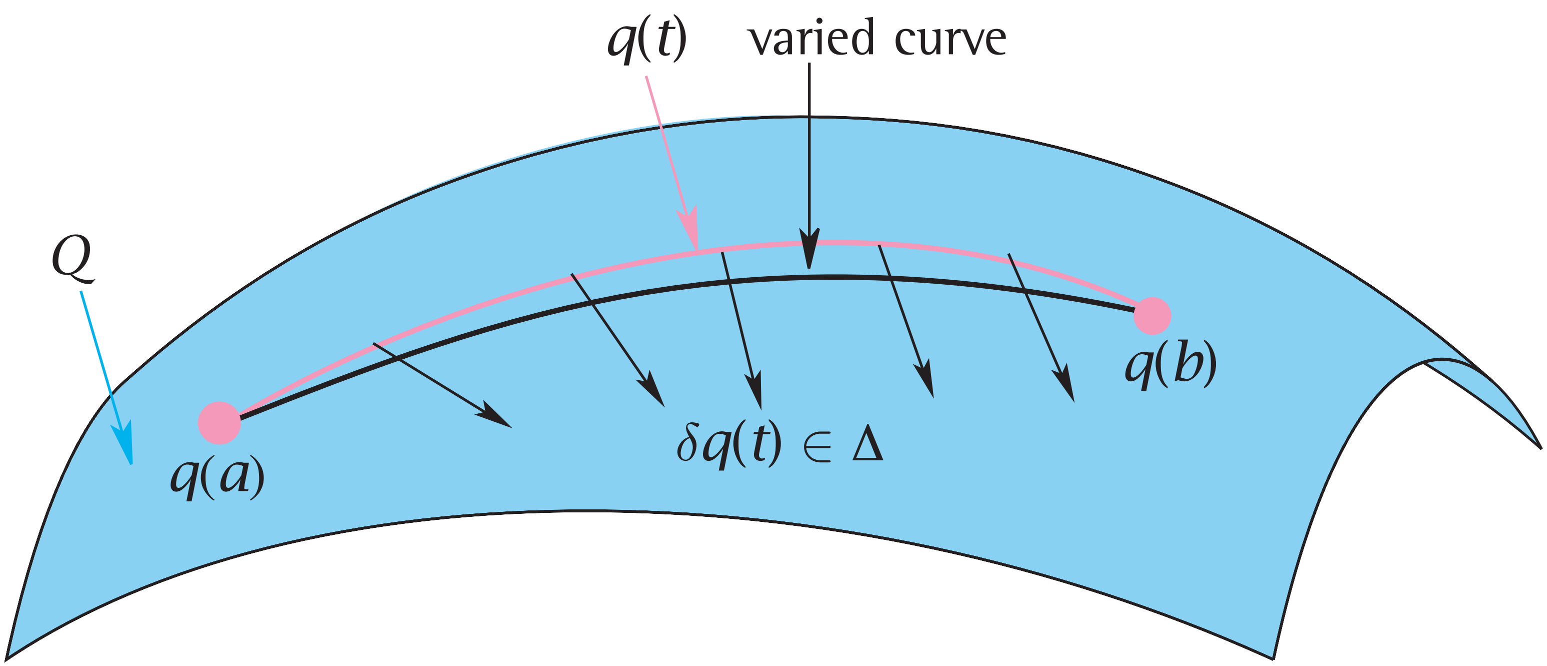
for all $\delta q \in \Delta$. One also impose the condition $\dot{q} \in \Delta$.

Nonholonomic Mechanical Systems

- Constraints such as rolling constraints. Dynamics governed by the *Lagrange–d'Alembert principle*: start with a distribution $\Delta \subset TQ$ and ask that

$$\delta \int L\, dt = 0$$

for all $\delta q \in \Delta$. One also impose the condition $\dot{q} \in \Delta$.



Nonholonomic Mechanical Systems

- Reduction theory for these systems started by Jair Koiller!

Nonholonomic Mechanical Systems

- Reduction theory for these systems started by Jair Koiller!
- In general, symmetries need not lead to conservation laws; *Momentum equation* discovered by BKMM. This, together with holonomy (geometric phases) plays a crucial role in locomotion (eg, how snakes move).

Nonholonomic Mechanical Systems

- Reduction theory for these systems started by Jair Koiller!
- In general, symmetries need not lead to conservation laws; *Momentum equation* discovered by BKMM. This, together with holonomy (geometric phases) plays a crucial role in locomotion (eg, how snakes move).
- The equations on the Hamiltonian side (assuming the Lagrangian is regular) are governed by an *almost Poisson structure* and, in a sense, by an almost symplectic structure. In fact, the Jacobiator is measured by the curvature of Δ.[5]

[5] Van der Schaft and Maschke, Bates and Sniatycki, Koon and JEM, Marle, ...

Nonholonomic Mechanical Systems

- Reduction theory for these systems started by Jair Koiller!
- In general, symmetries need not lead to conservation laws; *Momentum equation* discovered by BKMM. This, together with holonomy (geometric phases) plays a crucial role in locomotion (eg, how snakes move).
- The equations on the Hamiltonian side (assuming the Lagrangian is regular) are governed by an *almost Poisson structure* and, in a sense, by an almost symplectic structure. In fact, the Jacobiator is measured by the curvature of Δ.[5]
- Despite this, the system *is* described by a Dirac structure, as I will explain below.

[5] Van der Schaft and Maschke, Bates and Sniatycki, Koon and JEM, Marle, ...

Nonholonomic Mechanical Systems

- Despite the above, there is also an energy-momentum method for stability (applies to the classical examples: rattleback, the unicycle, ...); there is also a picture similar to the bundle picture above with a beautiful intrinsic geometric description.[6]
- Resulting equations: the *Lagrange-d'Alembert-Poincaré* equations.
- (Not quite as bad as hemi-quasi-twisted-algebroids, ...)
- Circuits are typically not only nonholonomic (because of the Kirchhoff current laws), they are also often degenerate (giving *primary* constraints in the sense of Dirac).

[6]Obtained by Cendra, Marsden and Ratiu in 2001.

Variational methods are useful!

- Shell collisions: thin shell models using *multisymplectic variational methods, AVI + subdivision + collision methods* (JEM, Ortiz, Cirac–West)

Shell collision

A Question for you.

- The Euler–Lagrange equations come from Hamilton's principle
- Hamilton's equations come from Hamilton's phase space variational principle
- The Euler–Poincaré equations come from the Euler–Poincaré variational principle

A Question for you.

- The Euler–Lagrange equations come from Hamilton's principle
- Hamilton's equations come from Hamilton's phase space variational principle
- The Euler-Poincaré equations come from the Euler-Poincaré variational principle
- **Question:** what is it for the Lie-Poisson equations? (I can make the question harder and ask this for the Hamilton-Poincaré equations or the Lagrange-Poincaré equations).

A Question for you.

- The Euler–Lagrange equations come from Hamilton's principle
- Hamilton's equations come from Hamilton's phase space variational principle
- The Euler-Poincaré equations come from the Euler-Poincaré variational principle
- **Question:** what is it for the Lie-Poisson equations? (I can make the question harder and ask this for the Hamilton-Poincaré equations or the Lagrange-Poincaré equations).
- Euler-Poincaré equations on $\mathfrak{g}$

$$\frac{d}{dt}\frac{\delta l}{\delta v} = \mathrm{ad}_v^* \frac{\delta l}{\delta v},$$

A Question for you.

- Answer:[7]

$$\delta \int_a^b l\,(v\,(t))\,dt = 0$$

for *constrained* variations of the form

$$\delta v\,(t) = \dot{\eta}(t) + [v\,(t), \eta(t)],$$

where $\eta(t)$ has fixed endpoints.

[7]See, for example, *Mechanics and Symmetry* by JEM and Ratiu.

A Question for you.

- Answer:[7]

$$\delta \int_a^b l(v(t))\, dt = 0$$

for *constrained* variations of the form

$$\delta v(t) = \dot{\eta}(t) + [v(t), \eta(t)],$$

where $\eta(t)$ has fixed endpoints.

- Method of proof: reduce Hamilton's principle on TG

[7]See, for example, *Mechanics and Symmetry* by JEM and Ratiu.

A Question for you.

- Back to our question: how are the Lie-Poisson equations

$$\dot{\mu} = \mathrm{ad}^*_{\delta h/\delta\mu}\,\mu$$

on $\mathfrak{g}^*$ variational?

A Question for you.

- Back to our question: how are the Lie-Poisson equations

$$\dot{\mu} = \mathrm{ad}^*_{\delta h/\delta\mu}\,\mu$$

on $\mathfrak{g}^*$ variational?

- Answer: A Pontryagin type principle:

$$\delta \int_a^b \left(\langle \mu(t), v(t) \rangle - h\left(\mu(t)\right) \right) dt = 0$$

for variations of the form

$$\delta v(t) = \dot{\eta}(t) + [v(t), \eta(t)],$$

where $\eta(t)$ has fixed endpoints, but $\delta\mu$ are arbitrary.

A Question for you.

- Back to our question: how are the Lie-Poisson equations

$$\dot{\mu} = \mathrm{ad}^*_{\delta h/\delta \mu}\, \mu$$

  on $\mathfrak{g}^*$ variational?

- Answer: A Pontryagin type principle:

$$\delta \int_a^b \left(\langle \mu(t), v(t) \rangle - h\left(\mu(t)\right) \right) dt = 0$$

  for variations of the form

$$\delta v(t) = \dot{\eta}(t) + [v(t), \eta(t)],$$

  where $\eta(t)$ has fixed endpoints, but $\delta\mu$ are arbitrary.

- The Legendre transformation $v = \delta h/\delta\mu$ is part of the variational principle! Neat! We will need this sort of thing.

- Method: reduce Hamilton's phase space principle.

Implicit Hamiltonian Systems

- Recall: a *Dirac structure* on a manifold R is: a subbundle $D \subset TR \times T^*R$ such that $D = D^\perp$, where the perp is with respect to the natural pairing

$$\langle\langle (u, \alpha), (v, \beta) \rangle\rangle = \langle \beta, u \rangle + \langle \alpha, v \rangle$$

Implicit Hamiltonian Systems

- Recall: a *Dirac structure* on a manifold R is: a subbundle $D \subset TR \times T^*R$ such that $D = D^\perp$, where the perp is with respect to the natural pairing

$$\langle\langle (u, \alpha), (v, \beta) \rangle\rangle = \langle \beta, u \rangle + \langle \alpha, v \rangle$$

- Standard examples: graph of an (almost) symplectic or (almost) Poisson structure. [Not enough for nonholonomic systems—one needs to put in the constraints].

Implicit Hamiltonian Systems

- Recall: a *Dirac structure* on a manifold R is: a subbundle $D \subset TR \times T^*R$ such that $D = D^\perp$, where the perp is with respect to the natural pairing

$$\langle\langle (u, \alpha), (v, \beta) \rangle\rangle = \langle \beta, u \rangle + \langle \alpha, v \rangle$$

- Standard examples: graph of an (almost) symplectic or (almost) Poisson structure. [Not enough for nonholonomic systems—one needs to put in the constraints].
- For a symplectic or Poisson manifold, a *Hamiltonian vector field* X_H on R associated to a function H satisfies

$$(X, dH) \in D$$

  at each point of R. For good reason, Van der Schaft calls these things *implicit Hamiltonian systems*; they are an important part of his theory of "interconnected" and "port controlled" systems.

The Big Diagram

- Before defining an implicit Lagrangian system, we will need some more terminology and a "big diagram".[8]
- Namely, we need two natural diffeomorphisms; first there is the diffeomorphism

$$TT^*Q \to T^*T^*Q$$

associated with the canonical symplectic form on T^*Q.

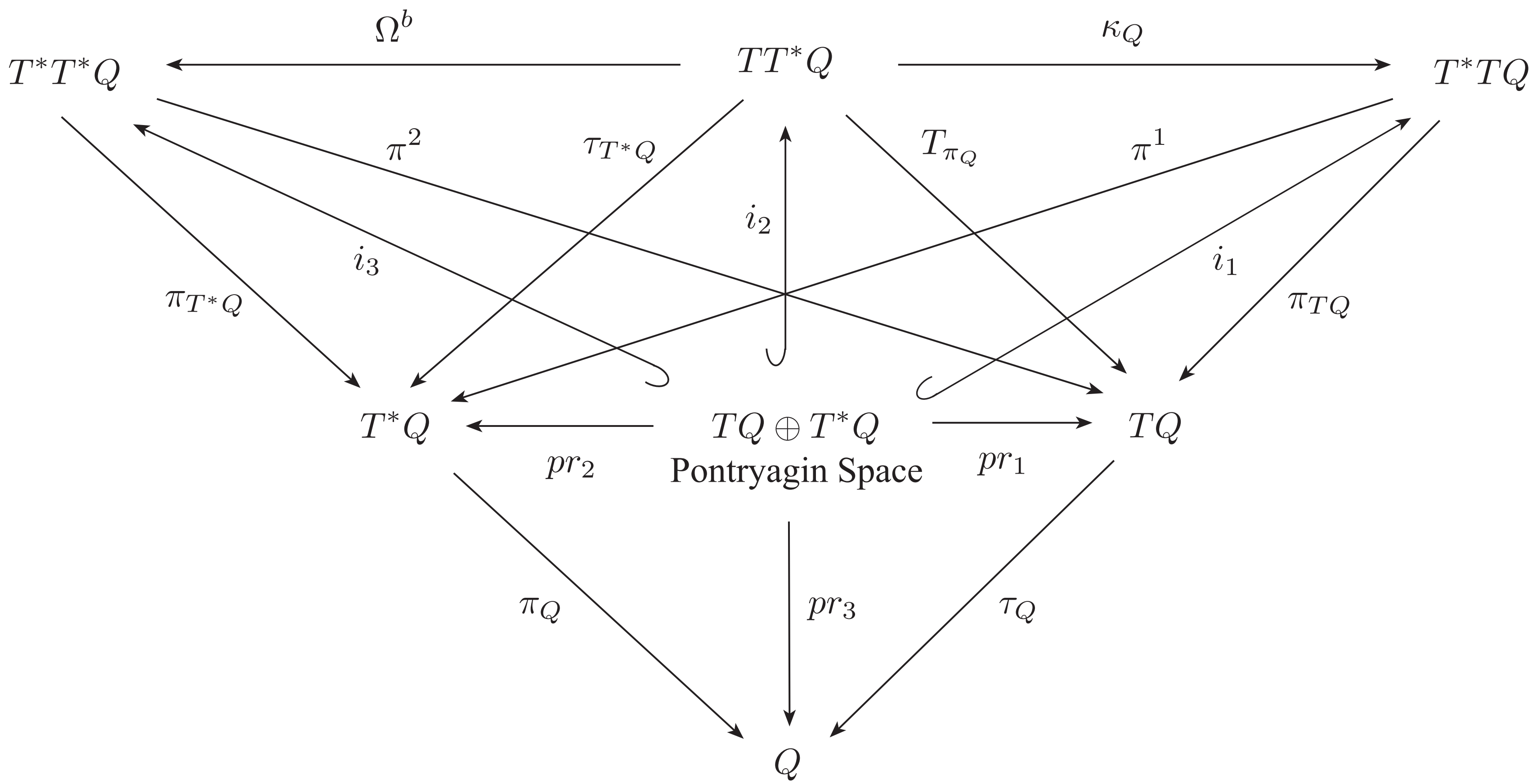
- Second, there is the natural diffeomorphism between

$$TT^*Q \to T^*TQ$$

given in coordinates by $(q, p, \delta q, \delta p) \mapsto (q, \delta q, \delta p, p)$ and determined intrinsically by the commutativity of the following "big diagram".

[8]Some parts of this picture are due to Tulczyjew; the full diagram was formulated by Cendra, JM and Ratiu.

The Big Diagram



Implicit Lagrangian Systems

- Let $D \subset TT^*Q \times T^*T^*Q$ be a given Dirac structure on T^*Q. Let $L : TQ \to \mathbb{R}$ be a given Lagrangian and let

$$\mathfrak{D}L \in T^*T^*Q$$

be $dL \in T^*TQ$ transferred over to T^*T^*Q by the canonical diffeomorphisms.

Implicit Lagrangian Systems

- Let $D \subset TT^*Q \times T^*T^*Q$ be a given Dirac structure on T^*Q. Let $L : TQ \to \mathbb{R}$ be a given Lagrangian and let

$$\mathfrak{D}L \in T^*T^*Q$$

be $dL \in T^*TQ$ transferred over to T^*T^*Q by the canonical diffeomorphisms.

- An *implicit Lagrangian system* relative to the given Dirac structure D is a vector field X on T^*Q satisfying

$$(X, \mathfrak{D}L) \in D$$

at each point of T^*Q.

Standard Lagrangian Systems

- Hamilton's principle may be rewritten so that it fits very well with the above definition.
- Write Hamilton's principle this way:

$$
\begin{aligned}
0 = \delta \int_a^b L(q(t)\,\dot{q}(t))\,dt
&= \int_a^b \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt \\
&= \int_a^b \left(\dot{p}\,\delta q - \dot{p}\,\delta q + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt \\
&= \int_a^b \left(-\dot{p}\,\delta q - p\,\delta \dot{q} + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt + p\,\delta q \Big|_a^b \\
&= -\int_a^b \left\{ \left(\dot{p} - \frac{\partial L}{\partial q} \right) \delta q + \left(p - \frac{\partial L}{\partial \dot{q}} \right) \delta \dot{q} \right\} dt + p\,\delta q \Big|_a^b .
\end{aligned}
$$

Comments

- This derivation can be given entirely intrinsically using objects in the big diagram.

Comments

- This derivation can be given entirely intrinsically using objects in the big diagram.
- Notice that the final equations include the Legendre transformation $p = \partial L/\partial \dot{q}$ as part of the equations. The boundary term, as is (now) standard, gives the canonical one-form.

Comments

- This derivation can be given entirely intrinsically using objects in the big diagram.
- Notice that the final equations include the Legendre transformation $p = \partial L/\partial \dot{q}$ as part of the equations. The boundary term, as is (now) standard, gives the canonical one-form.
- The equations (where $X(q,p) = (q,p,\dot{q},\dot{p})$) written this way are *exactly* the Dirac structure equations, namely, $(X, \mathfrak{D}L) \in D$, *including* the Legendre transform and also the correct identification of $\dot{q}$.

Nonholonomic Example

- Now we are ready to state the nonholonomic equations in terms of Dirac structures.
- Given the constraint distribution $\Delta \subset TQ$, we will now define an *associated Dirac structure* D_Δ on T^*Q.
- It will save time if we define D_Δ when $Q = V$, a vector space. Then at each point $q \in V$, $\Delta \subset V$ and

$$D_\Delta \subset TT^*Q \times T^*T^*Q$$

becomes, at each fiber point $(q, p) \in V \times V^*$,

$$D_\Delta \subset (V \times V^*) \times (V^* \times V)$$

Let

$$D_\Delta = \{(v, \beta), (\gamma, -v) \mid v \in \Delta, \gamma - \beta \in \Delta^o\}$$

where $\Delta^o \subset V^*$ is the polar of Δ.

Nonholonomic Example

- It is easy to check that D_Δ is a Dirac structure.

Nonholonomic Example

- It is easy to check that D_Δ is a Dirac structure.
- Then we have

Theorem. *The statement that* $(X, \mathfrak{D}L) \in D_\Delta$ *is equivalent to the Lagrange-d'Alembert equations, including the dynamic equations, the condition* $\dot{q} \in \Delta$, *and the constraints.*

Nonholonomic Example

- It is easy to check that D_Δ is a Dirac structure.
- Then we have

Theorem. *The statement that* $(X, \mathfrak{D}L) \in D_\Delta$ *is equivalent to the Lagrange-d'Alembert equations, including the dynamic equations, the condition* $\dot{q} \in \Delta$*, and the constraints.*

- Of course this is also directly tied to the Lagrange-d'Alembert variational structure of the equations.

Concluding Remarks

- You are seeing the tip of a big iceberg. Much is left to uncover.

Concluding Remarks

- You are seeing the tip of a big iceberg. Much is left to uncover.
- Conjectures are easy to make: Reduction of Dirac structures should produce, for instance, a Dirac structure for the Lagrange-d'Alembert-Poincaré equations as a special case; that is, Lagrange-d'Alembert reduction should be consistent with Dirac reduction. Ditto for the Lie-Poisson variational principle described in the "question to you". A *metaprinciple* emerges.

Concluding Remarks

- You are seeing the tip of a big iceberg. Much is left to uncover.
- Conjectures are easy to make: Reduction of Dirac structures should produce, for instance, a Dirac structure for the Lagrange-d'Alembert-Poincaré equations as a special case; that is, Lagrange-d'Alembert reduction should be consistent with Dirac reduction. Ditto for the Lie-Poisson variational principle described in the "question to you". A *metaprinciple* emerges.
- In particular, the reduction of the standard Dirac structure on T^*Q is *not* the standard Dirac structure on $\mathfrak{g}^*$, but rather should be one on $\mathfrak{g} \times \mathfrak{g}^*$ associated with the reduced "Pontryagin" space.

Concluding Remarks

- In doing PDE, relativistic field theories or even quantum mechanics, multisymplectic and multipoisson structures are the way to do. What is a *multi-Dirac* structure? (Think of a rolling ball of jello, or squishy tires on a road to motivate nonholonomic PDE's).

Concluding Remarks

- In doing PDE, relativistic field theories or even quantum mechanics, multisymplectic and multipoisson structures are the way to do. What is a *multi-Dirac* structure? (Think of a rolling ball of jello, or squishy tires on a road to motivate nonholonomic PDE's).
- The Lagrangian perspective on circuits provide a beautiful set of examples and show how Dirac structures can handle situations that are of the sort considered by Dirac.

Concluding Remarks

- In doing PDE, relativistic field theories or even quantum mechanics, multisymplectic and multipoisson structures are the way to do. What is a *multi-Dirac* structure? (Think of a rolling ball of jello, or squishy tires on a road to motivate nonholonomic PDE's).
- The Lagrangian perspective on circuits provide a beautiful set of examples and show how Dirac structures can handle situations that are of the sort considered by Dirac.
- Hopefully Dirac himself would be pleased with all this.

Concluding Remarks

- In doing PDE, relativistic field theories or even quantum mechanics, multisymplectic and multipoisson structures are the way to do. What is a *multi-Dirac* structure? (Think of a rolling ball of jello, or squishy tires on a road to motivate nonholonomic PDE's).
- The Lagrangian perspective on circuits provide a beautiful set of examples and show how Dirac structures can handle situations that are of the sort considered by Dirac.
- Hopefully Dirac himself would be pleased with all this.

◦ Go to my home page for free templates for making TeX slides like these ones. Click on "Marslides".

The End

TYPESETTING SOFTWARE: TEX, *Textures*, LATEX, hyperref, texpower, Adobe Acrobat 4.05
GRAPHICS SOFTWARE: Adobe Illustrator 10.0.1
LATEX SLIDE MACRO PACKAGES: Wendy McKay, Ross Moore