Vol. 63 (2009) *REPORTS ON MATHEMATICAL PHYSICS* No. 1

COVARIANT BALANCE LAWS IN CONTINUA WITH MICROSTRUCTURE

ARASH YAVARI*

School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA
(e-mail: arash.yavari@ce.gatech.edu)

and

JERROLD E. MARSDEN†

Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125, USA

(*Received February 4, 2008 – Revised August 21, 2008*)

The purpose of this paper is to extend the Green–Naghdi–Rivlin balance of energy method to continua with microstructure. The key idea is to replace the group of Galilean transformations with the group of diffeomorphisms of the ambient space. A key advantage is that one obtains in a natural way all the needed balance laws on both the macro and micro levels along with two Doyle–Erickson formulas.

We model a structured continuum as a triplet of Riemannian manifolds: a material manifold, the ambient space manifold of material particles and a director field manifold. The Green–Naghdi–Rivlin theorem and its extensions for structured continua are critically reviewed. We show that when the ambient space is Euclidean and when the microstructure manifold is the tangent space of the ambient space manifold, postulating a single balance of energy law and its invariance under time-dependent isometries of the ambient space, one obtains conservation of mass, balances of linear and angular momenta but *not* a separate balance of linear momentum.

We develop a covariant elasticity theory for structured continua by postulating that energy balance is invariant under time-dependent spatial diffeomorphisms of the ambient space, which in this case is the product of two Riemannian manifolds. We then introduce two types of constrained continua in which microstructure manifold is linked to the reference and ambient space manifolds. In the case when at every material point, the microstructure manifold is the tangent space of the ambient space manifold at the image of the material point, we show that the assumption of covariance leads to balances of linear and angular momenta with contributions from both forces and micro-forces along with two Doyle–Ericksen formulas. We show that generalized covariance leads to two balances of linear momentum and a single coupled balance of angular momentum.

Using this theory, we covariantly obtain the balance laws for two specific examples, namely elastic solids with distributed voids and mixtures. Finally, the Lagrangian field theory of structured elasticity is revisited and a connection is made between covariance and Noether's theorem.

Keywords: continuum mechanics, elasticity, generalized continua, couple stress, energy balance.

*Research supported by the Georgia Institute of Technology.
†Research partially supported by the California Institute of Technology and NSF-ITR Grant ACI-0204932.

1. Introduction

The idea of generalized continua goes back to the work of Cosserat brothers [8]. The main idea in generalized continua is to consider extra degrees of freedom for material points in order to be able to better model materials with microstructure in the framework of continuum mechanics. Many developments have been reported since the seminal work of the Cosserat brothers. Depending on the specific choice of kinematics, generalized continua are called polar, micropolar, micromorphic, Cosserat, multipolar, oriented, complex, etc. (see Green and Rivlin [17], Kafadar and Eringen [22], Toupin [35, 36], Mindlin [29] and references therein). The more recent developments can be seen in Capriz [6], Capriz and Mariano [7], de Fabritiis and Mariano [11], Epstein and de Leon [12], Muschik et al. [30], Sławianowski [34] and references therein. For a recent review see Mariano and Stazi [25].

By choosing a specific form for the kinetic energy density of directors, Cowin [9] obtained the balance laws of a Cosserat continuum with three directors by imposing invariance of energy balance under rigid translations and rotations in the current configuration. A similar work was done by Buggisch [4]. Capriz et al. [5] obtained the balance laws for a continuum with the so-called affine microstructure by postulating invariance of balance of energy under time-dependent rigid translations and rotations of the deformed configuration. The main assumption there is that the orthogonal second-order tensor representing the affine microdeformations remains unchanged under a rigid translation but is transformed like a two-point tensor under a rigid rotation in the deformed configuration. Accepting this assumption, one obtains conservation of mass, the standard balance of linear momentum and balance of angular momentum, which in this case states that the sum of Cauchy stress and some new terms is symmetric. Recently, de Fabritiis and Mariano [11] conducted an interesting study of the geometric structure of complex continua and studied different geometric aspects of continua with microstructure. Capriz and Mariano [7] studied the Lagrangian field theory of Coserrat continua and obtained the Euler–Lagrange equations for standard and microstructure deformation mappings. However, in their Lagrangian density they did not consider an explicit dependence on the metric of the order-parameter manifold. In this paper, we will consider an explicit dependence of the Lagrangian density on metrics of both standard and microstructure manifolds. One should remember that the original developments in the theory of generalized continua in the Sixties were variational [35, 36]. However, revisiting the Lagrangian field theory of structured continua in the language of modern geometric mechanics may be worthwhile.

It is believed that kinematics of a structured continuum can be described by two independent maps, one mapping material points to their current positions and one mapping the material points to their directors [27]. Looking at the literature one can see that for a Cosserat continuum (and even for multipolar continua [16, 17]), the only balance laws are the standard balances of linear and angular momenta; couple stresses do not enter into balance of linear momentum but do enter into balance of angular momentum and make the Cauchy stress unsymmetric. This is indeed different from the situation in the so-called complex continua or continua with microstructure [6, 7, 11], where one sees separate balance laws for microstresses. Marsden and

Hughes [27] postulated two balances of linear momenta. However, it is not clear why, in general, one should see two balances of linear momentum and only one balance of angular momentum. In other words, why do standard and microstructure forces interact only in the balance of angular momentum? It should be noted that in all the existing generalizations of Green–Naghdi–Rivlin (GNR) Theorem (see Green and Rivlin [16]) to generalized continua the standard Galilei group $\mathcal{G}$ is considered. It is always assumed that rigid translations leave the micro-kinematical variables and their corresponding forces unchanged (with no rigorous justification) and these quantities come into play only when rigid rotations are considered.

It is known that the traditional formulation of balance laws of continuum mechanics are not intrinsically meaningful and heavily depend on the linear structure of Euclidean space. Marsden and Hughes [27] resolved this shortcoming of the traditional formulation by postulating a balance of energy, which is intrinsically defined even on manifolds, and its invariance under spatial changes of frame. This results in conservation of mass, balance of linear and angular momenta and the Doyle–Ericksen formula. Similar ideas had been proposed in Green and Rivlin [16] for deriving balance laws by postulating energy balance invariance under Galilean transformations. For more details and discussions on material changes of frame see Yavari et al. [39]. See also Yavari [40], Yavari and Ozakin [41], and Yavari and Marsden [42] for similar discussions. A natural question to ask is whether it is possible to develop covariant theories of elasticity for structured continua. As we will see shortly, the answer is affirmative.

Similar to Noether's theorem that makes a connection between conserved quantities and symmetries of a Lagrangian density, GNR theorem makes a connection between balance laws and invariance properties of balance of energy. One major difference between the two theorems is that in GNR theorem one looks at balance of energy for a finite subbody, i.e., a global quantity, and its invariance, while in Noether's theorem symmetries are local properties of the Lagrangian density.

In some applications, e.g., recent applications of continuum mechanics to biology, one may need to enlarge the configuration manifold of the continuum to take into account the fact that changes in material points, e.g., rearrangements of microstructure, etc., should somehow be considered in the continuum theory, at least in an average sense. This was a motivation for various developments for generalized continuum theories in the last few decades. In a structured continuum, in addition to the standard deformation mapping, one introduces some extra fields that represent the underlying microstructure. In the nondissipative case, assuming the existence of a Lagrangian density that depends on all the fields, using Hamilton's principle of least action one obtains new Euler–Lagrange equations corresponding to microstructural fields [35, 36, 7]. However, to our best knowledge, it is not clear in the literature how one can obtain these extra balance laws by postulating a single energy balance and its invariance under some groups of transformations. This is the main motivation of the present work.

To summarize, looking at the literature of generalized continua, one sees that the structure of balance laws is not completely clear. It is observed that there is always a standard balance of linear momentum with only macro-quantities and a balance of angular momentum, which has contributions from both macro- and micro-forces. In some treatments there is no balance of micro-linear momentum (see Toupin [35, 36], Capriz et al. [5], Ericksen [13]) while sometimes there is one, as in Green and Naghdi [19], Capriz [6]. In particular, we can mention the work of Leslie [23] on liquid crystals in which he starts by postulating a balance of energy and a linear momentum balance for micro-forces. In his work, he realizes that the balance of micro-linear momentum cannot be obtained from invariance of energy balance. To date, there have been several works on relating balance laws of structured continua to invariance of energy balance under some group of transformations. These efforts will be reviewed in detail in the sequel.

This paper is organized as follows. In Section 2 geometry of continua with microstructure is discussed. Section 3 discusses the previous efforts in generalizing Green–Naghdi–Rivlin Theorem for generalized continua. Assuming that the ambient space is Euclidean and assuming that the microstructure manifold at every material point is the tangent space of $\mathbb{R}^3$ at the spatial image of the material point, we generalize GNR theorem. Section 4 develops a covariant theory of elasticity for those structured continua for which microstructure manifold is completely independent of the ambient space manifold in the sense that ambient space and microstructure manifolds can have separate changes of frame. We then develop a covariant theory of elasticity for those structured continua in which microstructure manifold is somewhat linked to the ambient space manifold. In particular, we study the case where microstructure manifold is the tangent bundle of the ambient space manifold. We also introduce a generalized notion of covariance in which one postulates energy balance invariance under two diffeomorphisms that act separately on micro and macro quantities simultaneously. We study consequences of this generalized covariance. In Section 5, we look at two concrete examples of structured continua, namely elastic solids with distributed voids and mixtures. In both cases, we obtain the balance laws covariantly. Section 6 presents a Lagrangian field theory formulation of structured continua. Noether's theorem and its connection with covariance is also investigated. Concluding remarks are given in Section 7.

2. Geometry of continua with microstructure

A structured continuum is a generalization of a standard continuum in which the internal structure of the material points is taken into account by assigning to them some independent internal variables or order parameters. For the sake of simplicity, let us assume that each material point $\mathbf{X}$ has a corresponding microstructure (director) field $\mathbf{p}$, which lies in a Riemannian manifold $(\mathcal{M}, \mathbf{g}_{\mathcal{M}})$. Note that $\mathbf{p}$, in general, could be a tensor field. In general, one may have a collection of director fields and the microstructure manifold may not be Riemannian. However, these assumptions are general enough to cover many problems of interest. In this case our structured continuum has a configuration manifold that consists of a pair of mappings $(\varphi_t, \widetilde{\varphi}_t)$ [27, 11], where $\mathbf{x} = \varphi_t(\mathbf{X})$ represents the standard motion and $\mathbf{p} = \widetilde{\varphi}_t(\mathbf{X})$ is the motion of the microstructure. Both φ_t and $\widetilde{\varphi}_t$ are understood as fields. As in the

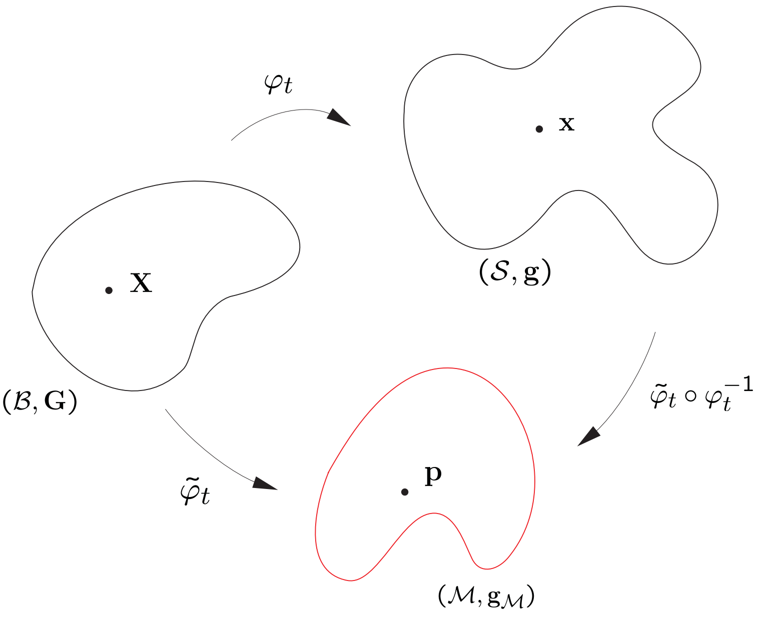

Fig. 2.1. Deformation mappings of a continuum with microstructure.

geometric treatment of standard continua, the current configuration lies in an embedding space $\mathcal{S}$, which is a Riemannian manifold with a metric $\mathbf{g}$. Note that ambient space for the structured continuum is $\overline{\mathcal{S}} = \mathcal{S} \times \mathcal{M}$ and for every $\mathbf{X} \in \mathcal{B}$, $\widetilde{\varphi}(\mathbf{X})$ lies in a separate copy of $\mathcal{M}$. Here, we have assumed that the structured continuum is microstructurally homogeneous in the sense that directors of two material points $\mathbf{X}_1$ and $\mathbf{X}_2$ lie in two copies of the same microstructure manifold $\mathcal{M}$ (see Fig. 2.1).

More precisely, kinematics of a structured continuum is described using fiber bundles (see, for instance, Epstein and de Leon [12]). Deformation of a structured continuum is a bundle map from the zero section of the trivial bundle $\mathcal{B} \times \mathcal{M}_0$ (for some manifold $\mathcal{M}_0$) to the trivial bundle $\mathcal{S} \times \mathcal{M}$ (see Fig. 2.2). Corresponding to the two maps φ_t and $\widetilde{\varphi}_t$, there are two velocities, which have the following material forms,

$$\mathbf{V}(\mathbf{X}, t) = \frac{\partial \varphi_t(\mathbf{X})}{\partial t} \in T_{\mathbf{x}}\mathcal{S}, \qquad \widetilde{\mathbf{V}}(\mathbf{X}, t) = \frac{\partial \widetilde{\varphi}_t(\mathbf{X})}{\partial t} \in T_{\mathbf{p}}\mathcal{M}. \tag{2.1}$$

Let us choose local coordinates $\{X^A\}, \{x^a\}$, and $\{p^\alpha\}$ on $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{M}$, respectively. In these coordinates

$$\mathbf{V}(\mathbf{X}, t) = V^a \mathbf{e}_a, \qquad \widetilde{\mathbf{V}}(\mathbf{X}, t) = \widetilde{V}^\alpha \ \widetilde{\mathbf{e}}_\alpha, \tag{2.2}$$

where $\{\mathbf{e}_a\}$ and $\{\widetilde{\mathbf{e}}_\alpha\}$ are bases for $T_{\mathbf{x}}\mathcal{S}$ and $T_{\mathbf{p}}\mathcal{M}$, respectively, and

$$V^a = \frac{\partial \varphi^a}{\partial t}, \qquad \widetilde{V}^\alpha = \frac{\partial \widetilde{\varphi}^\alpha}{\partial t}. \tag{2.3}$$

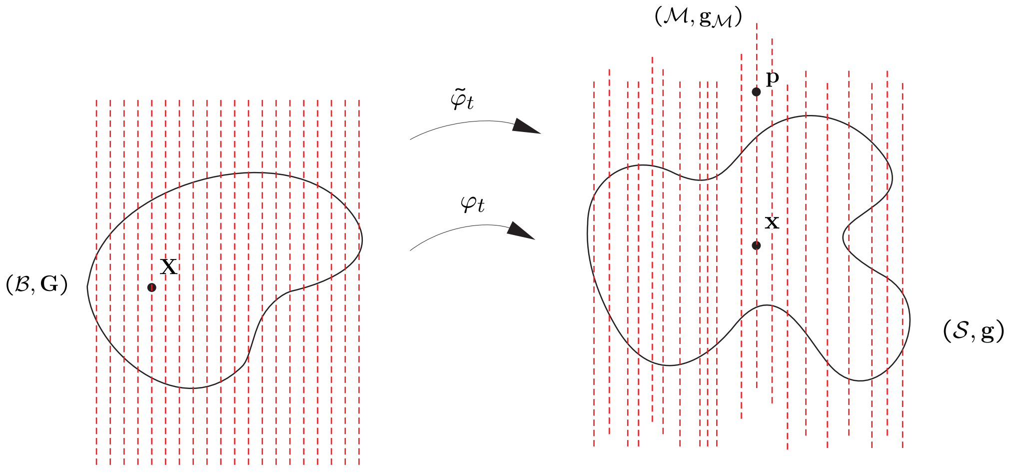

Fig. 2.2. Deformation of a continuum with microstructure can be understood as a bundle map between two trivial bundles. Here all is needed is the zero-section of the reference bundle, i.e. the material manifold.

In spatial coordinates

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V} \circ \varphi_t^{-1}, \qquad \widetilde{\mathbf{v}}(\mathbf{x}, t) = \widetilde{\mathbf{V}} \circ \varphi_t^{-1}. \tag{2.4}$$

In a local coordinate chart

$$\mathbf{v}(\mathbf{x}, t) = v^a \mathbf{e}_a, \qquad \widetilde{\mathbf{v}}(\mathbf{x}, t) = \widetilde{v}^\alpha \ \widetilde{\mathbf{e}}_\alpha. \tag{2.5}$$

Here, for the sake of simplicity, we have assumed that our structured continuum has one director field, which is assumed to be a vector field. As was mentioned earlier, this is not the most general possibility and in general one may need to work with several director fields or even with a tensor-valued director field. Generalization to these cases is straightforward.

Marsden and Hughes [27] chose the classical viewpoint in taking $\mathbb{R}^3$ to be the ambient space for material particles and postulated the integral form of balances of linear and angular momenta. The more natural approach would be to start from balance of energy and look at consequences of its invariance under some transformations. This is the approach we choose in this paper. Note that the two maps φ_t and $\widetilde{\varphi}_t$, in general, are independent and interact only in the balance of energy, i.e. power has contributions from both deformation maps. The other important observation is that balance of energy is written on an arbitrary subset $\varphi_t(\mathcal{U}) \subset \mathcal{S}$.

3. The Green–Naghdi–Rivlin Theorem for a continuum with microstructure

In most theories of generalized continua, macro and micro-forces enter the same balance of angular momentum because the ambient space manifold and the manifold of microstructure are somewhat related. Now the important question is the following:

how can one obtain two sets of balance of linear momentum, one for micro-forces and one for marco-forces in such cases starting from first principles? Of course, one can always postulate as many balance laws as one needs in a theory. However, a fundamental understanding of balance laws is crucial in any theory. Accepting a Lagrangian viewpoint, one has two sets of Euler–Lagrange equations as there are two independent macro and micro kinematic variables (see Toupin [35, 36], de Fabritiis and Mariano [11]). Then, assuming that these equations are satisfied, Noether's theorem leads us to expect that any conserved quantity of the system corresponds to some symmetry of the Lagrangian density. The Lagrangian density can be invariant under groups of transformations that act on the ambient and microstructure manifolds simultaneously. For example, if one assumes that an arbitrary element of $SO(3)$ acts simultaneously on $\mathcal{S}$ and $\mathcal{M}$ and Lagrangian density remains invariant, then the conserved quantity is nothing but angular momentum with some extra terms representing the effect of microstructure. However, another possibility would be a symmetry in which an arbitrary element of $SO(3)$ acts only on $\mathcal{M}$. Now one may ask why the Lagrangian density should be invariant under simultaneous actions of $SO(3)$ on $\mathcal{S}$ and $\mathcal{M}$.

A way out of this difficulty may be to look for a generalization of the Green–Naghdi–Rivlin theorem for continua with microstructure. There have been several attempts in the literature to generalize this theorem. In all the existing generalizations, it is assumed that in a Galilean transformation, micro-forces and micro-displacements remain unchanged under a rigid translation while under a rigid rotation both micro and macro quantities transform. Postulating invariance of balance of energy under an arbitrary element of the Galilean group and accepting this assumption, one obtains conservation of mass, the standard balance of linear momentum and balance of angular momentum with some extra terms that represent the effect of microstruture. However, this does not give a micro-linear momentum balance. So, it is seen that the link between energy balance invariance and balance of micro-linear momentum is missing.

It should be noted that in most of the treatments of continua with microstructure, the microstructure manifold $\mathcal{M}$ may not be completely independent of the ambient space manifold $\mathcal{S}$ and this may be a key point in understanding the structure of balance laws. From a geometric point of view this means that spatial and microstructure changes of frame may not be independent, in general.

There have been several attempts in the literature to obtain balance laws of generalized continua by energy invariance arguments. Capriz et al. [5] start from balance of energy and postulate its invariance under rigid translations and rotations of the current configuration. They assume that microstructure quantities (kinematic and kinetic) remain unchanged under rigid translations while under rigid rotations micro-forces transform exactly like their macro counterparts. This somehow implies that the microstructure manifold is not independent of the standard ambient space. Under a rigid translation, each microstructure manifold (fiber) translates rigidly and hence micro-forces and directors remain unchanged. Under a rigid rotation directors and their corresponding micro-forces transform exactly like their macro counterparts because rotating a representative volume element its director goes through the same rotation. This invariance postulate results in the standard conservation of mass and balance of linear and angular momenta. Balance of linear momentum has its standard form while balance of angular momentum has contributions from both forces and micro-forces. However, this invariance argument does not lead to a separate balance of micro-linear momentum.

Gurtin and Podio-Guidugli [21] introduce a fine structure for each material point. They then postulate two balances of energy, one in the macro scale and one in the fine scale. The fine structure is characterized by the limit $\epsilon \to 0$ of some scale parameter ϵ. Postulating invariance of these two balance laws under rigid translations and rotations they obtain two sets of balance of linear and angular momenta. They emphasize that balance of micro-angular momentum only introduces a micro-couple and offers nothing essential.

Green and Naghdi [19] and Green and Naghdi [20] start from balance of energy and assume that it is invariant under the transformation $\mathbf{v} \to \mathbf{v} + \mathbf{c}$, where $\mathbf{v}$ is the spatial velocity field and $\mathbf{c}$ is an arbitrary constant vector field. This gives the conservation of mass and balance of linear momentum. Then they obtain a local form for balance of energy and assume it remains invariant under rigid translations and rotations. In the case of a Cosserat continuum they assume invariance of energy balance under $\mathbf{v} \to \mathbf{v} + \mathbf{c}_1$ and $\mathbf{w} \to \mathbf{w} + \mathbf{c}_2$, where $\mathbf{w}$ is the spatial microstructure velocity field and $\mathbf{c}_1$ and $\mathbf{c}_2$ are arbitrary constant vectors. However, it is not clear what it means to replace $\mathbf{w}$ by $\mathbf{w} + \mathbf{c}_2$ in terms of transformations of the ambient space and microstructure manifolds. In other words, what group of transformations lead to this replacement and why they should not affect the macro-velocity field. This seems to be more or less an assumption convenient for obtaining the desired balance laws. This assumption leads to conservation of mass and balance of macro and micro-linear momenta. Then, again they postulate invariance of local balance of energy under rigid translations and rotations that transform micro and macro forces simultaneously. This gives a local form for balance of angular momentum.

The Green–Naghdi–Rivlin Theorem for structured continua in Euclidean space. Let us now study the consequences of postulating invariance of energy balance under time-dependent isomorphisms of the ambient Euclidean space with constant velocity for a structured continuum. Consider balance of energy for $\varphi_t(\mathcal{U}) \subset \varphi_t(\mathcal{B})$ that reads

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho \left(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) dv = \int_{\varphi_t(\mathcal{U})} \rho \left(\mathbf{b}\cdot\mathbf{v} + \widetilde{\mathbf{b}}\cdot\widetilde{\mathbf{v}} + r\right) dv + \int_{\partial\varphi_t(\mathcal{U})} \left(\mathbf{t}\cdot\mathbf{v} + \widetilde{\mathbf{t}}\cdot\widetilde{\mathbf{v}} + h\right) da, \tag{3.1}$$

where for the sake of simplicity, we have ignored the microstructure inertia. Here e is the internal energy density, $\mathbf{b}$ is the body force per unit of mass in the deformed configuration, $\widetilde{\mathbf{b}}$ is the micro-body force per unit of mass in the deformed configuration, r is heat supply per unit mass of the deformed configuration, $\mathbf{t}$ is traction, $\widetilde{\mathbf{t}}$ is micro-traction, and h is the heat flux. Let us assume that the ambient space is Euclidean, i.e., $\mathcal{S}=\mathbb{R}^3$. Consider a rigid translation of the ambient space of the form

$$\mathbf{x}'=\xi_t(\mathbf{x})=\mathbf{x}+(t-t_0)\mathbf{c}, \tag{3.2}$$

where $\mathbf{c}$ is a constant vector field on $\mathcal{S}=\mathbb{R}^3$. Let us also assume that the director field is a vector field on $\mathbb{R}^3$. We know that for any $\mathbf{x}\in\mathbb{R}^3$, $T_{\mathbf{x}}\mathbb{R}^3$ can be identified with $\mathbb{R}^3$ itself. So, we assume that for $\mathbf{x}=\varphi_t(\mathbf{X})\in\mathbb{R}^3$, $\mathcal{M}_{\varphi_t(\mathbf{X})}=T_{\mathbf{x}}\mathbb{R}^3\simeq\mathbb{R}^3$. Note that for a rigid translation of the ambient space

$$T\xi_t=\mathrm{id}, \tag{3.3}$$

where id is the identity map. Therefore, a rigid translation does not affect the microstructure quantities. Assuming invariance of balance of energy under arbitrary rigid translations implies the existence of Cauchy stress and the usual conservation of mass and balance of energy, i.e.

$$\dot{\rho}+\rho\,\mathrm{div}\,\mathbf{v}=0, \tag{3.4}$$

$$\mathrm{div}\,\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\mathbf{a}. \tag{3.5}$$

Next, let us consider a rigid rotation of $\mathcal{S}=\mathbb{R}^3$ of the form

$$\mathbf{x}'=\xi_t(\mathbf{x})=e^{\Omega(t-t_0)}\mathbf{x}, \tag{3.6}$$

where Ω is a skew-symmetric matrix. Note that

$$T\xi_t=e^{\Omega(t-t_0)},\qquad TT\xi_t=0. \tag{3.7}$$

We know that

$$\mathbf{p}'=\xi_{t*}\mathbf{p}=T\xi_t\cdot\mathbf{p}. \tag{3.8}$$

Thus

$$\widetilde{\mathbf{V}}'=\frac{\partial}{\partial t}\Big|_{\mathbf{X}}\mathbf{p}'=\Omega e^{\Omega(t-t_0)}\mathbf{p}+e^{\Omega(t-t_0)}\frac{\partial}{\partial t}\Big|_{\mathbf{X}}\mathbf{p}. \tag{3.9}$$

This means that at $t=t_0$

$$\widetilde{\mathbf{V}}'=\widetilde{\mathbf{V}}+\Omega\mathbf{p}. \tag{3.10}$$

Subtracting balance of energy for $\varphi_t(\mathcal{U})$ from that of $\varphi_t'(\mathcal{U})$ at $t=t_0$, we obtain

$$\int_{\varphi_t(\mathcal{U})}\rho\mathbf{a}\cdot\Omega\mathbf{x}\,dv=\int_{\varphi_t(\mathcal{U})}\rho\mathbf{b}\cdot\Omega\mathbf{x}\,dv+\int_{\partial\varphi_t(\mathcal{U})}\mathbf{t}\cdot\Omega\mathbf{x}\,da+\int_{\varphi_t(\mathcal{U})}\rho\widetilde{\mathbf{b}}\cdot\Omega\mathbf{p}\,dv+\int_{\partial\varphi_t(\mathcal{U})}\widetilde{\mathbf{t}}\cdot\Omega\mathbf{p}\,da. \tag{3.11}$$

We know that

$$\int_{\partial\varphi_t(\mathcal{U})}\mathbf{t}\cdot\Omega\mathbf{x}\,da=\int_{\varphi_t(\mathcal{U})}(\mathrm{div}\,\boldsymbol{\sigma}\cdot\Omega\mathbf{x}+\boldsymbol{\sigma}:\Omega)\,dv, \tag{3.12}$$

$$\int_{\partial\varphi_t(\mathcal{U})}\widetilde{\mathbf{t}}\cdot\Omega\mathbf{p}\,da=\int_{\varphi_t(\mathcal{U})}\left[\mathrm{div}\,\widetilde{\boldsymbol{\sigma}}\otimes\mathbf{p}+\widetilde{\boldsymbol{\sigma}}\cdot\nabla\mathbf{p}\right]:\Omega\,dv. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11) and using the local form of balance of linear momentum, we obtain

$$\int_{\varphi_t(\mathcal{U})} \left[\boldsymbol{\sigma} + \operatorname{div} \tilde{\boldsymbol{\sigma}} \otimes \mathbf{p} + \tilde{\boldsymbol{\sigma}} \cdot \nabla \mathbf{p}\right] : \Omega \, dv = 0. \tag{3.14}$$

Because $\mathcal{U}$ is arbitrary, we conclude that

$$\left[\boldsymbol{\sigma} + \operatorname{div}(\tilde{\boldsymbol{\sigma}} \otimes \mathbf{p})\right]^{\mathsf{T}} = \boldsymbol{\sigma} + \operatorname{div}(\tilde{\boldsymbol{\sigma}} \otimes \mathbf{p}). \tag{3.15}$$

In components this reads as follows:

$$\sigma^{ab} + \tilde{\sigma}^{ac}{}_{,c} \; p^b + \tilde{\sigma}^{ac} p^b{}_{,c} = \kappa^{ab} = \kappa^{ba}. \tag{3.16}$$

It is seen that the rigid structure of $\mathbb{R}^3$ and its isometries does not allow one to obtain a separate balance of microstructure linear momentum. We will show in the sequel that when the ambient space is $\mathbb{R}^3$ or, more generally a Riemannian manifold, a generalized covariance can give us such a separate balance of microstructure linear momentum. We will also see that for a structured continuum with a scalar microstructure field, e.g., an elastic solid with distributed voids, one can covariantly obtain a separate scalar balance of micro-linear momentum.

4. A covariant theory of elasticity for structured continua with free microstructure manifold

In this section we develop a covariant theory of elasticity for those structured continua for which one can change the spatial and microstructure frames independently. An example of such continua is a continuum with voids or a continuum with distributed "damage", which will be studied in detail in Section 5. Let us first review some important concepts from geometric continuum mechanics.

The reference configuration $\mathcal{B}$ is a submanifold of the reference configuration manifold $(\mathfrak{B}, \mathbf{G})$, which is a Riemannian manifold. Motion is thought of as an embedding $\varphi_t : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{S}, \mathbf{g})$ is the ambient space manifold. An element $d\mathbf{X} \in T_{\mathbf{X}}\mathcal{B}$ is mapped to $d\mathbf{x} \in T_{\mathbf{x}}\mathcal{S}$ by the deformation gradient

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}. \tag{4.1}$$

The length of $d\mathbf{x}$ is geometrically important as it represents the effect of deformation. Note that

$$\langle\!\langle d\mathbf{x}, d\mathbf{x} \rangle\!\rangle_{\mathbf{g}} = \langle\!\langle d\mathbf{X}, d\mathbf{X} \rangle\!\rangle_{\varphi_t^* \mathbf{g}} \, . \tag{4.2}$$

In this sense the pulled-back metric $\mathbf{C} = \varphi_t^* \mathbf{g}$ is a measure of deformation. The material free energy density has the following form,

$$\Psi = \Psi\left(\mathbf{X}, \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \varphi_t\right). \tag{4.3}$$

Let us define the spatial free energy density as

$$\psi(t, \mathbf{x}, \mathbf{g}) = \Psi\left(\varphi_t^{-1}, \mathbf{F} \circ \varphi_t^{-1}, \mathbf{G} \circ \varphi_t^{-1}, \mathbf{g}\right). \tag{4.4}$$

Similarly, internal energy density has the following form

$$e = e(t, \mathbf{x}, \mathbf{g}). \tag{4.5}$$

This means that fixing a deformation mapping φ_t, internal energy density explicitly depends on time, current position of the material point and the metric tensor at the current position of the material point. Note also that e is supported on $\varphi_t(\mathcal{B})$, i.e. $e = 0$ in $\mathcal{S} \setminus \varphi_t(\mathcal{B})$.

Now let us look at internal energy density for an elastic body with substructure in which free energy density has the following form

$$\Psi = \Psi\left(\mathbf{X}, \mathbf{F}, \widetilde{\varphi}_t, \widetilde{\mathbf{F}}, \mathbf{G}, \mathbf{g} \circ \varphi_t, \mathbf{g}_{\mathcal{M}} \circ \widetilde{\varphi}_t\right). \tag{4.6}$$

For a given deformation mapping $(\varphi_t, \widetilde{\varphi}_t)$ define

$$\begin{aligned} &\psi(t, \mathbf{x}, \mathbf{g}, \mathbf{p}, \widetilde{\mathbf{g}}_{\mathcal{M}}) \\ &\quad = \Psi\left(\varphi_t^{-1}, \mathbf{F} \circ \varphi_t^{-1}, \widetilde{\varphi}_t \circ \varphi_t^{-1}, \widetilde{\mathbf{F}} \circ \varphi_t^{-1}, \mathbf{G} \circ \varphi_t^{-1}, \mathbf{g}, \mathbf{p} \circ \varphi_t^{-1}, \mathbf{g}_{\mathcal{M}} \circ \widetilde{\varphi}_t \circ \varphi_t^{-1}\right), \end{aligned} \tag{4.7}$$

where $\widetilde{\mathbf{g}}_{\mathcal{M}} = \mathbf{g}_{\mathcal{M}} \circ \widetilde{\varphi} \circ \varphi_t^{-1}$. Similarly, internal energy density has the following form

$$e = e(t, \mathbf{x}, \mathbf{g}, \mathbf{p}, \widetilde{\mathbf{g}}_{\mathcal{M}}). \tag{4.8}$$

Balance of energy for $\varphi_t(\mathcal{U}) \subset \mathcal{S}$ is written as

$$\begin{aligned} &\frac{d}{dt} \int_{\varphi_t(\mathcal{U})} \rho(\mathbf{x}, t) \left[e(t, \mathbf{x}, \mathbf{g}, \mathbf{p}, \widetilde{\mathbf{g}}_{\mathcal{M}}) + \frac{1}{2} \langle\langle \mathbf{v}, \mathbf{v} \rangle\rangle_{\mathbf{g}} + \kappa(\mathbf{p}, \widetilde{\mathbf{v}}) \right] \\ &\quad = \int_{\varphi_t(\mathcal{U})} \rho(\mathbf{x}, t) \left(\langle\langle \mathbf{b}, \mathbf{v} \rangle\rangle_{\mathbf{g}} + \langle\langle \widetilde{\mathbf{b}}, \widetilde{\mathbf{v}} \rangle\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + r \right) + \int_{\partial\varphi_t(\mathcal{U})} \left(\langle\langle \mathbf{t}, \mathbf{v} \rangle\rangle_{\mathbf{g}} + \langle\langle \widetilde{\mathbf{t}}, \widetilde{\mathbf{v}} \rangle\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + h \right) da, \end{aligned} \tag{4.9}$$

where we think of $\rho(\mathbf{x}, t)$ as a 3-form and $\widetilde{\mathbf{b}}$ and $\widetilde{\mathbf{t}}$ are microstructure body force and traction vector fields, respectively. For the sake of simplicity, let us assume that the microstructure kinetic energy has the following form

$$\kappa(\mathbf{p}, \widetilde{\mathbf{v}}) = \frac{1}{2} j \,\langle\langle \widetilde{\mathbf{v}}, \widetilde{\mathbf{v}} \rangle\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}}, \tag{4.10}$$

where we assume the microstructure inertia j is a scalar.

All the physical processes happen in $\mathcal{S}$ and thus balance of energy is written on subsets of $\varphi_t(\mathcal{B}) \subset \mathcal{S}$. Standard traction is a vector field on $\mathcal{S}$ and the microstructure traction is a vector field on $\mathcal{M}$. The standard and microstructure tractions have the following coordinate representations

$$\mathbf{t}(\mathbf{x}, t) = t^a \mathbf{e}_a, \qquad \widetilde{\mathbf{t}}(\mathbf{x}, t) = \widetilde{t}^{\alpha} \widetilde{\mathbf{e}}_{\alpha}, \tag{4.11}$$

where $\{\mathbf{e}_a\}$ and $\{\widetilde{\mathbf{e}}_a\}$ are bases for $T_{\mathbf{x}}\mathcal{S}$ and $T_{\mathbf{p}}\mathcal{M}$, respectively. Similarly, the stress tensors have the following local representations

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \sigma^{ab} \; \mathbf{e}_a \otimes \mathbf{e}_b, \qquad \widetilde{\boldsymbol{\sigma}}(\mathbf{x}, t) = \widetilde{\sigma}^{\alpha b} \; \widetilde{\mathbf{e}}_{\alpha} \otimes \mathbf{e}_b. \tag{4.12}$$

The first Piola Kirchhoff stresses for the standard deformation and the microstructure deformation are obtained by the following Piola transformations

$$P^{aA} = J(\mathbf{F}^{-1})^A{}_b \ \sigma^{ab}, \qquad \widetilde{P}^{\alpha A} = J(\mathbf{F}^{-1})^A{}_b \ \widetilde{\sigma}^{\alpha b}, \tag{4.13}$$

where $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \ \det \mathbf{F}$. These transformations ensure that

$$\mathbf{t}\, da = \mathbf{T}\, dA \qquad \text{and} \qquad \widetilde{\mathbf{t}}\, da = \widetilde{\mathbf{T}}\, dA. \tag{4.14}$$

Now this means that in terms of contributions of tractions to balance of energy we have

$$\langle\langle \mathbf{t}, \mathbf{v} \rangle\rangle_{\mathbf{g}}\, da = \langle\langle \mathbf{T}, \mathbf{V} \rangle\rangle_{\mathbf{g}}\, dA \qquad \text{and} \qquad \left\langle\left\langle \widetilde{\mathbf{t}}, \widetilde{\mathbf{v}} \right\rangle\right\rangle_{\mathbf{g}_{\mathcal{M}}} da = \left\langle\left\langle \widetilde{\mathbf{T}}, \widetilde{\mathbf{V}} \right\rangle\right\rangle_{\mathbf{g}_{\mathcal{M}}} dA. \tag{4.15}$$

For $\mathcal{U} \subset \mathcal{B}$, material energy balance can be written as

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0(\mathbf{X},t) \left[E(t,\mathbf{X},\mathbf{g},\mathbf{g}_{\mathcal{M}}) + \frac{1}{2} \langle\langle \mathbf{V}, \mathbf{V} \rangle\rangle_{\mathbf{g}} + \frac{1}{2} J \left\langle\left\langle \widetilde{\mathbf{V}}, \widetilde{\mathbf{V}} \right\rangle\right\rangle_{\mathbf{g}_{\mathcal{M}}} \right]$$
$$= \int_{\mathcal{U}} \rho_0(\mathbf{X},t) \left(\langle\langle \mathbf{B}, \mathbf{V} \rangle\rangle_{\mathbf{g}} + \left\langle\left\langle \widetilde{\mathbf{B}}, \widetilde{\mathbf{V}} \right\rangle\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + R \right) + \int_{\partial\mathcal{U}} \left(\langle\langle \mathbf{T}, \mathbf{V} \rangle\rangle_{\mathbf{g}} + \left\langle\left\langle \widetilde{\mathbf{T}}, \widetilde{\mathbf{V}} \right\rangle\right\rangle_{\mathbf{g}_{\mathcal{M}}} + H \right) dA, \tag{4.16}$$

where again ρ_0 is a 3-form.

4.1. Covariance of energy balance

Let us assume that for each $\mathbf{x} \in \mathcal{S}$, the microstructure manifold is completely independent of $\mathcal{S}$. In other words, a change of frame in $\mathcal{S}$(or $\mathcal{M}$) does not affect $\mathcal{M}$(or $\mathcal{S}$) and quantities defined on it. An example of a structured continuum with this type of microstructure manifold is a structured continuum with a scalar director field, although there are other possibilities. We show in this subsection that postulating energy balance and its invariance under time-dependent changes of frame in $\mathcal{S}$ and $\mathcal{M}$ results in conservation of mass and micro-inertia, two balances of linear and angular momenta, and two Doyle–Ericksen formulas, one for the Cauchy stress and one for the micro-Cauchy stress.

THEOREM 4.1. *If balance of energy holds and if it is invariant under arbitrary spatial and microstructure diffeomorhisms $\xi_t : \mathcal{S} \to \mathcal{S}$ and $\eta_t : \mathcal{M} \to \mathcal{M}$, then there exist second-order tensors $\boldsymbol{\sigma}$ and $\widetilde{\boldsymbol{\sigma}}$ such that*

$$\mathbf{t} = \langle\langle \boldsymbol{\sigma}, \mathbf{n} \rangle\rangle_{\mathbf{g}} \qquad \textit{and} \qquad \widetilde{\mathbf{t}} = \langle\langle \widetilde{\boldsymbol{\sigma}}, \mathbf{n} \rangle\rangle_{\mathbf{g}}, \tag{4.17}$$

and

$$\mathbf{L}_{\mathbf{v}}\rho = 0, \tag{4.18}$$

$$\mathbf{L}_{\mathbf{v}} j = 0, \tag{4.19}$$

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}, \tag{4.20}$$

$$\operatorname{div} \widetilde{\boldsymbol{\sigma}} + \rho \widetilde{\mathbf{b}} = \rho j \widetilde{\mathbf{a}}, \tag{4.21}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \tag{4.22}$$

$$(\mathbf{F}_0\widetilde{\boldsymbol{\sigma}})^T = \mathbf{F}_0\widetilde{\boldsymbol{\sigma}}, \tag{4.23}$$

$$2\rho\frac{\partial e}{\partial \mathbf{g}} = \boldsymbol{\sigma}, \tag{4.24}$$

$$\mathbf{F}_0\widetilde{\boldsymbol{\sigma}} = 2\rho\frac{\partial e}{\partial \widetilde{\mathbf{g}}_{\mathcal{M}}}, \tag{4.25}$$

where div *is divergence with respect to the metric* $\mathbf{g}$, $\mathbf{F}_0 = \widetilde{\mathbf{F}}\mathbf{F}^{-1}$ *and* η_t *acts on all the microstructure fibers simultaneously.*

Proof: Let us consider spatial and microstructure diffeomorphisms separately.

Microstructure covariance of energy balance. Consider a microstructure diffeomorphism $\eta_t : \mathcal{M} \to \mathcal{M}$ (see Fig. 4.1) and assume that

$$\eta_t\Big|_{t=t_0} = \mathrm{id}. \tag{4.26}$$

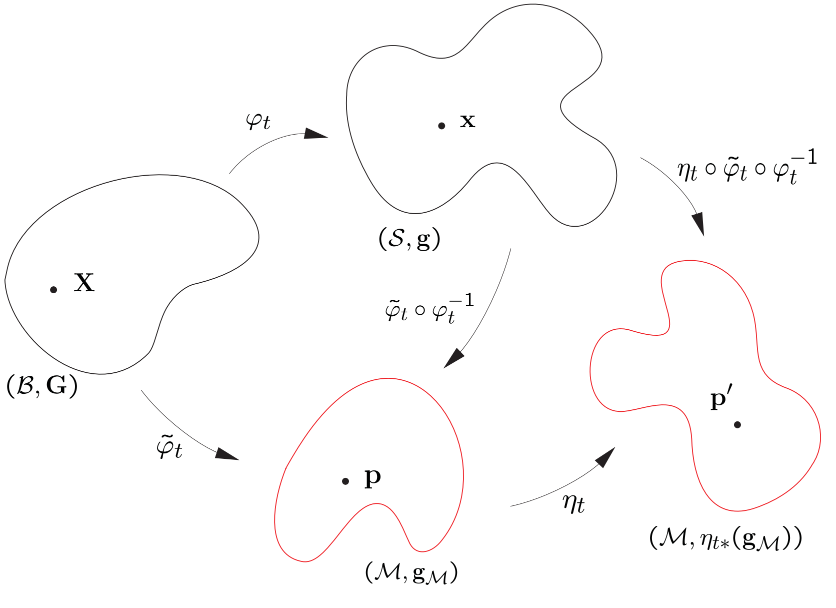

Fig. 4.1. A microstructure change of frame.

Invariance of energy balance under $\eta_t : \mathcal{M} \to \mathcal{M}$ means that balance of energy in the new frame has the following form

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho(\mathbf{x}, t)\left[e'(t, \mathbf{x}, \mathbf{g}, \mathbf{p}', \widetilde{\mathbf{g}}_{\mathcal{M}}) + \frac{1}{2}\langle\langle \mathbf{v}, \mathbf{v}\rangle\rangle_{\mathbf{g}} + \frac{1}{2}j'\left\langle\!\left\langle \widetilde{\mathbf{v}}', \widetilde{\mathbf{v}}'\right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}}\right]$$
$$= \int_{\varphi_t(\mathcal{U})} \rho(\mathbf{x}, t)\left(\langle\langle \mathbf{b}, \mathbf{v}\rangle\rangle_{\mathbf{g}} + \left\langle\!\left\langle \widetilde{\mathbf{b}}', \widetilde{\mathbf{v}}'\right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + r\right) + \int_{\partial\varphi_t(\mathcal{U})}\left(\langle\langle \mathbf{t}, \mathbf{v}\rangle\rangle_{\mathbf{g}} + \left\langle\!\left\langle \widetilde{\mathbf{t}}', \widetilde{\mathbf{v}}'\right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + h\right) da. \tag{4.27}$$

Note that

$$e'(t, \mathbf{x}, \mathbf{g}, \mathbf{p}', \widetilde{\mathbf{g}}_{\mathcal{M}}) = e(t, \mathbf{x}, \mathbf{g}, \mathbf{p}, \eta_t^* \widetilde{\mathbf{g}}_{\mathcal{M}}). \tag{4.28}$$

Thus

$$\frac{d}{dt}\Big|_{t=t_0} = \dot{e} + \frac{\partial e}{\partial \widetilde{\mathbf{g}}_{\mathcal{M}}} : \mathfrak{L}_{\mathbf{z}} \widetilde{\mathbf{g}}_{\mathcal{M}}, \tag{4.29}$$

where

$$\mathbf{z} = \frac{\partial}{\partial t}\Big|_{t=t_0} \eta_t. \tag{4.30}$$

Note also that

$$\widetilde{\mathbf{v}}'\Big|_{t=t_0} = \widetilde{\mathbf{v}} + \mathbf{z}. \tag{4.31}$$

Assuming that $\widetilde{\mathbf{b}}' - j'\widetilde{\mathbf{a}}' = \eta_{t*}(\widetilde{\mathbf{b}} - j\widetilde{\mathbf{a}})$, at $t = t_0$ we obtain

$$\begin{aligned}
&\int_{\varphi_t(\mathcal{U})} \mathbf{L}_{\mathbf{v}} \rho \left(e + \langle\langle \mathbf{v}, \mathbf{v} \rangle\rangle_{\mathbf{g}} + \frac{1}{2} j \, \langle\langle \widetilde{\mathbf{v}} + \mathbf{z}, \widetilde{\mathbf{v}} + \mathbf{z} \rangle\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} \right) \\
&\quad + \int_{\varphi_t(\mathcal{U})} \rho \left(\dot{e} + \frac{\partial e}{\partial \widetilde{\mathbf{g}}_{\mathcal{M}}} : \mathfrak{L}_{\mathbf{z}} \widetilde{\mathbf{g}}_{\mathcal{M}} + j \, \langle\langle \widetilde{\mathbf{a}}, \mathbf{z} \rangle\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + \frac{1}{2} \mathbf{L}_{\mathbf{v}} j \, \langle\langle \widetilde{\mathbf{v}} + \mathbf{z}, \widetilde{\mathbf{v}} + \mathbf{z} \rangle\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} \right) \\
&= \int_{\varphi_t(\mathcal{U})} \rho \left(\langle\langle \mathbf{b}, \mathbf{v} \rangle\rangle_{\mathbf{g}} + \left\langle\!\left\langle \widetilde{\mathbf{b}}, \widetilde{\mathbf{v}} + \mathbf{z} \right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + r \right) + \int_{\partial\varphi_t(\mathcal{U})} \left(\langle\langle \mathbf{t}, \mathbf{v} \rangle\rangle_{\mathbf{g}} + \left\langle\!\left\langle \widetilde{\mathbf{t}}, \widetilde{\mathbf{v}} + \mathbf{z} \right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + h \right) da.
\end{aligned} \tag{4.32}$$

Replacing ρ by ρdv and subtracting balance of energy (4.9) from the above identity and considering the fact that $\mathbf{z}$ and $\mathcal{U}$ are arbitrary, one obtains

$$\mathbf{L}_{\mathbf{v}}(\rho j) = 0, \tag{4.33}$$

$$\int_{\varphi_t(\mathcal{U})} \rho \frac{\partial e}{\partial \widetilde{\mathbf{g}}_{\mathcal{M}}} : \mathfrak{L}_{\mathbf{z}} \widetilde{\mathbf{g}}_{\mathcal{M}} \, dv = \int_{\varphi_t(\mathcal{U})} \rho \left\langle\!\left\langle \widetilde{\mathbf{b}}, \mathbf{z} \right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} dv + \int_{\partial\varphi_t(\mathcal{U})} \left\langle\!\left\langle \widetilde{\mathbf{t}}, \mathbf{z} \right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} da. \tag{4.34}$$

Applying Cauchy's theorem (see Marsden and Hughes [27]) to (4.34), one concludes that there exists a second-order tensor $\widetilde{\boldsymbol{\sigma}}$ such that

$$\widetilde{\mathbf{t}} = \langle\langle \widetilde{\boldsymbol{\sigma}}, \mathbf{n} \rangle\rangle_{\mathbf{g}}. \tag{4.35}$$

Now let us simplify the surface integral.

LEMMA 4.2. *The contribution of microstructure traction has the following simplified form*

$$\int_{\partial\varphi_t(\mathcal{U})} \left\langle\!\left\langle \widetilde{\mathbf{t}}, \mathbf{z} \right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} da = \int_{\varphi_t(\mathcal{U})} \left[\langle\langle \operatorname{div} \widetilde{\boldsymbol{\sigma}}, \mathbf{z} \rangle\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + \mathbf{F}_0 \widetilde{\boldsymbol{\sigma}} : \frac{1}{2} \mathfrak{L}_{\mathbf{z}} \widetilde{\mathbf{g}}_{\mathcal{M}} + \mathbf{F}_0 \widetilde{\boldsymbol{\sigma}} : \boldsymbol{\omega}_{\mathcal{M}} \right] dv. \tag{4.36}$$

Proof:

$$\int_{\partial\varphi_t(\mathcal{U})} \left\langle\!\left\langle \widetilde{\mathbf{t}}, \mathbf{z} \right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} = \int_{\partial\varphi_t(\mathcal{U})} \sigma^{\alpha b} n^c g_{bc} z^{\beta} (g_{\mathcal{M}})_{\alpha\beta} \, da = \int_{\varphi_t(\mathcal{U})} \left[\sigma^{\alpha b} z^{\beta} (g_{\mathcal{M}})_{\alpha\beta} \right]_{|b} dv. \tag{4.37}$$

But because $(g_{\mathcal{M}})_{\alpha\beta|b} = (g_{\mathcal{M}})_{\alpha\beta|\gamma}(F_0)^{\gamma}{}_b = 0$, we have

$$\left[\sigma^{\alpha b} z^{\beta} (g_{\mathcal{M}})_{\alpha\beta}\right]_{|b} = \left[\sigma^{\alpha b} z^{\beta}\right]_{|b} (g_{\mathcal{M}})_{\alpha\beta} = \sigma^{\alpha b}{}_{|b} z^{\beta} (g_{\mathcal{M}})_{\alpha\beta} + z^{\beta}{}_{|b} \sigma^{\alpha b} (g_{\mathcal{M}})_{\alpha\beta}. \tag{4.38}$$

Note that

$$z^{\beta}{}_{|b} (g_{\mathcal{M}})_{\alpha\beta} = z_{\alpha|\gamma} \, (F_0)^{\lambda}{}_b. \tag{4.39}$$

□

Now, because $\mathbf{z}$ and $\mathcal{U}$ are arbitrary from (4.34) one obtains

$$\mathbf{F}_0 \widetilde{\boldsymbol{\sigma}} = 2\rho \frac{\partial e}{\partial \widetilde{\mathbf{g}}_{\mathcal{M}}}, \tag{4.40}$$

$$(\mathbf{F}_0 \widetilde{\boldsymbol{\sigma}})^{\mathsf{T}} = \mathbf{F}_0 \widetilde{\boldsymbol{\sigma}}, \tag{4.41}$$

$$\operatorname{div} \widetilde{\boldsymbol{\sigma}} + \rho \widetilde{\mathbf{b}} = \rho j \widetilde{\mathbf{a}}. \tag{4.42}$$

Spatial covariance of energy balance. Invariance of energy balance under an arbitrary diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$ means that (see Fig. 4.2)

$$\begin{aligned} \frac{d}{dt} \int_{\varphi'_t(\mathcal{U})} \rho'(\mathbf{x}', t) \left[e'(t, \mathbf{x}', \mathbf{g}, \mathbf{g}_{\mathcal{M}}) + \frac{1}{2} \langle\!\langle \mathbf{v}', \mathbf{v}' \rangle\!\rangle_{\mathbf{g}} + \frac{1}{2} j' \langle\!\langle \widetilde{\mathbf{v}}', \widetilde{\mathbf{v}}' \rangle\!\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} \right] \\ = \int_{\varphi'_t(\mathcal{U})} \rho'(\mathbf{x}', t) \left(\langle\!\langle \mathbf{b}', \mathbf{v}' \rangle\!\rangle_{\mathbf{g}} + \langle\!\langle \widetilde{\mathbf{b}}', \widetilde{\mathbf{v}}' \rangle\!\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + r' \right) \\ + \int_{\partial\varphi'_t(\mathcal{U})} \left(\langle\!\langle \mathbf{t}', \mathbf{v}' \rangle\!\rangle_{\mathbf{g}} + \langle\!\langle \widetilde{\mathbf{t}}', \widetilde{\mathbf{v}}' \rangle\!\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + h' \right) da', \end{aligned} \tag{4.43}$$

where $\varphi'_t = \xi_t \circ \varphi_t$. We also assume that

$$\xi_t \big|_{t=t_0} = \mathrm{id}. \tag{4.44}$$

The relation between primed and unprimed quantities are dictated by Cartan's spacetime theory, i.e.,

$$\rho'(\mathbf{x}', t) = \xi_* \rho(\mathbf{x}, t), \quad \mathbf{t}' = \xi_* \mathbf{t}, \quad \widetilde{\mathbf{t}}' = \xi_* \widetilde{\mathbf{t}}, \quad r'(\mathbf{x}', t) = r(\mathbf{x}, t), \quad h'(\mathbf{x}', t) = h(\mathbf{x}, t). \tag{4.45}$$

The internal energy density has the following transformation

$$e'(t, \mathbf{x}', \mathbf{g}, \widetilde{\mathbf{g}}_{\mathcal{M}}) = e(t, \mathbf{x}, \xi^* \mathbf{g}, \mathbf{p}, \widetilde{\mathbf{g}}_{\mathcal{M}}). \tag{4.46}$$

Thus

$$\frac{d}{dt}\bigg|_{t=t_0} e' = \dot{e} + \frac{\partial e}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{w}} \mathbf{g}, \tag{4.47}$$

where

$$\mathbf{w} = \frac{\partial}{\partial t}\bigg|_{t=t_0} \xi_t. \tag{4.48}$$

Spatial velocity has the following transformation

$$\mathbf{v}' = \xi_* \mathbf{v} + \mathbf{w}_t. \tag{4.49}$$

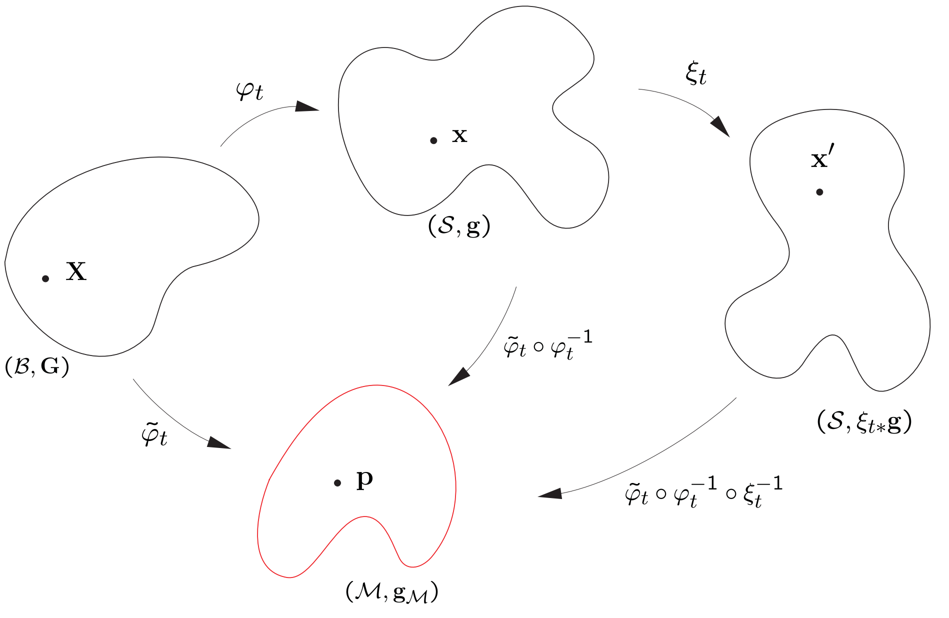

Fig. 4.2. A spatial change of frame in a continuum with microstructure.

Thus, at $t=t_0$, $\mathbf{v}'=\mathbf{v}+\mathbf{w}$. Also

$$\widetilde{\mathbf{v}}'=\widetilde{\mathbf{V}}\circ\varphi_t^{-1}\circ\xi_t^{-1}=\widetilde{\mathbf{v}}\circ\xi_t^{-1}. \tag{4.50}$$

Therefore, at $t=t_0$

$$\widetilde{\mathbf{v}}'=\widetilde{\mathbf{v}}. \tag{4.51}$$

Assuming that $\mathbf{b}'-\mathbf{a}'=\xi_{t*}(\mathbf{b}-\mathbf{a})$ [27] and noting that $\widetilde{\mathbf{b}}'-\widetilde{\mathbf{a}}'=\widetilde{\mathbf{b}}-\widetilde{\mathbf{a}}$, balance of energy in the new frame at $t=t_0$ reads

$$\begin{aligned}
&\int_{\varphi_t(\mathcal{U})}\mathbf{L}_{\mathbf{v}}\rho\left(e+\frac{1}{2}\left\langle\!\left\langle\mathbf{v}+\mathbf{w},\mathbf{v}+\mathbf{w}\right\rangle\!\right\rangle_{\mathbf{g}}+\frac{1}{2}j\left\langle\!\left\langle\widetilde{\mathbf{v}},\widetilde{\mathbf{v}}\right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}}\right)\\
&\quad+\int_{\varphi_t(\mathcal{U})}\rho\left(\dot{e}+\frac{\partial e}{\partial\mathbf{g}}:\mathfrak{L}_{\mathbf{w}}\mathbf{g}+\left\langle\!\left\langle\mathbf{v}+\mathbf{w},\mathbf{a}\right\rangle\!\right\rangle_{\mathbf{g}}+j\left\langle\!\left\langle\widetilde{\mathbf{v}},\widetilde{\mathbf{a}}\right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}}+\frac{1}{2}\mathbf{L}_{\mathbf{v}}j\left\langle\!\left\langle\widetilde{\mathbf{v}},\widetilde{\mathbf{v}}\right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}}\right)\\
&\qquad=\int_{\varphi_t(\mathcal{U})}\rho\left(\left\langle\!\left\langle\mathbf{b},\mathbf{v}+\mathbf{w}\right\rangle\!\right\rangle_{\mathbf{g}}+\left\langle\!\left\langle\widetilde{\mathbf{b}},\widetilde{\mathbf{v}}\right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}}+r\right)\\
&\qquad\quad+\int_{\partial\varphi_t(\mathcal{U})}\left(\left\langle\!\left\langle\mathbf{t},\mathbf{v}+\mathbf{w}\right\rangle\!\right\rangle_{\mathbf{g}}+\left\langle\!\left\langle\widetilde{\mathbf{t}},\widetilde{\mathbf{v}}\right\rangle\!\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}}+h\right)da.
\end{aligned} \tag{4.52}$$

Subtracting (4.9) from (4.52) and considering the fact that $\mathbf{w}$ and $\mathcal{U}$ are arbitrary, we obtain conservation of mass $\mathbf{L}_{\mathbf{v}}\rho=0$ and using it in (4.33) we obtain balance of microstructure inertia,

$$\mathbf{L}_{\mathbf{v}}j=0. \tag{4.53}$$

Now using conservation of mass and microstructure inertia, and replacing ρ by ρdv in (4.52), one obtains

$$\int_{\varphi_t(\mathcal{U})} \rho \left(\frac{\partial e}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{w}} \mathbf{g} + \langle\langle \mathbf{w}, \mathbf{a} \rangle\rangle_{\mathbf{g}} \right) dv = \int_{\varphi_t(\mathcal{U})} \rho \left(\langle\langle \mathbf{b}, \mathbf{w} \rangle\rangle_{\mathbf{g}} \right) dv + \int_{\partial\varphi_t(\mathcal{U})} \left(\langle\langle \mathbf{t}, \mathbf{w} \rangle\rangle_{\mathbf{g}} \right) da. \tag{4.54}$$

Applying Cauchy's theorem to the above identity and considering (4.35) shows that there exists a second-order tensor $\boldsymbol{\sigma}$ such that

$$\mathbf{t} = \langle\langle \boldsymbol{\sigma}, \mathbf{n} \rangle\rangle_{\mathbf{g}} . \tag{4.55}$$

Now let us look at the surface integral in (4.54). This surface integral is simplified to read

$$\int_{\partial\varphi_t(\mathcal{U})} \langle\langle \mathbf{t}, \mathbf{w} \rangle\rangle_{\mathbf{g}} \, da = \int_{\varphi_t(\mathcal{U})} \langle\langle \operatorname{div} \boldsymbol{\sigma}, \mathbf{w} \rangle\rangle_{\mathbf{g}} \, dv + \int_{\varphi_t(\mathcal{U})} \left(\boldsymbol{\sigma} : \frac{1}{2} \mathfrak{L}_{\mathbf{w}} \mathbf{g} + \boldsymbol{\sigma} : \boldsymbol{\omega} \right) dv, \tag{4.56}$$

where $\boldsymbol{\omega}$ has the coordinate representation $\omega_{ab} = \frac{1}{2}(w_{a|b} - w_{b|a})$. Substituting (4.56) into (4.54) yields

$$\int_{\varphi_t(\mathcal{U})} \left(2\rho \frac{\partial e}{\partial \mathbf{g}} - \boldsymbol{\sigma} \right) : \frac{1}{2} \mathfrak{L}_{\mathbf{w}} \mathbf{g} \, dv + \int_{\varphi_t(\mathcal{U})} \boldsymbol{\sigma} : \boldsymbol{\omega} \, dv - \int_{\varphi_t(\mathcal{U})} \langle\langle \operatorname{div} \boldsymbol{\sigma} + \rho \left(\mathbf{b} - \mathbf{a} \right), \mathbf{w} \rangle\rangle_{\mathbf{g}} \, dv = 0. \tag{4.57}$$

Because $\mathcal{U}$ and $\mathbf{w}$ are arbitrary we conclude that

$$2\rho \frac{\partial e}{\partial \mathbf{g}} = \boldsymbol{\sigma}, \tag{4.58}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}, \tag{4.59}$$

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}. \tag{4.60}$$

□

Next, we study the effect of material diffeomorphisms on balance of energy.

4.2. Transformation of energy balance under material diffeomorphisms

It was shown in Yavari et al. [39] that, in general, energy balance cannot be invariant under diffeomorphisms of the reference configuration and what one should be looking for instead is the way in which energy balance transforms under material diffeomorphisms. In this subsection we first obtain such a transformation formula for a continuum with microstructure under an arbitrary time-dependent material diffeomorphism (see Eq. (4.99)) and then obtain the conditions under which balance of energy can be materially covariant.

The material energy balance transformation formula. Let us begin with a discussion of how energy balance transforms under material diffeomorphisms. Let us define

$$E(t,\mathbf{X},\mathbf{G}) = E\left(\mathbf{X},\mathbf{F}(\mathbf{X}),\widetilde{\varphi}_t(\mathbf{X}),\widetilde{\mathbf{F}}(\mathbf{X}),\mathbf{g}(\varphi_t(\mathbf{X})),\mathbf{g}_{\mathcal{M}}(\widetilde{\varphi}_t(\mathbf{X})),\mathbf{G}\right), \tag{4.61}$$

where E is the material internal energy density per unit of undeformed mass. Material (Lagrangian) energy balance (4.16) can be simplified to read

$$\int_{\mathcal{U}} \frac{d}{dt}\left[\rho_0\left(E(t,\mathbf{X},\mathbf{G}) + \frac{1}{2}\langle\langle \mathbf{V},\mathbf{V}\rangle\rangle_{\mathbf{g}} + \frac{1}{2}J\left\langle\left\langle\widetilde{\mathbf{V}},\widetilde{\mathbf{V}}\right\rangle\right\rangle_{\mathbf{g}_{\mathcal{M}}}\right)\right]$$
$$= \int_{\mathcal{U}}\rho_0\left(\langle\langle\mathbf{B},\mathbf{V}\rangle\rangle_{\mathbf{g}} + \left\langle\left\langle\widetilde{\mathbf{B}},\widetilde{\mathbf{V}}\right\rangle\right\rangle_{\widetilde{\mathbf{g}}_{\mathcal{M}}} + R\right) + \int_{\partial\mathcal{U}}\left(\langle\langle\mathbf{T},\mathbf{V}\rangle\rangle_{\mathbf{g}} + \left\langle\left\langle\widetilde{\mathbf{T}},\widetilde{\mathbf{V}}\right\rangle\right\rangle_{\mathbf{g}_{\mathcal{M}}} + H\right)dA, \tag{4.62}$$

where $\mathcal{U}$ is an arbitrary nice subset of the reference configuration $\mathcal{B}$, $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ are body force and microstructure body force, respectively, per unit undeformed mass, $\mathbf{V}(\mathbf{X},t)$ and $\widetilde{\mathbf{V}}(\mathbf{X},t)$ are the material velocity and microstructure material velocity, respectively, $\rho_0(\mathbf{X},t)$ is the material density, $R(\mathbf{X},t)$ is the heat supply per unit undeformed mass, and $H(\mathbf{X},t,\hat{\mathbf{N}})$ is the heat flux across a surface with normal $\hat{\mathbf{N}}$ in the undeformed configuration (normal to $\partial\mathcal{U}$ at $\mathbf{X}\in\partial\mathcal{U}$).

Change of reference frame. A material change of frame is a diffeomorphism

$$\Xi_t : (\mathfrak{B},\mathbf{G}) \to (\mathfrak{B},\mathbf{G}'). \tag{4.63}$$

A change of frame can be thought of as a change of coordinates in the reference configuration (passive definition) or a rearrangement of microstructure (active definition). Under such a framing, a nice subset $\mathcal{U}$ is mapped to another nice subset $\mathcal{U}' = \Xi_t(\mathcal{U})$ and a material point $\mathbf{X}$ is mapped to $\mathbf{X}' = \Xi_t(\mathbf{X})$ (see Fig. 4.3). The deformation mappings for the new reference configuration are $\varphi_t' = \varphi_t \circ \Xi_t^{-1}$ and $\widetilde{\varphi}_t' = \widetilde{\varphi}_t \circ \Xi_t^{-1}$. This can be clearly seen in Fig. 4.3. The material velocity in $\mathcal{U}'$ is

$$\mathbf{V}'(\mathbf{X}',t) = \frac{\partial}{\partial t}\varphi_t'(\mathbf{X}') = \frac{\partial\varphi_t}{\partial t}\circ\Xi_t^{-1}(\mathbf{X}') + T\varphi_t\circ\frac{\partial\,\Xi_t^{-1}}{\partial t}(\mathbf{X}'), \tag{4.64}$$

where partial derivatives are calculated for fixed $\mathbf{X}'$. We assume that

$$\Xi_t\Big|_{t=t_0} = \mathrm{id}, \qquad \frac{\partial\,\Xi_t}{\partial t}(\mathbf{X}) = \mathbf{W}(\mathbf{X},t). \tag{4.65}$$

Note that $\mathbf{W}$ is the infinitesimal generator of the rearrangement Ξ_t. It is an easy exercise to show that

$$\mathbf{V}' = \mathbf{V}\circ\Xi_t^{-1} - \mathbf{F}\mathbf{F}_{\Xi}^{-1}\cdot\mathbf{W}\circ\Xi_t^{-1}. \tag{4.66}$$

Thus, at $t = t_0$

$$\mathbf{V}' = \mathbf{V} - \mathbf{F}\mathbf{W}. \tag{4.67}$$

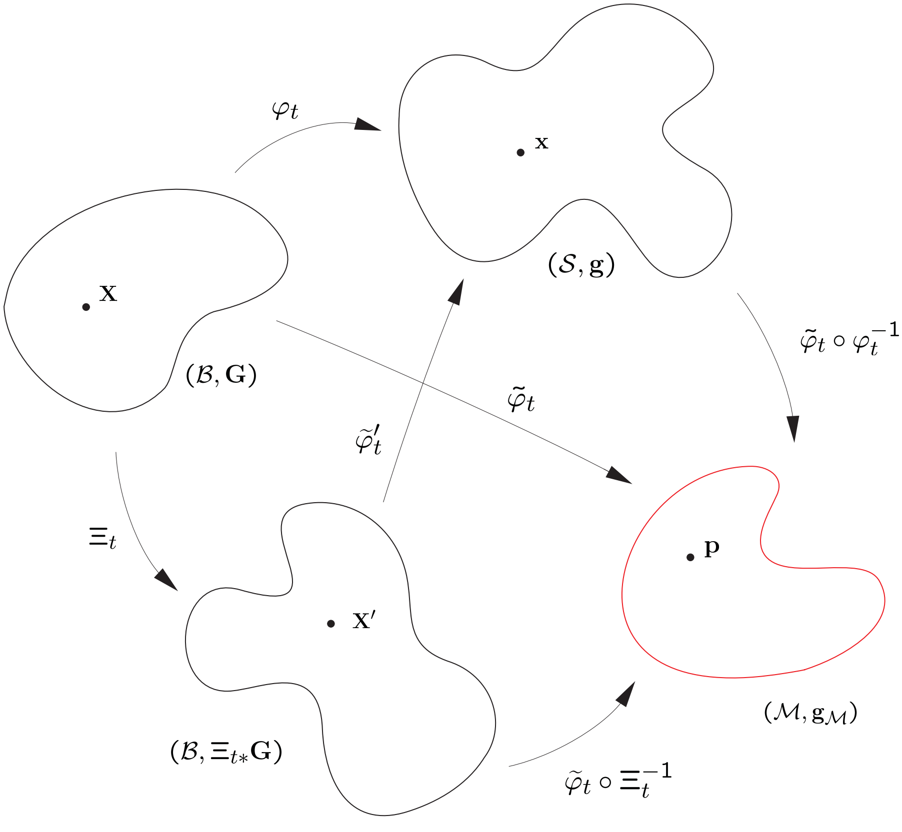

Fig. 4.3. Referential change of frame in a continuum with microstructure.

Similarly

$$\widetilde{\mathbf{V}}' = \widetilde{\mathbf{V}} - \widetilde{\mathbf{F}}\mathbf{W}. \tag{4.68}$$

Note that

$$\begin{aligned}\mathbf{G}' &= (\varphi_t \circ \Xi_t^{-1})^* \circ \varphi_{t*}\mathbf{G} = (\Xi_t^{-1})^* \circ \varphi_t^* \circ \varphi_{t*}\mathbf{G} = (\Xi_t^{-1})^*\mathbf{G} = \Xi_{t*}\mathbf{G} \\ &= (T\,\Xi_t)^{-*}\,\mathbf{G}\,(T\,\Xi_t)^{-1}, \end{aligned}\tag{4.69}$$

and

$$\mathbf{F}' = \Xi_{t*}\mathbf{F} = \mathbf{F} \circ (T\,\Xi_t)^{-1}. \tag{4.70}$$

The material internal energy density is assumed to transform tensorially, i.e.

$$E'(t, \mathbf{X}', \mathbf{G}') = E(t, \mathbf{X}, \mathbf{G}). \tag{4.71}$$

This means that internal energy density at $\mathbf{X}'$ evaluated by the transformed metric $\mathbf{G}'$ is equal to the internal energy density at $\mathbf{X}$ evaluated by the metric $\mathbf{G}$. We know that $\mathbf{G}' = \Xi_{t*}\mathbf{G}$, and thus

$$E'(t, \mathbf{X}', \mathbf{G}) = E(t, \mathbf{X}, \Xi_t^*\mathbf{G}). \tag{4.72}$$

Therefore

$$\frac{d}{dt}\Big|_{t=t_0} E'(t, \mathbf{X}', \mathbf{G}) = \frac{\partial E}{\partial t} + \frac{\partial E}{\partial \mathbf{G}} : \mathfrak{L}_{\mathbf{W}}\mathbf{G}. \tag{4.73}$$

Balance of energy for reframings of the reference configuration. Consider a deformation mapping $\varphi_t : \mathcal{B} \to \mathcal{S}$ and a referential diffeomorphism $\Xi_t : \mathfrak{B} \to \mathfrak{B}$. The mappings $\varphi_t' = \varphi_t \circ \Xi_t^{-1} : \mathcal{B}' \to \mathcal{S}$ and $\widetilde{\varphi}_t' = \widetilde{\varphi}_t \circ \Xi_t^{-1} : \mathcal{B}' \to \mathcal{M}$, where $\mathcal{B}' = \Xi_t(\mathcal{B})$, represent the deformation of the new (evolved) reference configuration. Balance of energy for $\Xi_t(\mathcal{U})$ should include the following two groups of terms:

i) Looking at $(\varphi_t', \widetilde{\varphi}_t')$ as the deformation of $\mathcal{B}'$ in $\mathcal{S} \times \mathcal{M}$, one has the usual material energy balance for $\Xi_t(\mathcal{U})$. Transformation of fields from $(\mathfrak{B}, \mathbf{G})$ to $(\mathfrak{B}, \mathbf{G}')$ follows Cartan's space-time theory.

ii) Nonstandard terms may appear to represent the energy associated with the material evolution.

We expect to see some new terms that are work-conjugate to $\mathbf{W}_t = \frac{\partial}{\partial t}\Xi_t$. Let us denote the volume and surface forces conjugate to $\mathbf{W}$ by $\mathbf{B}_0$ and $\mathbf{T}_0$, respectively.

Instead of looking at spatial framings, let us fix the deformed configuration and look at framings of the reference configuration. We postulate that energy balance for each nice subset $\mathcal{U}'$ has the following form,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathcal{U}'} \rho_0' \left(E' + \frac{1}{2}\langle\langle \mathbf{V}', \mathbf{V}' \rangle\rangle + \frac{1}{2} J' \langle\langle \widetilde{\mathbf{V}}', \widetilde{\mathbf{V}}' \rangle\rangle \right) dV' \\ &\quad = \int_{\mathcal{U}'} \rho_0' \left(\langle\langle \mathbf{B}', \mathbf{V}' \rangle\rangle + \langle\langle \widetilde{\mathbf{B}}', \widetilde{\mathbf{V}}' \rangle\rangle + R' \right) dV' + \int_{\partial\mathcal{U}'} \left(\langle\langle \mathbf{T}', \mathbf{V}' \rangle\rangle + \langle\langle \widetilde{\mathbf{T}}', \widetilde{\mathbf{V}}' \rangle\rangle + H' \right) dA' \\ &\qquad + \int_{\mathcal{U}'} \langle\langle \mathbf{B}_0', \mathbf{W}_t \rangle\rangle \, dV' + \int_{\partial\mathcal{U}'} \langle\langle \mathbf{T}_0', \mathbf{W}_t \rangle\rangle \, dA', \end{aligned} \tag{4.74}$$

where $\mathcal{U}' = \Xi_t(\mathcal{U})$ and $\mathbf{B}_0'$ and $\mathbf{T}_0'$ are unknown vector fields at this point. Using Cartan's spacetime theory, it is assumed that the primed quantities have the following relation with the unprimed quantities,

$$\begin{aligned} &dV' = \Xi_{t*} dV, \qquad R'(\mathbf{X}', t) = R(\mathbf{X}, t), \qquad \rho_0'(\mathbf{X}', t) = \rho_0(\mathbf{X}), \\ &H'(\mathbf{X}', \hat{\mathbf{N}}', t) = H(\mathbf{X}, \hat{\mathbf{N}}, t), \qquad J' = J, \\ &\mathbf{T}'(\mathbf{X}', \hat{\mathbf{N}}', t) = \mathbf{T}(\mathbf{X}, \hat{\mathbf{N}}, t), \qquad \widetilde{\mathbf{T}}'(\mathbf{X}', \hat{\mathbf{N}}', t) = \widetilde{\mathbf{T}}(\mathbf{X}, \hat{\mathbf{N}}, t). \end{aligned} \tag{4.75}$$

We assume that body force is transformed in such a way that

$$\mathbf{B}' - \mathbf{A}' = \Xi_{t*}(\mathbf{B} - \mathbf{A}), \qquad \widetilde{\mathbf{B}}' - \widetilde{\mathbf{A}}' = \Xi_{t*}(\widetilde{\mathbf{B}} - \widetilde{\mathbf{A}}). \tag{4.76}$$

Thus

$$(\mathbf{B}' - \mathbf{A}')\big|_{t=t_0} = \mathbf{B} - \mathbf{A}, \qquad (\widetilde{\mathbf{B}}' - \widetilde{\mathbf{A}}')\big|_{t=t_0} = \widetilde{\mathbf{B}} - \widetilde{\mathbf{A}}. \tag{4.77}$$

Note that if α is a 3-form on $\mathcal{U}$, then

$$\frac{d}{dt}\Big|_{t=t_0}\int_{\mathcal{U}'}\alpha' = \int_{\mathcal{U}}\frac{d}{dt}\Big|_{t=t_0}\left(\Xi_t^*\alpha'\right), \tag{4.78}$$

where $\mathcal{U}'=\Xi_t(\mathcal{U})$. Thus

$$\frac{d}{dt}\Big|_{t=t_0}\int_{\mathcal{U}'}E'dV' = \int_{\mathcal{U}}\frac{d}{dt}\Big|_{t=t_0}\left(\Xi_t^*E'\right)dV = \int_{\mathcal{U}}\left(\frac{\partial E}{\partial t}+\frac{\partial E}{\partial \mathbf{G}}:\mathfrak{L}_{\mathbf{W}}\mathbf{G}\right)dV. \tag{4.79}$$

Material energy balance for $\mathcal{U}'\subset\mathcal{B}'$ at $t=t_0$ reads

$$\begin{aligned}
&\int_{\mathcal{U}}\frac{\partial\rho_0}{\partial t}\left(E+\frac{1}{2}\langle\langle\mathbf{V}-\mathbf{F}\mathbf{W},\mathbf{V}-\mathbf{F}\mathbf{W}\rangle\rangle+\frac{1}{2}J\left\langle\!\left\langle\widetilde{\mathbf{V}}-\widetilde{\mathbf{F}}\mathbf{W},\widetilde{\mathbf{V}}-\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle\right)dV\\
&\quad+\int_{\mathcal{U}}\rho_0\Bigg(\frac{\partial E}{\partial t}+\frac{\partial E}{\partial\mathbf{G}}:\mathfrak{L}_{\mathbf{W}}\mathbf{G}+\left\langle\!\left\langle\mathbf{V}-\mathbf{F}\mathbf{W},\mathbf{A}'\big|_{t=t_0}\right\rangle\!\right\rangle+J\left\langle\!\left\langle\widetilde{\mathbf{V}}-\widetilde{\mathbf{F}}\mathbf{W},\widetilde{\mathbf{A}}'\big|_{t=t_0}\right\rangle\!\right\rangle\\
&\quad+\frac{1}{2}\frac{\partial J}{\partial t}\left\langle\!\left\langle\widetilde{\mathbf{V}}-\widetilde{\mathbf{F}}\mathbf{W},\widetilde{\mathbf{V}}-\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle\Bigg)dV=\int_{\mathcal{U}}\rho_0\left(\left\langle\!\left\langle\mathbf{B}'\big|_{t=t_0},\mathbf{V}-\mathbf{F}\mathbf{W}\right\rangle\!\right\rangle+R\right)dV\\
&\quad+\int_{\mathcal{U}}\rho_0\left\langle\!\left\langle\widetilde{\mathbf{B}}'\big|_{t=t_0},\widetilde{\mathbf{V}}-\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle dV+\int_{\partial\mathcal{U}}\left(\langle\langle\mathbf{T},\mathbf{V}-\mathbf{F}\mathbf{W}\rangle\rangle+H\right)dA\\
&\quad+\int_{\partial\mathcal{U}}\left\langle\!\left\langle\widetilde{\mathbf{T}},\widetilde{\mathbf{V}}-\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle dA+\int_{\mathcal{U}}\langle\langle\mathbf{B}_0,\mathbf{W}\rangle\rangle dV+\int_{\partial\mathcal{U}}\langle\langle\mathbf{T}_0,\mathbf{W}\rangle\rangle dA.
\end{aligned}\tag{4.80}$$

We know that $\mathbf{T}_0$ and $\mathbf{B}_0$ are defined on $\mathcal{B}$ and $\mathbf{T}_0'$ and $\mathbf{B}_0'$ are the corresponding quantities defined on $\Xi_t(\mathcal{B})$. Here we assume that

$$\mathbf{T}_0'=\Xi_{t*}\mathbf{T}_0 \qquad\text{and}\qquad \mathbf{B}_0'=\Xi_{t*}\mathbf{B}_0. \tag{4.81}$$

Subtracting balance of energy for $\mathcal{U}$ from this and noting that $\left(\mathbf{A}'-\mathbf{B}'\right)_{t=t_0}=\mathbf{A}-\mathbf{B}$ and $\left(\widetilde{\mathbf{A}}'-\widetilde{\mathbf{B}}'\right)_{t=t_0}=\widetilde{\mathbf{A}}-\widetilde{\mathbf{B}}$ one obtains

$$\begin{aligned}
&\int_{\mathcal{U}}\frac{\partial\rho_0}{\partial t}\left(-\langle\langle\mathbf{V},\mathbf{F}\mathbf{W}\rangle\rangle+\frac{1}{2}\langle\langle\mathbf{F}\mathbf{W},\mathbf{F}\mathbf{W}\rangle\rangle-J\left\langle\!\left\langle\widetilde{\mathbf{V}},\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle+\frac{1}{2}J\left\langle\!\left\langle\widetilde{\mathbf{F}}\mathbf{W},\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle\right)dV\\
&+\int_{\mathcal{U}}\rho_0\left[\frac{\partial E}{\partial\mathbf{G}}:\mathfrak{L}_{\mathbf{W}}\mathbf{G}-\langle\langle\mathbf{F}\mathbf{W},\mathbf{A}\rangle\rangle-\left\langle\!\left\langle\widetilde{\mathbf{F}}\mathbf{W},J\widetilde{\mathbf{A}}\right\rangle\!\right\rangle+\frac{\partial J}{\partial t}\left(-\left\langle\!\left\langle\widetilde{\mathbf{V}},\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle+\frac{1}{2}\left\langle\!\left\langle\widetilde{\mathbf{F}}\mathbf{W},\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle\right)\right]dV\\
&\qquad=-\int_{\mathcal{U}}\langle\langle\rho_0\mathbf{B},\mathbf{F}\mathbf{W}\rangle\rangle dV-\int_{\partial\mathcal{U}}\langle\langle\mathbf{T},\mathbf{F}\mathbf{W}\rangle\rangle dA-\int_{\mathcal{U}}\left\langle\!\left\langle\rho_0\widetilde{\mathbf{B}},\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle dV\\
&\qquad\quad-\int_{\partial\mathcal{U}}\left\langle\!\left\langle\widetilde{\mathbf{T}},\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle dA+\int_{\mathcal{U}}\langle\langle\mathbf{B}_0,\mathbf{W}\rangle\rangle dV+\int_{\partial\mathcal{U}}\langle\langle\mathbf{T}_0,\mathbf{W}\rangle\rangle dA.
\end{aligned}\tag{4.82}$$

We know that

$$\langle\langle\mathbf{T},\mathbf{F}\mathbf{W}\rangle\rangle=\left\langle\!\left\langle\mathbf{F}\mathbf{W},\left\langle\!\left\langle\mathbf{P},\hat{\mathbf{N}}\right\rangle\!\right\rangle\right\rangle\!\right\rangle,\qquad \left\langle\!\left\langle\widetilde{\mathbf{T}},\widetilde{\mathbf{F}}\mathbf{W}\right\rangle\!\right\rangle=\left\langle\!\left\langle\widetilde{\mathbf{F}}\mathbf{W},\left\langle\!\left\langle\widetilde{\mathbf{P}},\hat{\mathbf{N}}\right\rangle\!\right\rangle\right\rangle\!\right\rangle, \tag{4.83}$$

where **P** is the first Piola–Kirchhoff stress tensor. Thus, substituting (4.83) into (4.82), Cauchy's theorem implies that

$$\mathbf{T}_0 = \left\langle\!\left\langle \mathbf{P}_0, \hat{\mathbf{N}} \right\rangle\!\right\rangle, \tag{4.84}$$

for some second-order tensor $\mathbf{P}_0$. The surface integrals in material energy balance have the following transformations (see Yavari et al. [39] for a proof)

$$\begin{aligned}\int_{\partial\mathcal{U}} \left\langle\!\left\langle \mathbf{F}^{\mathsf{T}}\mathbf{T}, \mathbf{W} \right\rangle\!\right\rangle dA &= \int_{\mathcal{U}} \mathrm{Div} \left\langle\!\left\langle \mathbf{F}^{\mathsf{T}}\mathbf{P}, \mathbf{W} \right\rangle\!\right\rangle dV \\ &= \int_{\mathcal{U}} \left[\left\langle\!\left\langle \mathrm{Div}(\mathbf{F}^{\mathsf{T}}\mathbf{P}), \mathbf{W} \right\rangle\!\right\rangle + \mathbf{F}^{\mathsf{T}}\mathbf{P} : \Omega + \mathbf{F}^{\mathsf{T}}\mathbf{P} : \mathbf{K} \right] dV. \end{aligned} \tag{4.85}$$

And

$$\begin{aligned}\int_{\partial\mathcal{U}} \left\langle\!\left\langle \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{T}}, \mathbf{W} \right\rangle\!\right\rangle dA &= \int_{\mathcal{U}} \mathrm{Div} \left\langle\!\left\langle \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}}, \mathbf{W} \right\rangle\!\right\rangle dV \\ &= \int_{\mathcal{U}} \left[\left\langle\!\left\langle \mathrm{Div}(\widetilde{\mathbf{F}}^{\mathsf{T}}\mathbf{P}), \mathbf{W} \right\rangle\!\right\rangle + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} : \Omega + \mathbf{F}^{\mathsf{T}}\mathbf{P} : \mathbf{K} \right] dV, \end{aligned} \tag{4.86}$$

where

$$\Omega_{IJ} = \frac{1}{2}\left(G_{IK}W^K{}_{|J} - G_{JK}W^K{}_{|I}\right) = \frac{1}{2}\left(W_{I|J} - W_{J|I}\right), \tag{4.87}$$

$$\mathbf{K}_{IJ} = \frac{1}{2}\left(G_{IK}W^K{}_{|J} + G_{JK}W^K{}_{|I}\right) = \frac{1}{2}\left(W_{I|J} + W_{J|I}\right), \quad \mathbf{K} = \frac{1}{2}\mathfrak{L}_{\mathbf{W}}\mathbf{G}. \tag{4.88}$$

Similarly

$$\begin{aligned}\int_{\partial\mathcal{U}} \langle\langle \mathbf{T}_0, \mathbf{W} \rangle\rangle\, dA &= \int_{\mathcal{U}} \mathrm{Div}\, \langle\langle \mathbf{P}_0, \mathbf{W} \rangle\rangle\, dV \\ &= \int_{\mathcal{U}} \left[\langle\langle \mathrm{Div}\, \mathbf{P}_0, \mathbf{W} \rangle\rangle + \mathbf{P}_0 : \Omega + \mathbf{P}_0 : \mathbf{K} \right] dV. \end{aligned} \tag{4.89}$$

At time $t = t_0$ the transformed balance of energy should be the same as the balance of energy for $\mathcal{U}$. Thus, subtracting the material balance of energy for $\mathcal{U}$ from the above balance law and considering conservation of mass and micro-inertia, one obtains

$$\begin{aligned}&\int_{\mathcal{U}} \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \mathfrak{L}_{\mathbf{W}}\mathbf{G}\, dV + \int_{\mathcal{U}} \left\langle\!\left\langle \rho_0 \mathbf{F}^{\mathsf{T}} \left(\mathbf{B} - \mathbf{A}\right), \mathbf{W} \right\rangle\!\right\rangle dV + \int_{\mathcal{U}} \left\langle\!\left\langle \rho_0 \widetilde{\mathbf{F}}^{\mathsf{T}} \left(\widetilde{\mathbf{B}} - \widetilde{\mathbf{A}}\right), \mathbf{W} \right\rangle\!\right\rangle dV \\ &\quad - \int_{\mathcal{U}} \langle\langle \rho_0 \mathbf{B}_0, \mathbf{W} \rangle\rangle\, dV + \int_{\partial\mathcal{U}} \left\langle\!\left\langle \mathbf{F}^{\mathsf{T}}\mathbf{T} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{T}} - \mathbf{T}_0, \mathbf{W} \right\rangle\!\right\rangle dA = 0. \end{aligned} \tag{4.90}$$

Therefore

$$\int_{\mathcal{U}} \left(2\rho_0 \frac{\partial E}{\partial \mathbf{G}} + \mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} - \mathbf{P}_0\right) : \frac{1}{2}\mathfrak{L}_{\mathbf{W}}\mathbf{G}\, dV + \int_{\mathcal{U}} \left(\mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} - \mathbf{P}_0\right) : \Omega\, dV$$
$$+ \int_{\mathcal{U}} \left\langle\!\!\left\langle \rho_0 \mathbf{F}^{\mathsf{T}} (\mathbf{B} - \mathbf{A}) + \rho_0 \widetilde{\mathbf{F}}^{\mathsf{T}} \left(\widetilde{\mathbf{B}} - \widetilde{\mathbf{A}}\right) - \mathbf{B}_0 + \mathrm{Div}\left(\mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}}\right) - \mathrm{Div}\,\mathbf{P}_0, \mathbf{W} \right\rangle\!\!\right\rangle dV = 0. \tag{4.91}$$

Using balance of linear and micro-linear momenta, (4.91) is simplified to read

$$\int_{\mathcal{U}} \left(2\rho_0 \frac{\partial E}{\partial \mathbf{G}} + \mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} - \mathbf{P}_0\right) : \frac{1}{2}\mathfrak{L}_{\mathbf{W}}\mathbf{G}\, dV + \int_{\mathcal{U}} \left(\mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} - \mathbf{P}_0\right) : \Omega\, dV$$
$$+ \int_{\mathcal{U}} \left\langle\!\!\left\langle \mathrm{Div}\left(\mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} - \mathbf{P}_0\right) - \mathbf{F}^{\mathsf{T}}\,\mathrm{Div}\,\mathbf{P} - \widetilde{\mathbf{F}}^{\mathsf{T}}\,\mathrm{Div}\,\widetilde{\mathbf{P}} - \mathbf{B}_0, \mathbf{W} \right\rangle\!\!\right\rangle dV = 0. \tag{4.92}$$

Because $\mathcal{U}$ and $\mathbf{W}$ are arbitrary, one obtains

$$\mathbf{P}_0 = 2\rho_0 \frac{\partial E}{\partial \mathbf{G}} + \mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}}, \tag{4.93}$$
$$\left(\mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} - \mathbf{P}_0\right)^{\mathsf{T}} = \mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} - \mathbf{P}_0, \tag{4.94}$$
$$\mathbf{B}_0 = \mathrm{Div}\left(\mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} - \mathbf{P}_0\right) - \mathbf{F}^{\mathsf{T}}\,\mathrm{Div}\,\mathbf{P} - \widetilde{\mathbf{F}}^{\mathsf{T}}\,\mathrm{Div}\,\widetilde{\mathbf{P}}. \tag{4.95}$$

Note that (4.94) is trivially satisfied after having (4.93). Thus, we have

$$\mathbf{P}_0 = 2\rho_0 \frac{\partial E}{\partial \mathbf{G}} + \mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}}, \tag{4.96}$$
$$\mathbf{B}_0 = \mathrm{Div}\left(\mathbf{F}^{\mathsf{T}}\mathbf{P} + \widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}} - \mathbf{P}_0\right) - \mathbf{F}^{\mathsf{T}}\,\mathrm{Div}\,\mathbf{P} - \widetilde{\mathbf{F}}^{\mathsf{T}}\,\mathrm{Div}\,\widetilde{\mathbf{P}}. \tag{4.97}$$

REMARK. Note that $\mathbf{B}_0$ and $\mathbf{P}_0$ are material tensors and hence the transformation (4.81) makes sense.

In summary, we have proven the following theorem.

THEOREM 4.3. *Under a referential diffeomorphism $\Xi_t : \mathfrak{B} \to \mathfrak{B}$, and assuming that material energy density transforms tensorially, i.e.*

$$E'(t, \mathbf{X}', \mathbf{G}) = E(t, \mathbf{X}, \Xi_t^* \mathbf{G}), \tag{4.98}$$

material energy balance has the following transformation

$$\frac{d}{dt}\int_{\Xi_t(\mathcal{U})}\rho_0'\left(E'+\frac{1}{2}\langle\!\langle \mathbf{V}',\mathbf{V}'\rangle\!\rangle+\frac{1}{2}J'\langle\!\langle \widetilde{\mathbf{V}}',\widetilde{\mathbf{V}}'\rangle\!\rangle\right)dV'$$
$$=\int_{\Xi_t(\mathcal{U})}\rho_0'\left(\langle\!\langle \mathbf{B}',\mathbf{V}'\rangle\!\rangle+\langle\!\langle \widetilde{\mathbf{B}}',\widetilde{\mathbf{V}}'\rangle\!\rangle+R'\right)dV'$$
$$+\int_{\partial\Xi_t(\mathcal{U})}\left(\langle\!\langle \mathbf{T}',\mathbf{V}'\rangle\!\rangle+\langle\!\langle \widetilde{\mathbf{T}}',\widetilde{\mathbf{V}}'\rangle\!\rangle+H'\right)dA'$$
$$+\int_{\Xi_t(\mathcal{U})}\langle\!\langle \mathbf{B}_0',\mathbf{W}_t\rangle\!\rangle dV'+\int_{\partial\Xi_t(\mathcal{U})}\langle\!\langle \mathbf{T}_0',\mathbf{W}_t\rangle\!\rangle dA', \quad (4.99)$$

where

$$\mathbf{T}_0'=\Xi_{t*}\left[\left\langle\!\!\left\langle 2\rho_0\frac{\partial E}{\partial \mathbf{G}}+\mathbf{F}^T\mathbf{P}+\widetilde{\mathbf{F}}^T\widetilde{\mathbf{P}},\hat{\mathbf{N}}\right\rangle\!\!\right\rangle\right], \quad (4.100)$$

$$\mathbf{B}_0'=\Xi_{t*}\left[\mathrm{Div}\left(\mathbf{F}^T\mathbf{P}+\widetilde{\mathbf{F}}^T\widetilde{\mathbf{P}}-\mathbf{P}_0\right)-\mathbf{F}^T\,\mathrm{Div}\,\mathbf{P}-\widetilde{\mathbf{F}}^T\,\mathrm{Div}\,\widetilde{\mathbf{P}}\right], \quad (4.101)$$

and the other quantities are already defined.

Consequences of assuming invariance of energy balance. Let us now study the consequences of assuming material covariance of energy balance. Material energy balance is invariant under material diffeomorphisms if and only if the following relations hold between the nonstandard terms

$$\mathbf{P}_0=\mathbf{0} \qquad \text{or} \qquad 2\rho_0\frac{\partial E}{\partial \mathbf{G}}=-\mathbf{F}^{\mathsf{T}}\mathbf{P}-\widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}}, \quad (4.102)$$

$$\mathbf{B}_0=\mathbf{0} \qquad \text{or} \qquad \mathrm{Div}\left(\mathbf{F}^{\mathsf{T}}\mathbf{P}+\widetilde{\mathbf{F}}^{\mathsf{T}}\widetilde{\mathbf{P}}\right)=\mathbf{F}^{\mathsf{T}}\,\mathrm{Div}\,\mathbf{P}+\widetilde{\mathbf{F}}^{\mathsf{T}}\,\mathrm{Div}\,\widetilde{\mathbf{P}}. \quad (4.103)$$

4.3. Covariant elasticity for a special class of structured continua

In this subsection, we consider two special types of structured continua in which microstructure manifold is linked to reference and ambient space manifolds. In the first example, we assume that for any $\mathbf{X}\in\mathcal{B}$, microstructure manifold is $(T_{\mathbf{X}}\mathcal{B},\mathbf{G})$. For such a continuum, directors are "attached" to material points. We call this continuum a *referentially constrained structured* (RCS) continuum. In the second example, we assume that in the deformed configuration, microstructure manifold for $\mathbf{x}=\varphi_t(\mathbf{X})$ is $(T_{\mathbf{x}}\mathcal{S},\mathbf{g})$. We call such a continuum a *spatially constrained structured* (SCS) continuum. For RCS continua we look at both referential and spatial covariance of energy balance. This is a concrete example of what we earlier called a structured continuum with free microstructue. For SCS continua we look at spatial covariance of energy balance.

As was mentioned earlier, in most treatments of continua with microstructure, one has two balances of linear momenta; one for standard forces and one for microstructure forces, and one balance of angular momentum, which has contributions from both standard and micro-forces. In this subsection, we show that in a special case when microstructure manifold is the tangent space of the ambient space manifold, one can obtain all the balance laws covariantly using a single balance of energy. Interestingly, there will be two balances of linear momenta and one balance of angular momentum. We will also see that there are different possibilities for defining "covariance" and depending on what one calls "covariance", balance laws have different forms.

Materially constrained structured continua. Given $\mathbf{X} \in \mathcal{B}$, and $\mathcal{M} = T_{\mathbf{X}}\mathcal{B}$, director velocity is defined as

$$\widetilde{\mathbf{V}} = \frac{\partial \widetilde{\varphi}_t(\mathbf{X})}{\partial t}. \tag{4.104}$$

For writing energy balance in $\mathcal{S}$ we need to push-forward the director velocity. The spatial director velocity is defined as

$$\widetilde{\mathbf{v}} = \varphi_{t*}\widetilde{\mathbf{V}} = \mathbf{F}\widetilde{\mathbf{V}}. \tag{4.105}$$

Micro-traction $\widetilde{\mathbf{T}}$ has the coordinate representation

$$\widetilde{\mathbf{T}} = \widetilde{T}^A \mathbf{E}_A. \tag{4.106}$$

Internal energy density has the form $e = e(t, \mathbf{x}, \mathbf{p} \circ \varphi_t^{-1}, \mathbf{g}, \mathbf{G} \circ \varphi_t^{-1})$. Spatial and microstructure diffeomorphisms act on macro and micro-forces independently as was explained in Section 4.1. The resulting governing equations are exactly similar to those obtained previously and thus we leave the details.

Spatially constrained structured continua. In the previous section we assumed that the standard ambient space and the microstructure manifolds are independent in the sense that they can have independent changes of frame. It seems that this is not the case for most materials with microstructure and this is perhaps why one sees only one balance of angular momentum, e.g. in liquid crystals [13, 23]. Here, we present an example of a structured continuum in which the microstructure manifold is linked to the standard ambient space manifold. We assume that for each $\mathbf{x} \in \mathcal{S}$, the director at $\mathbf{x}$, i.e. $\mathbf{p}(\mathbf{x})$ is an element of $T_{\mathbf{x}}\mathcal{S}$. In other words

$$\mathcal{M}_{\mathbf{x}} = T_{\mathbf{x}}\mathcal{S} \qquad \forall\ \mathbf{x} \in \varphi_t(\mathcal{B}), \tag{4.107}$$

i.e. for each $\mathbf{x}$ microstructure manifold is $T_{\mathbf{x}}\mathcal{S}$ and $\widetilde{\varphi}$ is a time-dependent vector in $T_{\mathbf{x}}\mathcal{S}$. In the fiber bundle representation schematically shown in Fig. 2.2, this means that microstructure bundle is $T\mathcal{S}$, i.e. the tangent bundle of the ambient space manifold.

Here we assume that the director field is a single vector field. Generalization of the results to cases where the director is a tensor field would be straightforward. The microstructure deformation gradient has the following representation

$$\widetilde{\mathbf{F}} = T\widetilde{\varphi}_t \circ \mathbf{F}, \qquad \widetilde{\mathbf{F}} : T_{\mathbf{x}}\mathcal{S} \to T_{\mathbf{p}(\mathbf{x})}T_{\mathbf{x}}\mathcal{S}. \tag{4.108}$$

In components

$$\widetilde{\mathbf{F}} = \widetilde{F}^a{}_b\ \mathbf{e}_a \otimes \mathbf{e}^b. \tag{4.109}$$

Microstructure velocity is defined as

$$\widetilde{\mathbf{v}}(\mathbf{x}, t) = \frac{\partial}{\partial t}\Big|_{\mathbf{X}} \widetilde{\varphi}_t(\mathbf{x}). \tag{4.110}$$

In components

$$\widetilde{v}^a = \frac{\partial p^a}{\partial t} + \frac{\partial p^a}{\partial x^b} v^b + \gamma^a_{bc} v^b p^c. \tag{4.111}$$

Or

$$\widetilde{\mathbf{v}} = \dot{\mathbf{p}} = \frac{\partial \mathbf{p}}{\partial t} + \nabla_{\mathbf{v}} \mathbf{p}. \tag{4.112}$$

Now let us consider a spatial change of frame, i.e. $\xi_t : \mathcal{S} \to \mathcal{S}$. Note that $\varphi'_t = \xi_t \circ \varphi_t$ and because $\widetilde{\varphi} \in T_{\mathbf{x}}\mathcal{S}$ we have

$$\widetilde{\varphi}'_t(\mathbf{x}') = T\xi_t \cdot \widetilde{\varphi}_t(\mathbf{x}). \tag{4.113}$$

Microstructure velocity in the new frame is defined as

$$\widetilde{\mathbf{v}}' = \frac{\partial \mathbf{p}'}{\partial t} + \nabla_{\mathbf{v}'} \mathbf{p}'. \tag{4.114}$$

Noting that $\mathbf{p}' = \xi_{t*}\mathbf{p}$ and $\mathbf{v}' = \xi_{t*}\mathbf{v} + \mathbf{w}_t$, we obtain

$$\widetilde{\mathbf{v}}' = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}'} (\xi_{t*}\mathbf{p}) + \xi_{t*} (\nabla_{\mathbf{v}} \mathbf{p}) + \nabla_{\mathbf{w}} (\xi_{t*}\mathbf{p}). \tag{4.115}$$

Note that[1]

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}'} (\xi_{t*}\mathbf{p}) = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} (\xi_{t*}\mathbf{p}) - \nabla_{\mathbf{w}} (\xi_{t*}\mathbf{p}). \tag{4.117}$$

Thus

$$\widetilde{\mathbf{v}}' = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} (\xi_{t*}\mathbf{p}) + \xi_{t*} (\nabla_{\mathbf{v}} \mathbf{p}). \tag{4.118}$$

Note also that

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} (\xi_{t*}\mathbf{p}) = \xi_{t*} \left(\frac{\partial \mathbf{p}}{\partial t}\right) + \nabla_{\xi_{t*}\mathbf{p}} \mathbf{w}. \tag{4.119}$$

Therefore

$$\widetilde{\mathbf{v}}' = \xi_{t*}\widetilde{\mathbf{v}} + \nabla_{\xi_{t*}\mathbf{p}} \mathbf{w}. \tag{4.120}$$

This means that at time $t = t_0$

$$\widetilde{\mathbf{v}}' = \widetilde{\mathbf{v}} + \nabla_{\mathbf{p}} \mathbf{w}. \tag{4.121}$$

We assume that microstructure body forces transform such that $\widetilde{\mathbf{a}}' - \widetilde{\mathbf{b}}' = \xi_{t*}(\widetilde{\mathbf{a}} - \widetilde{\mathbf{b}})$.

For this structured continuum we assume that, in addition to metric, internal energy density explicitly depends on a connection too, i.e.[2]

$$e = e(t, \mathbf{x}, \mathbf{p}, \mathbf{g}, \nabla). \tag{4.122}$$

[1] This can be proved as follows,

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} \mathbf{p}' = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}'} \mathbf{p}' + \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} \left[p'^{\alpha}(\xi(\mathbf{x}))\mathbf{e}_{\alpha}(\xi(\mathbf{x}))\right] = \left(\frac{\partial p'^{\alpha}}{\partial \xi^{\beta}} + \gamma^{\alpha}_{\lambda\beta} p'^{\lambda}\right) w^{\beta} \mathbf{e}_{\alpha} = \nabla_{\mathbf{w}} \mathbf{p}'. \tag{4.116}$$

[2] Note that this is similar to Palatini's formulation of general relativity [37], where both metric and connection are assumed to be fields.

The connection ∇ is assumed to be metric compatible, i.e $\nabla\mathbf{g}=\mathbf{0}$ but not necessarily torsion-free, i.e. ∇ is not necessarily the Levi-Civita connection. Therefore, under a change of frame we have the following transformation of internal energy density

$$e'(t,\mathbf{x}',\mathbf{p}',\mathbf{g},\nabla)=e(t,\mathbf{x},\mathbf{p},\xi_t^*\mathbf{g},\xi_t^*\nabla). \tag{4.123}$$

Thus, at $t=t_0$

$$\dot{\overline{e'}}=\dot{e}+\frac{\partial e}{\partial\mathbf{g}}:\mathfrak{L}_{\mathbf{w}}\mathbf{g}+\frac{\partial e}{\partial\nabla}:\mathfrak{L}_{\mathbf{w}}\nabla. \tag{4.124}$$

We know that for a given connection ∇ [27]

$$\mathfrak{L}_{\mathbf{w}}\nabla=\nabla\nabla\mathbf{w}+\boldsymbol{\mathcal{R}}\cdot\mathbf{w}. \tag{4.125}$$

Or in coordinates

$$(\mathfrak{L}_{\mathbf{w}}\nabla)^a{}_{bc}=w^a{}_{b|c}+\mathcal{R}^a{}_{dbc}w^d, \tag{4.126}$$

where $\boldsymbol{\mathcal{R}}$ is the curvature tensor of $(\mathcal{S},\mathbf{g})$.

Balance of energy for $\varphi_t(\mathcal{U})\subset\mathcal{S}$ is written as

$$\begin{aligned}\frac{d}{dt}\int_{\varphi_t(\mathcal{U})}\rho(\mathbf{x},t)\left[e(t,\mathbf{x},\mathbf{p},\mathbf{g},\nabla)+\frac{1}{2}\langle\langle\mathbf{v},\mathbf{v}\rangle\rangle+\frac{1}{2}j\,\langle\langle\widetilde{\mathbf{v}},\widetilde{\mathbf{v}}\rangle\rangle\right]\\ =\int_{\varphi_t(\mathcal{U})}\rho(\mathbf{x},t)\left(\langle\langle\mathbf{b},\mathbf{v}\rangle\rangle+\left\langle\left\langle\widetilde{\mathbf{b}},\widetilde{\mathbf{v}}\right\rangle\right\rangle+r\right)+\int_{\partial\varphi_t(\mathcal{U})}\left(\langle\langle\mathbf{t},\mathbf{v}\rangle\rangle+\left\langle\left\langle\widetilde{\mathbf{t}},\widetilde{\mathbf{v}}\right\rangle\right\rangle+h\right)da.\end{aligned} \tag{4.127}$$

Let us postulate that energy balance is invariant under arbitrary spatial changes of frame $\xi_t:\mathcal{S}\to\mathcal{S}$, i.e.

$$\begin{aligned}\frac{d}{dt}\int_{\varphi'_t(\mathcal{U})}\rho'(\mathbf{x}',t)\left[e'(t,\mathbf{x}',\mathbf{p}',\mathbf{g},\nabla)+\frac{1}{2}\left\langle\left\langle\mathbf{v}',\mathbf{v}'\right\rangle\right\rangle+\frac{1}{2}j'\left\langle\left\langle\widetilde{\mathbf{v}}',\widetilde{\mathbf{v}}'\right\rangle\right\rangle\right]\\ =\int_{\varphi'_t(\mathcal{U})}\rho'(\mathbf{x}',t)\left(\left\langle\left\langle\mathbf{b}',\mathbf{v}'\right\rangle\right\rangle+\left\langle\left\langle\widetilde{\mathbf{b}}',\widetilde{\mathbf{v}}'\right\rangle\right\rangle+r'\right)+\int_{\partial\varphi'_t(\mathcal{U})}\left(\left\langle\left\langle\mathbf{t}',\mathbf{v}'\right\rangle\right\rangle+\left\langle\left\langle\widetilde{\mathbf{t}}',\widetilde{\mathbf{v}}'\right\rangle\right\rangle+h'\right)da.\end{aligned} \tag{4.128}$$

We know that

$$e'(t,\mathbf{x}',\mathbf{p}',\mathbf{g},\nabla)=e(t,\mathbf{x},\mathbf{p},\xi_t^*\mathbf{g},\xi_t^*\nabla),\quad r'=r,\quad h'=h, \tag{4.129}$$

$$\rho'(\mathbf{x}',t)=\xi_{t*}\rho(\mathbf{x},t),\quad \mathbf{v}'=\xi_{t*}\mathbf{v}+\mathbf{w},\quad \mathbf{b}'-\mathbf{a}'=\xi_{t*}(\mathbf{b}-\mathbf{a}), \tag{4.130}$$

$$\mathbf{t}'=\xi_{t*}\mathbf{t},\quad \widetilde{\mathbf{t}}'=\xi_{t*}\widetilde{\mathbf{t}},\quad \widetilde{\mathbf{b}}'-\widetilde{\mathbf{a}}'=\xi_{t*}(\widetilde{\mathbf{b}}-\widetilde{\mathbf{a}}). \tag{4.131}$$

Subtracting balance of energy for $\varphi_t(\mathcal{U})$ from that of $\varphi'_t(\mathcal{U})$ at $t = t_0$, we obtain

$$\int_{\varphi_t(\mathcal{U})} \Bigg[\mathbf{L_v}\rho \left(\frac{1}{2} \langle\langle \mathbf{w}, \mathbf{w} \rangle\rangle + \langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle \right) + \mathbf{L_v}(\rho j) \left(\frac{1}{2} \langle\langle \nabla\mathbf{w} \cdot \mathbf{p}, \nabla\mathbf{w} \cdot \mathbf{p} \rangle\rangle + \langle\langle \widetilde{\mathbf{v}}, \nabla\mathbf{w} \cdot \mathbf{p} \rangle\rangle \right)$$
$$+ \rho \left(\frac{\partial e}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{w}}\mathbf{g} + \frac{\partial e}{\partial \nabla} : (\nabla\nabla\mathbf{w} + \boldsymbol{\mathcal{R}} \cdot \mathbf{w}) + \langle\langle \mathbf{a}, \mathbf{w} \rangle\rangle + j \, \langle\langle \widetilde{\mathbf{a}}, \nabla\mathbf{w} \cdot \mathbf{p} \rangle\rangle \right) \Bigg]$$
$$= \int_{\varphi_t(\mathcal{U})} \rho \, \langle\langle \mathbf{b}, \mathbf{w} \rangle\rangle + \int_{\partial\varphi_t(\mathcal{U})} \langle\langle \mathbf{t}, \mathbf{w} \rangle\rangle \, da + \int_{\varphi_t(\mathcal{U})} \left\langle\!\left\langle \rho\widetilde{\mathbf{b}}, \nabla\mathbf{w} \cdot \mathbf{p} \right\rangle\!\right\rangle + \int_{\partial\varphi_t(\mathcal{U})} \left\langle\!\left\langle \widetilde{\mathbf{t}}, \nabla\mathbf{w} \cdot \mathbf{p} \right\rangle\!\right\rangle da. \tag{4.132}$$

Assuming that ξ_t is such that $\widetilde{\mathbf{v}}'\big|_{t=t_0} - \widetilde{\mathbf{v}} = \mathbf{0}$, i.e. $\nabla\mathbf{w} = \mathbf{0}$, Cauchy's theorem applied to (4.132) implies that there is a second-order tensor $\boldsymbol{\sigma}$ such that $\mathbf{t} = \langle\langle \boldsymbol{\sigma}, \mathbf{n} \rangle\rangle$. Now applying Cauchy's theorem to (4.132) for an arbitrary ξ_t implies the existence of another second-order tensor $\widetilde{\boldsymbol{\sigma}}$ such that $\widetilde{\mathbf{t}} = \langle\langle \widetilde{\boldsymbol{\sigma}}, \mathbf{n} \rangle\rangle$.

REMARK. Microstructure manifold is the tangent space of the ambient space manifold at every point. However, microstructure is not related to the deformation mapping. This is why, unlike the so-called second-grade materials (see Fried and Gurtin [14]), two separate stress tensors exist.

As $\mathcal{U}$ and $\mathbf{w}$ are arbitrary, and replacing ρ by ρdv in (4.132), we conclude that

$$\mathbf{L_v}\rho = 0, \tag{4.133}$$

$$\mathbf{L_v}j = 0. \tag{4.134}$$

Now let us simplify the last two integrals in (4.132). The volume integral is simplified to read

$$\int_{\varphi_t(\mathcal{U})} \left\langle\!\left\langle \rho(\widetilde{\mathbf{b}} - j\widetilde{\mathbf{a}}), \nabla\mathbf{w} \cdot \mathbf{p} \right\rangle\!\right\rangle dv = \int_{\varphi_t(\mathcal{U})} \rho(\widetilde{\mathbf{b}} - j\widetilde{\mathbf{a}}) \otimes \mathbf{p} : \left(\frac{1}{2}\mathfrak{L}_{\mathbf{w}}\mathbf{g} + \boldsymbol{\omega} \right) dv. \tag{4.135}$$

The surface integral is simplified as

$$\int_{\partial\varphi_t(\mathcal{U})} \left\langle\!\left\langle \widetilde{\mathbf{t}}, \nabla\mathbf{w} \cdot \mathbf{p} \right\rangle\!\right\rangle da = \int_{\varphi_t(\mathcal{U})} \left(\widetilde{\sigma}^{ad} p^c w_{a|c} \right)_{|d} dv$$
$$= \int_{\varphi_t(\mathcal{U})} [(\operatorname{div} \widetilde{\boldsymbol{\sigma}}) \otimes \mathbf{p} + \widetilde{\boldsymbol{\sigma}} \cdot \nabla\mathbf{p}] : \left(\frac{1}{2}\mathfrak{L}_{\mathbf{w}}\mathbf{g} + \boldsymbol{\omega} \right) dv + \int_{\varphi_t(\mathcal{U})} \widetilde{\sigma}^{ad} p^c w_{a|c|d} \, dv,$$
$$= \int_{\varphi_t(\mathcal{U})} [(\operatorname{div} \widetilde{\boldsymbol{\sigma}}) \otimes \mathbf{p} + \widetilde{\boldsymbol{\sigma}} \cdot \nabla\mathbf{p}] : \left(\frac{1}{2}\mathfrak{L}_{\mathbf{w}}\mathbf{g} + \boldsymbol{\omega} \right) dv + \int_{\varphi_t(\mathcal{U})} \widetilde{\boldsymbol{\sigma}} \otimes \widetilde{\mathbf{p}} : \nabla\nabla\mathbf{w} \, dv. \tag{4.136}$$

Thus

$$\int_{\varphi_t(\mathcal{U})}\left(-2\rho\frac{\partial e}{\partial \mathbf{g}}+\boldsymbol{\sigma}+(\operatorname{div}\widetilde{\boldsymbol{\sigma}})\otimes\mathbf{p}+\widetilde{\boldsymbol{\sigma}}\cdot\nabla\mathbf{p}+\rho(\widetilde{\mathbf{b}}-j\widetilde{\mathbf{a}})\otimes\mathbf{p}\right):\frac{1}{2}\mathfrak{L}_{\mathbf{w}}\mathbf{g}\,dv$$
$$+\int_{\varphi_t(\mathcal{U})}\left(-2\rho\frac{\partial e}{\partial \mathbf{g}}+\boldsymbol{\sigma}+(\operatorname{div}\widetilde{\boldsymbol{\sigma}})\otimes\mathbf{p}+\widetilde{\boldsymbol{\sigma}}\cdot\nabla\mathbf{p}+\rho(\widetilde{\mathbf{b}}-j\widetilde{\mathbf{a}})\otimes\mathbf{p}\right):\boldsymbol{\omega}\,dv$$
$$+\int_{\varphi_t(\mathcal{U})}\left\langle\!\!\left\langle-\rho\mathbf{a}+\rho\mathbf{b}+\operatorname{div}\boldsymbol{\sigma}-\rho\frac{\partial e}{\partial \boldsymbol{\nabla}}:\boldsymbol{\mathcal{R}},\mathbf{w}\right\rangle\!\!\right\rangle dv,$$
$$+\int_{\varphi_t(\mathcal{U})}\left(-\rho\frac{\partial e}{\partial \boldsymbol{\nabla}}+\widetilde{\boldsymbol{\sigma}}\otimes\widetilde{\mathbf{p}}\right):\boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{w}\,dv=0. \tag{4.137}$$

Therefore, because $\mathcal{U}$, $\mathbf{w}$, and $\mathbf{z}$ are arbitrary we finally obtain

$$\mathbf{L}_{\mathbf{v}}\rho=0, \tag{4.138}$$

$$\mathbf{L}_{\mathbf{v}}j=0, \tag{4.139}$$

$$\operatorname{div}\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\mathbf{a}+\rho\frac{\partial e}{\partial \boldsymbol{\nabla}}:\boldsymbol{\mathcal{R}}, \tag{4.140}$$

$$2\rho\frac{\partial e}{\partial \mathbf{g}}=\boldsymbol{\sigma}+(\operatorname{div}\widetilde{\boldsymbol{\sigma}})\otimes\mathbf{p}+\widetilde{\boldsymbol{\sigma}}\cdot\nabla\mathbf{p}+\rho(\widetilde{\mathbf{b}}-j\widetilde{\mathbf{a}})\otimes\mathbf{p}, \tag{4.141}$$

$$\left[\boldsymbol{\sigma}+(\operatorname{div}\widetilde{\boldsymbol{\sigma}})\otimes\mathbf{p}+\widetilde{\boldsymbol{\sigma}}\cdot\nabla\mathbf{p}+\rho\widetilde{\mathbf{b}}\otimes\mathbf{p}\right]^{\mathsf{T}}=\boldsymbol{\sigma}+(\operatorname{div}\widetilde{\boldsymbol{\sigma}})\otimes\mathbf{p}+\widetilde{\boldsymbol{\sigma}}\cdot\nabla\mathbf{p}+\rho(\widetilde{\mathbf{b}}-j\widetilde{\mathbf{a}})\otimes\mathbf{p}, \tag{4.142}$$

$$\rho\frac{\partial e}{\partial \boldsymbol{\nabla}}=\widetilde{\boldsymbol{\sigma}}\otimes\mathbf{p}. \tag{4.143}$$

In component form, (4.141) reads

$$2\rho\frac{\partial e}{\partial g_{ab}}=\sigma^{ab}+\widetilde{\sigma}^{ac}{}_{|c}p^b+\widetilde{\sigma}^{ac}p^b{}_{|c}+\rho(\widetilde{b}^a-\widetilde{a}^a)p^b=\sigma^{ab}+\rho(\widetilde{b}^a-\widetilde{a}^a)p^b+\left(\widetilde{\sigma}^{ac}p^b\right)_{|c}. \tag{4.144}$$

Note that combining (4.140) and (4.143), one can write balance of linear momentum as

$$\operatorname{div}\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\mathbf{a}+(\widetilde{\boldsymbol{\sigma}}\otimes\mathbf{p}):\boldsymbol{\mathcal{R}}. \tag{4.145}$$

This means that both stress and micro-stress tensors contribute to balance of linear momentum. It is seen that there is a single balance of linear momentum, a single balance of angular momentum both with contributions from forces and micro-forces, and two Doyle–Ericksen formulas.

We should mention that Toupin [35, 36] showed that for elastic materials for which energy depends on gradient of the deformation gradient, i.e. the second derivative of deformation mapping, balance of linear momentum and angular momentum are both coupled for micro and macro forces. However, as was mentioned earlier, here we are not considering second-grade materials.

Generalized covariance of energy balance for spatially constrained structured continua. In all the previous examples we observed that covariance under a single spatial diffeomorphism cannot lead to a separate balance of micro-linear momentum. Let us consider two diffeomorphisms $\xi_t, \eta_t : \mathcal{S} \to \mathcal{S}$ such that both are identity at $t = t_0$ and

$$\mathbf{z} \neq \mathbf{w}, \qquad \nabla \mathbf{z} \neq \nabla \mathbf{w}, \qquad \nabla\nabla \mathbf{z} = \nabla\nabla \mathbf{w}, \tag{4.146}$$

where

$$\mathbf{w} = \frac{\partial}{\partial t}\Big|_{t=t_0} \xi_t, \qquad \mathbf{z} = \frac{\partial}{\partial t}\Big|_{t=t_0} \eta_t. \tag{4.147}$$

We assume that under the simultaneous actions of these two diffeomorphisms, η_t acts on micro-quantities and ξ_t acts on the remaining quantities (including metric and connection). Thus, in the new frame

$$\mathbf{p}' = \eta_{t*}\mathbf{p}, \qquad \widetilde{\mathbf{v}}' = \widetilde{\mathbf{v}} + \nabla_{\mathbf{p}}\mathbf{z}, \qquad \widetilde{\mathbf{a}}' - \widetilde{\mathbf{b}}' = \eta_{t*}(\widetilde{\mathbf{a}} - \widetilde{\mathbf{b}}). \tag{4.148}$$

We assume that energy balance is invariant under the simultaneous actions of ξ_t and η_t and call this a *generalized covariance*. Therefore, generalized covariance implies that at time $t = t_0$

$$\begin{aligned} \int_{\varphi_t(\mathcal{U})} \Bigg[\mathbf{L_v}\rho \left(\frac{1}{2} \langle\langle \mathbf{w}, \mathbf{w} \rangle\rangle + \langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle \right) + \mathbf{L_v}(\rho j) \left(\frac{1}{2} \langle\langle \nabla\mathbf{z}\cdot\mathbf{p}, \nabla\mathbf{z}\cdot\mathbf{p} \rangle\rangle + \langle\langle \widetilde{\mathbf{v}}, \nabla\mathbf{z}\cdot\mathbf{p} \rangle\rangle \right) \\ + \rho \left(\frac{\partial e}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{w}}\mathbf{g} + \frac{\partial e}{\partial \nabla} : (\nabla\nabla\mathbf{w} + \boldsymbol{\mathcal{R}}\cdot\mathbf{w}) + \langle\langle \mathbf{a}, \mathbf{w} \rangle\rangle + \langle\langle j\widetilde{\mathbf{a}}, \nabla\mathbf{z}\cdot\mathbf{p} \rangle\rangle \right) \Bigg] \\ = \int_{\varphi_t(\mathcal{U})} \rho \langle\langle \mathbf{b}, \mathbf{w} \rangle\rangle + \int_{\partial\varphi_t(\mathcal{U})} \langle\langle \mathbf{t}, \mathbf{w} \rangle\rangle\, da + \int_{\varphi_t(\mathcal{U})} \left\langle\!\left\langle \rho\widetilde{\mathbf{b}}, \nabla\mathbf{z}\cdot\mathbf{p} \right\rangle\!\right\rangle + \int_{\partial\varphi_t(\mathcal{U})} \left\langle\!\left\langle \widetilde{\mathbf{t}}, \nabla\mathbf{z}\cdot\mathbf{p} \right\rangle\!\right\rangle da. \end{aligned} \tag{4.149}$$

Arbitrariness of $\mathbf{w}$ and $\mathbf{z}$ gives us conservation of mass $\mathbf{L_v}\rho = 0$, conservation of microstructure inertia $\mathbf{L_v} j = 0$, and the existence of stress tensors $\boldsymbol{\sigma}$ and $\widetilde{\boldsymbol{\sigma}}$. Thus

$$\begin{aligned} \int_{\varphi_t(\mathcal{U})} \rho \left(\frac{\partial e}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{w}}\mathbf{g} + \frac{\partial e}{\partial \nabla} : (\nabla\nabla\mathbf{w} + \boldsymbol{\mathcal{R}}\cdot\mathbf{w}) + \langle\langle \mathbf{a}, \mathbf{w} \rangle\rangle + \langle\langle j\widetilde{\mathbf{a}}, \nabla\mathbf{z}\cdot\mathbf{p} \rangle\rangle \right) \\ = \int_{\varphi_t(\mathcal{U})} \rho \langle\langle \mathbf{b}, \mathbf{w} \rangle\rangle + \int_{\partial\varphi_t(\mathcal{U})} \langle\langle \mathbf{t}, \mathbf{w} \rangle\rangle\, da + \int_{\varphi_t(\mathcal{U})} \left\langle\!\left\langle \rho\widetilde{\mathbf{b}}, \nabla\mathbf{z}\cdot\mathbf{p} \right\rangle\!\right\rangle \\ + \int_{\varphi_t(\mathcal{U})} [(\operatorname{div}\widetilde{\boldsymbol{\sigma}}) \otimes \mathbf{p} + \widetilde{\boldsymbol{\sigma}}\cdot\nabla\mathbf{p}] : \nabla\mathbf{z}\, dv + \int_{\varphi_t(\mathcal{U})} \widetilde{\boldsymbol{\sigma}} \otimes \widetilde{\mathbf{p}} : \nabla\nabla\mathbf{z}\, dv. \end{aligned} \tag{4.150}$$

Arbitrariness of $\mathbf{z}, \mathbf{w}$, and $\mathcal{U}$, and noting that $\nabla\nabla\mathbf{z} = \nabla\nabla\mathbf{w}$, one obtains

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a} + \rho\frac{\partial e}{\partial\nabla} : \boldsymbol{\mathcal{R}}, \tag{4.151}$$

$$2\rho\frac{\partial e}{\partial\mathbf{g}} = \boldsymbol{\sigma}, \tag{4.152}$$

$$\boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma},$$

$$\operatorname{div}(\widetilde{\boldsymbol{\sigma}} \otimes \mathbf{p}) + \rho\widetilde{\boldsymbol{b}} \otimes \mathbf{p} = \rho\widetilde{\boldsymbol{a}} \otimes \mathbf{p}, \tag{4.153}$$

$$\rho\frac{\partial e}{\partial\nabla} = \widetilde{\boldsymbol{\sigma}} \otimes \mathbf{p}. \tag{4.154}$$

It is seen that generalized covariance gives a separate balance of micro-linear momentum, i.e. Eq. (4.153).

5. Examples of continua with microstructure

In this section, we present two examples of continua with microstructure and obtain their governing equations covariantly. We first look at a theory of elastic solids with voids (see Nunziato and Cowin [31]), which is a structured continuum with a one-dimensional microstructure manifold. We show that microstructure covariance in this case gives all the balance laws and a scalar Doyle–Ericksen formula. We then geometrically study the classical theory of mixtures (see Bowen [3], Bedford and Drumheller [2], Green and Naghdi [18], Sampaio [32], Williams [38]) and obtain the governing equations covariantly.

5.1. A geometric theory of elastic solids with distributed voids

An elastic solid with distributed voids can be thought of as a structured continuum with a scalar microstructure kinematical variable, as in Capriz [6]; here, we follow Nunziato and Cowin [31]. In addition to the standard deformation mapping, it is assumed that mass density has the following multiplicative decomposition

$$\rho_0(\mathbf{X}) = \overline{\rho}_0(\mathbf{X}, t)\nu_0(\mathbf{X}, t), \tag{5.1}$$

where $\overline{\rho}_0$ is the density of the matrix material and ν_0 is the matrix volume fraction and $0 < \nu_0 \leq 1$. Deformation is a pair of mappings $(\varphi_t, \widetilde{\varphi}_t) : \mathcal{B} \times \mathcal{B} \to \mathcal{S} \times \mathbb{R}$. Material void velocity and void deformation gradient (a one-form on $\mathcal{B}$) are defined as

$$\widetilde{V}(\mathbf{X}, t) = \frac{\partial\nu_0(\mathbf{X}, t)}{\partial t}, \qquad \widetilde{\mathbf{F}}(\mathbf{X}, t) = \frac{\partial\nu_0(\mathbf{X}, t)}{\partial\mathbf{X}}. \tag{5.2}$$

Spatial void velocity is defined as $\widetilde{v} = \widetilde{V} \circ \varphi^{-1}$. Internal energy density at $\mathbf{x} \in \mathcal{S}$ has the following form

$$e = e(t, \mathbf{x}, \mathbf{g}, \nu, T\nu), \tag{5.3}$$

where $\nu = \nu_0 \circ \varphi$ and hence

$$(T\nu)_a = \frac{\partial\nu}{\partial x^a} = F^{-A}{}_a\frac{\partial\nu}{\partial X^A}. \tag{5.4}$$

For a subset $\varphi_t(\mathcal{U}) \subset \mathcal{S}$, balance of energy reads

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho(\mathbf{x},t)\left(e(t,\mathbf{x},\mathbf{g},\nu,T\nu)+\frac{1}{2}\langle\langle\mathbf{v},\mathbf{v}\rangle\rangle+\frac{1}{2}\kappa\ \widetilde{v}^{\ 2}\right)dv$$
$$=\int_{\varphi_t(\mathcal{U})} \rho(\mathbf{x},t)\left(\langle\langle\mathbf{b},\mathbf{v}\rangle\rangle+\widetilde{b}\ \widetilde{v}+r\right)dv+\int_{\partial\varphi_t(\mathcal{U})}\left(\langle\langle\mathbf{t},\mathbf{v}\rangle\rangle+\widetilde{t}\ \widetilde{v}+h\right)da, \quad (5.5)$$

where $\kappa=\kappa(\mathbf{x},t)$ is the so-called equilibrated inertia [31], and $\widetilde{b}$ and $\widetilde{t}$ are the void body force and traction, respectively, and both are scalars.

Let us first consider a time-dependent spatial change of frame $\xi_t:\mathcal{S}\to\mathcal{S}$ such that at $t=t_0$, $\xi_{t_0}=\text{id}$. Under this change of frame $\nu'(\mathbf{x}')=\nu(\mathbf{x})$ and hence

$$e'=e'(t,\mathbf{x}',\mathbf{g},\nu',T\nu')=e(t,\mathbf{x},\xi_t^*\mathbf{g},\nu,T\nu). \quad (5.6)$$

Therefore, at $t=t_0$

$$\dot{\overline{e'}}=\dot{e}+\frac{\partial e}{\partial\mathbf{g}}:\mathfrak{L}_{\mathbf{w}}\mathbf{g}, \quad (5.7)$$

where $\mathbf{w}=\frac{\partial}{\partial t}\xi_t\big|_{t=t_0}$. Subtracting balance of energy for $\varphi_t(\mathcal{U})$ from that of $\varphi'_t(\mathcal{U})$ at $t=t_0$, gives the existence of Cauchy stress and the standard balance laws [39].

Let us now consider a microstructure change of frame $\eta_t:(0,1]\to(0,1]$ such that $\eta_t\big|_{t=t_0}=\text{id}$ and

$$\frac{\partial\eta_t(\nu)}{\partial t}=z_t(\nu). \quad (5.8)$$

Void velocity in the new frame has the following form,

$$\widetilde{v}\ '=\frac{\partial}{\partial t}\eta_t\circ\nu=\eta_{t*}\widetilde{v}+z_t. \quad (5.9)$$

Thus, at $t=t_0$, $\widetilde{v}\ '(\nu)=\widetilde{v}(\nu)+z(\nu)$. Under the void change of frame, we have

$$e'(t,\mathbf{x},\mathbf{g},\nu',T\nu')=e(t,\mathbf{x},\mathbf{g},\nu,T\eta_t\cdot T\nu). \quad (5.10)$$

Note that

$$\frac{d}{dt}(T\eta_t\cdot T\nu)=\frac{d}{dt}\left(\frac{\partial\eta_t}{\partial\nu}\right)\frac{\partial\nu}{\partial\mathbf{X}}+\frac{\partial\eta_t}{\partial\nu}\frac{\partial\widetilde{v}}{\partial\mathbf{X}}=\frac{\partial z}{\partial\nu}\frac{\partial\nu}{\partial\mathbf{X}}+\frac{\partial^2\eta_t(\nu)}{\partial\nu^2}\frac{\partial\nu}{\partial\mathbf{X}}+\frac{\partial\eta_t}{\partial\nu}\frac{\partial\widetilde{v}}{\partial\mathbf{X}}. \quad (5.11)$$

Thus, at $t=t_0$

$$\frac{d}{dt}(T\eta_t\cdot T\nu)=z'(\nu)\frac{\partial\nu}{\partial\mathbf{X}}+\frac{\partial\eta_t}{\partial\nu}\frac{\partial\widetilde{v}}{\partial\mathbf{X}}. \quad (5.12)$$

Therefore, at $t=t_0$

$$\dot{\overline{e'}}=\dot{e}+\frac{\partial e}{\partial\nu_{,A}}\nu_{,A}z'(\nu). \quad (5.13)$$

Balance of energy in the new void frame at $t = t_0$ reads

$$\int_{\varphi_t(\mathcal{U})} \rho \left[\dot{e} + \frac{\partial e}{\partial \nu_{,A}} \nu_{,A} z'(\nu) + \langle\langle \mathbf{v}, \mathbf{a} \rangle\rangle + \frac{1}{2} \dot{\kappa} (\widetilde{v} + z)^2 + \kappa \widetilde{a} (\widetilde{v} + z) \right] dv$$
$$= \int_{\varphi_t(\mathcal{U})} \rho \left(\langle\langle \mathbf{b}, \mathbf{v} \rangle\rangle + \widetilde{b}(\widetilde{v} + z) + r \right) dv + \int_{\partial\varphi_t(\mathcal{U})} \left(\langle\langle \mathbf{t}, \mathbf{v} \rangle\rangle + \widetilde{t}(\widetilde{v} + z) + h \right) da. \quad (5.14)$$

Subtracting (5.5) from (5.14), one obtains

$$\int_{\varphi_t(\mathcal{U})} \rho \left[\frac{\partial e}{\partial \nu_{,A}} \nu_{,A} z'(\nu) + \frac{1}{2} \dot{\kappa} (2\widetilde{v} z + z^2) + \kappa \widetilde{a} z \right] dv = \int_{\varphi_t(\mathcal{U})} \rho \widetilde{b} z dv + \int_{\partial\varphi_t(\mathcal{U})} \widetilde{t} z da. \quad (5.15)$$

Because z and $\mathcal{U}$ are arbitrary, we conclude that $\dot{\kappa} = 0$, which is the balance of equilibrated inertia [15]. Using Cauchy's theorem in the above identity, we conclude that there exists a vector field $\widetilde{\boldsymbol{\sigma}}$ (void Cauchy stress), such that $\widetilde{t} = \widetilde{\sigma}^a \hat{n}_a$. Therefore, the surface integral in (5.15) can be simplified to read

$$\int_{\partial\varphi_t(\mathcal{U})} \widetilde{t} z da = \int_{\varphi_t(\mathcal{U})} \left[(\operatorname{div} \widetilde{\boldsymbol{\sigma}}) z + F^{-A}{}_a \widetilde{\sigma}^a \nu_{,A} z' \right] dv. \quad (5.16)$$

Now, (5.15) can be rewritten as

$$\int_{\varphi_t(\mathcal{U})} \left(\rho \frac{\partial e}{\partial \nu_{,A}} \nu_{,A} - F^{-A}{}_a \ \widetilde{\sigma}^a \nu_{,A} \right) z'(\nu) dv - \int_{\varphi_t(\mathcal{U})} \left(\operatorname{div} \widetilde{\boldsymbol{\sigma}} + \rho \widetilde{b} - \rho \widetilde{a} \right) z(\nu) dv = 0. \quad (5.17)$$

Because z and z' can be chosen independently and $\mathcal{U}$ is arbitrary, we conclude that

$$\operatorname{div} \widetilde{\boldsymbol{\sigma}} + \rho \widetilde{b} = \rho \widetilde{a}, \quad (5.18)$$

$$F^{-A}{}_a \ \widetilde{\sigma}^a \nu_{,A} = \rho \frac{\partial e}{\partial \nu_{,A}} \nu_{,A}. \quad (5.19)$$

Eq. (5.18) is balance of equilibrated linear momentum [31] and Eq. (5.19) is a scalar Doyle–Ericksen formula.

5.2. A geometric theory of mixtures

In mixture theory, one is given a finite number of bodies (constituents) that can penetrate into one another with the understanding that there is no self penetration within a given constituent. Here, for the sake of simplicity, we ignore diffusion as our goal is to demonstrate the power of covariance arguments in deriving the balance laws. We assume that in our mixture M there are two constituents; generalization of our results to the case of N constituents is straightforward. We denote the constituents by $\mathsf{1}$ and $\mathsf{2}$. We should mention that recently Mariano [24] studied some invariance/covariance ideas for mixtures. Our approach is slightly different as will be explained in the sequel.

Each constituent is assumed to have its own reference manifold $\left({}^{\mathsf{i}}\mathcal{B}, {}^{\mathsf{i}}\mathbf{G}\right)$, $\mathsf{i}=1,2$. Deformation of M is defined by two deformation mappings ${}^{\mathsf{i}}\varphi_t$, $\mathsf{i}=1,2$ such that

$$ {}^{\mathsf{i}}\varphi_t : \left({}^{\mathsf{i}}\mathcal{B}, {}^{\mathsf{i}}\mathbf{G}\right) \to \left(\mathcal{S}, {}^{\mathsf{i}}\mathbf{g}\right), \qquad \mathsf{i}=1,2, \tag{5.20} $$

i.e., it is assumed that the ambient space manifold $\mathcal{S}$ is equipped with two different metrics ${}^{1}\mathbf{g}$ and ${}^{2}\mathbf{g}$.[3] Material and spatial velocities are defined as

$$ {}^{\mathsf{i}}\mathbf{V}(\mathbf{X}_{\mathsf{i}}, t) = \frac{\partial\ {}^{\mathsf{i}}\varphi_t(\mathbf{X}_{\mathsf{i}}, t)}{\partial t}, \qquad {}^{\mathsf{i}}\mathbf{v} = {}^{\mathsf{i}}\mathbf{V} \circ {}^{\mathsf{i}}\varphi_t^{-1}, \qquad \mathsf{i}=1,2. \tag{5.21} $$

Deformation gradients are tangent maps of the two deformation mappings, i.e., ${}^{\mathsf{i}}\mathbf{F} = T\ {}^{\mathsf{i}}\varphi$, $\mathsf{i}=1,2$.

Given $\mathbf{x} \in {}^{\mathsf{i}}\varphi_t(\mathcal{B}_{\mathsf{i}})$, it is assumed that this point is occupied by particles from both $\mathcal{B}_1$ and $\mathcal{B}_2$, i.e., given a time t_0, $\mathbf{x}$ is the pre-image of particles $\mathbf{X}_1$ and $\mathbf{X}_2$ defined as

$$ \mathbf{X}_1 = {}^{1}\varphi_{t_0}^{-1}(\mathbf{x}) \qquad \text{and} \qquad \mathbf{X}_2 = {}^{2}\varphi_{t_0}^{-1}(\mathbf{x}). \tag{5.22} $$

Thus, at a later time t

$$ {}^{1}\varphi_t(\mathbf{X}_1) = {}^{1}\varphi_t \circ {}^{1}\varphi_{t_0}^{-1}(\mathbf{x}) \neq {}^{2}\varphi_t(\mathbf{X}_2) = {}^{2}\varphi_t \circ {}^{2}\varphi_{t_0}^{-1}(\mathbf{x}), \tag{5.23} $$

i.e., in general, the two particles $\mathbf{X}_1$ and $\mathbf{X}_2$ will occupy two different points of $\mathcal{S}$ at time t. This means that one can have spatial changes of frame that act separately on different constituents.[4]

In the traditional formulation of mixture theories, for each constituent, one assumes the existence of an internal energy density and a "growth" of internal energy density. Here, we assume that each constituent has an internal energy that depends on all the spatial metrics. For our two-phase mixture M this means that

$$ e_1 = e_1\left(t, \mathbf{x}, {}^{1}\mathbf{g}, {}^{2}\mathbf{g}\right) \qquad \text{and} \qquad e_2 = e_2\left(t, \mathbf{x}, {}^{1}\mathbf{g}, {}^{2}\mathbf{g}\right). \tag{5.24} $$

Dependence of each internal energy density on both the spatial metrics accounts for the interaction of constituents. Each constituent is assumed to have its own mass density ρ_{i}, $\mathsf{i}=1,2$, and mass density at point $\mathbf{x}$ is defined as

$$ \rho(\mathbf{x}, t) = \nu_1(\mathbf{x}, t)\rho_1(\mathbf{x}, t) + \nu_2(\mathbf{x}, t)\rho_2(\mathbf{x}, t), \tag{5.25} $$

where ν_{i} are volume fractions of the constituents, although at this point we do not need to define ρ.

Balance of energy for a subset $\mathcal{U}_t = {}^{1}\varphi_t(\mathcal{U}_1) = {}^{2}\varphi_t(\mathcal{U}_2) \subset \mathcal{S}$ is written as

$$ \begin{aligned} &\frac{d}{dt}\int_{\mathcal{U}_t} \sum_{\mathsf{i}} \rho_{\mathsf{i}}(\mathbf{x}, t)\left[e_{\mathsf{i}}\left(\mathbf{x}, t, {}^{1}\mathbf{g}, {}^{2}\mathbf{g}\right) + \frac{1}{2}\left\langle\!\left\langle {}^{\mathsf{i}}\mathbf{v}, {}^{\mathsf{i}}\mathbf{v}\right\rangle\!\right\rangle_{\mathsf{i}}\right] \\ &\qquad = \int_{\mathcal{U}_t} \sum_{\mathsf{i}} \rho_{\mathsf{i}}(\mathbf{x}, t)\left(\left\langle\!\left\langle {}^{\mathsf{i}}\mathbf{b}, {}^{\mathsf{i}}\mathbf{v}\right\rangle\!\right\rangle_{\mathsf{i}} + r_{\mathsf{i}}\right) + \int_{\partial\mathcal{U}_t} \sum_{\mathsf{i}} \left(\left\langle\!\left\langle {}^{\mathsf{i}}\mathbf{t}, {}^{\mathsf{i}}\mathbf{v}\right\rangle\!\right\rangle_{\mathsf{i}} + h_{\mathsf{i}}\right) da, \end{aligned} \tag{5.26} $$

---

[3]This is similar to what Mariano [24] does when postulating covariance of energy balance.

[4]This is closely related to what Mariano [24] does in his energy balance covariance argument.

where $\langle\langle .,. \rangle\rangle_{\mathsf{i}}$ is the inner product induced from the metric ${}^{\mathsf{i}}\mathbf{g}$ and all the other quantities have the obvious meanings. Balance of energy can be simplified to read

$$\int_{\mathcal{U}_t} \sum_{\mathsf{i}} \mathbf{L}_{\ {}^{\mathsf{i}}\mathbf{v}}\ \rho_{\mathsf{i}}(\mathbf{x},t)\left[e_{\mathsf{i}}\left(\mathbf{x},t,{}^1\mathbf{g},{}^2\mathbf{g}\right)+\frac{1}{2}\left\langle\!\left\langle {}^{\mathsf{i}}\mathbf{v},{}^{\mathsf{i}}\mathbf{v}\right\rangle\!\right\rangle_{\mathsf{i}}\right]$$
$$+\int_{\mathcal{U}_t} \sum_{\mathsf{i}} \rho_{\mathsf{i}}(\mathbf{x},t)\left[\dot{e}_{\mathsf{i}}\left(\mathbf{x},t,{}^1\mathbf{g},{}^2\mathbf{g}\right)+\left\langle\!\left\langle {}^{\mathsf{i}}\mathbf{v},{}^{\mathsf{i}}\mathbf{a}\right\rangle\!\right\rangle_{\mathsf{i}}\right]$$
$$=\int_{\mathcal{U}_t} \sum_{\mathsf{i}} \rho_{\mathsf{i}}(\mathbf{x},t)\left(\left\langle\!\left\langle {}^{\mathsf{i}}\mathbf{b},{}^{\mathsf{i}}\mathbf{v}\right\rangle\!\right\rangle_{\mathsf{i}}+r_{\mathsf{i}}\right)+\int_{\partial\mathcal{U}_t} \sum_{\mathsf{i}} \left(\left\langle\!\left\langle {}^{\mathsf{i}}\mathbf{t},{}^{\mathsf{i}}\mathbf{v}\right\rangle\!\right\rangle_{\mathsf{i}}+h_{\mathsf{i}}\right)da. \quad (5.27)$$

Traditionally, a separate balance of energy is postulated for each constituent [24]. Here, we only postulate a balance of energy for the whole mixture.

We now consider a spatial diffeomorphism $\xi_t:\mathcal{S}\to\mathcal{S}$ that acts only on $(\mathcal{S},{}^1\mathbf{g})$ and is the identity map at $t=t_0$. We postulate covariance of energy balance, i.e., in the new spatial frame energy balance reads

$$\frac{d}{dt}\int_{\mathcal{U}'_t} \rho'_1(\mathbf{x}',t)\left[e'_1\left(\mathbf{x}',t,{}^1\mathbf{g},{}^2\mathbf{g}\right)+\frac{1}{2}\langle\langle {}^1\mathbf{v}',{}^1\mathbf{v}'\rangle\rangle_1\right]$$
$$+\frac{d}{dt}\int_{\mathcal{U}'_t} \rho'_2(\mathbf{x}',t)\left[e'_2\left(\mathbf{x}',t,{}^1\mathbf{g},{}^2\mathbf{g}\right)+\frac{1}{2}\langle\langle {}^2\mathbf{v}',{}^2\mathbf{v}'\rangle\rangle_2\right]$$
$$=\int_{\mathcal{U}'_t} \rho'_1(\mathbf{x}',t)\left(\langle\langle {}^1\mathbf{b}',{}^1\mathbf{v}'\rangle\rangle_1+r'_1\right)+\int_{\partial\mathcal{U}'_t}\left(\langle\langle {}^1\mathbf{t}',{}^1\mathbf{v}'\rangle\rangle_1+h'\right)da',$$
$$+\int_{\mathcal{U}'_t} \rho'_2(\mathbf{x}',t)\left(\langle\langle {}^2\mathbf{b}',{}^2\mathbf{v}'\rangle\rangle_2+r'_2\right)+\int_{\partial\mathcal{U}'_t}\left(\langle\langle {}^2\mathbf{t}',{}^2\mathbf{v}'\rangle\rangle_2+h'_2\right)da'. \quad (5.28)$$

Spatial velocities have the following transformations,

$${}^1\mathbf{v}'=\xi_{t*}{}^1\mathbf{v}+\mathbf{w}_t \qquad \text{and} \qquad {}^2\mathbf{v}'={}^2\mathbf{v}. \quad (5.29)$$

We assume that ${}^1\mathbf{b}$ is transformed such that [27]

$${}^1\mathbf{b}'-{}^1\mathbf{a}'=\xi_{t*}\left({}^1\mathbf{b}-{}^1\mathbf{a}\right). \quad (5.30)$$

Note also that

$$e'_1\left(\mathbf{x}',t,{}^1\mathbf{g},{}^2\mathbf{g}\right)=e_1\left(\mathbf{x},t,\xi_t^*\ {}^1\mathbf{g},{}^2\mathbf{g}\right), \quad (5.31)$$
$$e'_2\left(\mathbf{x}',t,{}^1\mathbf{g},{}^2\mathbf{g}\right)=e_2\left(\mathbf{x},t,\xi_t^*\ {}^1\mathbf{g},{}^2\mathbf{g}\right). \quad (5.32)$$

Thus, at $t=t_0$,

$$\dot{\overline{e'_1}}=\dot{e}_1+\frac{\partial e_1}{\partial\ {}^1\mathbf{g}}:\mathfrak{L}_{\mathbf{w}}{}^1\mathbf{g}, \quad (5.33)$$
$$\dot{\overline{e'_2}}=\dot{e}_2+\frac{\partial e_2}{\partial\ {}^1\mathbf{g}}:\mathfrak{L}_{\mathbf{w}}{}^1\mathbf{g}. \quad (5.34)$$

Subtracting the energy balance (5.27) from (5.28) evaluated at $t = t_0$ yields

$$\int_{\mathcal{U}_t} \mathbf{L}_{\ {}^1\mathbf{v}}\ \rho_1(\mathbf{x}, t) \left[\left\langle\!\left\langle \mathbf{w}, {}^1\mathbf{v} \right\rangle\!\right\rangle_1 + \frac{1}{2} \langle\langle \mathbf{w}, \mathbf{w} \rangle\rangle_1 \right] + \int_{\mathcal{U}_t} \rho_1(\mathbf{x}, t) \left(\frac{\partial e_1}{\partial\ {}^1\mathbf{g}} + \frac{\partial e_2}{\partial\ {}^1\mathbf{g}} \right) : \mathfrak{L}_{\mathbf{w}}\mathbf{g}$$
$$= \int_{\mathcal{U}_t} \rho_1(\mathbf{x}, t) \left(\left\langle\!\left\langle {}^1\mathbf{b} - {}^1\mathbf{a}, \mathbf{w} \right\rangle\!\right\rangle_1 \right) + \int_{\partial\mathcal{U}_t} \left(\left\langle\!\left\langle {}^1\mathbf{t}, \mathbf{w} \right\rangle\!\right\rangle_1 \right) da. \quad (5.35)$$

Arbitrariness of $\mathcal{U}_t$ and $\mathbf{w}$ would guarantee the existence of a Cauchy stress ${}^1\boldsymbol{\sigma}$ such that ${}^1\mathbf{t} = \langle\langle {}^1\boldsymbol{\sigma}, \hat{\mathbf{n}} \rangle\rangle_1$ and also will give the following after replacing ρ_1 by $\rho_1 dv$,

$$\mathbf{L}_{\ {}^1\mathbf{v}}\ \rho_1 = 0, \quad (5.36)$$

$$\mathrm{div}_1\, {}^1\boldsymbol{\sigma} + \rho_1 {}^1\mathbf{b} = \rho_1 {}^1\mathbf{a}, \quad (5.37)$$

$${}^1\boldsymbol{\sigma} = {}^1\boldsymbol{\sigma}^{\mathsf{T}}, \quad (5.38)$$

$${}^1\boldsymbol{\sigma} = 2\rho_1 \frac{\partial(e_1 + e_2)}{\partial\ {}^1\mathbf{g}}. \quad (5.39)$$

Similarly, assuming that $\xi_t : \mathcal{S} \to \mathcal{S}$ acts only on $(\mathcal{S}, {}^2\mathbf{g})$ and postulating energy balance covariance will give the following balance laws,

$$\mathbf{L}_{\ {}^2\mathbf{v}}\ \rho_2 = 0, \quad (5.40)$$

$$\mathrm{div}_2\, {}^2\boldsymbol{\sigma} + \rho_2 {}^2\mathbf{b} = \rho_2 {}^2\mathbf{a}, \quad (5.41)$$

$${}^2\boldsymbol{\sigma} = {}^2\boldsymbol{\sigma}^{\mathsf{T}}, \quad (5.42)$$

$${}^2\boldsymbol{\sigma} = 2\rho_2 \frac{\partial(e_1 + e_2)}{\partial\ {}^2\mathbf{g}}. \quad (5.43)$$

Note the coupling in the Doyle–Ericksen formulas. Note also that these balance laws can be pulled back to either $\mathcal{B}_1$ or $\mathcal{B}_2$.

6. Lagrangian field theory of continua with microstructure, noether's theorem and covariance

The original formulations of Cosserat continua were mainly variational [35, 36]. There have also been recent geometric formulations in the literature [7, 11]. In this section we consider a Lagrangian density that depends explicitly on metrics and look at the corresponding Euler–Lagrange equations. Then an explicit relation between covariance and Noether's theorem is established. Similar to the ambiguity encountered in covariant energy balance in terms of the link of the microstructure manifold with the ambient space manifold, here we will see that this ambiguity shows up in the action of a given flow on different independent variables of the Lagrangian density.

The Lagrangian may be regarded as a map $L : T\mathcal{C} \to \mathbb{R}$, where $\mathcal{C}$ is the space of some sections[5], associated to the Lagrangian density $\mathcal{L}$ and a volume element

[5] See Marsden and Hughes [27] for details in the case of standard continua. The case of structured continua would be a straightforward generalization.

$dV(X)$ on $\mathcal{B}$ and is defined as

$$L(\varphi,\dot{\varphi},\widetilde{\varphi},\dot{\widetilde{\varphi}}) = \int_{\mathcal{B}} \mathcal{L}\Big(X,\varphi(X),\dot{\varphi}(X),\mathbf{F}(X),\mathbf{G}(X),\mathbf{g}(\varphi(X)),\widetilde{\varphi}(X),\dot{\widetilde{\varphi}}(X),\widetilde{\mathbf{F}}(X),\widetilde{\mathbf{g}}(\varphi(X))\Big)dV(X). \tag{6.1}$$

Here φ and $\widetilde{\varphi}$ are understood as fields representing standard and microstructure deformations, respectively. Note that, in general, one may need to consider more than one microstructure field with possibly different tensorial properties. Note also that in this material representation, the two maps φ and $\widetilde{\varphi}$ have the same role and it is not clear from the Lagrangian density which one is the standard deformation map. However, having the coordinate representation for these two maps and their tangent maps, one can see which one is the microstructure map. Note also that $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ are background metrics with no dynamics.

The *action function* is defined as

$$S(\varphi) = \int_{t_0}^{t_1} L(\varphi,\dot{\varphi},\widetilde{\varphi},\dot{\widetilde{\varphi}})dt. \tag{6.2}$$

Hamilton's principle states that the physical configuration $(\varphi,\widetilde{\varphi})$ is the critical point of the action, i.e.

$$\delta S = \mathbf{d}S(\varphi,\widetilde{\varphi})\cdot(\delta\varphi,\delta\widetilde{\varphi}) = 0. \tag{6.3}$$

This can be simplified to read

$$\int_{t_0}^{t_1}\int_{\mathcal{B}}\Big(\frac{\partial\mathcal{L}}{\partial\varphi}\cdot\delta\varphi + \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\cdot\delta\dot{\varphi} + \frac{\partial\mathcal{L}}{\partial\mathbf{F}}:\delta\mathbf{F} + \frac{\partial\mathcal{L}}{\partial\mathbf{g}}:\delta\mathbf{g} + \frac{\partial\mathcal{L}}{\partial\widetilde{\varphi}}\cdot\delta\widetilde{\varphi} + \frac{\partial\mathcal{L}}{\partial\dot{\widetilde{\varphi}}}\cdot\delta\dot{\widetilde{\varphi}} + \frac{\partial\mathcal{L}}{\partial\widetilde{\mathbf{F}}}:\delta\widetilde{\mathbf{F}} + \frac{\partial\mathcal{L}}{\partial\widetilde{\mathbf{g}}}:\delta\widetilde{\mathbf{g}}\Big)dV(X)dt = 0. \tag{6.4}$$

As $\delta\varphi$ and $\delta\widetilde{\varphi}$ are independent, we obtain the following Euler–Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\varphi^a} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}^a} - \left(\frac{\partial\mathcal{L}}{\partial F^a{}_A}\right)_{|A} - \frac{\partial\mathcal{L}}{\partial F^b{}_A}F^c{}_A\gamma^b_{ac} + 2\frac{\partial\mathcal{L}}{\partial g_{cd}}\ g_{bd}\gamma^b_{ac} = 0, \tag{6.5}$$

$$\frac{\partial\mathcal{L}}{\partial\widetilde{\varphi}^\alpha} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\widetilde{\varphi}}^\alpha} - \left(\frac{\partial\mathcal{L}}{\partial\widetilde{F}^\alpha{}_A}\right)_{|A} - \frac{\partial\mathcal{L}}{\partial\widetilde{F}^\beta{}_A}\widetilde{F}^\mu{}_A\widetilde{\gamma}^\beta_{\alpha\mu} + 2\frac{\partial\mathcal{L}}{\partial\widetilde{g}_{\mu\lambda}}\ \widetilde{g}_{\beta\lambda}\widetilde{\gamma}^\beta_{\alpha\mu} = 0. \tag{6.6}$$

We know that because of material-frame-indifference, $\mathcal{L}$ depends on $\mathbf{F}$ and $\mathbf{g}$ through $\mathbf{C}$. Thus, Euler–Lagrange equations for the standard deformation mapping is simplified to read

$$P_a{}^A{}_{|A} + \frac{\partial\mathcal{L}}{\partial\varphi^a} = \rho_0 g_{ab}A^b, \tag{6.7}$$

$$\frac{\partial\mathcal{L}}{\partial\widetilde{\varphi}^\alpha} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\widetilde{\varphi}}^\alpha} + \widetilde{P}_\alpha{}^A{}_{|A} + \widetilde{P}_\beta{}^A\widetilde{F}^\mu{}_A\widetilde{\gamma}^\beta_{\alpha\mu} + 2\frac{\partial\mathcal{L}}{\partial\widetilde{g}_{\mu\lambda}}\ \widetilde{g}_{\beta\lambda}\widetilde{\gamma}^\beta_{\alpha\mu} = 0, \tag{6.8}$$

where

$$P_a{}^A = -\frac{\partial \mathcal{L}}{\partial F^a{}_A}, \qquad \widetilde{P}_\alpha{}^A = -\frac{\partial \mathcal{L}}{\partial \widetilde{F}^\alpha{}_A}. \tag{6.9}$$

When Euler–Lagrange equations are satisfied, given a symmetry of the Lagrangian density **Noether's theorem** tells us what its corresponding conserved quantity is. Suppose ψ_s is a flow on $\mathcal{S}$ generated by a vector field $\mathbf{w}$, i.e.

$$\frac{d}{ds}\Big|_{s=0} \psi_s \circ \varphi = \mathbf{w} \circ \varphi. \tag{6.10}$$

Now if we assume that this flow leaves the microstructure quantities unchanged, i.e., if we assume that the ambient space manifold and the microstructure manifold are independent, then invariance of the Lagrangian density means that

$$\mathcal{L}\left(X^A, \psi_s^a(\varphi), \frac{\partial \psi_s^a}{\partial x^b}\dot{\varphi}^b, \frac{\partial \psi_s^a}{\partial x^b} F^b{}_A, G_{AB}, -\frac{\partial \psi_s^c}{\partial x^a}\frac{\partial \psi_s^d}{\partial x^b} g_{cd}, \widetilde{\varphi}^\alpha, \dot{\widetilde{\varphi}}^\alpha, \widetilde{F}^\alpha{}_A, \widetilde{g}_{\alpha\beta}\right)$$
$$= \mathcal{L}\left(X^A, \varphi^a, \dot{\varphi}^a, F^a{}_A, G_{AB}, g_{ab}, \widetilde{\varphi}^\alpha, \dot{\widetilde{\varphi}}^\alpha, \widetilde{F}^\alpha{}_A, \widetilde{g}_{\alpha\beta}\right). \tag{6.11}$$

Yavari et al. [39] proved that this implies the following two conditions

$$2\frac{\partial \mathcal{L}}{\partial g_{ab}} = g^{bc}\frac{\partial \mathcal{L}}{\partial F^c{}_A} F^a{}_A + g^{bc}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}^c}\dot{\varphi}^a, \tag{6.12}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi^a} = 0, \tag{6.13}$$

i.e., the Doyle–Ericksen formula and spatial homogeneity of the Lagrangian density. Now, suppose η_s is a flow on $\mathcal{M}$ generated by a vector field $\mathbf{z}$, i.e.

$$\frac{d}{ds}\Big|_{s=0} \eta_s \circ \widetilde{\varphi} = \mathbf{z} \circ \widetilde{\varphi}. \tag{6.14}$$

Invariance of the Lagrangian density with respect to η_s means that

$$\mathcal{L}\left(X^A, \varphi^a, \dot{\varphi}^a, F^a{}_A, G_{AB}, g_{ab}, \eta_s^\alpha(\widetilde{\varphi}), \frac{\partial \eta_s^\alpha}{\partial p^\beta}\ \dot{\widetilde{\varphi}}^\beta, \frac{\partial \eta_s^\alpha}{\partial p^\beta}\ \widetilde{F}^\beta{}_A, -\frac{\partial \eta_s^\mu}{\partial p^\alpha}\frac{\partial \eta_s^\lambda}{\partial p^\beta}\ \widetilde{g}_{\mu\lambda}\right)$$
$$= \mathcal{L}\left(X^A, \varphi^a, \dot{\varphi}^a, F^a{}_A, G_{AB}, g_{ab}, \widetilde{\varphi}^\alpha, \dot{\widetilde{\varphi}}^\alpha, \widetilde{F}^\alpha{}_A, \widetilde{g}_{\alpha\beta}\right). \tag{6.15}$$

Differentiating the above identity with respect to s and evaluating it for $s = 0$, after some lengthy manipulations we obtain

$$2\frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\alpha\beta}} = \widetilde{F}^\alpha{}_A\ \widetilde{g}^{\beta\mu}\frac{\partial \mathcal{L}}{\partial \widetilde{F}^\mu{}_A} + g^{\beta\mu}\frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^\mu}\dot{\widetilde{\varphi}}^\alpha, \tag{6.16}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^\alpha} + \left(\frac{\partial \mathcal{L}}{\partial \widetilde{F}^\alpha{}_A}\right)_{|A} - \frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^\lambda}\ \widetilde{\gamma}^\lambda_{\alpha\mu}\dot{\widetilde{\varphi}}^\mu = 0. \tag{6.17}$$

Assuming that $\mathcal{L}$ has the following splitting in terms of internal energy density and kinetic energy

$$\mathcal{L} = \rho_0 e + \frac{1}{2}\rho_0 \left\langle\left\langle \mathbf{V}, \mathbf{V} \right\rangle\right\rangle_{\mathbf{g}} + \frac{1}{2}\widetilde{\rho}_0 \left\langle\left\langle \widetilde{\mathbf{V}}, \widetilde{\mathbf{V}} \right\rangle\right\rangle_{\widetilde{\mathbf{g}}}, \tag{6.18}$$

(6.16) is simplified to read

$$2\rho_0 \frac{\partial e}{\partial \widetilde{g}_{\alpha\beta}} = \widetilde{F}^{\alpha}{}_{A} \ \widetilde{g}^{\beta\mu} \frac{\partial \mathcal{L}}{\partial \widetilde{F}^{\mu}{}_{A}} = \widetilde{F}^{\alpha}{}_{A} \widetilde{P}^{\beta A}. \tag{6.19}$$

Now let us simplify this relation and show that it is exactly equivalent to (4.40). Note that

$$\widetilde{P}^{\alpha A} = J \left(\mathbf{F}^{-1}\right)^{A}{}_{b} \ \widetilde{\sigma}^{\alpha b}. \tag{6.20}$$

Thus

$$\widetilde{F}^{\alpha}{}_{A} \widetilde{P}^{\beta A} = J \left(\widetilde{\mathbf{F}}\mathbf{F}^{-1}\right)^{\alpha}{}_{b} \ \widetilde{\sigma}^{\beta b}. \tag{6.21}$$

Hence

$$2\rho \frac{\partial e}{\partial \widetilde{g}_{\alpha\beta}} = (\mathbf{F}_0)^{\alpha}{}_{b} \ \widetilde{\sigma}^{\beta b}. \tag{6.22}$$

This means that (6.16) is equivalent to (4.40)!

Following Yavari et al. [39], it can be shown that using Euler–Lagrange equations and some lengthy manipulations, (6.17) can be simplified to read

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^{\alpha}} = 0. \tag{6.23}$$

This means that if Lagrangian density is microstructurally covariant, then it has to be microstructurally homogenous and a micro-Doyle–Ericksen formula should be satisfied.

de Fabritiis and Mariano [11] study invariance of Lagrangian density of a structured continuum under different groups of transformations. In particular, they require invariance of the Lagrangian density when the same copy of $SO(3)$ acts on ambient space and microstructure manifolds in order to obtain balance of angular momentum. This seems to be a matter of choice at first sight but can also be understood as an interpretation of balance of angular momenta for a special class of structured continua. In the following, we study a similar symmetry of the Lagrangian density.

Constrained microstructure manifold. Now let us assume that $\mathcal{M}(\mathbf{X}) = T_{\varphi_t(\mathbf{X})}\mathcal{S}$. In this case the Euler–Lagrange equations are

$$P_a{}^{A}{}_{|A} + \frac{\partial \mathcal{L}}{\partial \varphi^a} = \rho_0 g_{ab} A^b, \tag{6.24}$$

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^a} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^a} - \left(\frac{\partial \mathcal{L}}{\partial \widetilde{F}^a{}_A}\right)_{|A} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^b{}_A}\widetilde{F}^c{}_A \gamma^b_{ac} = 0. \tag{6.25}$$

Now a flow on $\mathcal{S}$ would affect the microstructure quantities too. In this case invariance of the Lagrangian density means that

$$\mathcal{L}\left(X^A,\, \psi_s^a(\varphi),\, \frac{\partial \psi_s^a}{\partial x^b}\dot{\varphi}^b,\, \frac{\partial \psi_s^a}{\partial x^b}F^b{}_A,\, G_{AB},\, -\frac{\partial \psi_s^c}{\partial x^a}\frac{\partial \psi_s^d}{\partial x^b}g_{cd},\, \frac{\partial \psi_s^a}{\partial x^b}\widetilde{\varphi}^b,\, \frac{\partial \psi_s^a}{\partial x^b}\dot{\widetilde{\varphi}}^b,\, \frac{\partial \psi_s^a}{\partial x^b}\widetilde{F}^b{}_A\right)$$
$$= \mathcal{L}\left(X^A, \varphi^a, \dot{\varphi}^a, F^a{}_A, G_{AB}, g_{ab}, \widetilde{\varphi}^a, \dot{\widetilde{\varphi}}^a, \widetilde{F}^a{}_A\right). \quad (6.26)$$

Differentiating the above identity with respect to s and evaluating it at $s = 0$ yields

$$2\frac{\partial \mathcal{L}}{\partial g_{ab}} = g^{bc}\left(\frac{\partial \mathcal{L}}{\partial F^c{}_A}F^a{}_A + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^c}\dot{\varphi}^a + \frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^c}\widetilde{\varphi}^a + \frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^c}\dot{\widetilde{\varphi}}^a + \frac{\partial \mathcal{L}}{\partial \widetilde{F}^c{}_A}\widetilde{F}^a{}_A\right), \quad (6.27)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi^a} - \left(\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^c}\widetilde{\varphi}^a + \frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^c}\dot{\widetilde{\varphi}}^a + \frac{\partial \mathcal{L}}{\partial \widetilde{F}^c{}_A}\widetilde{F}^a{}_A\right)\gamma^c_{ab} = 0. \quad (6.28)$$

7. Concluding remarks

This paper first critically reviewed the geometry of structured continua. Similar to classical continuum mechanics, one assumes the existence of a well-defined reference configuration and each material point is mapped to its current position in the ambient space by the standard deformation mapping. In addition to this, each material point is given a director, which lies in a microstructure manifold. A separate map, the microstructure deformation mapping, maps each material point to its director, which could be a scalar field, a vector field, or in general a tensor field.

The Green–Naghdi–Rivlin Theorem relates balance laws to invariance of balance of energy under some groups of transformations. Previous attempts to extend this theorem to structured continua were critically reviewed. It was explained that any generalization of this theorem explicitly depends on the nature of the microstructure manifold. It turns out that in most continua with microstructure, the microstructure manifold is linked to the ambient space manifold. We gave a concrete example of a structured continuum, in which the ambient space is Euclidean, for which the microstructure manifold is again $\mathbb{R}^3$ but thought of as the tangent space of $\mathbb{R}^3$ at a given point. Postulating balance of energy and its invariance under isometries of $\mathbb{R}^3$, we obtained conservation of mass, balance of linear momentum and balance of angular momentum with contributions from both macro and micro-forces. Limiting oneself to rigid motions does not allow one to obtain a separate balance of micro-linear momentum. This leads one to think about investigating covariant balance laws for structured continua.

We first assumed that the structured continuum is such that the ambient space and the macrostructure manifold can have independent reframings. We showed that postulating energy balance and its invariance under spatial and microstructure diffeomorphisms gives conservation of mass, existence of Cauchy stress and micro-Cauchy stress, balance of linear and micro-linear momenta, balance of angular and micro-angular momenta and two Doyle–Ericksen formulas. We then considered structured continua for which the microstructure manifold is somewhat constrained in the sense that a spatial change of frame affects the microstructure quantities too. As concrete examples, we defined materially and spatially constrained structured continua. In a spatially constrained continuum the microstructure bundle is the tangent bundle of the ambient space manifold. In a materially constrained structured continuum, microstructure manifold at a given point $\mathbf{X} \in \mathcal{B}$ is $T_{\mathbf{X}}\mathcal{B}$. We showed that postulating energy balance and its invariance under spatial diffeomorphisms for a MCS continuum gives conservation of mass, two balances of linear momentum, two balances of angular momentum and two Doyle–Ericksen formulas. For a SCS continuum, spatial covariance gives balances of linear and angular momenta, which both have contributions from macro and micro forces. We then defined a generalized covariance in which two separate maps act on macro and micro quantities simultaneously. Under some assumptions, we showed that generalized covariance can give a coupled balance of angular momentum and two separate balances of linear momentum for macro and micro forces.

As concrete examples of structured continua, we looked at elastic solids with distributed voids and mixtures and obtained their balance laws covariantly.

In the last part of the paper, we reviewed the Lagrangian field theory of structured continua, when both ambient space and microstructure manifolds are equipped with their own metrics. Assuming that standard deformation mapping and microstructure deformation mapping are independent, they would have independent variations and hence Hamilton's principle of least action gives us two sets of Euler–Lagrange equations. We then studied the connection between Noether's theorem and covariance. It was observed that there is some ambiguity in making this connection. The ambiguity arises from the fact that there are different possibilities in defining covariance for a Lagrangian density. One choice is to assume that the Lagrangian density is covariant under independent actions of spatial and microstructure flows. We showed that this results in Doyle–Ericksen formulas identical to those obtained from covariant energy balance for structured continua with free microstructure manifolds.

Acknowledgements

AY benefited from discussions with J. D. Clayton, P. M. Mariano and A. Ozakin.

REFERENCES

[1] R. Abraham, J. E. Marsden and T. Ratiu: *Manifolds, Tensor Analysis, and Applications*, Springer, New York, 1988.

[2] A. Bedford and D. S. Drumheller: Theories of immiscible and structured mixtures, *Int. J. Eng. Sci.* **21** (1983), 863–960.

[3] R. M. Bowen: Toward a thermodynamics and mechanics of mixtures, *Arch. Rational Mech. Anal.* **24** (1967), 370–403.

[4] H. Buggisch: Zur theorie elasto-plastischer rotationsschalen, *ZAMM* **68** (1973), T68–T69.

[5] G. Capriz, P. Podio-Guidugli and W. Williams: On balance equations for materials with affine structure, *Mechanica* **17** (1982), 80–84.

[6] G. Capriz: *Continua with Microstructure*, Springer, Berlin 1989.

[7] G. Capriz and P. M. Mariano: Symmetries and Hamiltonian formalism for complex materials, *J. Elasticity* **72** (2003), 57–70.

[8] E. Cosserat and F. Cosserat: *Sur la théorie des corps deformables*, Dunod, Paris 1909.

[9] S. C. Cowin: On local inertia in Cosserat continua, *J. Appl. Math. Physics (ZAMP)* **26** (1975), 51–59.

[10] S .C. Cowin and J. W. Nunziato: Linear elastic-materials with voids, *J. Elasticity* **13** (1983), 125–147.

[11] C. de Fabritiis and P. M. Mariano: Geometry of interactions in complex bodies, *J. Geom. and Physics* **54** (2005), 301–323.

[12] M. Epstein and M. de Leon: Geometrical theory of uniform Cosserat media, *J. Geom. and Physics* **26** (1998), 127-170.
[13] J. L. Ericksen: Conservation laws for liquid crystals, *Transactions of the Society of Theology* **5** (1961), 23–34.
[14] E. Fried and M. E. Gurtin: Tractions, balances, and boundary conditions for nonsimple materials with application to liquid flow at small-length scales, *Arch. Rational Mech. Anal.* **182** (2006), 513–554.
[15] M. A. Goodman and S. C. Cowin: Continuum theory for granular materials, *Arch. Rational Mech. Anal.* **44** (1972), 249–266.
[16] A. E. Green and R. S. Rivlin: On Cauchy's equations of motion, *ZAMP* **15** (1964), 290–293.
[17] A. E. Green and R. S. Rivlin: Multipolar continuum mechanics, *Arch. Rational Mech. Anal.* **17** (1964), 113–147.
[18] A. E. Green and P. M. Naghdi: A theory of mixtures, *Arch. Rational Mech. Anal.* **24** (1967), 243–263.
[19] A. E. Green and P. M. Naghdi: A unified procedure for construction of theories of deformable media. I. Classical continuum physics, *Proc Royal Soc. of London A* **448** (1995a), 335–356.
[20] A. E. Green and P. M. Naghdi: A unified procedure for construction of theories of deformable media. II. Generalized continua, *Proc. Royal Soc. of London A* **448** (1995b), 357–377.
[21] M. E. Gurtin and P. Podio-Guidugli: On the formulation of mechanical balance laws for structured continua, *ZAMP* **43** (1992), 181–190.
[22] C. B. Kafadar and A. C. Eringen: Micropolar media–I The classical theory, *Int J. Eng. Sci.* **9** (1971), 271–305.
[23] F. M. Leslie: Some constitutive equations for liquid crystals. *Arch. Rational Mech. Anal.* **28** (1968), 265–283.
[24] P. M. Mariano: SO(3) invariance and covariance in mixtures of simple bodies, *Int. J. Non-Linear Mechanics* **40** (2005), 1023–1030.
[25] P. M. Mariano and F. L. Stazi: Computational aspects of the mechanics of complex materials, *Arch. Comp. Meth. Eng.* **12** (2005), 391–478.
[26] M. Marin: Some basic theorems in elastostatics of micropolar materials with voids, *J. Computational App. Math.* **70** (1996), 115–126.
[27] J. E. Marsden and T. J. R. Hughes: *Mathematical Foundations of Elasticity*, Dover, New York 1983.
[28] G. A. Maugin: On the structure of the theory of polar elasticity, *Philosophical Transactions of the Royal Society of London* **A356** (1998), 1367–1395.
[29] R. D. Mindlin: Micro-structure in linear elasticity, *Arch. Rational Mech. Anal.* **16** (1964), 51–78.
[30] W. Muschik, C. Papenfuss and H. Ehrentraut: A sketch of continuum thermodynamics, *J. Non-Newtonian Fluid Mechanics* **96** (2001), 255–290.
[31] J. W. Nunziato and S. C. Cowin: Non-linear theory of elastic-materials with voids, *Arch. Rational Mech. Anal.* **72** (1979), 175–201.
[32] R. Sampaio: Axiomatic theory of mixtures with diffusion, *Arch. Rational Mech. Anal.* **62** (1976), 99–116.
[33] J. C. Simo and J. E. Marsden: On the rotated stress tensor and the material version of the Doyle–Ericksen formula, *Arch. Rational Mech. Anal.* **86** (1984), 213–231.
[34] J. J. Sławianowski: Classical and quantized affine models of structured media, *Meccanica* **40** (2005), 365–387.
[35] R. A. Toupin: Elastic materials with couple-stresses, *Arch. Rational Mech. Anal.* **11** (1962), 385–414.
[36] R. A. Toupin: Theories of elasticity with couple stresses, *Arch. Rational Mech. Anal.* **17** (1964), 85–112.
[37] R. M. Wald: *General Relativity*, The University of Chicago Press, Chicago 1984.
[38] W. D. Williams: Theory of mixtures, *Arch. Rational Mech. Anal.* **51** (1973), 239–260.
[39] A. Yavari, J. E. Marsden and M. Ortiz: On the spatial and material covariant balance laws in elasticity, *J. Math. Phys.* **47** 042903 (2006), 85–112.
[40] A. Yavari: On geometric discretization of elasticity, *J. Math. Phys.* **49** 022901 (2008), 1–36.
[41] A. Yavari and A. Ozakin: Covariance in linearized elasticity, *ZAMP*, DOI 10.1007/s00033-007-7127-2 (2008).
[42] A. Yavari and J. E. Marsden: Energy balance invariance for interacting particle systems, to appear in *ZAMP* (2008).