The Averaged Euler Equations and Computational Mechanics

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OUTLINE

- Geometric methods play a vital role in the development of new continuum models suited for computation, as well as numerical algorithms that preserve structure at the discrete level.
- Recent work on the averaged Euler equations and computational mechanics is surveyed.
- The averaged Euler equations may be regarded as geodesic equations for the H^1 metric on the volume preserving diffeomorphism group, as Arnold did with the L^2 metric for the Euler equations.
- We present some of the analytical theorems including convergence as viscosity tends to zero, even in the presence of boundaries.
- We indicate some interesting computational aspects of the equations and how this relates to current trends in computational mechanics.

INTRODUCTION

Heritage of Poincaré

- Poincaré [1901] gave the **Euler–Poincaré equations**—the Lie group reduction of the Euler-Lagrange equations on a Lie group. He understood how the rigid body and ideal fluid equations are special cases.
- The first insights into the geometric formulation of hydrodynamics; combining dynamical systems methods, Lie groups, mechanics, geometry, analysis, bifurcation theory, etc.
- Interesting applications to rotating fluid masses, including the precession of the Earth, stability theory, bifurcations of rotating fluid masses and many, many other things.

Some Notations

- **Velocity Field.** Consider $u(x,t)$, a time dependent divergence free vector field on a compact Riemannian n-manifold M, possibly with boundary, with u parallel to the boundary.

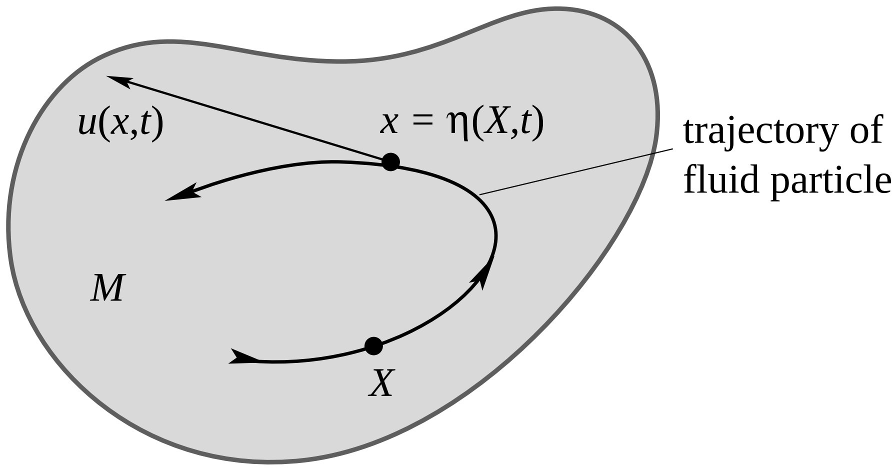

FIGURE 1: The velocity field and the particle paths for a fluid motion.

- **Particle paths.** Let $\eta(X,t)$ be the volume preserving flow of u:

$$\frac{d}{dt}\eta(X,t) = u(\eta(X,t),t)$$

- **Diffeomorphism group.** For each fixed t, the map η belongs to $\mathrm{Diff}_{\mathrm{vol}}(M)$, the group of volume preserving diffeomorphisms mapping M to M.
- ***Euler equations* for ideal flow**:

$$\frac{\partial u}{\partial t} + \nabla_u u = -\mathrm{grad}\, p$$

(∇ is the Levi-Civita connection).

- **Equivalent form of the Euler equations**:

$$\frac{\partial u^\flat}{\partial t} + \pounds_u u^\flat = -dp'$$

(Notation: $\pounds$ = Lie derivative, $u^\flat$ = one-form associated to u via the Riemannian structure and $p' = p - \|u\|^2/2$).

- **The Euler equations in $\mathbb{R}^3$:**

$$\frac{\partial u^i}{\partial t} + \sum_{j=1}^{3} u^j \frac{\partial u^i}{\partial x^j} = -\frac{\partial p}{\partial x^i}\,.$$

Implicit in Poincaré:

The Euler equations are the Euler–Poincaré equations on the Lie algebra of divergence free vector fields, the Lie algebra of $\mathrm{Diff}_{\mathrm{vol}}(M)$.

Proved in Arnold [1966]:

u *satisfies the Euler equations if and only if the curve* $t \mapsto \eta(\cdot, t)$ *is an* L^2 *geodesic in* $\mathrm{Diff}_{\mathrm{vol}}(M)$.

Proved in Ebin and Marsden [1970]:

- **Smoothness of the spray.** *The geodesic spray of the L^2 right invariant metric on* $\mathrm{Diff}^s_{\mathrm{vol}}(M)$, *the group of volume preserving Sobolev H^s diffeomorphisms is C^∞.* Here, $s > (n/2) + 1$ where n is the dimension of the underlying manifold M. This implies well-posedness of the Euler equations (and many other things).
- **Limit of zero viscosity.** Solutions of the Navier-Stokes equations converge to solutions of the Euler equations as the viscosity goes to zero when M has no boundary.
- **Product formulas.** These are Trotter product type formulas interleaving the divergence constraint, the unconstrained dynamics, & the dissipation. Useful in some numerical algorithms.

Related Developments

- Hydrodynamic stability (the **energy-Casimir method**) with applications to plasma physics and stellar dynamics
- **Bifurcations** such as rotating fluid masses, rotating liquid drops, symmetry induced instabilities (e.g., non-axial perturbations).
- **Riemannian geometry** of the group of diffeomorphisms. (Arnold, Misiolek, and Shkoller).
- The **energy-momentum method** (generalization of the energy-Casimir method) for stability and bifurcation in mechanical systems with symmetry.
- **Geometric phases** applied to, e.g., vortex dynamics.
- Development of **Lagrangian reduction theory** (Marsden and Scheurle), and in particular, new insight into the ***Euler–Poincaré***

equations (continuous and discrete):

$$\boxed{\frac{d}{dt}\frac{\partial l}{\partial \xi} = \mathrm{ad}^*_\xi \frac{\partial l}{\partial \xi}.}$$

Notations:

- $\mathfrak{g}$ is a **given Lie algebra**
- $l : \mathfrak{g} \longrightarrow \mathbb{R}$ is the given **Lagrangian**.
- $\mathrm{ad}_\xi : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the ***adjoint map*** $\zeta \mapsto [\xi, \zeta]$,
- $\mathrm{ad}^*_\xi : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ is its dual.
- **Example:** $\mathfrak{g} = \mathrm{so}(3) \cong \mathbb{R}^3$; $l : \mathbb{R}^3 \longrightarrow \mathbb{R}$; the Euler–Poincaré equations become the **rigid body equations**:

$$\frac{d}{dt}\frac{\partial l}{\partial \mathbf{\Omega}} = \frac{\partial l}{\partial \mathbf{\Omega}} \times \mathbf{\Omega}.$$

- Both the **fluid Euler equations** and the **Euler rigid body equations** are examples of the general Euler–Poincaré equations, as are many other systems, including the main example of this talk, the **averaged Euler equations**.

Where We Are Headed

- Instead of L^2 geodesics, we will be now looking at H^1 geodesics on $\mathrm{Diff}_{\mathrm{vol}}(M)$.
- What is **mathematically interesting** is also **physically and computationally interesting**: the resulting equations have a physical interpretation in terms of ***an averaging of the Euler equations over small scales***.
- This H^1 theory started historically with an **integrable shallow water model** and later the link with averaging was made.

A SHALLOW WATER EQUATION & H^1 GEODESICS

Shallow water equation in one spatial dimension:

$$u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx},$$

or equivalently,

$$\frac{\partial v}{\partial t} + uv_x + 2vu_x = 0$$

where $v = u - u_{xx}$.

- This is a **completely integrable bi-Hamiltonian system** (Fokas and Fuchsteiner [1981] and Camassa and Holm [1993])
- Has **non-smooth solitons** (peakons).
- Also has interesting associated **algebraic geometry** (Alber, Camassa, Holm and Marsden [1994,5]).
- Has an interpretation in terms of H^1 **geodesics** on $\mathrm{Diff}(S^1)$ (Camassa and Holm [1993], Misiolek [1998], Kouranbaeva [1999]).

- Shkoller [1998] proved smoothness of the spray and the well posedness in $H^{3/2+\epsilon}(S^1)$.
- These results are related to the fact that the **KdV equations** are of Euler–Poincaré form on the Virasoro Lie algebra and are L^2 geodesics on the Bott–Virasoro group (Khesin and Ovsienko, 1988).

AVERAGED EULER EQUATIONS

- Equations due to Holm, Marsden and Ratiu [1998], who developed the "Poincaré–Arnold view" of fluid mechanics and applied this view it to many other types of fluid equations, such as those in geophysics.
- The equations may be described in two mathematically equivalent ways: **as given PDE's** or as H^1 **geodesics on** $\mathrm{Diff}_{\mathrm{vol}}(M)$**.**

THE AVERAGED EULER EQUATIONS AS PDEs

- **Averaged Euler equations** in Euclidean coordinates:

$$\frac{\partial v^i}{\partial t} + u^j \frac{\partial v^i}{\partial x^j} - \alpha^2 \left[\frac{\partial u^j}{\partial x^i}\right] \triangle u_j = -\frac{\partial p}{\partial x^i},$$

  - α is a constant (a **length scale** in the averaging process),

$$v = u - \alpha^2 \triangle u,$$

  - $\triangle$ = **Laplacian**; summation over repeated indices.
  - p is determined from incompressibility: $\operatorname{div} u = 0$.

- Two choices of boundary conditions:

1. No slip: $u = 0$ on the boundary. **Compatible with the usual Navier-Stokes boundary conditions**.

2. u parallel to the boundary **and**

$$\nabla_n u = S_n(u)$$

at points of ∂M; $S_n : T_xM \to T_xM$ is the **second fundamental form** of the boundary.

- The first set of boundary conditions corresponds to the subgroup of $\mathrm{Diff}_{\mathrm{vol}}(M)$ leaving the boundary **pointwise fixed**, while the second corresponds to the subgroup that maps the **normal direction to the normal direction**. These are interesting subgroups of the diffeomorphism group.
- If α tends to zero, we formally recover the usual Euler equations.
- Equations are the same as those of a certain **second grade fluid**, although the physics of our derivation is different.

- **Geometric form** of the equations:

$$\boxed{\frac{\partial v^\flat}{\partial t} + \pounds_u v^\flat = -dp}$$

where, as before, $v = (1 - \alpha^2 \Delta) u$.

- Applying d, one gets a **vorticity formulation**.
- **Conserved H^1 energy:**

$$\frac{1}{2} \|u\|^2_{H^1} = \frac{1}{2} \int_M \langle\langle u, v \rangle\rangle \, d\mu$$

BASIC LINK WITH GEODESICS

- **Theorem:** *The flow $\eta_t(\cdot) := \eta(\cdot, t)$ of the time dependent vector field u is a geodesic in a subgroup of $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ with respect to the right invariant H^1 metric iff the vector field u satisfies the averaged Euler equations.*

- **Proved** by appealing to the Euler–Poincaré equations.
- Relates the Lagrangian dynamics on the large configuration space $\mathrm{Diff}_{\mathrm{vol}}(M)$ with the **reduced equations** on the Lie algebra.
- Part of the general theory of **Lagrangian reduction** for mechanical systems with symmetry.

DERIVATION OF THE EQUATIONS

Lagrangian Mean–Fluctuating Decomposition

- For each length scale α, assume that actual configuration of the fluid η can be described by a composition:

$$\eta_t = \xi_t^\alpha \circ \eta_t^\alpha .$$

- η_t^α defines the mean fluid configuration, (corresponding to an average over length scales smaller than α), and ξ_t^α denotes the Lagrangian (material) fluctuations of the fluid about the mean. When $\alpha = 0$, $\xi_t^0 = e$, the identity map, and $\eta_t^0 = \eta_t$.

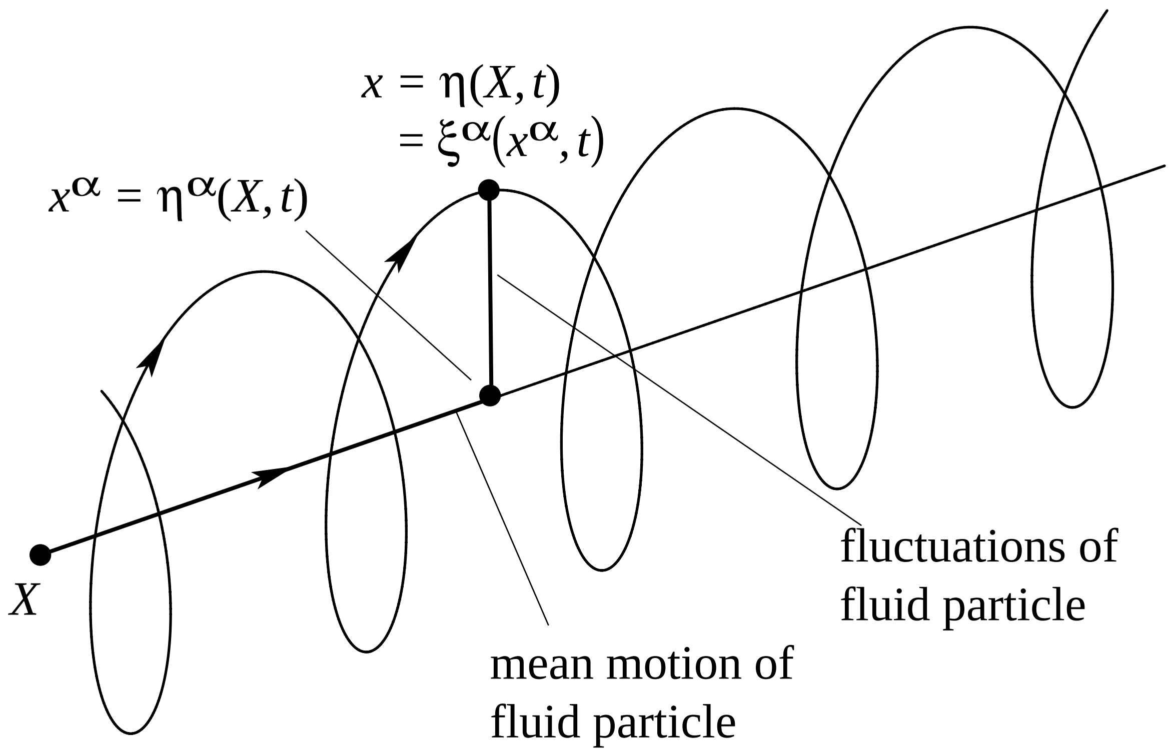

FIGURE 2: Mean-fluctuating decomposition of the particle placement field.

Expansion of the Velocity Field

- The Lagrangian decomposition implies a decomposition of the spatial velocity field u:

$$u_t \circ \eta_t = w_t^\alpha \circ \eta_t + \mathbf{D}\xi_t^\alpha \circ u_t^\alpha \circ \eta_t^\alpha,$$

where w_t^α is the spatial velocity field of the fluctuations.

- Effect of the fluctuations to first order in α: Let

$$\xi_t' = \left.\frac{d}{d\alpha}\right|_{\alpha=0} \xi_t^\alpha.$$

- Expand in α: $\quad u_t = w_t^\alpha + u_t^\alpha - \alpha \pounds_{\xi_t'} u_t^\alpha + O(\alpha^2),.$
- Lie derivative–covariant derivative relation:

$$\pounds_Y X = \nabla_Y X - \nabla_X Y.$$

- Consequence:

$$u_t = w_t^\alpha + u_t^\alpha - \alpha \nabla_{\xi'} u_t^\alpha + \alpha \nabla_{u_t^\alpha} \xi' + O(\alpha^2).$$

Now we make the following

Taylor Hypothesis. *Assume that the fluctuations are advected by parallel transport along the mean flow.*

$$\dot{\xi}' + \nabla_{u_t^\alpha} \xi' = O(\alpha)$$

- Consequence:

$$u_t = u_t^\alpha - \alpha \nabla_{\xi'} u_t^\alpha + O(\alpha^2).$$

Averaging the Lagrangian

- Average the Lagrangian for Euler flow with respect to the fluctuations to produce the Lagrangian for the averaged Euler equations.
- Lagrangian for the Euler equations:

$$L(u) = \frac{1}{2} \int_M \|u\|^2 \mu$$

- Substitute the decomposition into this Lagrangian:

$$L(u) = \frac{1}{2} \int_M \|u_t^\alpha - \alpha \nabla_{\xi'} u_t^\alpha + O(\alpha^2)\|^2 \mu$$

- Average with respect to the fluctuations, producing a new averaged Lagrangian $\bar{L}(u^\alpha)$.
- One has to be extremely careful with the higher order terms here.
- Assume that $\overline{\xi'} = 0$.

- Terms appearing in $\bar{L}$ containing factors linear in ξ' vanish when averaged
- Averaged Lagrangian becomes

$$\bar{L}(u^\alpha) = \frac{1}{2}\int_M \left\{ \langle\!\langle u^\alpha, u^\alpha \rangle\!\rangle + \alpha^2 \left\langle\!\!\left\langle \overline{\xi' \otimes \xi'} \nabla u^\alpha, \nabla u^\alpha \right\rangle\!\!\right\rangle \right\} \mu.$$

  Since $\overline{\xi' \otimes \xi'}$ is positive symmetric, this Lagrangian is that of an H^1 metric.

- One can think of the averaged Euler equations as a **conservative regularization** of the Euler equations, similar to the way KdV is a dispersive regularization of the inviscid Burger's equation.

SOME ANALYTICAL RESULTS

Main Result A: Well-Posedness

- **Theorem:** *The geodesic spray of the equations is smooth (in the sense of a smooth vector field on an infinite dimensional Hilbert manifold), just as in the case of the Euler equations.*
- That is, for $s > n/2 + 1$, the geodesic spray is a C^1 vector field on $T\,\mathrm{Diff}^s_{\mathrm{vol}}(M)$.
- Smoothness yields a **local existence and uniqueness theorem** and other analytical results parallel to those for the Euler equations.
- For example, the geodesic exponential map **covers a neighborhood of the identity** while the Lie group exponential map does not.

- Another result: **automatic smoothness** of solutions in time, even if the initial data has finite smoothness.
- **viscous analog** of the averaged Euler equations:

$$\frac{\partial v^i}{\partial t} - \nu \triangle u^i + u^j \frac{\partial v^i}{\partial x^j} - \alpha^2 \left[\frac{\partial u^j}{\partial x^i} \right] \triangle u_j = -\frac{\partial p}{\partial x^i},$$

which are the **averaged Navier-Stokes equations**, or the Navier-Stokes-α equations.

- Same equations as those for a second-grade fluid, but the physics is quite different!

Main Result B: Limit of Zero Viscosity

- **Theorem:** *The solutions for the corresponding viscous problem converge to those for the ideal problem,* **as the viscosity goes to zero** *(the* **infinite Reynolds number limit***), even* *in the presence of boundaries on uniform time intervals* $[0, T]$, *for* $T > 0$, *independent of the viscosity. The size of the interval* $[0, T]$ *is governed by the time of existence for the averaged Euler equations with the initial data fixed.*

- The inclusion of boundaries is a **major difference** from the situation with the usual Navier-Stokes equations and the Euler equations, where convergence is believed to **not** hold because of the **generation of vorticity at the boundary**.
- This provides a context in which one sees that *on the average* one gets convergence to the averaged Euler flow in the infinite Reynolds number limit, (conjectured by Ebin–Marsden–Fischer & Chorin–Barenblatt).
- This result may be relevant to the role of viscosity in turbulence theory, a subject going back to Onsager.

- **Product Formulas.** The smoothness of the spray leads to interesting representation formulas; e.g.,

$$E_t = \lim_{n\to\infty} \left(P^\alpha \circ G_{t/n} \right)^n ,$$

  - E_t is the flow on the space of divergence free velocity fields u of the averaged Euler equations (with, say, zero boundary conditions),
  - P^α is the H^2-orthogonal projection onto the divergence free vector fields zero on the boundary
  - G_t is the unconstrained H^1 spray—that is, the problem with the incompressibility condition dropped.

- In the case of the viscous version of the equations, *one does not require the vorticity creation operator to correct the boundary* *term.* The form of this product formula is

$$\begin{aligned} F_t &= \lim_{n\to\infty} \left(S_{t/n} \circ E_{t/n} \right)^n \\ &= \lim_{n\to\infty} \left(S_{t/n} \circ P^\alpha \circ G_{t/n} \right)^n \end{aligned}$$

where S_t is the Stokes-α flow.

NUMERICAL SIMULATIONS

3D LES Model–Flow in a Periodic Box

Allows one to compute with much higher Reynolds numbers than is possible with the usual NSE and still get the features of interest computed correctly.

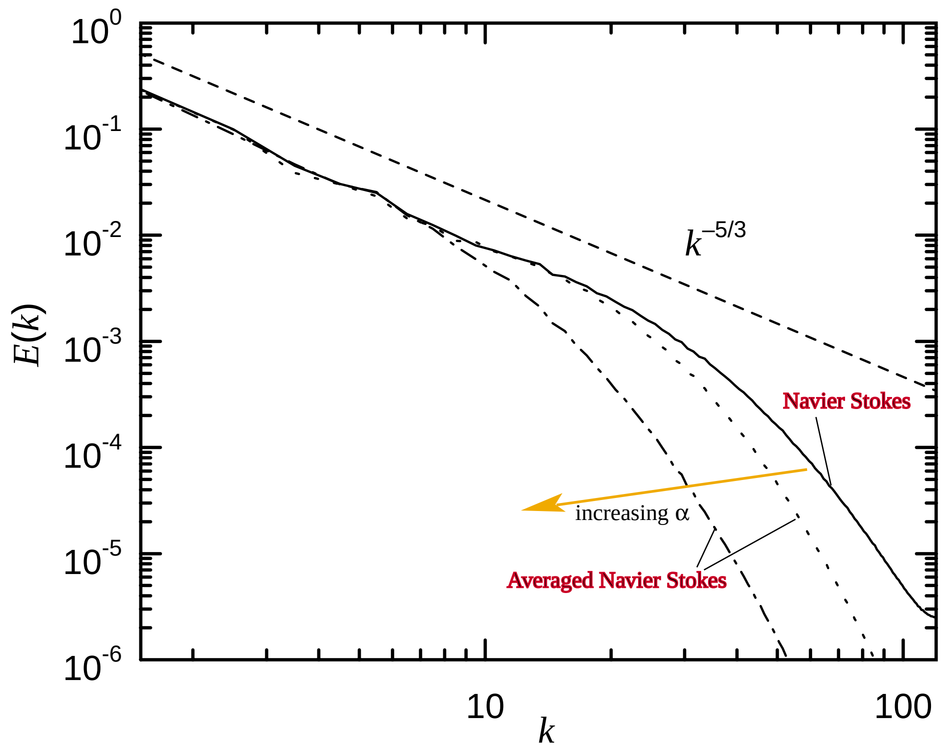

FIGURE 3: Energy spectrum for Navier-Stokes-α: from Chen et al. [1999].

Treating flows with boundary layers in the context of recent work on **transient instabilities** (Farell, Trefethen, Bamieh, Dhaleh, etc) and **HOT systems** (robust yet fragile) (Doyle, Carleson) is under investigation.

Vortex Merger

- Simulations done with a fully-dealiased **pseudospectral scheme** using **170 modes** consisting of 85 sines and cosines on a $2\pi \times 2\pi$ periodic square. (Embedded Runge-Kutta Cash-Karp discretization in time; Nadiga and Shkoller [1999]).
- Initial conditions:

$$\omega_0 = \big(\sin(x^1) + \sin(2x^1)\big) * \big(\sin(x^2) + \sin(2x^2)\big).$$

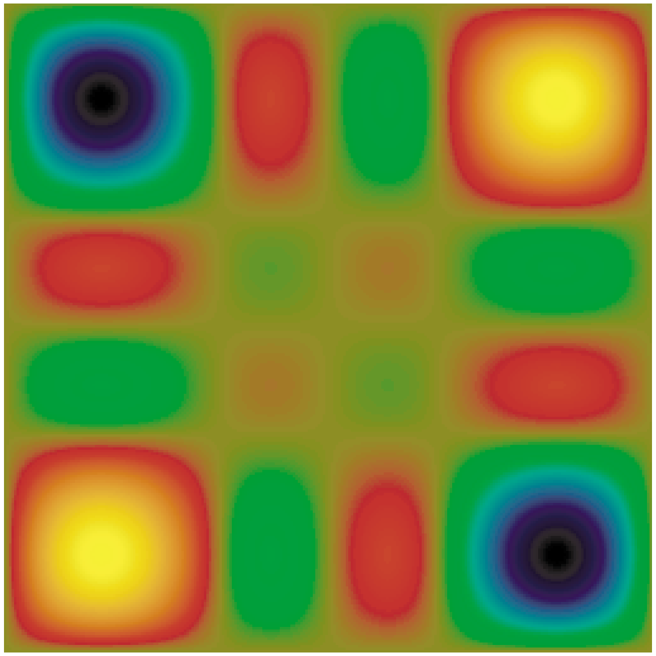

FIGURE 4: The Initial Conditions.

- Simulation of the **Euler equations**:

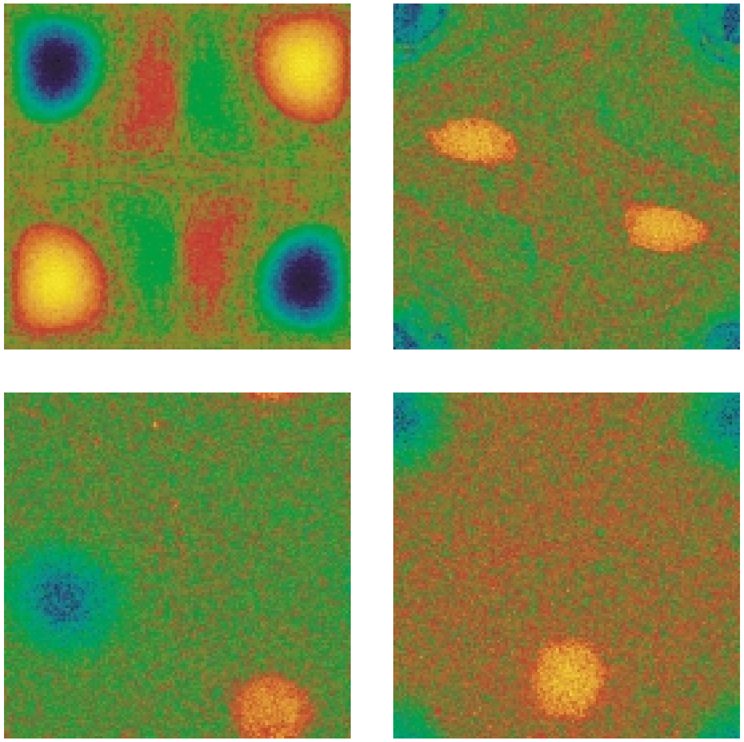

FIGURE 5: Euler equation simulation.

- Energy and enstrophy are conserved. **Fragmentation** occurs.

- Simulation of the **Navier-Stokes Equations**:
  - $\nu = 1.e^{-5}$

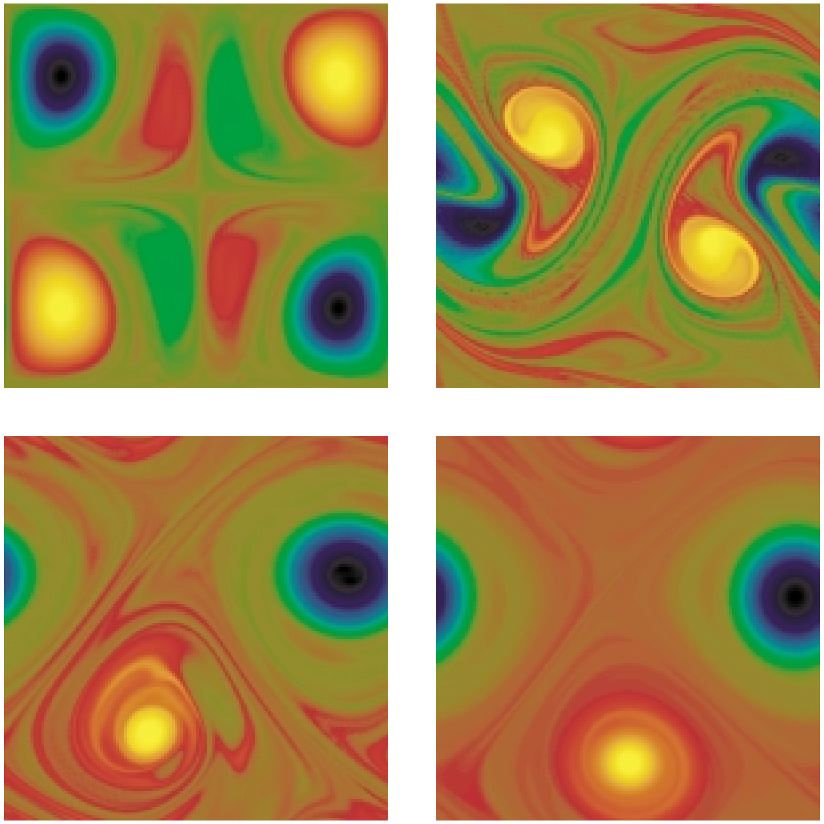

FIGURE 6: Navier Stokes simulation.

- Energy Spectrum for $\nu = 1.e^{-7}$ (dashed line); $1.e^{-6}$ (dot-dashed line); $1.e^{-5}$ (dot-dot-dot-dashed line); $5.e^{-5}$ (solid line).

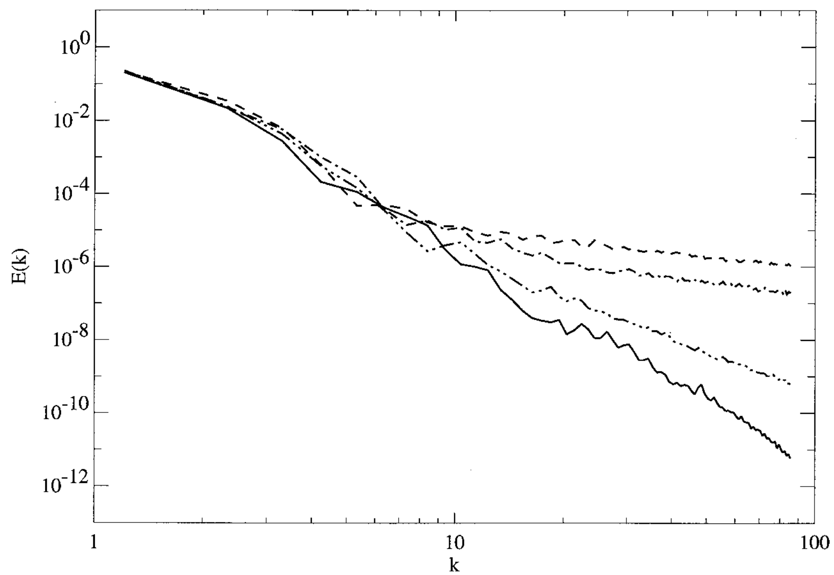

FIGURE 7: The instantaneous **energy spectrum** at time 92 (92 eddy turnover times) for the Navier-Stokes simulation with four different values of the viscosity.

- $\|\omega\|_{L^2}$ for $\nu = 1.e^{-7}$ (dashed line); $1.e^{-6}$ (dot-dashed line); $1.e^{-5}$ (dot-dot-dot-dashed line); $5.e^{-5}$ (solid line).

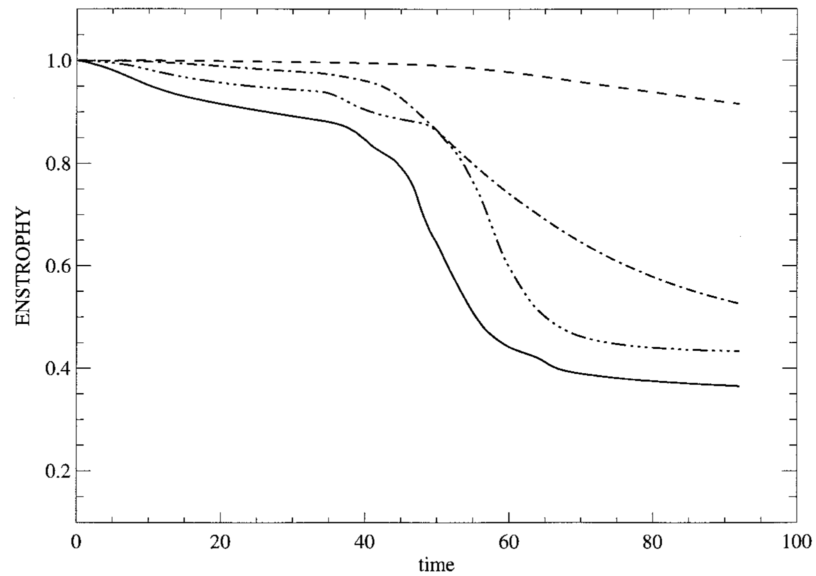

FIGURE 8: **Evolution of enstrophy** for the Navier-Stokes simulations with four different values of the viscosity.

- The drop in enstrophy during vertex merger is one of the **key features of 2D turbulence.**
- NSE does a nice job modeling the **vortex merger**, but requires $\nu = 1.e^{-5}$ for which there is a 10% **drop in energy.** In many applications, such as geophysics, this is **not acceptable**. Usually one **fudges** one's way out of this.
- For **longtime** simulations, **too much** dissipation can **destroy** relevant features (structures) of the solution.
- But the averaged Euler equations provide a simple and elegant solution to this, **not requiring viscosity!**

- Simulation of the **Euler-α Equations**:
  - $\alpha = .1$; filters-out scales smaller than 1.6% of 2π

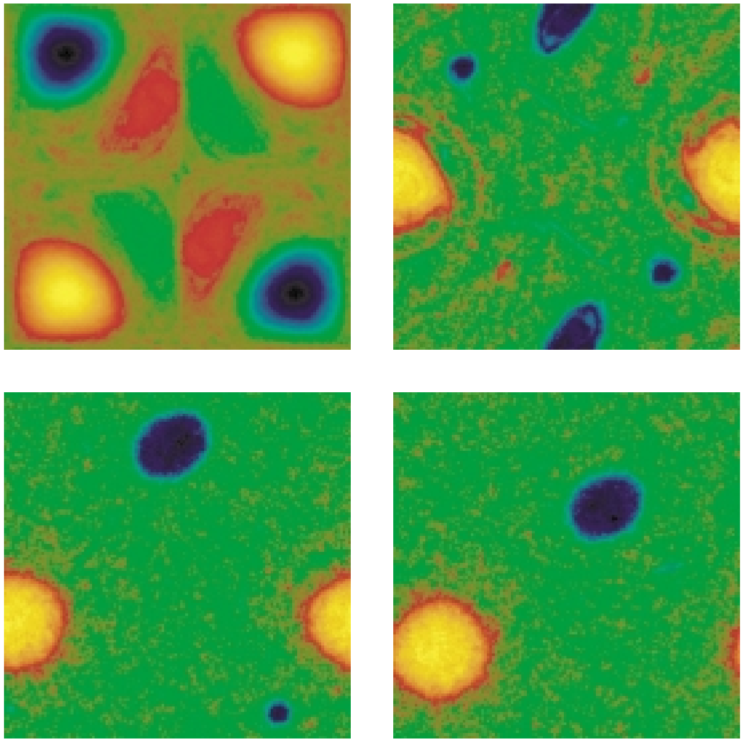

FIGURE 9: Vortex merger using the Euler-alpha equations.

- $\|\omega^\alpha\|_{L^2}$ for $\alpha = 0, .1, .2, .4.$

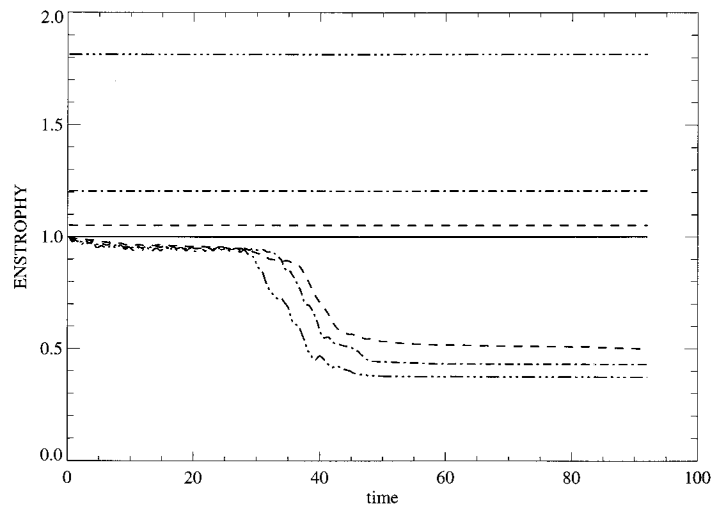

FIGURE 10: Evolution of norms of enstrophies for the averaged Euler equations (the H^1 norm is conserved, but the L^2 norm is not); the solid line is $\alpha = 0$, the dashed line is $\alpha = 0.1$, the dot-dashed line is $\alpha = 0.2$ and the dot-dot-dot-dashed line is $\alpha = 0.4$.

- The **conservative** Euler-α simulation captures the vortex merger phenomenon without the addition of any artificial viscosity.

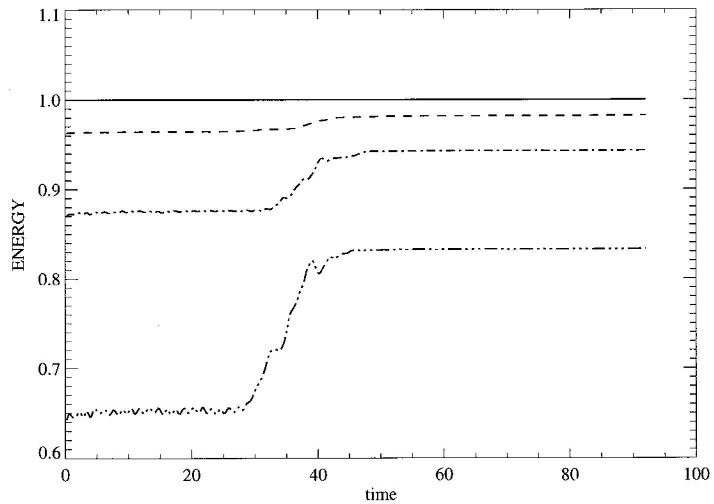

FIGURE 11: Evolution of energies. The H^1 energy is conserved, but the L^2 energy is not. The solid line is $\alpha = 0$, the dashed line is $\alpha = 0.1$, the dot-dashed line is $\alpha = 0.2$ and the dot-dot-dot-dashed line is $\alpha = 0.4$.

- Remarkably, while H^1_α energy is conserved, the L^2 kinetic energy increases – fluctuations are adding energy into the system.
- Averaged equations behave like a statistical theory. Enstrophy decay appears to be built in. More work needed here.

Variational Integrators

Multisymplectic Integrators

- Integration schemes used above, eg, the spectral truncation, do not preserve the mechanical structure. **Structure preserving discretization schemes** can improve the numerical simulation by keeping the conservation laws inherent in the physics.
- Much more needs to be done with variational multisymplectic integrators as in Marsden, Patrick and Shkoller [1998] on the long time integration of soliton equations, for example.
- We are developing extensions of these techniques to the context of **classical field theory** (electromagnetism, fluids, elasticity) using **multisymplectic geometry**.
- Should also work for the averaged Euler equations.

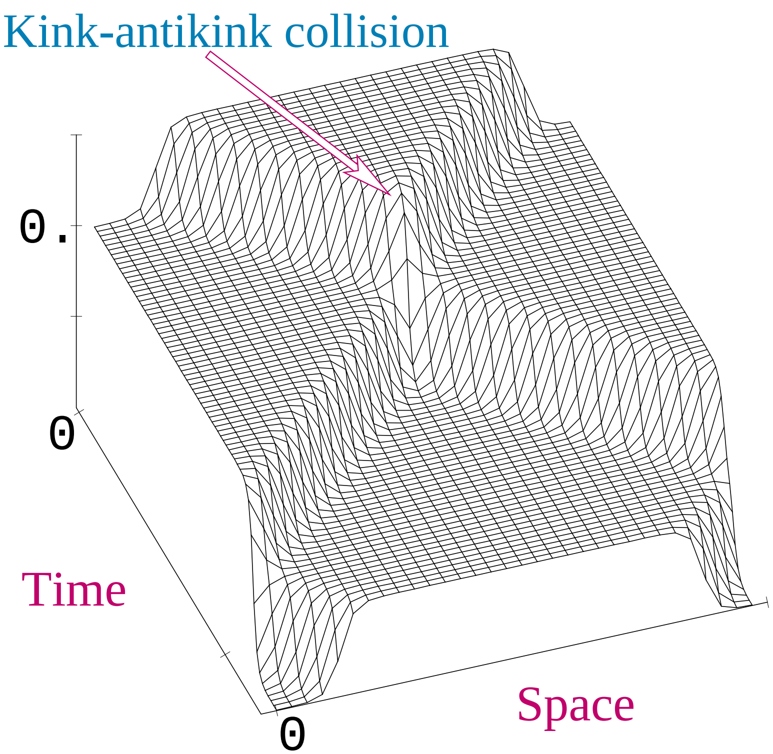

FIGURE 12: Collision of solitons in the sine-Gordon equation.

Variational Integrators for ODE's.

- There has been much recent progress in the area of **variational integrators** which build integrators out of the variational structure. This is based on the **Veselov** method for discrete mechanics. When designed this way, the integrators are automatically symplectic, momentum preserving and have excellent energy behavior.
- Idea is to update pairs of points

$$(q_k, q_{k+1}) \to (q_{k+1}, q_{k+2})$$

rather than position-velocity information. This leads to a discrete form of **Hamilton's principle** and associated **discrete Euler-Lagrange equations.**

- Such integration algorithms are automatically **symplectic.**

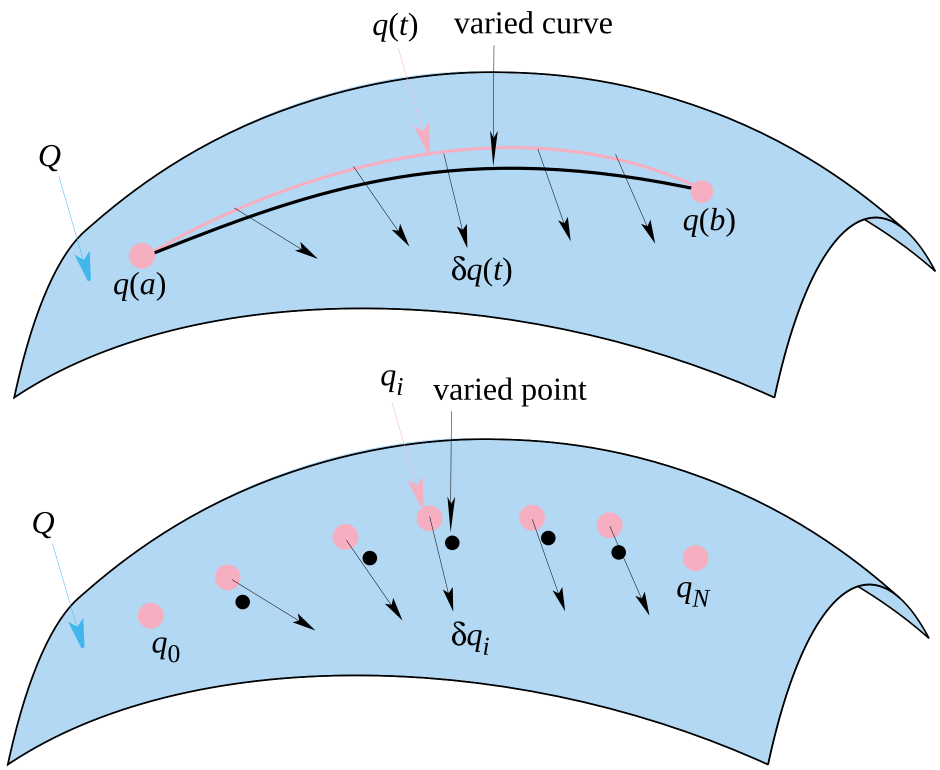

FIGURE 13: Discrete form of the variational principle.

- One of the most widely used time stepping algorithms in structural mechanics is the **Newmark scheme**—***it is variational*** (and hence symplectic; Kane, Marsden, Ortiz, West [1999]).

- It has also been understood how to properly incorporate dissipative and forcing into the algorithm via a discrete Lagrange d'Alembert type principle—it gets the changes in energy correct!

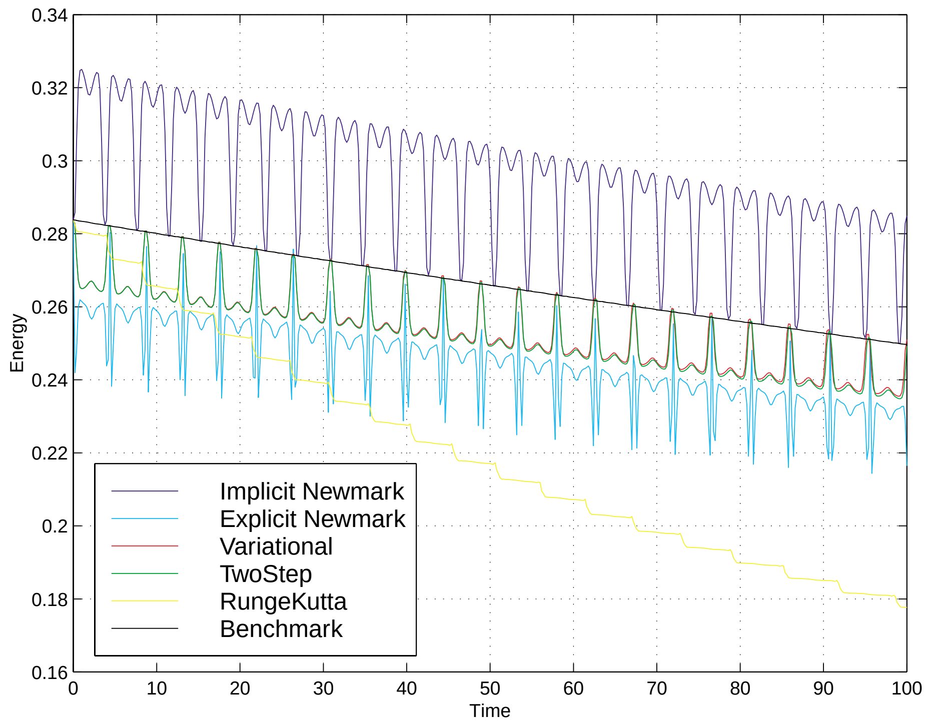

FIGURE 14: Energy behavior of integrators for a dissipative system. The variational integrators accurately simulate energy decay, unlike standard methods such as Runge-Kutta.

- These algorithms can also be extended to the context of collisions of rigid and elastic bodies—again the variational structure is critical as are the techniques of nonsmooth analysis (eg, as in Clarke). This is recent and ongoing work of Kane, Ortiz, Marsden and Pandolfi.

Movie

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A BIASED GLIMPSE AT THE LITERATURE

Era of Poincaré

- Clebsch, A. [1859] Über die Integration der hydrodynamischen Gleichungen, *Z. Reine Angew. Math.* **56**, 1–10. Typifies many of the references before Poincaré.
- Poincaré, H. [1890] *Théorie des tourbillons*, Reprinted by Éditions Jacques Gabay, Paris.
- Poincaré, H. [1892] Les formes d'équilibre d'une masse fluide en rotation, *Revue Générale des Sciences* **3**, 809–815 and *Philosophical Transactions A* **198**, 333–373, 1901.
- Poincaré, H. [1901] Sur une forme nouvelle des équations de la méchanique, *C.R. Acad. Sci.* **132**, 369–371. (Discovery of the **Euler–Poincaré equations.**)
- Poincaré, H. [1910] Sur la precession des corps deformables. *Bull Astron* **27**, 321–356.
- Ehrenfest, P. [circa 1904] PhD thesis showing that solutions of the fluid equations satisfy a Gauss principle of least curvature. See Klein, M. [1970] *Paul Ehrenfest.* North-Holland.

Post Poincaré

- Arnold, V.I., Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits, *Ann. Inst. Grenoble*, **16**, (1966), 319–361.

- Ebin, D and J. Marsden [1970] Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.*, **92**, 102–163.

Energy-Casimir Method for Hydrodynamical Stability

- Arnold, V.I. [1966] On an a priori estimate in the theory of hydrodynamical stability. *Izv. Vyssh. Uchebn. Zaved. Mat. Nauk* **54**, 3–5; English Translation: *Amer. Math. Soc. Transl.* **79** [1969], 267–269.

- Holm, D.D., J.E. Marsden, T.S. Ratiu, and A. Weinstein [1985] Nonlinear stability of fluid and plasma equilibria, *Phys. Rep.* **123**, 1–116.

- Simo, J.C., D.R. Lewis, and J.E. Marsden [1991] Stability of relative equilibria I: The reduced energy momentum method, *Arch. Rat. Mech. Anal.* **115**, 15-59.

Shallow Water Equation

- Fuchsteiner, B. and A.S. Fokas [1981], *Physica*, **4D**, 47–66.
- Camassa, R. and D.D. Holm [1993] An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71**, 1661-1664 and (with Hyman) *Adv. Appl. Mech.*, **31**, 1–33.
- Alber, M.S., R. Camassa, D.D. Holm and J.E. Marsden [1994], The geometry of peaked solitons and billiard solutions of a class of integrable pde's, *Lett. Math. Phys.* **32**, 137–151 and *Proc. Roy. Soc* **450**, 677–692, 1995.
- Misiolek, G., [1998] A shallow water equation as a geodesic flow on the Bott-Virasoro group. *J. Geom. Phys.*, **24**, 203–208.
- Kouranbaeva, S. [1999], The Camassa-Holm equation as a geodesic flow on the diffeomorphism group, *J. Math. Phys.* **40**, 857–868.
- Shkoller, S., *Geometry and curvature of diffeomorphism groups with H^1 metric and mean hydrodynamics*, J. Func. Anal., **160**, (1998), 337–365.

Averaged Euler and Navier–Stokes Equations

- Holm, D. D., J. E. Marsden and T. S. Ratiu, [1998] Euler-Poincaré models of ideal fluids with nonlinear dispersion, *Phys. Rev. Lett.* **349**, 4173-4177.
- Holm, D.D., J. E. Marsden and T. S. Ratiu, [1998] The Euler–Poincaré equations and semidirect products with applications to continuum theories, *Adv. in Math.*, **137**, 1-81.
- Shkoller, S., *Geometry and curvature of diffeomorphism groups with H^1 metric and mean hydrodynamics*, J. Func. Anal., **160**, (1998), 337–365.
- Marsden, J.E., T.S. Ratiu, and S. Shkoller, [1999] The geometry and analysis of the averaged Euler equations with normal boundary conditions, *Geom. Func. Anal.*, (to appear).
- Chen, S.Y., C. Foias, D.D. Holm, E.J. Olson, E.S. Titi and S. Wynne, [1999] The Camassa-Holm equations as a closure model for turbulent channel and pipe flow, *Phys. Rev. Lett.,* **81**, 5338–5341.
- Foias, C., D.D. Holm and E.S. Titi [1999], in preparation (Well-posedness of the averaged Navier-Stokes equations and attractor estimates).

Second Grade Fluids

- Cioranescu, D. and E.H. Ouazar, Existence and uniqueness for fluids of second grade, In *Non-linear Partial Differential Equations*, **109**, pp. 178–197. Collège de France Seminar, Pitman (1984).
- Cioranescu, D. and V. Girault [1997], Weak and classical solutions of a family of second grade fluids, *Inter. J. Non-Linear Mech.*, **32**, 317–335.
- Dunn, J.E. and R.L. Fosdick [1974], Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade, *Arch. Rat. Mech. Anal.*, **56**, 191–252.

Limit of Zero Viscosity

- Onsager, L. [1945] The distribution of energy in turbulence. *Phys. Rev.*, **68**, 286.
- Marsden, J.E., D.G. Ebin, and A. Fischer [1972] Diffeomorphism groups, hydrodynamics and relativity. *Proceedings 13th Biennial Seminar on Canadian Mathematics Congress*, 135–279.
- Barenblatt, G.I. and A.J. Chorin [1998], New Perspectives in turbulence: scaling laws, asymptotics, and intermittency, *SIAM Rev.*, **40**, (1998), 265–291.

Numerical Simulations

- Nadiga, B. and S. Shkoller [1999], On a conservative numerical scheme for vortex merger (*preprint.*)
- Chen, S.Y., C. D.D. Holm, L. Margolin, and R. Zhang [1999] Direct numerical simulations of the Navier-Stokes alpha model (*preprint.*)

Variational Integrators & Discrete Mechanics

- Moser, J. and A.P. Veselov [1991] Discrete versions of some classical integrable systems and factorization of matrix polynomials. *Comm. Math. Phys.* **139**, 217–243.
- Wendlandt, J.M. and J.E. Marsden [1997] Mechanical integrators derived from a discrete variational principle, *Physica D* **106**, 223–246.
- Marsden, J.E. S. Pekarsky, and S. Shkoller [1999] Discrete Euler-Poincaré and Lie Poisson equations (*preprint.*)
- Kane, C, J.E. Marsden, and M. Ortiz [1999] Symplectic energy momentum integrators, *J. Math. Phys.*, **40**, 3353–3371.
- Kane, C, J.E. Marsden, M. Ortiz and M. West [1999] Variational Integrators and the Newmark Algorithm for Conservative and Dissipative Mechanical Systems (*preprint.*)

- Kane, C, E.A. Repetto, M. Ortiz and J.E. Marsden [1998] Finite element analysis of nonsmooth contact (*Computer Meth. in Appl. Mech. and Eng., to appear*).
- Kane, C, J.E. Marsden, M. Ortiz and A. Pandolfi [1999] Finite element analysis of nonsmooth contact with friction (in preparation).

Multisymplectic Geometry & Integrators

- Gotay, M., J. Isenberg, and J.E. Marsden [1997] *Momentum Maps and the Hamiltonian Structure of Classical Relativistic Field Theories, I.*.
- Marsden, J.E. and S. Shkoller [1999] Multisymplectic geometry, covariant Hamiltonians and water waves, *Math. Proc. Camb. Phil. Soc.*, **125**.
- Marsden, J.E., G. Patrick, and S. Shkoller [1998] Multisymplectic geometry, variational integrators, and nonlinear PDEs, *Comm. Math. Phys.*, **199**, 351–395.