Mechanical Systems: Symmetry and Reduction

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1 Glossary and Notation

Lie Group Action. A process by which a Lie group, acting as a symmetry, moves points in a space. When points in the space that are related by a group element are identified, one obtains the quotient space.

Free Action. An action that moves every point under any nontrivial group element.

Proper Action. An action that obeys a compactness condition.

Momentum Mapping. A dynamically conserved quantity that is associated with the symmetry of a mechanical system. An example is angular momentum, which is associated with rotational symmetry.

Symplectic Reduction. A process of reducing the dimension of the phase space of a mechanical system by restricting to the level set of a momentum map and also identifying phase space points that are related by a symmetry.

Poisson Reduction. A process of reducing the dimension of the phase space of a mechanical system by identifying phase space points that are related by a symmetry.

Equivariance. Equivariance of a momentum map is a property that reflects the consistency of the mapping with a group action on its domain and range.

Momentum Cocycle. A measure of the lack of equivariance of a momentum mapping.

Singular Reduction. A reduction process that leads to non-smooth reduced spaces. Often associated with non-free group actions.

Coadjoint Orbit. The orbit of an element of the dual of the Lie algebra under the natural action of the group.

KKS (Kostant-Kirillov-Souriau) Form. The natural symplectic form on coadjoint orbits.

Cotangent Bundle. A mechanical phase space that has a structure that distinguishes configurations and momenta. The momenta lie in the dual to the space of velocity vectors of configurations.

Shape Space. The space obtained by taking the quotient of the configuration space of a mechanical system by the symmetry group.

Principal Connection. A mathematical object that describes the geometry of how a configuration space is related to its shape space. Related to geometric phases through the subject of holonomy. In turn related to locomotion in mechanical systems.

Mechanical Connection. A special (principal) connection that is built out of the kinetic energy and momentum map of a mechanical system with symmetry.

Magnetic Terms. These are expressions that are built out of the curvature of a connection. They are so named because terms of this form occur in the equations of a particle moving in a magnetic field.

The Primary Literature. We shall be referring to a few sources often, so it will be convenient to have abbreviations for them:

- We refer to Abraham and Marsden [1978] as [FofM].
- We refer to Abraham, Marsden, and Ratiu [1988] as [MTA].
- We refer to Marsden and Ratiu [1999] as [MandS].
- We refer to Marsden [1992] as [LonM].
- We refer to Marsden, Misiolek, Perlmutter, and Ratiu [1998a] as [MMPR].
- We refer to Ortega and Ratiu [2004a] as [HRed].
- We refer to Marsden, ek, Ortega, Perlmutter, and Ratiu [2007] as [HStages]

These main references may be consulted as needed for further details and for proofs of the theorems.

Notation. To keep things reasonably systematic, we have adopted the following universal conventions for some common maps and other objects:

- ***Configuration space of a mechanical system***: Q
- ***Phase space***: P

- ***Cotangent bundle projection***: $\pi_Q : T^*Q \to Q$
- ***Tangent bundle projection***: $\tau_Q : TQ \to Q$
- ***Quotient projection***: $\pi_{P,G} : P \to P/G$
- ***Tangent map***: $T\varphi : TM \to TN$ for the tangent of a map $\varphi : M \to N$

Thus, for example, the symbol $\pi_{T^*Q,G}$ denotes the quotient projection from T^*Q to $(T^*Q)/G$.

- The Lie algebra of a Lie group G is denoted $\mathfrak{g}$.
- Actions of G on a space is denoted by concatenation. For example, the action of a group element g on a point $q \in Q$ is written as gq
- The infinitesimal generator of a Lie algebra element $\xi \in \mathfrak{g}$ for an action of G on P is denoted ξ_P, a vector field on P
- Momentum maps are denoted $\mathbf{J} : P \to \mathfrak{g}^*$.
- Pairings between vector spaces and their duals are denoted by simple angular brackets: for example, the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is denoted $\langle \mu, \xi \rangle$ for $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$
- Inner products are denoted with double angular brackets: $\langle\langle u, v \rangle\rangle$.

2 Definition

Reduction theory is concerned with mechanical systems with symmetries. It constructs a lower dimensional *reduced space* in which associated conservation laws are taken out and symmetries are "factored out" and studies the relation between the dynamics of the given system with the dynamics on the reduced space. This subject is important in many areas, such as stability of relative equilibria, geometric phases and integrable systems.

2 Introduction

Geometric mechanics has developed in the last 30 years or so into a mature subject in its own right, and its applications to problems in Engineering, Physics and other physical sciences has been impressive. One of the important aspects of this subject has to do with symmetry; even things as apparently simple as the symmetry of a system such as the n-body problem under the group of rotations in space (the Euclidean group) or a wheeled vehicle under the planar Euclidean group turns out to have profound consequences. Symmetry often gives *conservation laws* through Noether's theorem and these conservation laws can be used to *reduce* the dimension of a system.

In fact, reduction theory is an old and time-honored subject, going back to the early roots of mechanics through the works of Euler, Lagrange, Poisson, Liouville, Jacobi, Hamilton, Riemann, Routh, Noether, Poincaré, and others. These founding masters regarded reduction theory as a useful tool for simplifying and studying concrete mechanical systems, such as the use of Jacobi's *elimination of the node* in the study of the n-body problem to deal with the overall rotational symmetry of the problem. Likewise, Liouville and Routh used the elimination of cyclic variables (what we would call today an Abelian symmetry group) to simplify problems and it was in this setting that the *Routh stability method* was developed.

The modern form of symplectic reduction theory begins with the works of Arnold [1966], Smale [1970], Meyer [1973], and Marsden and Weinstein [1974]. A more detailed survey of the history of reduction theory can be found in the first sections of this article. As was the case with Routh, this theory has close connections with the stability theory of *relative equilibria*, as in Arnold [1969] and Simo, Lewis, and Marsden [1991]. The symplectic reduction method is, in fact, by now so well known that it is used as a standard tool, often without much mention. It has also entered many textbooks on geometric mechanics and symplectic geometry, such as Abraham and Marsden [1978], Arnold [1989], Guillemin and Sternberg [1984], Libermann and Marle [1987], and Mcduff and Salamon [1995]. Despite its relatively old age, research in reduction theory continues vigorously today.

It will be assumed that the reader is familiar with the basic concepts in [MandS]. For the statements of the bulk of the theorems, it is assumed that the manifolds involved are finite dimensional and are smooth unless otherwise stated. While many interesting examples are infinite-dimensional, the general theory in the infinite dimensional case is still not in ideal shape; see, for example, Chernoff and Marsden [1974], Marsden and Hughes [1983], and Mielke [1991] and examples and discussion in [HStages].

3 Symplectic Reduction

Roughly speaking, here is how symplectic reduction goes: given the symplectic action of a Lie group on a symplectic manifold that has a momentum map, one divides a level set of the momentum map by the action of a suitable subgroup to form a new symplectic manifold. Before the division step, one has a manifold (that can be singular if the points in the level set have symmetries) carrying a degenerate closed 2-form. Removing such a degeneracy by passing to a quotient space is a differential-geometric operation that was promoted by Cartan [1922].

The "suitable subgroup" related to a momentum mapping was identified by Smale [1970] in the special context of cotangent bundles. It was Smale's work that inspired the general symplectic construction by Meyer [1973] and the version we shall use, which makes explicit use of the properties of momentum maps, by Marsden and Weinstein [1974].

Momentum Maps. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ be its dual. Suppose that G acts symplectically on a symplectic manifold P with symplectic form denoted by Ω. We shall denote the infinitesimal generator associated with the Lie algebra element ξ by ξ_P and we shall let the Hamiltonian vector field associated to a function $f : P \to \mathbb{R}$ be denoted X_f.

A ***momentum map*** is a map $\mathbf{J} : P \to \mathfrak{g}^*$, which is defined by the condition

$$\xi_P = X_{\langle \mathbf{J}, \xi \rangle} \tag{3.1}$$

for all $\xi \in \mathfrak{g}$, and where $\langle \mathbf{J}, \xi \rangle : P \to \mathbb{R}$ is defined by the natural pointwise pairing. We call such a momentum map ***equivariant*** when it is equivariant with respect to the given action on P and the coadjoint action of G on $\mathfrak{g}^*$. That is,

$$\mathbf{J}(g \cdot z) = \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(z) \tag{3.2}$$

for every $g \in G$, $z \in P$, where $g \cdot z$ denotes the action of g on the point z, Ad denotes the adjoint action, and Ad^* the coadjoint action.[1] A quadruple $(P, \Omega, G, \mathbf{J})$, where (P, Ω) is a given symplectic manifold and $\mathbf{J} : P \to \mathfrak{g}^*$ is an equivariant momentum map for the symplectic action of a Lie group G, is sometimes called a ***Hamiltonian G-space***.

[1] Note that when we write $\mathrm{Ad}^*_{g^{-1}}$, we *literally* mean the adjoint of the linear map $\mathrm{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$. The inverse of g is necessary for this to be a *left* action on $\mathfrak{g}^*$. Some authors let that inverse be understood in the notation. However, such a convention would be a notational disaster since we need to deal with both *left* and *right* actions, a distinction that is essential in mechanics.

Taking the derivative of the equivariance identity (3.2) with respect to g at the identity yields the condition of ***infinitesimal equivariance***:

$$T_z\mathbf{J}(\xi_P(z)) = -\operatorname{ad}^*_\xi \mathbf{J}(z) \tag{3.3}$$

for any $\xi \in \mathfrak{g}$ and $z \in P$. Here, $\operatorname{ad}_\xi : \mathfrak{g} \to \mathfrak{g}; \eta \mapsto [\xi, \eta]$ is the adjoint map and $\operatorname{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is its dual. A computation shows that (3.3) is equivalent to

$$\langle \mathbf{J}, [\xi, \eta] \rangle = \{ \langle \mathbf{J}, \xi \rangle, \langle \mathbf{J}, \eta \rangle \} \tag{3.4}$$

for any $\xi, \eta \in \mathfrak{g}$, that is, $\langle \mathbf{J}, \cdot \rangle : \mathfrak{g} \to \mathcal{F}(P)$ is a Lie algebra homomorphism, where $\mathcal{F}(P)$ denotes the Poisson algebra of smooth functions on P. The converse is also true if the Lie group is connected, that is, if G is connected then an infinitesimally equivariant action is equivariant (see [MandS], §12.3).

The idea that an action of a Lie group G with Lie algebra $\mathfrak{g}$ on a symplectic manifold P should be accompanied by such an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ and the fact that the orbits of this action are themselves symplectic manifolds both occur already in Lie [1890]; the links with mechanics also rely on the work of Lagrange, Poisson, Jacobi and Noether. In modern form, the momentum map and its equivariance were rediscovered by Kostant [1966] and Souriau [1966, 1970] in the general symplectic case and by Smale [1970] for the case of the lifted action from a manifold Q to its cotangent bundle $P = T^*Q$. Recall that the equivariant momentum map in this case is given explicitly by

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle \, , \tag{3.5}$$

where $\alpha_q \in T^*_q Q$, $\xi \in \mathfrak{g}$, and where the angular brackets denote the natural pairing on the appropriate spaces.

Smale referred to $\mathbf{J}$ as the 'angular momentum" by generalization from the special case $G = \mathrm{SO}(3)$, while Souriau used the French word 'moment". Marsden and Weinstein [1974], followed usage emerging at that time and used the word "moment" for $\mathbf{J}$, but they were soon corrected by Richard Cushman and Hans Duistermaat, who suggested that the proper English translation of Souriau's French word was "momentum," which fit better with Smale's designation as well as standard usage in mechanics. Since 1976 or so, most people who have contact with mechanics use the term momentum map (or mapping). On the other hand, Guillemin and Sternberg popularized the continuing use of "moment" in English, and both words coexist today. It is a curious twist, as comes out in work on collective nuclear motion (Guillemin and Sternberg [1980]) and plasma physics (Marsden and Weinstein [1982] and Marsden, Weinstein, Ratiu, Schmid, and Spencer [1982]), that moments of inertia and moments of probability distributions can actually be the values of momentum maps! Mikami and Weinstein [1988] attempted a linguistic reconciliation between the usage of "moment" and "momentum" in the context of groupoids. See [MandS] for more information on the history of the momentum map and §5 for a more systematic review of general reduction theory.

Momentum Cocycles and Nonequivariant Momentum Maps. Consider a momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ that *need not be equivariant*, where P is a symplectic manifold on which a Lie group G acts symplectically. The map $\sigma : G \to \mathfrak{g}^*$ that is defined by

$$\sigma(g) := \mathbf{J}(g \cdot z) - \operatorname{Ad}^*_{g^{-1}} \mathbf{J}(z), \tag{3.6}$$

where $g \in G$ and $z \in P$ is called a ***nonequivariance or momentum one-cocycle***. Clearly, *σ is a measure of the lack of equivariance of the momentum map.*

We shall now prove a number of facts about σ. The first claim is that σ *does not depend on the point $z \in P$ provided that the symplectic manifold P is connected* (otherwise it is constant on connected components). To prove this, we first recall the following *equivariance identity for infinitesimal generators*:

$$T_q\Phi_g\left(\xi_P(q)\right) = (\mathrm{Ad}_g\,\xi)_P(g\cdot q); \quad \textit{i.e.}, \quad \Phi_g^*\xi_P = \left(\mathrm{Ad}_{g^{-1}}\,\xi\right)_P. \tag{3.7}$$

This is an easy Lie group identity that is proved, for example, in [MandS], Lemma 9.3.7.

One shows that $\sigma(g)$ is constant by showing that its Hamiltonian vector field vanishes. Using the fact that $\sigma(g)$ is independent of z along with the basic identity $\mathrm{Ad}_{gh} = \mathrm{Ad}_g\,\mathrm{Ad}_h$ and its consequence $\mathrm{Ad}^*_{(gh)^{-1}} = \mathrm{Ad}^*_{g^{-1}}\,\mathrm{Ad}^*_{h^{-1}}$, shows that σ satisfies the ***cocycle identity***

$$\sigma(gh) = \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\,\sigma(h) \tag{3.8}$$

for any $g, h \in G$. This identity shows that σ produces a new action $\Theta : G\times\mathfrak{g}^* \to \mathfrak{g}^*$ defined by

$$\Theta(g,\mu) := \mathrm{Ad}^*_{g^{-1}}\,\mu + \sigma(g) \tag{3.9}$$

with respect to which the momentum map $\mathbf{J}$ is obviously equivariant. This action Θ is not linear anymore—it is an ***affine action***.

For $\eta \in \mathfrak{g}$, let $\sigma_\eta(g) = \langle\sigma(g),\eta\rangle$. Differentiating the definition of σ, namely

$$\sigma_\eta(g) = \langle\mathbf{J}(g\cdot z),\eta\rangle - \left\langle\mathbf{J}(z),\mathrm{Ad}_{g^{-1}}\,\eta\right\rangle$$

with respect to g at the identity in the direction $\xi\in\mathfrak{g}$ shows that

$$T_e\sigma_\eta(\xi) = \Sigma(\xi,\,\eta), \tag{3.10}$$

where $\Sigma(\xi,\,\eta)$, which is called the ***infinitesimal nonequivariance two-cocycle*** is defined by

$$\Sigma(\xi,\eta) = \langle\mathbf{J},[\xi,\eta]\rangle - \{\langle\mathbf{J},\xi\rangle,\langle\mathbf{J},\eta\rangle\}. \tag{3.11}$$

Since σ does not depend on the point $z\in P$, neither does Σ. Also, it is clear from this definition that Σ *measures the lack of infinitesimal equivariance of* $\mathbf{J}$. Another way to look at this is to notice that from the derivation of equation (3.10), for $z\in P$ and $\xi\in\mathfrak{g}$, we have

$$T_z\mathbf{J}(\xi_P(z)) = -\,\mathrm{ad}^*_\xi\,\mathbf{J}(z) + \Sigma(\xi,\cdot). \tag{3.12}$$

Comparison of this relation with equation (3.3) also shows the relation between Σ and the infinitesimal equivariance of $\mathbf{J}$.

The map $\Sigma : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ is bilinear, skew-symmetric, and, as can be readily verified, satisfies the ***two-cocycle identity***

$$\Sigma([\xi,\eta],\zeta) + \Sigma([\eta,\zeta],\xi) + \Sigma([\zeta,\xi],\eta) = 0, \tag{3.13}$$

for all $\xi,\eta,\zeta\in\mathfrak{g}$.

The Symplectic Reduction Theorem. There are many precursors of symplectic reduction theory. When G is Abelian, the components of the momentum map form a system of functions in involution (i.e. the Poisson bracket of any two is zero). The use of k such functions to reduce a phase space to one having $2k$ fewer dimensions may be found already in the work of Lagrange, Poisson, Jacobi, and Routh; it is well described in, for example, Whittaker [1937].

In the nonabelian case, Smale [1970] noted that Jacobi's elimination of the node in SO(3) symmetric problems can be understood as division of a nonzero angular momentum level by the SO(2) subgroup which fixes the momentum value. In his setting of cotangent bundles, Smale clearly stated that the coadjoint isotropy group G_μ of $\mu \in \mathfrak{g}^*$ (defined to be the group of those $g \in G$ such that $g \cdot \mu = \mu$, where the dot indicates the coadjoint action), leaves the level set $\mathbf{J}^{-1}(\mu)$ invariant (Smale [1970], Corollary 4.5). However, he only divided by G_μ after fixing the total energy as well, in order to obtain the "minimal" manifold on which to analyze the reduced dynamics. The goal of his "topology and mechanics" program was to use topology, and specifically Morse theory, to study relative equilibria, which he did with great effectiveness.

Marsden and Weinstein [1974] combined Souriau's momentum map for general symplectic actions, Smale's idea of dividing the momentum level by the coadjoint isotropy group, and Cartan's idea of removing the degeneracy of a 2-form by passing to the leaf space of the form's null foliation. The key observation was that the leaves of the null foliation are precisely the (connected components of the) orbits of the coadjoint isotropy group (a fact we shall prove in the next section as the *reduction lemma*). An analogous observation was made in Meyer [1973], except that Meyer worked in terms of a basis for the Lie algebra $\mathfrak{g}$ and identified the subgroup G_μ as the group which left the momentum level set $\mathbf{J}^{-1}(\mu)$ invariant. In this way, he did not need to deal with the equivariance properties of the coadjoint representation.

In the more general setting of symplectic manifolds with an equivariant momentum map for a symplectic group action, the fact that G_μ acts on $\mathbf{J}^{-1}(\mu)$ follows directly from equivariance of $\mathbf{J}$. Thus, it makes sense to form the ***symplectic reduced space*** which is defined to be the quotient space

$$P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu. \tag{3.14}$$

Roughly speaking, the symplectic reduction theorem states that, under suitable hypotheses, P_μ is itself a symplectic manifold. To state this precisely, we need a short excursion on level sets of the momentum map and some facts about quotients.

Free and Proper Actions The action of a Lie group G on a manifold M is called a ***free action*** if $g \cdot m = m$ for some $g \in G$ and $m \in M$ implies that $g = e$, the identity element.

An action of G on M is called ***proper*** when the map $G \times M \to M \times M; (g, m) \mapsto (g \cdot m, m)$ is a proper map—that is, inverse images of compact sets are compact. This is equivalent to the statement that if m_k is a convergent sequence in M and if $g_k \cdot m_k$ converges in M, then g_k has a convergent subsequence in G.

As is shown in, for example, [MTA] and Duistermaat and Kolk [1999], freeness, together with properness implies that the quotient space M/G is a smooth manifold and that the projection map $\pi : M \to M/G$ is a smooth surjective submersion.

Locally Free Actions. An action of G on M is called ***infinitesimally free*** at a point $m \in M$ if $\xi_M(m) = 0$ implies that $\xi = 0$. An action of G on M is called ***locally free*** at a point $m \in M$ if there is a neighborhood U of the identity in G such that $g \in U$ and $g \cdot m = m$ implies $g = e$.

Proposition 3.1. *An action of a Lie group G on a manifold M is locally free at $m \in M$ if and only if it is infinitesimally free at m.*

A free action is obviously locally free. The converse is not true because the action of any discrete group is locally free, but need not be globally free. When one has an action that is locally free but not globally free, one is lead to the theory of orbifolds, as in Satake [1956]. In fact, quotients of manifolds by locally free and proper group actions are orbifolds, which follows by the use of the Palais slice theorem (see Palais [1957]). Orbifolds come up in a variety of interesting examples involving, for example, resonances; see, for instance, Cushman and Bates [1997] and Alber, Luther, Marsden, and Robbins [1998] for some specific examples.

Symmetry and Singularities. If μ is a regular value of $\mathbf{J}$ then we claim that the action is automatically locally free at the elements of the corresponding level set $\mathbf{J}^{-1}(\mu)$. In this context it is convenient to introduce the notion of the ***symmetry algebra*** at $z \in P$ defined by

$$\mathfrak{g}_z = \{\xi \in \mathfrak{g} \mid \xi_P(z) = 0\}.$$

The symmetry algebra $\mathfrak{g}_z$ is the Lie algebra of the ***isotropy subgroup*** G_z of $z \in P$ defined by

$$G_z = \{g \in G \mid g \cdot z = z\}.$$

The following result (due to Smale [1970] in the special case of cotangent bundles and in general to Arms, Marsden, and Moncrief [1981]), is important for the recognition of regular as well as singular points in the reduction process.

Proposition 3.2. *An element $\mu \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$ if and only if $\mathfrak{g}_z = 0$ for all $z \in \mathbf{J}^{-1}(\mu)$.*

In other words, *points are regular points precisely when they have trivial symmetry algebra.* In examples, this gives an easy way to recognize regular points. For example, for the double spherical pendulum (see, for example, Marsden and Scheurle [1993a] or [LonM], one can say right away that the only singular points are those with *both* pendula pointing vertically (either straight down or straight up). This result holds whether or not $\mathbf{J}$ is equivariant.

This result, connecting the symmetry of z with the regularity of μ, suggests that *points with symmetry are bifurcation points of* $\mathbf{J}$. This observation turns out to have many important consequences, including some related key convexity theorems.

Now we are ready to state the symplectic reduction theorem. We make one of the following two sets of hypotheses; other variants are discussed in the next section.

SR. (P, Ω) is a symplectic manifold, G is a Lie group that acts symplectically on P and has an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$.

SRFree. G acts freely and properly on P.

SRRegular. Assume that $\mu \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$ and that the action of G_μ on $\mathbf{J}^{-1}(\mu)$ is free and proper

From the previous discussion, note that condition **SRFree** implies condition **SRRegular**. The real difference is that **SRRegular** assumes local freeness of the action of G (which is equivalent to μ being a regular value, as we have seen), while **SRFree** assumes global freeness (on all of P).

Theorem 3.3 (Symplectic Reduction Theorem)**.** *Assume that condition* **SR** *and that either the condition* **SRFree** *or the condition* **SRRegular** *holds. Then P_μ is a symplectic manifold, and is equipped with the* ***reduced symplectic form*** *Ω_μ that is uniquely characterized by the condition*

$$\pi_\mu^* \Omega_\mu = i_\mu^* \Omega, \tag{3.15}$$

where $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ is the projection to the quotient space and where $i_\mu : \mathbf{J}^{-1}(\mu) \to P$ is the inclusion.

The above procedure is often called *point reduction* because one is fixing the value of the momentum map at a point $\mu \in \mathfrak{g}^*$. An equivalent reduction method called *orbit reduction* will be discussed shortly.

Coadjoint Orbits. A standard example (due to Marsden and Weinstein [1974]) that we shall derive in detail in the next section, is the construction of the coadjoint orbits in $\mathfrak{g}^*$ of a group G by reduction of the cotangent bundle T^*G with its canonical symplectic structure and with G acting on T^*G by the cotangent lift of left (resp. right) group multiplication. In this case, one finds that $(T^*G)_\mu = \mathcal{O}_\mu$, the coadjoint orbit through $\mu \in \mathfrak{g}^*$. The reduced symplectic form is given by the ***Kostant, Kirillov, Souriau coadjoint form***, also referred to as the ***KKS form***:

$$\omega^{\mp}_{\mathcal{O}_\mu}(\nu)(\operatorname{ad}^*_\xi \nu, \, \operatorname{ad}^*_\eta \nu) = \mp\langle \nu, \, [\xi, \, \eta]\rangle, \tag{3.16}$$

where $\xi, \eta \in \mathfrak{g}$, $\nu \in \mathcal{O}_\mu$, $\operatorname{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is the adjoint operator defined by $\operatorname{ad}_\xi \eta := [\xi, \eta]$ and $\operatorname{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is its dual. In this formula, one uses the minus sign for the left action and the plus sign for the right action. We recall that coadjoint orbits, like any group orbit is always an immersed manifold. Thus, one arrives at the following result (see also Theorem 4.3):

Corollary 3.4. *Given a Lie group G with Lie algebra $\mathfrak{g}$ and any point $\mu \in \mathfrak{g}^*$, the reduced space $(T^*G)_\mu$ is the coadjoint orbit $\mathcal{O}_\mu$ through the point μ; it is a symplectic manifold with symplectic form given by (3.16).*

This example, which "explains" Kostant, Kirillov and Souriau's formula for this structure, is typical of many of the ensuing applications, in which the reduction procedure is applied to a "trivial" symplectic manifold to produce something interesting.

Orbit Reduction. An important variant of the symplectic reduction theorem is called ***orbit reduction*** and, roughly speaking, it constructs $\mathbf{J}^{-1}(\mathcal{O})/G$, where $\mathcal{O}$ is a coadjoint orbit in $\mathfrak{g}^*$. In the next section—see Theorem 4.4—we show that orbit reduction is equivalent to the point reduction considered above.

Cotangent Bundle Reduction. The theory of cotangent bundle reduction is a very important special case of general reduction theory. Notice that the reduction of T^*G above to give a coadjoint orbit is a special case of the more general procedure in which G is replaced by a configuration manifold Q. The theory of cotangent bundle reduction will be outlined in the historical overview in this chapter, and then treated in some detail in the following chapter.

Mathematical Physics Links. Another example in Marsden and Weinstein [1974] came from general relativity, namely the reduction of the cotangent bundle of the space of Riemannian metrics on a manifold M by the action of the group of diffeomorphisms of M. In this case, restriction to the zero momentum level is the divergence constraint of general relativity, and so one is led to a construction of a symplectic structure on a space closely related to the space of solutions of the Einstein equations, a question revisited in Fischer, Marsden, and Moncrief [1980] and Arms, Marsden, and Moncrief [1982]. Here one sees a precursor of an idea of Atiyahf. and Bott [1982], which has led to some of the most spectacular applications of reduction in mathematical physics and related areas of pure mathematics, especially low-dimensional topology.

Singular Reduction. In the preceding discussion, we have been making hypotheses that ensure the momentum levels and their quotients are smooth manifolds. Of course, this is not always the case, as was already noted in Smale [1970] and analyzed (even in the infinite-dimensional case) in Arms, Marsden, and Moncrief [1981]. We give a review of some of the current literature and history on this singular case in §5. For an outline of this subject, see Ortega and Ratiu [2006c] and for a complete account of the technical details, see [HRed].

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Reduction of Dynamics. Along with the geometry of reduction, there is also a theory of *reduction of dynamics*. The main idea is that a G-invariant Hamiltonian H on P induces a Hamiltonian H_μ on each of the reduced spaces, and the corresponding Hamiltonian vector fields X_H and X_{H_μ} are π_μ-related. The reverse of reduction is reconstruction and this leads one to the theory of classical geometric phases (Hannay–Berry phases); see Marsden, Montgomery, and Ratiu [1990].

Reduction theory has many interesting connections with the theory of integrable systems; we just mention some selected references Kazhdan, Kostant, and Sternberg [1978]; Ratiu [1980a,c,b]; Bobenko, Reyman, and Semenov-Tian-Shansky [1989]; Pedroni [1995]; Marsden and Ratiu [1986]; Vanhaecke [1996]; Bloch, Crouch, Marsden, and Ratiu [2002], which the reader can consult for further information.

4 Symplectic Reduction—Further Discussion

The symplectic reduction theorem leans on a few key lemmas that we just state. The first refers to the reflexivity of the operation of taking the symplectic orthogonal complement.

Lemma 4.1. *Let (V, Ω) be a finite dimensional symplectic vector space and $W \subset V$ be a subspace. Define the* ***symplectic orthogonal*** *to W by*

$$W^\Omega = \{v \in V \mid \Omega(v, w) = 0 \text{ for all } w \in W\}.$$

Then

$$(W^\Omega)^\Omega = W. \tag{4.1}$$

In what follows, we denote by $G \cdot z$ and $G_\mu \cdot z$ the G and G_μ-orbits through the point $z \in P$; note that if $z \in \mathbf{J}^{-1}(\mu)$ then $G_\mu \cdot z \subset \mathbf{J}^{-1}(\mu)$.

The key lemma that is central for the symplectic reduction theorem is the following.

Lemma 4.2 (Reduction Lemma)**.** *Let P be a Poisson manifold and let $\mathbf{J} : P \to \mathfrak{g}^*$ be an equivariant momentum map of a Lie group action by Poisson maps of G on P. Let $G \cdot \mu$ denote the coadjoint orbit through a regular value $\mu \in \mathfrak{g}^*$ of $\mathbf{J}$. Then*

(i) $\mathbf{J}^{-1}(G \cdot \mu) = G \cdot \mathbf{J}^{-1}(\mu) = \{g \cdot z \mid g \in G \text{ and } \mathbf{J}(z) = \mu\}$*;*

(ii) $G_\mu \cdot z = (G \cdot z) \cap \mathbf{J}^{-1}(\mu)$*;*

(iii) $\mathbf{J}^{-1}(\mu)$ *and* $G \cdot z$ ***intersect cleanly****, i.e.,*

$$T_z(G_\mu \cdot z) = T_z(G \cdot z) \cap T_z(\mathbf{J}^{-1}(\mu));$$

(iv) *if (P, Ω) is symplectic, then $T_z(\mathbf{J}^{-1}(\mu)) = (T_z(G \cdot z))^\Omega$; i.e., the sets*

$$T_z(\mathbf{J}^{-1}(\mu)) \quad \text{and} \quad T_z(G \cdot z)$$

are Ω-orthogonal complements of each other.

Refer to Figure 4.1 for one way of visualizing the geometry associated with the reduction lemma. As it suggests, the two manifolds $\mathbf{J}^{-1}(\mu)$ and $G \cdot z$ intersect in the orbit of the isotropy group $G_\mu \cdot z$ and their tangent spaces $T_z\mathbf{J}^{-1}(\mu)$ and $T_z(G \cdot z)$ are symplectically orthogonal and intersect in the space $T_z\,(G_\mu \cdot z)$.

Notice from the statement (iv) that $T_z(\mathbf{J}^{-1}(\mu))^\Omega \subset T_z(\mathbf{J}^{-1}(\mu))$ provided that $G_\mu \cdot z = G \cdot z$. Thus, $\mathbf{J}^{-1}(\mu)$ *is* coisotropic *if* $G_\mu = G$; for example, this happens if $\mu = 0$ or if G is Abelian.

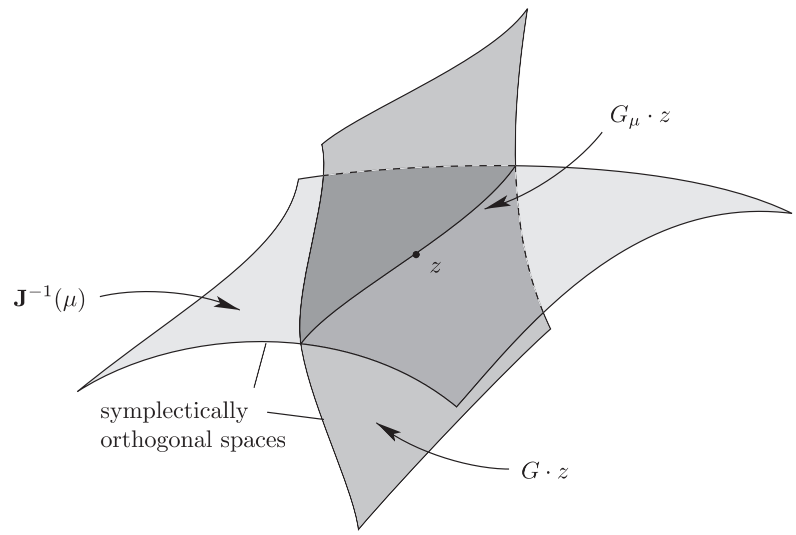

Figure 4.1: The geometry of the reduction lemma.

Remarks on the Reduction Theorem.

1. Even if Ω is exact; say $\Omega = -\mathbf{d}\Theta$ and the action of G leaves Θ invariant, Ω_μ *need not be exact*. Perhaps the simplest example is a nontrivial coadjoint orbit of SO(3), which is a sphere with symplectic form given by the area form (by Stokes' theorem, it cannot be exact). That this is a symplectic reduced space of $T^*\operatorname{SO}(3)$ (with the canonical symplectic structure, so is exact) is shown in Theorem 4.3 below.

2. Continuing with the previous remark, assume that $\Omega = -\mathbf{d}\Theta$ and that the G_μ principal bundle $\mathbf{J}^{-1}(\mu) \to P_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$ is trivializable; that is, it admits a *global* section $s : P_\mu \to \mathbf{J}^{-1}(\mu)$. Let $\Theta_\mu := s^* i_\mu^* \Theta \in \Omega^1(P_\mu)$. Then the reduced symplectic form $\Omega_\mu = -\mathbf{d}\Theta_\mu$ is exact. This statement *does not imply that the one-form* Θ *descends to the reduced space*, only that the reduced symplectic form is exact and one if its primitives is Θ_μ. In fact, if one changes the global section, another primitive of Ω_μ is found which differs from Θ_μ by a closed one-form on P_μ.

3. The assumption that μ is a regular value of $\mathbf{J}$ can be relaxed. *The only hypothesis needed is that* μ *be a clean value of* $\mathbf{J}$, *i.e.*, $\mathbf{J}^{-1}(\mu)$ is a manifold and $T_z(\mathbf{J}^{-1}(\mu)) = \ker T_z\mathbf{J}$. This generalization applies, for instance, for zero angular momentum in the three dimensional two body problem, as was noted by Marsden and Weinstein [1974] and Kazhdan, Kostant, and Sternberg [1978]; see also Guillemin and Sternberg [1984].

   Here are the general definitions: If $f : M \to N$ is a smooth map, a point $y \in N$ is called a ***clean value*** if $f^{-1}(y)$ is a submanifold and for each $x \in f^{-1}(y)$, $T_x f^{-1}(y) = \ker T_x f$. We say that f intersects a submanifold $L \subset N$ ***cleanly*** if $f^{-1}(L)$ is a submanifold of M and $T_x(f^{-1}(L)) = (T_x f)^{-1}(T_{f(x)}L)$. Note that *regular values of* f *are clean values and that if* f *intersects the submanifold* L *transversally, then it intersects it cleanly.* Also note that the definition of clean intersection of two manifolds is equivalent to the statement that the inclusion map of either one of them intersects the other cleanly. The reduction lemma is an example of this situation.

4. The freeness and properness of the G_μ action on $\mathbf{J}^{-1}(\mu)$ are used only to guarantee that P_μ is a manifold; these hypotheses can thus be replaced by the requirement that

P_μ is a manifold and $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ a submersion; the proof of the symplectic reduction theorem remains unchanged.

5. Even if μ is a regular value (in the sense of a regular value of the mapping $\mathbf{J}$), it need not be a ***regular point*** (also called a ***generic point***) in $\mathfrak{g}^*$; that is, a point whose coadjoint orbit is of maximal dimension. The reduction theorem *does not require that μ be a regular point*. For example, if G acts on itself on the left by group multiplication and if we lift this to an action on T^*G by the cotangent lift, then the action is free and so all μ are regular values, but such values (for instance, the zero element in $\mathfrak{so}(3)$) need not be regular. On the other hand, in many important stability considerations, a regularity assumption on the point μ is required; see, for instance, Patrick [1992], Ortega and Ratiu [1999b] and Patrick, Roberts, and Wulff [2004].

Nonequivariant Reduction. We now describe how one can carry out reduction for a *nonequivariant momentum map*.

If $\mathbf{J} : P \to \mathfrak{g}^*$ is a nonequivariant momentum map on the connected symplectic manifold P with nonequivariance group one-cocycle σ consider the affine action (3.9) and let $\widetilde{G}_\mu$ be the isotropy subgroup of $\mu \in \mathfrak{g}^*$ relative to this action. Then, under the same regularity assumptions (for example, assume that G acts freely and properly on P, or that μ is a regular value of $\mathbf{J}$ and that $\widetilde{G}_\mu$ acts freely and properly on $\mathbf{J}^{-1}(\mu)$), the quotient manifold $P_\mu := \mathbf{J}^{-1}(\mu)/\widetilde{G}_\mu$ is a symplectic manifold whose symplectic form is uniquely determined by the relation $i_\mu^*\Omega = \pi_\mu^*\Omega_\mu$. The proof of this statement is identical to the one given above with the obvious changes in the meaning of the symbols.

When using nonequivariant reduction, one has to remember that G acts on $\mathfrak{g}^*$ in an *affine* and not a linear manner. For example, while the coadjoint isotropy subgroup at the origin is equal to G; that is, $G_0 = G$, this is no longer the case for the affine action, where $\widetilde{G}_0$ in general does not equal G.

Coadjoint Orbits as Symplectic Reduced Spaces. We now examine Corollary 3.4—that is, that coadjoint orbits may be realized as reduced spaces—a little more closely. Realizing them as reduced spaces shows that they are symplectic manifolds[2] Historically, a direct argument was found first, by Kirillov, Kostant and Souriau in the early 1960's and the (minus) coadjoint symplectic structure was found to be

$$\omega_\nu^-(\operatorname{ad}_\xi^*\nu, \operatorname{ad}_\eta^*\nu) = -\langle \nu, [\xi, \eta]\rangle \tag{4.2}$$

Interestingly, this is the symplectic structure on the symplectic leaves of the Lie–Poisson bracket, as is shown in, for example, [MandS]. (See the historical overview in §5 below and specifically, see equation (5.1) for a quick review of the Lie–Poisson bracket.)

The strategy of the reduction proof, as mentioned in the discussion in the last section, is to show that the coadjoint symplectic form on a coadjoint orbit $\mathcal{O}_\mu$ of the point μ, at a point $\nu \in \mathcal{O}$, may be obtained by symplectically reducing T^*G at the value μ. The following theorem (due to Marsden and Weinstein [1974]), and which is an elaboration on the result in Corollary 3.4, formulates the result for left actions; of course there is a similar one for right actions, with the minus sign replaced by a plus sign.

Theorem 4.3 (Reduction to Coadjoint Orbits)**.** *Let G be a Lie group and let G act on G (and hence on T^*G by cotangent lift) by left multiplication. Let $\mu \in \mathfrak{g}^*$ and let $\mathbf{J}_L : T^*G \to \mathfrak{g}^*$ be the momentum map for the left action. Then μ is a regular value of $\mathbf{J}_L$, the action of G is free and proper, the symplectic reduced space $\mathbf{J}_L^{-1}(\mu)/G_\mu$ is identified via left translation*

[2]See [MandS], Chapter 14 for a "direct" or "bare hands" argument.

with $\mathcal{O}_\mu$, the coadjoint orbit through μ, and the reduced symplectic form coincides with ω^- given in equation (4.2).

Remarks.

1. Notice that, as in the general Symplectic Reduction Theorem 3.3, this result does *not* require μ to be a regular (or generic) *point* in $\mathfrak{g}^*$; that is, arbitrarily nearby coadjoint orbits may have a different dimension.

2. The form ω^- on the orbit need not be exact even though Ω is. An example that shows this is SO(3), whose coadjoint orbits are spheres and whose symplectic structure is, as shown in [MandS], a multiple of the area element, which is not exact by Stokes' Theorem.

Orbit Reduction. So far, we have presented what is usually called *point reduction*. There is another point of view that is called *orbit reduction*, which we now summarize. We assume the same set up as in the symplectic reduction theorem, with P connected, G acting symplectically, freely, and properly on P with an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$.

The connected components of the point reduced spaces P_μ can be regarded as the symplectic leaves of the Poisson manifold $\left(P/G, \{\cdot, \cdot\}_{P/G}\right)$ in the following way. Form a map $[i_\mu] : P_\mu \to P/G$ defined by selecting an equivalence class $[z]_{G_\mu}$ for $z \in \mathbf{J}^{-1}(\mu)$ and sending it to the class $[z]_G$. This map is checked to be well-defined and smooth. We then have the commutative diagram

$$\begin{array}{ccc} \mathbf{J}^{-1}(\mu) & \xrightarrow{\ i_\mu\ } & P \\ \downarrow{\scriptstyle \pi_\mu} & & \downarrow{\scriptstyle \pi} \\ P_\mu & \xrightarrow{\ [i_\mu]\ } & P/G \end{array}$$

One then checks that $[i_\mu]$ is a Poisson injective immersion. Moreover, the $[i_\mu]$-images in P/G of the connected components of the symplectic manifolds (P_μ, Ω_μ) are its symplectic leaves (see [HRed] and references therein for details). As sets,

$$[i_\mu]\,(P_\mu) = \mathbf{J}^{-1}\left(\mathcal{O}_\mu\right)/G,$$

where $\mathcal{O}_\mu \subset \mathfrak{g}^*$ is the coadjoint orbit through $\mu \in \mathfrak{g}^*$. The set

$$P_{\mathcal{O}_\mu} := \mathbf{J}^{-1}\left(\mathcal{O}_\mu\right)/G$$

is called the ***orbit reduced space*** associated to the orbit $\mathcal{O}_\mu$. The smooth manifold structure (and hence the topology) on $P_{\mathcal{O}_\mu}$ is the one that makes the map $[i_\mu] : P_\mu \to P_{\mathcal{O}_\mu}$ into a diffeomorphism.

For the next theorem, which characterizes the symplectic form and the Hamiltonian dynamics on $P_{\mathcal{O}_\mu}$, recall the coadjoint orbit symplectic structure of Kirillov, Kostant and Souriau that was established in the preceding Theorem 4.3:

$$\omega^-_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = -\langle \nu, [\xi, \eta] \rangle, \tag{4.3}$$

for $\xi, \eta \in \mathfrak{g}$ and $\nu \in \mathcal{O}_\mu$.

We also recall that an injectively immersed submanifold of S of Q is called an ***initial submanifold*** of Q when for any smooth manifold P, a map $g : P \to S$ is smooth if and only if $\iota \circ g : P \to Q$ is smooth, where $\iota : S \hookrightarrow Q$ is the inclusion.

Theorem 4.4 (Symplectic Orbit Reduction Theorem)**.** *In the setup explained above:*

(i) *The momentum map* $\mathbf{J}$ *is transverse to the coadjoint orbit* $\mathcal{O}_\mu$ *and hence* $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ *is an initial submanifold of* P*. Moreover, the projection* $\pi_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to P_{\mathcal{O}_\mu}$ *is a surjective submersion.*

(ii) $P_{\mathcal{O}_\mu}$ *is a symplectic manifold with the symplectic form* $\Omega_{\mathcal{O}\mu}$ *uniquely characterized by the relation*

$$\pi^*_{\mathcal{O}_\mu}\Omega_{\mathcal{O}\mu} = \mathbf{J}^*_{\mathcal{O}_\mu}\omega^-_{\mathcal{O}_\mu} + i^*_{\mathcal{O}_\mu}\Omega, \tag{4.4}$$

where $\mathbf{J}_{\mathcal{O}_\mu}$ *is the restriction of* $\mathbf{J}$ *to* $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ *and* $i_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \hookrightarrow P$ *is the inclusion.*

(iii) *The map* $[i_\mu] : P_\mu \to P_{\mathcal{O}_\mu}$ *is a symplectic diffeomorphism.*

(iv) (Dynamics.) *Let* H *be a* G*-invariant function on* P *and define* $\widetilde{H} : P/G \to \mathbb{R}$ *by* $H = \widetilde{H} \circ \pi$*. Then the Hamiltonian vector field* X_H *is also* G*-invariant and hence induces a vector field on* P/G*, which coincides with the Hamiltonian vector field* $X_{\widetilde{H}}$*. Moreover, the flow of* $X_{\widetilde{H}}$ *leaves the symplectic leaves* $P_{\mathcal{O}_\mu}$ *of* P/G *invariant. This flow restricted to the symplectic leaves is again Hamiltonian relative to the symplectic form* $\Omega_{\mathcal{O}\mu}$ *and the Hamiltonian function* $\widetilde{H}_{\mathcal{O}_\mu}$ *given by*

$$\widetilde{H}_{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu} = H \circ i_{\mathcal{O}_\mu}.$$

Note that if $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$ then $\mathbf{J}$ is transverse to $\mathcal{O}_\mu$ and hence $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is automatically an embedded submanifold of P.

The proof of this theorem when $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$ can be found in Marle [1976], Kazhdan, Kostant, and Sternberg [1978], with useful additions given in Marsden [1981] and Blaom [2001]. For nonfree actions and when $\mathcal{O}_\mu$ is not an embedded submanifold of $\mathfrak{g}^*$ see [HRed]. Further comments on the historical context of this result are given in the next section.

Remarks.

1. A similar result holds for right actions.

2. Freeness and properness of the G_μ-action on $\mathbf{J}^{-1}(\mu)$ are only needed indirectly. In fact these conditions are sufficient but not necessary for P_μ to be a manifold. All that is needed is for P_μ to be a manifold and π_μ to be a submersion and the above result remains unchanged.

3. Note that the description of the symplectic structure on $\mathbf{J}^{-1}(\mathcal{O})/G$ is not as simple as it was for $\mathbf{J}^{-1}(\mu)/G$, while the Poisson bracket description is simpler on $\mathbf{J}^{-1}(\mathcal{O})/G$. Of course, the symplectic structure depends only on the orbit $\mathcal{O}$ and not on the choice of a point μ on it.

Cotangent Bundle Reduction. Perhaps the most important and basic reduction theorem in addition to those already presented is the cotangent bundle reduction theorem. We shall give an exposition of the key aspects of this theory in §6 and give a historical account of its development, along with references in the next section.

At this point, to orient the reader, we note that one of the special cases is cotangent bundle reduction at zero (see Theorem 6.2). This result says that if one has, again for simplicity, a free and proper action of G on Q (which is then lifted to T^*Q by the cotangent lift), then the reduced space at zero of T^*Q is given by $T^*(Q/G)$, with its canonical symplectic structure. On the other hand, reduction at a nonzero value is a bit more complicated and gives rise to modifications of the standard symplectic structure; namely, one adds to the canonical structure, the pull-back of a closed two form on Q to T^*Q. Because of their physical interpretation (discussed, for example, in [MandS]), such extra terms are called magnetic terms. In §6, we state the basic cotangent bundle reduction theorems along with providing some of the other important notions, such as the mechanical connection and the locked inertia tensor. Other notions that are important in mechanics, such as the amended potential, can be found in [LonM].

5 Reduction Theory: Historical Overview

We have already given bits an pieces of the history of symplectic reduction and momentum maps. In this section we take a broader view of the subject to put things in historical and topical context.

History before 1960. In the preceding sections, reduction theory has been presented as a mathematical construction. Of course, these ideas are rooted in classical work on mechanical systems with symmetry by such masters as Euler, Lagrange, Hamilton, Jacobi, Routh, Riemann, Liouville, Lie, and Poincaré. The aim of their work was, to a large extent, to eliminate variables associated with symmetries in order to simplify calculations in concrete examples. Much of this work was done using coordinates, although the deep connection between mechanics and geometry was already evident. Whittaker [1937] gives a good picture of the theory as it existed up to about 1910.

A highlight of this early theory was the work of Routh [1860, 1884] who studied reduction of systems with cyclic variables and introduced the amended potential for the reduced system for the purpose of studying, for instance, the *stability of a uniformly rotating state*—what we would call today a *relative equilibrium*, terminology introduced later by Poincaré.[3] Routh's work was closely related to the reduction of systems with integrals in involution studied by Jacobi and Liouville around 1870; the Routh method corresponds to the modern theory of Lagrangian reduction for the action of Abelian groups.

The rigid body, whose equations were discovered by Euler around 1740, was a key example of reduction–what we would call today either reduction to coadjoint orbits or Lie–Poisson reduction on the Hamiltonian side, or Euler–Poincaré reduction on the Lagrangian side, depending on one's point of view. Lagrange [1788] already understood reduction of the rigid body equations by a method not so far from what one would do today with the symmetry group SO(3).

Many later authors, unfortunately, relied so much on coordinates (especially Euler angles) that there is little mention of SO(3) in classical mechanics books written before 1990, which by today's standards, seems rather surprising! In addition, there seemed to be little appreciation until recently for the role of topological notions; for example, the fact that one cannot globally split off cyclic variables for the S^1 action on the configuration space of the

[3] Smale [1970] eventually put the amended potential into a nice geometric setting.

heavy top. The Hopf fibration was patiently waiting to be discovered in the reduction theory for the classical rigid body, but it was only explicitly found later on by H. Hopf [1931]. Hopf was, apparently, unaware that this example is of great mechanical interest—the gap between workers in mechanics and geometers seems to have been particularly wide at that time.

Another noteworthy instance of reduction is Jacobi's elimination of the node for reducing the gravitational (or electrostatic) n-body problem by means of the group SE(3) of Euclidean motions, around 1860 or so. This example has, of course, been a mainstay of celestial mechanics. It is related to the work done by Riemann, Jacobi, Poincaré and others on rotating fluid masses held together by gravitational forces, such as stars. Hidden in these examples is much of the beauty of modern reduction, stability and bifurcation theory for mechanical systems with symmetry.

While both symplectic and Poisson geometry have their roots in the work of Lagrange and Jacobi, it matured considerably with the work of Lie [1890], who discovered many remarkably modern concepts such as the Lie–Poisson bracket on the dual of a Lie algebra.[4] How Lie could have viewed his wonderful discoveries so divorced from their roots in mechanics remains a mystery. We can only guess that he was inspired by Jacobi, Lagrange and Riemann and then, as mathematicians often do, he quickly abstracted the ideas, losing valuable scientific and historical connections along the way.

As we have already hinted, it was the famous paper Poincaré [1901] where we find what we call today the Euler–Poincaré equations–a generalization of the Euler equations for both fluids and the rigid body to general Lie algebras. (The Euler–Poincaré equations are treated in detail in [MandS]). It is curious that Poincaré did not stress either the symplectic ideas of Lie, nor the variational principles of mechanics of Lagrange and Hamilton—in fact, it is not clear to what extent he understood what we would call today Euler–Poincaré reduction. It was only with the development and physical application of the notion of a manifold, pioneered by Lie, Poincaré, Weyl, Cartan, Reeb, Synge and many others, that a more general and intrinsic view of mechanics was possible. By the late 1950's, the stage was set for an explosion in the field.

1960-1972. Beginning in the 1960's, the subject of geometric mechanics indeed did explode with the basic contributions of people such as (alphabetically and nonexhaustively) Abraham, Arnold, Kirillov, Kostant, Mackey, MacLane, Segal, Sternberg, Smale, and Souriau. Kirillov and Kostant found deep connections between mechanics and pure mathematics in their work on the orbit method in group representations, while Arnold, Smale, and Souriau were in closer touch with mechanics.

The modern vision of geometric mechanics combines strong links to important questions in mathematics with the traditional classical mechanics of particles, rigid bodies, fields, fluids, plasmas, and elastic solids, as well as quantum and relativistic theories. Symmetries in these theories vary from obvious translational and rotational symmetries to less obvious particle relabeling symmetries in fluids and plasmas, to the "hidden" symmetries underlying integrable systems. As we have already mentioned, reduction theory concerns the removal of variables using symmetries and their associated conservation laws. Variational principles, in addition to symplectic and Poisson geometry, provide fundamental tools for this endeavor. In fact, conservation of the momentum map associated with a symmetry group action is a geometric expression of the classical Noether theorem (discovered by variational, not symplectic methods).

[4]See Weinstein [1983] and Marsden and Ratiu [1999] for more details on the history.

Arnold and Smale. The modern era of reduction theory began with the fundamental papers of Arnold [1966] and Smale [1970]. Arnold focused on systems whose configuration manifold is a Lie group, while Smale focused on bifurcations of relative equilibria. Both Arnold and Smale linked their theory strongly with examples. For Arnold, they were the same examples as for Poincaré, namely the rigid body and fluids, for which he went on to develop powerful stability methods, as in Arnold [1969].

With hindsight, we can say that Arnold [1966] was picking up on the basic work of Poincaré for both rigid body motion and fluids. In the case of fluids, G is the group of (volume preserving) diffeomorphisms of a compact manifold (possibly with boundary). In this setting, one obtains the Euler equations for (incompressible) fluids by reduction from the Lagrangian formulation of the equations of motion, an idea exploited by Arnold [1966] and Ebin and Marsden [1970]. This sort of description of a fluid goes back to Poincaré (using the Euler–Poincaré equations) and to the thesis of Ehrenfest (as geodesics on the diffeomorphism group), written under the direction of Boltzmann.

For Smale, the motivating example was celestial mechanics, especially the study of the number and stability of relative equilibria by a topological study of the energy-momentum mapping. He gave an intrinsic geometric account of the amended potential and in doing so, discovered what later became known as the mechanical connection. (Smale appears to not to have recognized that the interesting object he called α is, in fact, a *principal connection*; this was first observed by Kummer [1981]). One of Smale's key ideas in studying relative equilibria was to link mechanics with topology via the fact that relative equilibria are critical points of the amended potential.

Besides giving a beautiful exposition of the momentum map, Smale also emphasized the connection between singularities and symmetry, observing that the symmetry group of a phase space point has positive dimension if and only if that point is not a regular point of the momentum map restricted to a fiber of the cotangent bundle (Smale [1970], Proposition 6.2)—a result we have proved in Proposition 3.2. He went on from here to develop his topology and mechanics program and to apply it to the planar n-body problem. The topology and mechanics program definitely involved reduction ideas, as in Smale's construction of the quotients of integral manifolds, as in $I_{c,p}/S^1$ (Smale [1970], page 320). He also understood Jacobi's elimination of the node in this context, although he did not attempt to give any general theory of reduction along these lines.

Smale thereby set the stage for symplectic reduction: he realized the importance of the momentum map and of quotient constructions, and he worked out explicit examples like the planar n-body problem with its S^1 symmetry group. (Interestingly, he pointed out that one should really use the nonabelian group SE(2); his feeling of unease with fixing the center of mass of an n-body system is remarkably perceptive.)

Synthesis. The problem of synthesizing the Lie algebra reduction methods of Arnold [1966] with the techniques of Smale [1970] on the reduction of cotangent bundles by Abelian groups, led to the development of reduction theory in the general context of symplectic manifolds and equivariant momentum maps in Marsden and Weinstein [1974] and Meyer [1973], as we described in the last section. Both of these papers were completed by 1972.

Poisson Manifolds. Meanwhile, things were also gestating from the viewpoint of Poisson brackets and the idea of a Poisson manifold was being initiated and developed, with much duplication and rediscovery (see [MandS] Section 10.1 for additional information).

A basic example of a noncanonical Poisson bracket is the Lie–Poisson bracket on $\mathfrak{g}^*$, the dual of a Lie algebra $\mathfrak{g}$. This bracket (which comes with a plus or minus sign) is given on two smooth functions on $\mathfrak{g}^*$ by

$$\{f, g\}_{\pm}(\mu) = \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle, \tag{5.1}$$

where $\delta f/\delta\mu$ is the derivative of f, but thought of as an element of $\mathfrak{g}$. These Poisson structures, including the coadjoint orbits as their symplectic leaves, were known to Lie [1890], although, as we mentioned previously, Lie does not seem to have recognized their importance in mechanics. It is also not clear whether or not Lie realized that the Lie Poisson bracket is the Poisson reduction of the canonical Poisson bracket on T^*G by the action of G. (See [MandS], Chapter 13 for an account of this theory). The first place we know of that has this clearly stated (but with no references, and no discussion of the context) is Bourbaki [1998], Chapter III, Section 4, Exercise 6. Remarkably, this exercise also contains an interesting proof of the Duflo–Vergne theorem (with no reference to the original paper, which appeared in 1969). Again, any hint of links with mechanics is missing.

This takes us up to about 1972.

Post 1972. An important contribution was made by Marle [1976], who divides the inverse image of an orbit by its characteristic foliation to obtain the product of an orbit and a reduced manifold. In particular, as we saw in Theorem 4.4, P_μ is symplectically diffeomorphic to an "orbit-reduced" space $P_\mu \cong J^{-1}(\mathcal{O}_\mu)/G$, where $\mathcal{O}_\mu$ is a coadjoint orbit of G. From this it follows that the P_μ are symplectic leaves in the Poisson space P/G. The related paper of Kazhdan, Kostant, and Sternberg [1978] was one of the first to notice deep links between reduction and integrable systems. In particular, they found that the Calogero–Moser systems could be obtained by reducing a system that was trivially integrable; in this way, reduction provided a method of producing an interesting integrable system from a simple one. This point of view was used again by, for example, Bobenko, Reyman, and Semenov-Tian-Shansky [1989] in their spectacular group theoretic explanation of the integrability of the Kowalewski top.

Noncanonical Poisson Brackets. The Hamiltonian description of many physical systems, such as rigid bodies and fluids in Eulerian variables, requires noncanonical Poisson brackets and constrained variational principles of the sort studied by Lie and Poincaré. As discussed above, a basic example of a noncanonical Poisson bracket is the Lie–Poisson bracket on the dual of a Lie algebra. From the mechanics perspective, the remarkably modern book (but which was, unfortunately, rather out of touch with the corresponding mathematical developments) by Sudarshan and Mukunda [1974] showed via explicit examples how systems such as the rigid body could be written in terms of noncanonical brackets, an idea going back to Pauli [1953], Martin [1959] and Nambu [1973]. Others in the physics community, such as Morrison and Greene [1980] also discovered noncanonical bracket formalisms for fluid and magnetohydrodynamic systems. In the 1980's, many fluid and plasma systems were shown to have a noncanonical Poisson formulation. It was Marsden and Weinstein [1982, 1983] who first applied reduction techniques to these systems.

The *reduction philosophy* concerning noncanonical brackets can be summarized by saying

> *Any mechanical system has its roots somewhere as a cotangent bundle and one can recover noncanonical brackets by the simple process of Poisson reduction. For example, in fluid mechanics, this reduction is implemented by the Lagrange-to-Euler map.*

This view ran contrary to the point of view, taken by some researchers, that one should proceed by analogy or guesswork to find Poisson structures and then to try to limit the guesses by the constraint of Jacobi's identity.

In the simplest version of the Poisson reduction process, one starts with a Poisson manifold P on which a group G acts by Poisson maps and then forms the quotient space P/G, which, if not singular, inherits a natural Poisson structure itself. Of course, the Lie–Poisson structure on $\mathfrak{g}^*$ is inherited in exactly this way from the canonical symplectic structure on T^*G. One of the attractions of this Poisson bracket formalism was its use in stability theory. This literature is now very large, but Holm, Marsden, Ratiu, and Weinstein [1985] is representative.

The way in which the Poisson structure on P_μ is related to that on P/G was clarified in a generalization of Poisson reduction due to Marsden and Ratiu [1986], a technique that has also proven useful in integrable systems (see, e.g., Pedroni [1995] and Vanhaecke [1996]).

Reduction theory for mechanical systems with symmetry has proven to be a powerful tool that has enabled key advances in stability theory (from the Arnold method to the energy-momentum method for relative equilibria) as well as in bifurcation theory of mechanical systems, geometric phases via reconstruction—the inverse of reduction—as well as uses in control theory from stabilization results to a deeper understanding of *locomotion*. For a general introduction to some of these ideas and for further references, see Marsden, Montgomery, and Ratiu [1990]; Simo, Lewis, and Marsden [1991]; Marsden and Ostrowski [1996]; Marsden and Ratiu [1999]; Montgomery [1988, 1990, 1991a,b, 1993]; Blaom [2000, 2001], and Kanso, Marsden, Rowley, and Melli-Huber [2005].

Tangent and Cotangent Bundle Reduction. The simplest case of cotangent bundle reduction is the case of reduction of $P = T^*Q$ at $\mu = 0$; the answer is simply $P_0 = T^*(Q/G)$ with the canonical symplectic form. Another basic case is when G is Abelian. Here, $(T^*Q)_\mu \cong T^*(Q/G)$, but the latter has a symplectic structure modified by magnetic terms, that is, by the curvature of the mechanical connection.

An Abelian version of cotangent bundle reduction was developed by Smale [1970]. Then Satzer [1977] studied the relatively simple, but important case of cotangent bundle reduction at the zero value of the momentum map. The full generalization of cotangent bundle reduction for nonabelian groups at arbitrary values of the momentum map appears for the first time in Abraham and Marsden [1978]. It was Kummer [1981] who first interpreted this result in terms of a connection, now called the ***mechanical connection***. The geometry of this situation was used to great effect in, for example, Guichardet [1984], Iwai [1987, 1990], and Montgomery [1984, 1990, 1991a]. We give an account of cotangent bundle reduction theory in the following section.

The Gauge Theory Viewpoint. Tangent and cotangent bundle reduction evolved into what we now term as the "bundle picture" or the "gauge theory of mechanics". This picture was first developed by Montgomery, Marsden, and Ratiu [1984] and Montgomery [1984, 1986]. That work was motivated and influenced by the work of Sternberg [1977] and Weinstein [1978] on a "Yang–Mills construction" which is, in turn, motivated by Wong's equations, i.e., the equations for a particle moving in a Yang–Mills field. The main result of the bundle picture gives a structure to the quotient spaces $(T^*Q)/G$ and $(TQ)/G$ when G acts by the cotangent and tangent lifted actions. The symplectic leaves in this picture were analyzed by Zaalani [1999], Cushman and Śniatycki [1999], and Marsden and Perlmutter [2000]. The work of Perlmutter and Ratiu [2005] gives a unified study of the Poisson bracket on $(T^*Q)/G$ in both the Sternberg and Weinstein realizations of the quotient.

As mentioned earlier, we shall review some of the basics of cotangent bundle reduction theory in §6. Further information on this theory may be found in [LonM], [FofM], and [HStages], as well as a number of the other references mentioned above.

Lagrangian Reduction. A key ingredient in Lagrangian reduction is the classical work of Poincaré [1901] in which the Euler–Poincaré equations were introduced. Poincaré realized that the equations of fluids, free rigid bodies, and heavy tops could all be described in Lie algebraic terms in a beautiful way. The importance of these equations was realized by Hamel [1904, 1949] and Chetayev [1941], but to a large extent, the work of Poincaré lay dormant until it was revived in the Russian literature in the 1980's.

The more recent developments of Lagrangian reduction were motivated by attempts to understand the relation between reduction, variational principles and Clebsch variables in Cendra and Marsden [1987] and Cendra, Ibort, and Marsden [1987]. In Marsden and Scheurle [1993b] it was shown that, for matrix groups, one could view the Euler–Poincaré equations via the reduction of Hamilton's variational principle from TG to $\mathfrak{g}$. The work of Bloch, Krishnaprasad, Marsden, and Ratiu [1996b] established the Euler–Poincaré variational structure for general Lie groups.

The paper of Marsden and Scheurle [1993b] also considered the case of more general configuration spaces Q on which a group G acts, which was motivated by both the Euler–Poincaré case as well as the work of Cendra and Marsden [1987] and Cendra, Ibort, and Marsden [1987]. The Euler–Poincaré equations correspond to the case $Q = G$. Related ideas stressing the groupoid point of view were given in Weinstein [1996]. The resulting reduced equations were called the *reduced Euler–Lagrange equations*. This work is the Lagrangian analogue of Poisson reduction, in the sense that no momentum map constraint is imposed.

Lagrangian reduction proceeds in a way that is very much in the spirit of the gauge theoretic point of view of mechanical systems with symmetry. It starts with Hamilton's variational principle for a Lagrangian system on a configuration manifold Q and with a symmetry group G acting on Q. The idea is to drop this variational principle to the quotient Q/G to derive a reduced variational principle. This theory has its origins in specific examples such as fluid mechanics (see, for example, Arnold [1966] and Bretherton [1970]), while the systematic theory of Lagrangian reduction was begun in Marsden and Scheurle [1993b] and further developed in Cendra, Marsden, and Ratiu [2001a]. The latter reference also introduced a connection to realize the space $(TQ)/G$ as the fiber product $T(Q/G) \times \tilde{\mathfrak{g}}$ of $T(Q/G)$ with the associated bundle formed using the adjoint action of G on $\mathfrak{g}$. The reduced equations associated to this construction are called the *Lagrange–Poincaré equations* and their geometry has been fairly well developed. Note that a G-invariant Lagrangian L on TQ induces a Lagrangian l on $(TQ)/G$.

Until recently, the Lagrangian side of the reduction story had lacked a general category that is the Lagrangian analogue of Poisson manifolds in which reduction can be repeated. One candidate is the category of Lie algebroids, as explained in Weinstein [1996]. Another is that of *Lagrange–Poincaré bundles*, developed in Cendra, Marsden, and Ratiu [2001a]. Both have tangent bundles and Lie algebras as basic examples. The latter work also develops the Lagrangian analogue of reduction for central extensions and, as in the case of symplectic reduction by stages, cocycles and curvatures enter in a natural way.

This bundle picture and Lagrangian reduction has proven very useful in control and optimal control problems. For example, it was used in Chang, Bloch, Leonard, Marsden, and Woolsey [2002] to develop a Lagrangian and Hamiltonian reduction theory for controlled mechanical systems and in Koon and Marsden [1997] to extend the falling cat theorem of Montgomery [1990] to the case of nonholonomic systems as well as to nonzero values of the momentum map.

Finally we mention that the paper Cendra, Marsden, Pekarsky, and Ratiu [2003] develops the reduction theory for Hamilton's *phase space principle* and the equations on the reduced space, along with a reduced variational principle, are developed and called *the Hamilton-Poincaré equations*. Even in the case $Q = G$, this collapses to an interesting variational principle for the Lie–Poisson equations on $\mathfrak{g}^*$.

Legendre Transformation. Of course the Lagrangian and Hamiltonian sides of the reduction story are linked by the Legendre transformation. This mapping descends at the appropriate points to give relations between the Lagrangian and the Hamiltonian sides of the theory. However, even in standard cases such as the heavy top, one must be careful with this approach, as is already explained in, for example, Holm, Marsden, and Ratiu [1998]. For field theories, such as the Maxwell-Vlasov equations, this issues is also important, as explained in Cendra, Holm, Hoyle, and Marsden [1998a] (see also Tulczyjew and Urbański [1999]).

Nonabelian Routh Reduction. Routh reduction for Lagrangian systems, which goes back Routh [1860, 1877, 1884] is classically associated with systems having cyclic variables (this is almost synonymous with having an Abelian symmetry group). Modern expositions of this classical theory can be found in Arnold, Koslov, and Neishtadt [1988] and in [MandS], §8.9. Routh Reduction may be thought of as the Lagrangian analog of symplectic reduction in that a momentum map is set equal to a constant. A key feature of Routh reduction is that when one drops the Euler–Lagrange equations to the quotient space associated with the symmetry, and when the momentum map is constrained to a specified value (i.e., when the cyclic variables and their velocities are eliminated using the given value of the momentum), then the resulting equations are in Euler–Lagrange form not with respect to the Lagrangian itself, but with respect to a modified function called the *Routhian*.

Routh [1877] applied his method to stability theory; this was a precursor to the energy-momentum method for stability that synthesizes Arnold's and Routh's methods (see Simo, Lewis, and Marsden [1991]). Routh's stability method is still widely used in mechanics.

The initial work on generalizing Routh reduction to the nonabelian case was that of Marsden and Scheurle [1993a]. This subject was further developed in Jalnapurkar and Marsden [2000] and Marsden, Ratiu, and Scheurle [2000]. The latter reference used this theory to give some nice formulas for geometric phases from the Lagrangian point of view.

Semidirect Product Reduction. In the simplest case of a semidirect product, one has a Lie group G that acts on a vector space V (and hence on its dual V^*) and then one forms the semidirect product $S = G \circledS V$, generalizing the semidirect product structure of the Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \circledS \mathbb{R}^3$.

Consider the isotropy group G_{a_0} for some $a_0 \in V^*$. The semidirect product reduction theorem states that each of the symplectic reduced spaces for the action of G_{a_0} on T^*G is symplectically diffeomorphic to a coadjoint orbit in $(\mathfrak{g} \circledS V)^*$, the dual of the Lie algebra of the semidirect product. This semidirect product theory was developed by Guillemin and Sternberg [1978, 1980], Ratiu [1980a, 1981, 1982], and Marsden, Ratiu, and Weinstein [1984a,b].

TheLagrangian reduction analog of semidirect product theory was developed by Holm, Marsden, and Ratiu [1998, 2002]. This construction is used in applications where one has advected quantities (such as the direction of gravity in the heavy top, density in compressible fluids and the magnetic field in MHD) as well as to geophysical flows. Cendra, Holm, Hoyle, and Marsden [1998a] applied this idea to the Maxwell–Vlasov equations of plasma physics. Cendra, Holm, Marsden, and Ratiu [1998b] showed how Lagrangian semidirect product theory fits into the general framework of Lagrangian reduction.

The semidirect product reduction theorem has been proved in Landsman [1995], Landsman [1998, Chapter 4] as an application of a stages theorem for his *special symplectic reduction method*. Even though special symplectic reduction generalizes Marsden-Weinstein reduction, the special reduction by stages theorem in Landsman [1995] studies a setup that, in general, is different to the ones in the reduction by stages theorems of [HStages].

Singular reduction. Singular reduction starts with the observation of Smale [1970] that we have already mentioned: $z \in P$ is a regular point of a momentum map $\mathbf{J}$ if and only if z has no continuous isotropy. Motivated by this, Arms, Marsden, and Moncrief [1981, 1982] showed that (under hypotheses which include the ellipticity of certain operators and which can be interpreted more or less, as playing the role of a properness assumption on the group action in the finite dimensional case) the level sets $\mathbf{J}^{-1}(0)$ of an equivariant momentum map $\mathbf{J}$ have quadratic singularities at points with continuous symmetry. While such a result is easy to prove for compact group actions on finite dimensional manifolds (using the equivariant Darboux theorem), the main examples of Arms, Marsden, and Moncrief [1981] were, in fact, infinite dimensional—both the phase space and the group. Singular points in the level sets of the momentum map are related to convexity properties of the momentum map in that the singular points in phase space map to corresponding singular points in the the image polytope.

The paper of Otto [1987] showed that if G is a Lie group acting properly on an almost Kähler manifold then the orbit space $\mathbf{J}^{-1}(\mu)/G_\mu$ decomposes into symplectic smooth manifolds constructed out of the orbit types of the G-action on P. In some related work, Huebschmann [1998] has made a careful study of the singularities of moduli spaces of flat connections.

The detailed structure of $\mathbf{J}^{-1}(0)/G$ for compact Lie groups acting on finite dimensional manifolds was determined by Sjamaar and Lerman [1991]; their work was extended to proper Lie group actions and to $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ by Bates and Lerman [1997], with the assumption that $\mathcal{O}_\mu$ be locally closed in $\mathfrak{g}^*$. Ortega [1998] and [HRed] redid the entire singular reduction theory for proper Lie group actions starting with the point reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ and also connected it to the more algebraic approach of Arms, Cushman, and Gotay [1991]. Specific examples of singular reduction, with further references, may be found in Lerman, Montgomery, and Sjamaar [1993] and Cushman and Bates [1997]. One of these, the "canoe" is given in detail in [HStages]. In fact, this is an example of singular reduction in the case of cotangent bundles, and much more can be said in this case; see Perlmutter, Rodríguez-Olmos, and Dias [2006, 2007]. Another approach to singular reduction based on the technique of blowing up singularities, and which was also designed for the case of singular cotangent bundle reduction, was started in Hernandez and Marsden [2005] and Birtea, Puta, Ratiu, and Tudoran [2005], a technique which requires further development.

Singular reduction has been extensively used in the study of the persistence, bifurcation, and stability of relative dynamical elements; see Chossat, Lewis, Ortega, and Ratiu [2003]; Chossat, Ortega, and Ratiu [2002]; Grabsi, Montaldi, and Ortega [2004]; Lerman and Singer [1998]; Lerman and Tokieda [1999]; Ortega [2003]; Ortega and Ratiu [1997, 1999a], Ortega and Ratiu [1999b, 2004b]; Patrick, Roberts, and Wulff [2004]; Roberts and de Sousa Dias [1997]; Roberts, Wulff, and Lamb [2002]; Wulff and Roberts [2002], and Wulff [2003].

Symplectic Reduction Without Momentum Maps. The reduction theory presented so far needs the existence of a momentum map. However, more primitive versions of this procedure based on foliation theory (see Cartan [1922] and Meyer [1973]) do not require the existence of this object. Working in this direction, but with a mathematical program that goes beyond the reduction problem, Condevaux, Dazord, and Molino [1988] introduced a concept that generalizes the momentum map. This object is defined via a connection that associates an additive holonomy group to each canonical action on a symplectic manifold. The existence of the momentum map is equivalent to the vanishing of this group. Symplectic reduction has been carried out using this generalized momentum map in Ortega and Ratiu [2006a] and Ortega and Ratiu [2006b].

Another approach to symplectic reduction that is able to avoid the possible non-existence of the momentum map is based on the optimal momentum map introduced and studied in Ortega and Ratiu [2002], Ortega [2002], and [HStages]. This distribution theoretical approach can also deal with reduction of Poisson manifolds, where the standard momentum map does not exist generically.

Reduction of Other Geometric Structures. Besides symplectic reduction, there are many other geometric structures on which one can perform similar constructions. For example, one can reduce Kähler, hyper-Kähler, Poisson, contact, Jacobi, etc. manifolds and this can be done either in the regular or singular cases. We refer to [HRed] for a survey of the literature for these topics.

The Method of Invariants. This method seeks to parametrize quotient spaces by group invariant functions. It has a rich history going back to Hilbert's invariant theory. It has been of great use in bifurcation with symmetry (see Golubitsky, Stewart, and Schaeffer [1988] for instance). In mechanics, the method was developed by Kummer, Cushman, Rod and coworkers in the 1980's; see, for example, Cushman and Rod [1982]. We will not attempt to give a literature survey here, other than to refer to Kummer [1990], Kirk, Marsden, and Silber [1996], Alber, Luther, Marsden, and Robbins [1998] and the book of Cushman and Bates [1997] for more details and references.

Nonholonomic Systems. Nonholonomic mechanical systems (such as systems with rolling constraints) provide a very interesting class of systems where the reduction procedure has to be modified. In fact this provides a class of systems that gives rise to an almost Poisson structure, i.e. a bracket which does not necessarily satisfy the Jacobi identity. Reduction theory for nonholonomic systems has made a lot of progress, but many interesting questions still remain. In these types of systems, there is a natural notion of a momentum map, but in general it is not conserved, but rather obeys a ***momentum equation*** as was discovered by Bloch, Krishnaprasad, Marsden, and Murray [1996a]. This means, in particular, that point reduction in such a situation may not be appropriate. Nevertheless, Poisson reduction in the almost Poisson and almost symplectic setting is interesting and from the mathematical point of view, point reduction is also interesting, although, as remarked, one has to be cautious with how it is applied to, for example, nonholonomic systems. A few references are Koiller [1992], Bates and Sniatycki [1993], Bloch, Krishnaprasad, Marsden, and Murray [1996a], Koon and Marsden [1998], Blankenstein and Van Der Schaft [2001], Cushman and Śniatycki [2002], Planas-Bielsa [2004], and Ortega and Planas-Bielsa [2004]. We refer to Cendra, Marsden, and Ratiu [2001b] and Bloch [2003] for a more detailed historical review.

Multisymplectic Reduction. Reduction theory is by no means completed. For example, for PDE's, the multisymplectic (as opposed to symplectic) framework seems appropriate, both for relativistic and nonrelativistic systems. In fact, this approach has experienced somewhat of a revival since it has been realized that it is rather useful for numerical computation (see Marsden, Patrick, and Shkoller [1998b]). Only a few instances and examples of multisymplectic and multi-Poisson reduction are really well understood (see Marsden, Montgomery, Morrison, and Thompson [1986]; Castrillón-López, Ratiu, and Shkoller [2000]; Castrillón López, Garcia Pérez, and Ratiu [2001]; Castrillón López and Ratiu [2003]; Castrillón-López and Marsden [2003]), so one can expect to see more activity in this area as well.

Discrete Mechanical Systems. Another emerging area, also motivated by numerical analysis, is that of discrete mechanics. Here the idea is to replace the velocity phase space TQ by $Q \times Q$, with the role of a velocity vector played by a pair of nearby points. This has been a powerful tool for numerical analysis, reproducing standard symplectic integration algorithms and much more. See, for example, Wendlandt and Marsden [1997]; Kane, Marsden, Ortiz, and West [2000]; Marsden and West [2001]; Lew, Marsden, Ortiz, and West [2004] and references therein. This subject, too, has its own reduction theory. See Marsden, Pekarsky, and Shkoller [1999], Bobenko and Suris [1999] and Jalnapurkar, Leok, Marsden, and West [2006]. Discrete mechanics also has some intriguing links with quantization, since Feynman himself first defined path integrals through a limiting process using the sort of discretization used in the discrete action principle (see Feynman and Hibbs [1965]).

6 Cotangent Bundle Reduction

As mentioned earlier, the cotangent bundle reduction theorems are amongst the most basic and useful of the symplectic reduction theorems. Here we only present the *regular versions* of the theorems. Cotangent bundle reduction theorems come in two forms—the *embedding cotangent bundle reduction theorem* and the *bundle cotangent bundle reduction* theorem. We start with a smooth, free, and proper, left action

$$\Phi : G \times Q \to Q$$

of the Lie group G on the configuration manifold Q and lift it to an action on T^*Q. This lifted action is symplectic with respect to the canonical symplectic form on T^*Q, which we denote Ω_{can}, and has an *equivariant* momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ given by

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle ,$$

where $\xi \in \mathfrak{g}$. Letting $\mu \in \mathfrak{g}^*$, the aim of this section is to determine the structure of the symplectic reduced space $((T^*Q)_\mu, \Omega_\mu)$, which, by Theorem 3.3, is a symplectic manifold. We are interested in particular in the question of to what extent $((T^*Q)_\mu, \Omega_\mu)$ is a synthesis of a cotangent bundles and a coadjoint orbit.

Cotangent Bundle Reduction: Embedding Version. In this version of the theorem, we first form the quotient manifold

$$Q_\mu := Q/G_\mu,$$

which we call the ***μ-shape space.*** Since the action of G on Q is smooth, free, and proper, so is the action of the isotropy subgroup G_μ and therefore, Q_μ is a smooth manifold and the canonical projection

$$\pi_{Q,G_\mu} : Q \to Q_\mu$$

is a surjective submersion.

Consider the G_μ-action on Q and its lift to T^*Q. This lifted action is of course also symplectic with respect to the canonical symplectic form Ω_{can} and has an equivariant momentum map $\mathbf{J}^\mu : T^*Q \to \mathfrak{g}_\mu^*$ obtained by restricting $\mathbf{J}$; that is, for $\alpha_q \in T_q^*Q$,

$$\mathbf{J}^\mu(\alpha_q) = \mathbf{J}(\alpha_q)|_{\mathfrak{g}_\mu}.$$

Let $\mu' := \mu|_{\mathfrak{g}_\mu} \in \mathfrak{g}_\mu^*$ be the restriction of μ to $\mathfrak{g}_\mu$. Notice that there is a natural inclusion of submanifolds

$$\mathbf{J}^{-1}(\mu) \subset (\mathbf{J}^\mu)^{-1}(\mu'). \tag{6.1}$$

Since the actions are free and proper, μ and μ' are regular values, so these sets are indeed smooth manifolds. Note that, by construction, μ' is G_μ-invariant.

There will be two key assumptions relevant to the embedding version of cotangent bundle reduction. Namely,

CBR1. *In the above setting, assume there is a* G_μ*-invariant one-form* α_μ *on* Q *with values in* $(\mathbf{J}^\mu)^{-1}(\mu')$.

and the condition (which by (6.1), is a stronger condition)

CBR2. *Assume that* α_μ *in* **CBR1** *takes values in* $\mathbf{J}^{-1}(\mu)$.

For $\xi \in \mathfrak{g}_\mu$ and $q \in Q$, notice that, under the condition **CBR1**,

$$(\mathbf{i}_{\xi_Q}\alpha_\mu)(q) = \langle \mathbf{J}(\alpha_\mu(q)), \xi\rangle = \langle \mu', \xi\rangle,$$

and so $\mathbf{i}_{\xi_Q}\alpha_\mu$ is a constant function on Q . Therefore, for $\xi \in \mathfrak{g}_\mu$,

$$\mathbf{i}_{\xi_Q}\mathbf{d}\alpha_\mu = \pounds_{\xi_Q}\alpha_\mu - \mathbf{d}\mathbf{i}_{\xi_Q}\alpha_\mu = 0, \tag{6.2}$$

since the Lie derivative is zero by G_μ-invariance of α_μ. It follows that

There is a unique two-form β_μ *on* Q_μ *such that*

$$\pi^*_{Q,G_\mu}\beta_\mu = \mathbf{d}\alpha_\mu.$$

Since π_{Q,G_μ} is a submersion, β_μ is closed (it need not be exact). Let

$$B_\mu = \pi^*_{Q_\mu}\beta_\mu,$$

where $\pi_{Q_\mu} : T^*Q_\mu \to Q_\mu$ is (following our general conventions for maps) the cotangent bundle projection. Also, to avoid confusion with the canonical symplectic form $\Omega_{\rm can}$ on T^*Q, we shall denote the canonical symplectic form on T^*Q_μ, the cotangent bundle of μ-shape space, by $\omega_{\rm can}$.

Theorem 6.1 (Cotangent Bundle Reduction—Embedding Version)**.**

(i) *If condition* **CBR1** *holds, then there is a symplectic embedding*

$$\varphi_\mu : ((T^*Q)_\mu, \Omega_\mu) \to (T^*Q_\mu, \omega_{\rm can} - B_\mu), \tag{6.3}$$

onto a submanifold of T^*Q_μ *covering the base* Q/G_μ.

(ii) *The map* φ_μ *in* (i) *gives a symplectic diffeomorphism of* $((T^*Q)_\mu, \Omega_\mu)$ *onto* $(T^*Q_\mu, \omega_{\rm can} - B_\mu)$ *if and only if* $\mathfrak{g} = \mathfrak{g}_\mu$.

(iii) *If* **CBR2** *holds, then the image of* φ_μ *equals the vector subbundle* $[T\pi_{Q,G_\mu}(V)]^\circ$ *of* T^*Q_μ*, where* $V \subset TQ$ *is the vector subbundle consisting of vectors tangent to the* G*-orbits, that is, its fiber at* $q \in Q$ *equals* $V_q = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}$*, and* $^\circ$ *denotes the annihilator relative to the natural duality pairing between* TQ_μ *and* T^*Q_μ.

Remarks.

1. A history of this result can be found in §5.
2. As shown in the appendix on Principal Connections (see Proposition A.2) the required one form α_μ may be constructed satisfying condition **CBR1** from a connection on the μ–shape space bundle $\pi_{Q,G_\mu} : Q \to Q/G_\mu$ and an α_μ satisfying **CBR2** can be constructed using a connection on the shape space bundle $\pi_{Q,G} : Q \to Q/G$.

3. Note that in the case of Abelian reduction, or, more generally, the case in which $G = G_\mu$, the reduced space is symplectically diffeomorphic to $T^*(Q/G)$ with the symplectic structure given by $\Omega_{\rm can} - B_\mu$. In particular, if $\mu = 0$, then the symplectic form on $T^*(Q/G)$ is the canonical one, since in this case one can choose $\alpha_\mu = 0$ which yields $B_\mu = 0$.

4. The term B_μ on T^*Q is usually called a ***magnetic term***, a ***gyroscopic term***, or a ***Coriolis term***. The terminology "magnetic" comes from the Hamiltonian description of a particle of charge e moving according to the Lorentz force law in $\mathbb{R}^3$ under the influence of a magnetic field B. This motion takes place in $T^*\mathbb{R}^3$ but with the nonstandard symplectic structure $\mathbf{d}q^i \wedge \mathbf{d}p_i - \frac{e}{c}B$, $i = 1, 2, 3$, where c is the speed of light and B is regarded as a closed two-form: $B = B_x \mathbf{d}y \wedge \mathbf{d}z - B_y \mathbf{d}x \wedge \mathbf{d}z + B_z \mathbf{d}x \wedge \mathbf{d}y$ (see [MandS], §6.7 for details).

The strategy for proving this theorem is to first deal with the case of reduction at zero and then to treat the general case using a momentum shift.

Reduction at Zero. The reduced space at $\mu = 0$ is, as a set,

$$(T^*Q)_0 = \mathbf{J}^{-1}(0)/G$$

since, for $\mu = 0$, $G_\mu = G$. Notice that in this case, there is no distinction between orbit reduction and symplectic reduction.

Theorem 6.2 (Reduction at Zero)**.** *Assume that the action of G on Q is free and proper, so that the quotient Q/G is a smooth manifold. Then 0 is a regular value of* $\mathbf{J}$ *and there is a symplectic diffeomorphism between $(T^*Q)_0$ and $T^*(Q/G)$ with its canonical symplectic structure.*

The Case $G = G_\mu$. If one is reducing at zero, then clearly $G = G_\mu$. However, this is an important special case of the general cotangent bundle reduction theorem that, for example, includes the case of Abelian reduction. The key assumption here is that $G = G_\mu$, which indeed is always the case if G is Abelian.

Theorem 6.3. *Assume that the action of G on Q is free and proper, so that the quotient Q/G is a smooth manifold. Let $\mu \in \mathfrak{g}^*$, assume that $G = G_\mu$, and assume that* **CBR2** *holds. Then μ is a regular value of* $\mathbf{J}$ *and there is a symplectic diffeomorphism between $(T^*Q)_\mu$ and $T^*(Q/G)$, the latter with the symplectic form $\omega_{\rm can} - B_\mu$; here, $\omega_{\rm can}$ is the canonical symplectic form on $T^*(Q/G)$ and $B_\mu = \pi^*_{Q/G}\beta_\mu$, where the two form β_μ on Q/G is defined by*

$$\pi^*_{Q,G}\beta_\mu = \mathbf{d}\alpha_\mu.$$

Example. Consider the reduction of a general cotangent bundle T^*Q by $G = \mathrm{SO}(3)$. Here $G_\mu \cong S^1$, if $\mu \neq 0$, and so the reduced space is embedded into the cotangent bundle $T^*(Q/S^1)$. A specific example is the case of $Q = \mathrm{SO}(3)$. Then the reduced space $(T^*\,\mathrm{SO}(3))_\mu$ is $S^2_{\|\mu\|}$, the sphere of radius $\|\mu\|$ which is a coadjoint orbit in $\mathfrak{so}(3)^*$. In this case, $Q/G_\mu = \mathrm{SO}(3)/S^1 \cong S^2_{\|\mu\|}$ and the embedding of $S^2_{\|\mu\|}$ into $T^*S^2_{\|\mu\|}$ is the zero section.

Magnetic Terms and Curvature. Using the results of the preceding section, we will now show how one can interpret the magnetic term B_μ as the curvature of a connection on a principal bundle.

We saw in the preamble to the Cotangent Bundle Reduction Theorem 6.1 that $\mathbf{i}_{\xi_Q}\mathbf{d}\alpha_\mu = 0$ for any $\xi \in \mathfrak{g}_\mu$, which was used to drop $\mathbf{d}\alpha_\mu$ to the quotient. In the language of principal bundles, this may be rephrased by saying that $\mathbf{d}\alpha_\mu$ is *horizontal* and thus, once a connection is introduced, *the covariant exterior derivative of* α_μ *coincides with* $\mathbf{d}\alpha_\mu$.

There are two methods to construct a form α_μ with the properties in Theorem 6.1. We continue to work under the general assumption that G acts on Q freely and properly.

First Method. *Construction of* α_μ *from a connection* $\mathcal{A}^\mu \in \Omega^1(Q;\mathfrak{g}_\mu)$ *on the principal bundle* $\pi_{Q,G_\mu} : Q \to Q/G_\mu$.

To carry this out, one shows that the choice

$$\alpha_\mu := \langle \mu', \mathcal{A}^\mu \rangle \in \Omega^1(Q)$$

satisfies the condition **CBR1** in Theorem 6.1, where, as above, $\mu' = \mu|_{\mathfrak{g}_\mu}$. The two-form $\mathbf{d}\alpha_\mu$ may be interpreted in terms of curvature. In fact, one shows that $\mathbf{d}\alpha_\mu$ is the μ'*-component of the curvature two-form*. We summarize these results in the following statement.

Proposition 6.4. *If the principal bundle* $\pi_{Q,G_\mu} : Q \to Q/G_\mu$ *with structure group* G_μ *has a connection* $\mathcal{A}^\mu$*, then* $\alpha_\mu(q)$ *can be taken to equal* $\mathcal{A}^\mu(q)^*\mu'$ *and* B_μ *is induced on* T^*Q_μ *by* $\mathbf{d}\alpha_\mu$ *(a two-form on* Q*), which equals the* μ'*-component of the curvature* $\mathcal{B}^\mu$ *of* $\mathcal{A}^\mu$*.*

Second Method. *Construction of* α_μ *from a connection* $\mathcal{A} \in \Omega^1(Q;\mathfrak{g})$ *on the principal bundle* $\pi_{Q,G} : Q \to Q/G$. One can show that the choice (A.1), that is,

$$\alpha_\mu := \langle \mu, \mathcal{A} \rangle \in \Omega^1(Q)$$

satisfies the condition **CBR2** in Theorem 6.1.

As with the first method, there is an interpretation of the two-form $\mathbf{d}\alpha_\mu$ in terms of curvature as follows.

Proposition 6.5. *If the principal bundle* $\pi_{Q,G} : Q \to Q/G$ *with structure group* G *has a connection* $\mathcal{A}$*, then* $\alpha_\mu(q)$ *can be taken to equal* $\mathcal{A}(q)^*\mu$ *and* B_μ *is the pull back to* T^*Q_μ *of* $\mathbf{d}\alpha_\mu \in \Omega^2(Q)$*, which equals the* μ*-component of the two form* $\mathcal{B} + [\mathcal{A},\mathcal{A}] \in \Omega^2(Q;\mathfrak{g})$*, where* $\mathcal{B}$ *is the curvature of* $\mathcal{A}$*.*

Coadjoint Orbits. We now apply the Cotangent Bundle Reduction Theorem 6.1 to the case $Q = G$ and with the G-action given by *left* translation. The *right* Maurer-Cartan form θ^R is a flat connection associated to this action (see Theorem A.13) and hence

$$\mathbf{d}\alpha_\mu(g)(u_g, v_g) = \left\langle \mu, [\theta^R, \theta^R](g)(u_g, v_g) \right\rangle = \left\langle \mu, [T_gR_{g^{-1}}u_g, T_gR_{g^{-1}}v_g] \right\rangle.$$

Recall from Theorem 4.3 that the reduced space $(T^*G)_\mu$ is the coadjoint orbit $\mathcal{O}_\mu$ endowed with the negative orbit symplectic form ω_μ^- and, according to the Cotangent Bundle Reduction Theorem, it symplectically embeds as the zero section into $(T^*\mathcal{O}_\mu, \omega_{\mathrm{can}} - B_\mu)$, where $B_\mu = \pi_{\mathcal{O}_\mu}^*\beta_\mu$, $\pi_{\mathcal{O}_\mu} : T^*\mathcal{O}_\mu \to \mathcal{O}_\mu$ is the cotangent bundle projection, $\pi_{G,G_\mu}^*\beta_\mu = \mathbf{d}\alpha_\mu$, and $\pi_{G,G_\mu} : G \to \mathcal{O}_\mu$ is given by $\pi_{G,G_\mu}(g) = \mathrm{Ad}_g^*\mu$. The derivative of π_{G,G_μ} is given by

$$T_g\pi_{G,G_\mu}(T_eL_g\xi) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{g\exp(t\xi)}\mu = \mathrm{ad}_\xi^*\,\mathrm{Ad}_g^*\mu$$

for any $\xi \in \mathfrak{g}$. Then a computation shows that $\beta_\mu = -\omega_\mu^-$. Thus, the embedding version of the cotangent bundle reduction theorem produces the following statement which, of course, can be easily checked directly.

Corollary 6.6. *The coadjoint orbit* $(\mathcal{O}_\mu, \omega_\mu^-)$ *symplectically embeds as the zero section into the symplectic manifold* $\left(T^*\mathcal{O}_\mu, \omega_{\mathrm{can}} + \pi_{\mathcal{O}_\mu}^*\omega_\mu^-\right)$.

Cotangent Bundle Reduction: Bundle Version The embedding version of the cotangent bundle reduction theorem presented in the preceding section states that $(T^*Q)_\mu$ *embeds* as a vector subbundle of $T^*(Q/G_\mu)$. The bundle version of this theorem says, roughly speaking, that $(T^*Q)_\mu$ is a *coadjoint orbit bundle* over $T^*(Q/G)$ with fiber the coadjoint orbit $\mathcal{O}$ through μ.

Again we utilize a choice of connection $\mathcal{A}$ on the shape space bundle $\pi_{Q,G} : Q \to Q/G$. A key step in the argument is to utilize orbit reduction and the identification $(T^*Q)_\mu \cong (T^*Q)_\mathcal{O}$.

Theorem 6.7 (Cotangent Bundle Reduction—Bundle Version)**.** *The reduced space* $(T^*Q)_\mu$ *is a locally trivial fiber bundle over* $T^*(Q/G)$ *with typical fiber* $\mathcal{O}$.

This point of view is explored further and the exact nature of the coadjoint orbit bundle is identified and its symplectic structure is elaborated in [HStages].

Poisson Version. This same type of argument as above shows the following, which we state slightly informally.

Theorem 6.8. *The Poisson reduced space* $(T^*Q)/G$ *is diffeomorphic to the coadjoint bundle of* $\pi_{Q,G} : Q \to Q/G$. *This diffeomorphism is implemented by a connection* $\mathcal{A} \in \Omega^1(Q; \mathfrak{g})$. *Thus the fiber of* $(T^*Q)/G \to T^*(Q/G)$ *is isomorphic to the Lie–Poisson space* $\mathfrak{g}^*$.

There is an interesting formula for the Poisson structure on $(T^*Q)/G$ that was originally computed in Montgomery, Marsden, and Ratiu [1984], Montgomery [1986]. Further developments in Cendra, Marsden, Pekarsky, and Ratiu [2003] and Perlmutter and Ratiu [2005] gives a unified study of the Poisson bracket on $(T^*Q)/G$ in both the Sternberg and Weinstein realizations of the quotient. Finally, we refer to, for instance, Lewis, Marsden, Montgomery, and Ratiu [1986] for an application of this result; in this case, the dynamics of fluid systems with free boundaries is studied.

Coadjoint Orbit Bundles. The details of the nature of the bundle and its associated symplectic structure that was sketched in Theorem 6.7 is due to Marsden and Perlmutter [2000]; see also Zaalani [1999], Cushman and Śniatycki [1999], and Perlmutter and Ratiu [2005]. An exposition may be found in [HStages].

7 Future Directions

One of the goals of reduction theory and geometric mechanics is to take the analysis of mechanical systems with symmetries to a deeper level of understanding. But much more needs to be done. As has already been explained, there is still a need to put many classical concepts, such as quasivelocities, into this context, with a resultant strengthening of the theory and its applications. In addition, links with Dirac structures, groupoids and algebroids is under development and should lead to further advances. Finally we mention that while much of this type of work has been applied to field theories (such as electromagnetism and gravity), greater insight is needed for many topics, stress-energy-momentum tensors being one example.

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A Appendix: Principal Connections

In preparation for the next section which gives a brief exposition of the cotangent bundle reduction theorem, we now give a review and summary of facts that we shall need about principal connections. An important thing to keep in mind is that the magnetic terms in the cotangent bundle reduction theorem will appear as the curvature of a connection.

Principal Connections Defined. We consider the following basic set up. Let Q be a manifold and let G be a Lie group acting freely and properly on the left on Q. Let

$$\pi_{Q,G} : Q \to Q/G$$

denote the bundle projection from the configuration manifold Q to ***shape space*** $S = Q/G$. We refer to $\pi_{Q,G} : Q \to Q/G$ as a ***principal bundle***.

One can alternatively use right actions, which is common in the principal bundle literature, but we shall stick with the case of left actions for the main exposition.

Vectors that are infinitesimal generators, namely those of the form $\xi_Q(q)$ are called ***vertical*** since they are sent to zero by the tangent of the projection map $\pi_{Q,G}$.

Definition A.1. *A* ***connection****, also called a* ***principal connection*** *on the bundle* $\pi_{Q,G} : Q \to Q/G$ *is a Lie algebra valued* 1*-form*

$$\mathcal{A} : TQ \to \mathfrak{g}$$

where $\mathfrak{g}$ *denotes the Lie algebra of* G*, with the following properties:*

(i) *the identity* $\mathcal{A}(\xi_Q(q)) = \xi$ *holds for all* $\xi \in \mathfrak{g}$*; that is,* $\mathcal{A}$ *takes infinitesimal generators of a given Lie algebra element to that same element, and*

(ii) *we have* ***equivariance****:* $\mathcal{A}(T_q\Phi_g(v)) = \mathrm{Ad}_g(\mathcal{A}(v))$

for all $v \in T_qQ$*, where* $\Phi_g : Q \to Q$ *denotes the given action for* $g \in G$ *and where* Ad_g *denotes the adjoint action of* G *on* $\mathfrak{g}$*.*

A remark is noteworthy at this point. The equivariance identity for infinitesimal generators noted previously (see (3.7)), namely,

$$T_q\Phi_g\left(\xi_Q(q)\right) = (\mathrm{Ad}_g\,\xi)_Q(g \cdot q),$$

shows that *if the first condition for a connection holds, then the second condition holds automatically on vertical vectors.*

If the G-action on Q is a *right action*, the equivariance condition (ii) in Definition A.1 needs to be changed to $\mathcal{A}(T_q\Phi_g(v)) = \mathrm{Ad}_{g^{-1}}(\mathcal{A}(v))$ for all $g \in G$ and $v \in T_qQ$.

Associated One-Forms. Since $\mathcal{A}$ is a Lie algebra valued 1-form, for each $q \in Q$, we get a linear map $\mathcal{A}(q) : T_qQ \to \mathfrak{g}$ and so we can form its dual $\mathcal{A}(q)^* : \mathfrak{g}^* \to T_q^*Q$. Evaluating this on μ produces an ordinary 1-form:

$$\alpha_\mu(q) = \mathcal{A}(q)^*(\mu). \tag{A.1}$$

This 1-form satisfies two important properties given in the next Proposition.

Proposition A.2. *For any connection* $\mathcal{A}$ *and* $\mu \in \mathfrak{g}^*$*, the corresponding* 1*-form* α_μ *defined by (A.1) takes values in* $\mathbf{J}^{-1}(\mu)$ *and satisfies the following* G*-equivariance property:*

$$\Phi_g^*\alpha_\mu = \alpha_{\mathrm{Ad}_g^*\mu}.$$

Notice in particular, if the group is Abelian or if μ is G-invariant, (for example, if $\mu = 0$), then α_μ is an *invariant* 1*-form.*

Horizontal and Vertical Spaces. Associated with any connection are vertical and horizontal spaces defined as follows.

Definition A.3. *Given the connection* $\mathcal{A}$*, its* ***horizontal space*** *at* $q \in Q$ *is defined by*

$$H_q = \{v_q \in T_qQ \mid \mathcal{A}(v_q) = 0\}$$

and the ***vertical space*** *at* $q \in Q$ *is, as above,*

$$V_q = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}.$$

The map

$$v_q \mapsto \mathrm{ver}_q(v_q) := [\mathcal{A}(q)(v_q)]_Q(q)$$

is called the ***vertical projection****, while the map*

$$v_q \mapsto \mathrm{hor}_q(v_q) := v_q - \mathrm{ver}_q(v_q)$$

is called the ***horizontal projection****.*

Because connections map infinitesimal generators of a Lie algebra elements to that same Lie algebra element, the vertical projection is indeed a projection for each fixed q onto the vertical space and likewise with the horizontal projection.

By construction, we have

$$v_q = \mathrm{ver}_q(v_q) + \mathrm{hor}_q(v_q)$$

and so

$$T_qQ = H_q \oplus V_q$$

and the maps hor_q and ver_q are projections onto these subspaces.

It is sometimes convenient to *define* a connection by the specification of a space H_q declared to be the horizontal space that is complementary to V_q at each point, varies smoothly with q and respects the group action in the sense that $H_{g\cdot q} = T_q\Phi_g(H_q)$. Clearly this alternative definition of a principal connection is equivalent to the definition given above.

Given a point $q \in Q$, the tangent of the projection map $\pi_{Q,G}$ restricted to the horizontal space H_q gives an isomorphism between H_q and $T_{[q]}(Q/G)$. Its inverse $\left[T_q\pi_{Q,G}|_{H_q}\right]^{-1}$: $T_{\pi_{Q,G}(q)}(Q/G) \to H_q$ is called the ***horizontal lift*** to $q \in Q$.

The Mechanical Connection. As an example of defining a connection by the specification of a horizontal space, suppose that the configuration manifold Q is a Riemannian manifold. Of course, the Riemannian structure will often be that defined by the kinetic energy of a given mechanical system.

Thus, assume that Q is a Riemannian manifold, with metric denoted $\langle\!\langle\, , \rangle\!\rangle$ and that G acts freely and properly on Q by isometries, so $\pi_{Q,G} : Q \to Q/G$ is a principal G-bundle.

In this context we may *define the horizontal space at a point simply to be the metric orthogonal to the vertical space*. This therefore defines a connection called the ***mechanical connection***.

Recall from the historical survey in the introduction that this connection was first introduced by Kummer [1981] following motivation from Smale [1970] and [FofM]. See also Guichardet [1984], who applied these ideas in an interesting way to molecular dynamics. The number of references since then making use of the mechanical connection is too large to survey here.

In Proposition A.5 we develop an explicit formula for the associated Lie algebra valued 1-form in terms of an inertia tensor and the momentum map. As a prelude to this formula, we show the following basic link with mechanics. In this context we write the momentum map on TQ simply as $\mathbf{J} : TQ \to \mathfrak{g}^*$.

Proposition A.4. *The horizontal space of the mechanical connection at a point $q \in Q$ consists of the set of vectors $v_q \in T_qQ$ such that $\mathbf{J}(v_q) = 0$.*

For each $q \in Q$, define the ***locked inertia tensor*** $\mathbb{I}(q)$ to be the linear map $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ defined by

$$\langle \mathbb{I}(q)\eta, \zeta\rangle = \langle\!\langle \eta_Q(q), \zeta_Q(q)\rangle\!\rangle \tag{A.2}$$

for any $\eta, \zeta \in \mathfrak{g}$. Since the action is free, $\mathbb{I}(q)$ is nondegenerate, so (A.2) defines an inner product. The terminology "locked inertia tensor" comes from the fact that for coupled rigid or elastic systems, $\mathbb{I}(q)$ is the classical moment of inertia tensor of the rigid body obtained by locking all the joints of the system. In coordinates,

$$I_{ab} = g_{ij}K^i_aK^j_b, \tag{A.3}$$

where $[\xi_Q(q)]^i = K^i_a(q)\xi^a$ define the ***action functions*** K^i_a.

Define the map $\mathcal{A} : TQ \to \mathfrak{g}$ which assigns to each $v_q \in T_qQ$ the corresponding *angular velocity of the locked system*:

$$\mathcal{A}(q)(v_q) = \mathbb{I}(q)^{-1}(\mathbf{J}(v_q)), \tag{A.4}$$

where L is the kinetic energy Lagrangian. In coordinates,

$$\mathcal{A}^a = I^{ab} g_{ij} K_b^i v^j \tag{A.5}$$

since $J_a(q,p) = p_i K_a^i(q)$.

We defined the mechanical connection by declaring its horizontal space to be the metric orthogonal to the vertical space. The next proposition shows that $\mathcal{A}$ is the associated connection one-form.

Proposition A.5. *The $\mathfrak{g}$-valued one-form defined by (A.4) is the mechanical connection on the principal G-bundle $\pi_{Q,G} : Q \to Q/G$.*

Given a general connection $\mathcal{A}$ and an element $\mu \in \mathfrak{g}^*$, we can define the μ-component of $\mathcal{A}$ to be the ordinary one-form α_μ given by

$$\alpha_\mu(q) = \mathcal{A}(q)^*\mu \in T_q^*Q; \quad \text{i.e.,} \quad \langle \alpha_\mu(q), v_q \rangle = \langle \mu, \mathcal{A}(q)(v_q) \rangle$$

for all $v_q \in T_qQ$. Note that α_μ is a G_μ-invariant one-form. It takes values in $\mathbf{J}^{-1}(\mu)$ since for any $\xi \in \mathfrak{g}$, we have

$$\langle \mathbf{J}(\alpha_\mu(q)), \xi \rangle = \langle \alpha_\mu(q), \xi_Q \rangle = \langle \mu, \mathcal{A}(q)(\xi_Q(q)) \rangle = \langle \mu, \xi \rangle .$$

In the Riemannian context, Smale [1970] constructed α_μ by a minimization process. Let $\alpha_q^\sharp \in T_qQ$ be the tangent vector that corresponds to $\alpha_q \in T_q^*Q$ via the metric $\langle\!\langle\,,\rangle\!\rangle$ on Q.

Proposition A.6. *The 1-form $\alpha_\mu(q) = \mathcal{A}(q)^*\mu \in T_q^*Q$ associated with the mechanical connection $\mathcal{A}$ given by (A.4) is characterized by*

$$K(\alpha_\mu(q)) = \inf\{K(\beta_q) \mid \beta_q \in \mathbf{J}^{-1}(\mu) \cap T_q^*Q\}, \tag{A.6}$$

where $K(\beta_q) = \frac{1}{2}\|\beta_q^\sharp\|^2$ is the kinetic energy function on T^*Q. See Figure A.1.

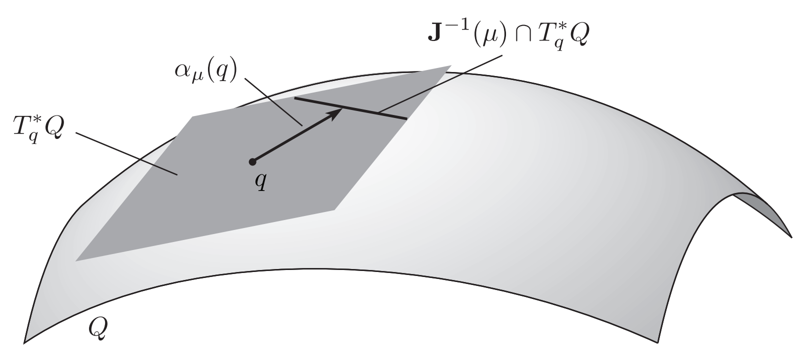

Figure A.1: The extremal characterization of the mechanical connection.

The proof is a direct verification. We do not give here it since this proposition will not be used later in this book. The original approach of Smale [1970] was to take (A.6) as the definition of α_μ. To prove from here that α_μ is a smooth one-form is a nontrivial fact;

see the proof in Smale [1970] or of Proposition 4.4.5 in [FofM]. Thus, one of the merits of the previous proposition is to show easily that this variational definition of α_μ does indeed yield a smooth one-form on Q with the desired properties. Note also that $\alpha_\mu(q)$ lies in the orthogonal space to $T_q^*Q \cap \mathbf{J}^{-1}(\mu)$ in the fiber T_q^*Q relative to the bundle metric on T^*Q defined by the Riemannian metric on Q. It also follows that $\alpha_\mu(q)$ is the unique critical point of the kinetic energy of the bundle metric on T^*Q restricted to the fiber $T_q^*Q \cap \mathbf{J}^{-1}(\mu)$.

Curvature. The curvature $\mathcal{B}$ of a connection $\mathcal{A}$ is defined as follows.

Definition A.7. *The **curvature** of a connection $\mathcal{A}$ is the Lie algebra valued two-form on Q defined by*

$$\mathcal{B}(q)(u_q, v_q) = \mathbf{d}\mathcal{A}(\mathrm{hor}_q(u_q), \mathrm{hor}_q(v_q)), \tag{A.7}$$

where $\mathbf{d}$ is the exterior derivative.

When one replaces vectors in the exterior derivative with their horizontal projections, then the result is called the ***exterior covariant derivative*** and one writes the preceding formula for $\mathcal{B}$ as

$$\mathcal{B} = \mathbf{d}^{\mathcal{A}}\mathcal{A}.$$

For a general Lie algebra valued k-form α on Q, the ***exterior covariant derivative*** is the $k+1$-form $\mathbf{d}^{\mathcal{A}}\alpha$ defined on tangent vectors $v_0, v_1, \ldots, v_k \in T_qQ$ by

$$\mathbf{d}^{\mathcal{A}}\alpha(v_0, v_1, \ldots, v_k) = \mathbf{d}\alpha\Big(\mathrm{hor}_q(v_0), \mathrm{hor}_q(v_1), \ldots, \mathrm{hor}_q(v_k)\Big). \tag{A.8}$$

Here, the symbol $\mathbf{d}^{\mathcal{A}}$ reminds us that it is like the exterior derivative but that it depends on the connection $\mathcal{A}$.

Curvature *measures the lack of integrability of the horizontal distribution* in the following sense.

Proposition A.8. *On two vector fields u, v on Q one has*

$$\mathcal{B}(u, v) = -\mathcal{A}([\mathrm{hor}(u), \mathrm{hor}(v)]).$$

Given a general distribution $\mathcal{D} \subset TQ$ on a manifold Q one can also define its curvature in an analogous way directly in terms of its lack of integrability. Define ***vertical vectors*** at $q \in Q$ to be the quotient space $T_qQ/\mathcal{D}_q$ and define the curvature acting on two ***horizontal vector fields*** u, v (that is, two vector fields that take their values in the distribution) to be the projection onto the quotient of their Jacobi–Lie bracket. One can check that this operation depends only on the point values of the vector fields, so indeed defines a two-form on horizontal vectors.

Cartan Structure Equations. We now derive an important formula for the curvature of a principal connection.

Theorem A.9 (Cartan Structure Equations)**.** *For any vector fields u, v on Q we have*

$$\mathcal{B}(u, v) = \mathbf{d}\mathcal{A}(u, v) - [\mathcal{A}(u), \mathcal{A}(v)], \tag{A.9}$$

where the bracket on the right hand side is the Lie bracket in $\mathfrak{g}$. We write this equation for short as

$$\mathcal{B} = \mathbf{d}\mathcal{A} - [\mathcal{A}, \mathcal{A}].$$

If the G-action on Q is a *right action*, then the Cartan Structure Equations read $\mathcal{B} = \mathbf{d}\mathcal{A} + [\mathcal{A}, \mathcal{A}]$.

The following Corollary shows how the Cartan Structure Equations yield a fundamental equivariance property of the curvature.

Corollary A.10. *For all $g \in G$ we have $\Phi_g^* \mathcal{B} = \mathrm{Ad}_g \circ \mathcal{B}$. If the G-action on Q is on the right, equivariance means $\Phi_g^* \mathcal{B} = \mathrm{Ad}_{g^{-1}} \circ \mathcal{B}$.*

Bianchi Identity. The ***Bianchi Identity***, which states that the exterior covariant derivative of the curvature is zero, is another important consequence of the Cartan Structure Equations.

Corollary A.11. *If $\mathcal{B} = \mathbf{d}^{\mathcal{A}}\mathcal{A} \in \Omega^2(Q; \mathfrak{g})$ is the curvature two-form of the connection $\mathcal{A}$, then the **Bianchi Identity** holds:*

$$\mathbf{d}^{\mathcal{A}}\mathcal{B} = 0.$$

This form of the Bianchi identity is implied by another version, namely

$$\mathbf{d}\mathcal{B} = [\mathcal{B}, \mathcal{A}]^{\wedge},$$

where the bracket on the right hand side is that of Lie algebra valued differential forms, a notion that we do not develop here; see the brief discussion at the end of §9.1 in [MandS]. The proof of the above form of the Bianchi identity can be found in, for example, Kobayashi and Nomizu [1963].

Curvature as a Two-Form on the Base. We now show how the curvature two-form drops to a two-form on the base with values in the adjoint bundle.

The associated bundle to the given left principal bundle $\pi_{Q,G} : Q \to Q/G$ via the adjoint action is called the ***adjoint bundle***. It is defined in the following way. Consider the free proper action $(g, (q, \xi)) \in G \times (Q \times \mathfrak{g}) \mapsto (g \cdot q, \mathrm{Ad}_g \xi) \in Q \times \mathfrak{g}$ and form the quotient $\tilde{\mathfrak{g}} := Q \times_G \mathfrak{g} := (Q \times \mathfrak{g})/G$ which is easily verified to be a vector bundle $\pi_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \to Q/G$, where $\pi_{\tilde{\mathfrak{g}}}(g, \xi) := \pi_{Q,G}(q)$. This vector bundle has an additional structure: it is a ***Lie algebra bundle***; that is, a vector bundle whose fibers are Lie algebras. In this case the bracket is defined pointwise:

$$[\pi_{\tilde{\mathfrak{g}}}(g, \xi), \pi_{\tilde{\mathfrak{g}}}(g, \eta)] := \pi_{\tilde{\mathfrak{g}}}(g, [\xi, \eta])$$

for all $g \in G$ and $\xi, \eta \in \mathfrak{g}$. It is easy to check that this defines a Lie bracket on every fiber and that this operation is smooth as a function of $\pi_{Q,G}(q)$.

The curvature two-form $\mathcal{B} \in \Omega^2(Q; \mathfrak{g})$ (the vector space of $\mathfrak{g}$-valued two-forms on Q) naturally induces a two-form $\overline{\mathcal{B}}$ on the base Q/G with values in $\tilde{\mathfrak{g}}$ by

$$\overline{\mathcal{B}}(\pi_{Q,G}(q)) \left(T_q\pi_{Q,G}(u), T_q\pi_{Q,G}(v)\right) := \pi_{\tilde{\mathfrak{g}}} \left(q, \mathcal{B}(u, v)\right) \tag{A.10}$$

for all $q \in Q$ and $u, v \in T_qQ$. One can check that $\overline{\mathcal{B}}$ is well defined.

Since (A.10) can be equivalently written as $\pi_{Q,G}^* \overline{\mathcal{B}} = \pi_{\tilde{\mathfrak{g}}} \circ (\mathrm{id}_Q \times \mathcal{B})$ and $\pi_{Q,G}$ is a surjective submersion, it follows that $\overline{\mathcal{B}}$ is indeed a smooth two-form on Q/G with values in $\tilde{\mathfrak{g}}$.

Associated Two-Forms. Since $\mathcal{B}$ is a $\mathfrak{g}$-valued two-form, in analogy with (A.1), for every $\mu \in \mathfrak{g}^*$ we can define the ***μ-component*** of $\mathcal{B}$, an ordinary two-form $\mathcal{B}_\mu \in \Omega^2(Q)$ on Q, by

$$\mathcal{B}_\mu(q)(u_q, v_q) := \langle \mu, \mathcal{B}(q)(u_q, v_q) \rangle \tag{A.11}$$

for all $q \in Q$ and $u_q, v_q \in T_qQ$.

The adjoint bundle valued curvature two-form $\overline{\mathcal{B}}$ induces an ordinary two-form on the base Q/G. To obtain it, we consider the dual $\tilde{\mathfrak{g}}^*$ of the adjoint bundle. This is a vector bundle over Q/G which is the associated bundle relative to the coadjoint action of the structure group G of the principal (left) bundle $\pi_{Q,G} : Q \to Q/G$ on $\mathfrak{g}^*$. This vector bundle has additional structure: each of its fibers is a Lie–Poisson space and the associated Poisson tensors on each fiber depend smoothly on the base, that is, $\pi_{\tilde{\mathfrak{g}}^*} : \tilde{\mathfrak{g}}^* \to Q/G$ is a ***Lie–Poisson bundle*** over Q/G.

Given $\mu \in \mathfrak{g}^*$, define the *ordinary two-form* $\overline{\mathcal{B}}_\mu$ *on* Q/G by

$$\begin{aligned}
&\overline{\mathcal{B}}_\mu \left(\pi_{Q,G}(q)\right)\left(T_q\pi_{Q,G}(u_q), T_q\pi_{Q,G}(v_q)\right) \\
&\quad := \left\langle \pi_{\tilde{\mathfrak{g}}^*}(q,\mu), \overline{\mathcal{B}}(\pi_{Q,G}(q))\left(T_q\pi_{Q,G}(u_q), T_q\pi_{Q,G}(v_q)\right)\right\rangle \\
&\quad = \langle \mu, \mathcal{B}(q)(u_q, v_q)\rangle = \mathcal{B}_\mu(q)(u_q, v_q),
\end{aligned} \tag{A.12}$$

where $q \in Q$, $u_q, v_q \in T_qQ$, and in the second equality $\langle\, , \rangle : \tilde{\mathfrak{g}}^* \times \tilde{\mathfrak{g}} \to \mathbb{R}$ is the duality pairing between the coadjoint and adjoint bundles. Since $\overline{\mathcal{B}}$ is well defined and smooth, so is $\overline{\mathcal{B}}_\mu$.

Proposition A.12. *Let $\mathcal{A} \in \Omega^1(Q;\mathfrak{g})$ be a connection one-form on the (left) principal bundle $\pi_{Q,G} : Q \to Q/G$ and $\mathcal{B} \in \Omega^2(Q;\mathfrak{g})$ its curvature two-form on Q. If $\mu \in \mathfrak{g}^*$, the corresponding two-forms $\mathcal{B}_\mu \in \Omega^2(Q)$ and $\overline{\mathcal{B}}_\mu \in \Omega^2(Q/G)$ defined by (A.11) and (A.12), respectively, are related by $\pi^*_{Q,G}\overline{\mathcal{B}}_\mu = \mathcal{B}_\mu$. In addition, $\mathcal{B}_\mu$ satisfies the following G-equivariance property:*

$$\Phi_g^*\mathcal{B}_\mu = \mathcal{B}_{\mathrm{Ad}_g^*\mu}.$$

Thus, if $G = G_\mu$ then $\mathbf{d}\alpha_\mu = \mathcal{B}_\mu = \pi^*_{Q,G}\overline{\mathcal{B}}_\mu$, where $\alpha_\mu(q) = \mathcal{A}(q)^*(\mu)$.

Further relations between α_μ and the μ-component of the curvature will be studied in the next section when discussing the magnetic terms appearing in cotangent bundle reduction.

The Maurer–Cartan Equations. A consequence of the structure equations relates curvature to the process of left and right trivialization and hence to momentum maps.

Theorem A.13 (Maurer–Cartan Equations)**.** *Let G be a Lie group and let $\theta^R : TG \to \mathfrak{g}$ be the map (called the right Maurer–Cartan form) that right translates vectors to the identity:*

$$\theta^R(v_g) = T_gR_{g^{-1}}(v_g).$$

Then

$$\mathbf{d}\theta^R - [\theta^R, \theta^R] = 0.$$

There is a similar result for the left trivialization θ^L, namely the identity

$$\mathbf{d}\theta^L + [\theta^L, \theta^L] = 0.$$

Of course there is much more to this subject, such as the link with classical connection theory, Riemannian geometry, etc. We refer to [HStages] for further basic information and references and to Bloch [2003] for applications to nonholonomic systems, and to Cendra, Marsden, and Ratiu [2001a] for applications to Lagrangian reduction.