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Controlled Lagrangian Methods and Tracking of Accelerated Motions

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***Abstract*— Matching techniques are applied to the problem of stabilization of uniformly accelerated motions of mechanical systems with symmetry. The theory is illustrated with a simple model—a wheel and pendulum system.**

I. Introduction

In this paper we apply the method of controlled Lagrangians to the problem of stabilization of accelerated motions of Lagrangian mechanical systems with symmetry. The method of controlled Lagrangians for stabilization of relative equilibria (steady state motions) originated in Bloch, Leonard, and Marsden [4] and was then developed in Auckly [1], Bloch, Leonard, and Marsden [5], [6], [7], Bloch, Chang, Leonard, and Marsden [8], and Hamberg [10], [11]. A similar approach for Hamiltonian controlled systems was introduced and further studied in the work of Blankenstein, Ortega, van der Schaft, Maschke and Spong and their collaborators (see [2], [14], [15], [16]) and the two methods were shown to be equivalent in [9]. A nonholonomic version of the method of controlled Lagrangians was studied in [3], [17], [18].

According to the method of controlled Lagrangians, the original controlled system is represented as a new, uncontrolled Lagrangian system for a suitable controlled Lagrangian. The energy associated with this controlled Lagrangian is designed to be positive or negative definite at the (relative) equilibrium to be stabilized. The time-invariant feedback control law is obtained from the equivalence requirement for the new and old systems of equations of motion. If asymptotic stabilization is desired, dissipation emulating terms are added to the control input.

In Bloch, Chang, Leonard, and Marsden [8], the problem of tracking was briefly discussed and, based on some numerical evidence and the study of some simple cases, it was proposed that tracking problems could be studied by means of the method of controlled Lagrangians. The idea is to create a time-dependent function that has a minimum at the point one wishes to track. The goal of the present paper is, in fact, to study the stabilization of a certain class of motions of mechanical systems with symmetry, which one may view as a special case of the general tracking problem. Assuming that the symmetry group is commutative, one can represent motions for which the component of acceleration in the group direction is constant as equilibria of the reduced system. Stabilization of such equilibria will thus produce orbitally stable accelerated trajectories.

In this paper we suggest a stabilization strategy using the framework of time-dependent Lagrangians. We expect this method to be applicable to more general tracking problems. In particular, we anticipate implementing our approach in problems of simultaneous tracking of a given trajectory in the symmetry group and stabilizing of an appropriate shape equilibrium.

The paper is organized as follows: In Section II we introduce a simple mechanical example—a wheel coupled with a pendulum—that demonstrates unstable accelerated dynamics. The main results are presented in Sections III and IV. In Section III we study a class of time-dependent Lagrangians with uniformly accelerated group dynamics represented by relative equilibria. We also discuss an energy-based stability analysis for these relative equilibria. In Section IV we derive the matching conditions and then illustrate the theory using the wheel and pendulum system.

In a future publication we intend to treat systems with noncommutative symmetry as well as systems with nonholonomic constraints.

II. The Mechanical Example

Consider a homogeneous vertical disk that is rolling without slipping along a horizontal straight line. A pendulum is attached to the center of the disk. The configuration coordinates are the angles (θ, ϕ); note that ϕ is measured from the rod. See Fig. 1 for details. This system is SO(2)-invariant; the group action is given by $\phi \mapsto \phi + \alpha$. See [12] and [13] for details about symmetry in mechanics.

We use the following notation for the parameters of the system:

$M =$ the mass of the disk,
$R =$ the radius of the disk,

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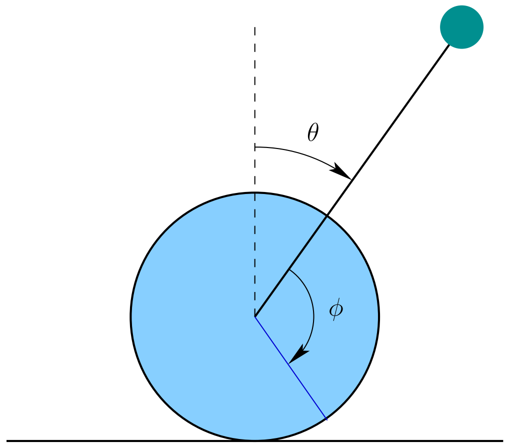

Fig. 1. The disk with inverted pendulum.

$$\begin{aligned} A &= \text{the principal moment of inertia of the disk,} \\ l &= \text{the rod length,} \\ m &= \text{the bob mass.} \end{aligned}$$

The kinetic energy of this system is

$$K = \frac{1}{2}\Big(\mathcal{A}(\theta)\dot\theta^2 + 2\mathcal{B}(\theta)\dot\theta\dot\phi + \mathcal{C}\dot\phi^2\Big),$$

where

$$\begin{aligned} \mathcal{A} &= A + MR^2 + m(R^2 + 2Rl\cos\theta + l^2), \\ \mathcal{B} &= A + MR^2 + m(R^2 + Rl\cos\theta), \\ \mathcal{C} &= A + MR^2 + mR^2. \end{aligned}$$

The potential energy is

$$V(\theta) = mgl\cos\theta.$$

The Lagrangian equals the kinetic minus potential energy, $K - V$. To simplify the exposition, we divide the Lagrangian by $\mathcal{C}$, *i.e.,* we put

$$l(\theta,\dot\theta,\dot\phi) = \frac{1}{2}\left(\alpha(\theta)\dot\theta^2 + 2\beta(\theta)\dot\theta\dot\phi + \dot\phi^2\right) - U(\theta),$$

where

$$\alpha(\theta) = \mathcal{A}(\theta)/\mathcal{C}, \quad \beta(\theta) = \mathcal{B}(\theta)/\mathcal{C}, \quad U(\theta) = V(\theta)/\mathcal{C}.$$

Assume there is a constant torque k applied to the disk. The equations of motion are

$$\frac{d}{dt}\frac{\partial l}{\partial\dot\theta} = \frac{\partial l}{\partial\theta}, \qquad \frac{d}{dt}\frac{\partial l}{\partial\dot\phi} = k.$$

This dynamics can be rewritten in the form of the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{\partial L}{\partial\theta}, \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot\phi} = 0$$

if a new, time-dependent Lagrangian

$$L(\theta,\dot\theta,\dot\phi,t) = l(\theta,\dot\theta,\dot\phi) - k\dot\phi t$$

is used instead of $l(\theta,\dot\theta,\dot\phi)$.

One can check that the system performs the uniformly accelerated motion

$$\theta = \theta_0, \qquad \ddot\phi = \ddot\phi_0 \tag{1}$$

if k and θ_0 satisfy the condition

$$k\beta(\theta_0) + U'(\theta_0) = 0.$$

The value of $\ddot\phi$ for such a motion is k. Straightforward computations confirm spectral instability of (1). Below we discuss how to design a torque actuator that stabilizes (1) with respect to θ.

III. Time-Dependent Lagrangians and Accelerated Dynamics

A. Accelerated Motions

Consider the following class of *moving system* Lagrangians:

$$\begin{aligned} L(r,\dot r,\dot s,t) = \frac{1}{2}\Big(&g_{\alpha\beta}(r)\dot r^\alpha\dot r^\beta + 2g_{\alpha a}(r)\dot r^\alpha\dot s^a + \delta_{ab}\dot s^a\dot s^b\Big) \\ &+ (a_\alpha(r)\dot r^\alpha + b_a(r)\dot s^a)t - U(r^\alpha). \end{aligned} \tag{2}$$

See [13] for details about moving systems.

Here and below $\alpha,\beta,\gamma,\ldots = 1,\ldots,m$ and $a,b,c,\ldots = 1,\ldots,n$, and summation over repeated indices is understood.

Remark. The variables s are cyclic. Without loss of generality, we assume that the quadratic form obtained from the kinetic energy of the system by setting $\dot r^\alpha = 0$ is $\frac{1}{2}\delta_{ab}\dot s^a\dot s^b$. (One can find an r-dependent basis in the commutative Lie algebra $\mathbb{R}^n$ that takes any positive-definite quadratic form $g_{ab}(r)\dot s^a\dot s^b$ to its canonical form $\delta_{ab}\dot s^a\dot s^b$.)

We intend to study here accelerated motions of the form

$$r = r_0, \qquad \ddot s = \ddot s_0. \tag{3}$$

The forces and/or torques that influence such motions are produced by the time-dependent terms in the Lagrangian.

The dynamics is governed by the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot r^\alpha} = \frac{\partial L}{\partial r^\alpha}, \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot s^a} = 0$$

or

$$\begin{aligned} g_{\alpha\beta}\ddot r^\beta + g_{\alpha a}\ddot s^a = {}& \left(\frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial r^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial r^\gamma}\right)\dot r^\beta\dot r^\gamma \\ &+ \left(\frac{\partial g_{\beta a}}{\partial r^\alpha} - \frac{\partial g_{\alpha a}}{\partial r^\beta}\right)\dot r^\beta\dot s^a \\ &+ \left(\frac{\partial a_\beta}{\partial r^\alpha} - \frac{\partial a_\alpha}{\partial r^\beta}\right)t\dot r^\beta \\ &+ \frac{\partial b_a}{\partial r^\alpha}t\dot s^a - a_\alpha(r) - \frac{\partial U}{\partial r^a}, \end{aligned} \tag{4}$$

$$g_{\alpha a}\dot r^\alpha + \delta_{ab}\dot s^b + b_a t = p_a. \tag{5}$$

Equation (5) represents the momentum conservation law.

Substituting (3) in (4) and (5), we obtain

$$\dot{s}^a = \delta^{ab}(p_b - b_b(r_0)t),$$
$$\ddot{s}^a_0 = -\delta^{ab} b_b(r_0),$$
$$-g_{\alpha a}(r_0)\delta^{ab} b_b(r_0) = \frac{\partial b_a}{\partial r^\alpha}\delta^{ab}(p_b - b_b(r_0)t)t - a_\alpha(r_0) - \frac{\partial U}{\partial r^\alpha}.$$

The last equation implies

$$\frac{\partial b_a}{\partial r^\alpha}(r_0) = 0.$$

The latter can be satisfied by setting $b_a(r) = -k_a = \text{const}$, which is assumed in the rest of the paper. The accelerated motions become

$$r = r_0, \qquad \ddot{s}^a = k_a, \tag{6}$$

where r_0 is determined from

$$g_{\alpha a}(r_0)\delta^{ab} k_b + a_\alpha(r_0) + \frac{\partial U}{\partial r^\alpha}(r_0) = 0.$$

Assuming that (6) is unstable, we impose a control input u in the group direction in order to stabilize (6) with respect to the shape variable r.

B. Reduced Dynamics and Stability Analysis

Recall that $b_a(r) = -k_a$. Since s^a are cyclic variables, the reduced dynamics is

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{r}^\alpha} = \frac{\partial R}{\partial r^\alpha}, \tag{7}$$

where the Routhian R is

$$R(r^\alpha, \dot{r}^\alpha, p_a, t) = \frac{1}{2}\left(g_{\alpha\beta} - \delta^{ab} g_{\alpha a} g_{\beta b}\right)\dot{r}^\alpha \dot{r}^\beta + \left(\delta^{ab} g_{\alpha a}(p_b + k_b t) + a_\alpha t\right)\dot{r}^\alpha - \frac{1}{2}\sum_a (p_a + k_a t)^2 - U. \tag{8}$$

Since p_a are the flow-invariant cyclic momenta, the terms $\frac{1}{2}\sum_a (p_a + k_a t)^2$ in the Routhian are independent of the reduced phase variables and thus can be safely omitted. This is assumed in the rest of the paper. The partial derivatives of the Routhian are computed below:

$$\frac{\partial R}{\partial \dot{r}^\alpha} = \left(g_{\alpha\beta} - \delta^{ab} g_{\alpha a} g_{\beta b}\right)\dot{r}^\beta + \delta^{ab} g_{\alpha a} p_b + (a_\alpha(r) + \delta^{ab} g_{\alpha a} b_b)t,$$
$$\frac{\partial R}{\partial r^\alpha} = \frac{1}{2}\frac{\partial}{\partial r^\alpha}\left(g_{\beta\gamma} - \delta^{bc} g_{\beta b} g_{\gamma c}\right)\dot{r}^\beta \dot{r}^\gamma + \frac{\partial}{\partial r^\alpha}\left(\delta^{ab} g_{\beta b}(p_a + k_a t) + a_\beta t\right)\dot{r}^\beta - \frac{\partial U}{\partial r^\alpha}.$$

The equilibria of (7) correspond to the accelerated motions (6). The accelerated motions are orbitally stable if the equilibria of the reduced system (7) are stable.

The reduced energy associated with (8) is

$$E = \dot{r}^\alpha \frac{\partial R}{\partial \dot{r}^\alpha} - R = \frac{1}{2}\left(g_{\alpha\beta} - \delta^{ab} g_{\alpha a} g_{\beta b}\right)\dot{r}^\alpha \dot{r}^\beta + U.$$

Its flow derivative $\dot{E}$ equals

$$-(a_\alpha + \delta^{ab} g_{\alpha a} k_b)\dot{r}^\alpha.$$

Assuming that the one-form

$$(a_\alpha + \delta^{ab} g_{\alpha a} k_b)\, dr^\alpha$$

is closed, define the modified energy by

$$\mathcal{E} = E + \int (a_\alpha + \delta^{ab} g_{\alpha a} k_b)\, dr^\alpha. \tag{9}$$

The modified energy is flow-invariant and thus can be used as a Lyapunov function.

IV. Matching and Stabilization of Uniformly Accelerated Motions

A. Matching Conditions

Given the Lagrangian (2), one writes the controlled dynamics as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^\alpha} = \frac{\partial L}{\partial r^\alpha}, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} = u_a. \tag{10}$$

Consider the controlled Lagrangian

$$\widetilde{L}(r, \dot{r}, \dot{s}, t) = \frac{1}{2}\Big(\widetilde{g}_{\alpha\beta}(r)\dot{r}^\alpha \dot{r}^\beta + 2\widetilde{g}_{\alpha a}(r)\dot{r}^\alpha \dot{s}^a + \delta_{ab}\dot{s}^a \dot{s}^b\Big) + (\widetilde{a}_\alpha(r)\dot{r}^\alpha - k_a \dot{s}^a)t - U(r^\alpha).$$

We require that the dynamics determined by $\widetilde{L}$,

$$\frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{r}^\alpha} = \frac{\partial \widetilde{L}}{\partial r^\alpha}, \qquad \frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{s}^a} = 0, \tag{11}$$

is equivalent to (10); this imposes certain conditions on the controlled kinetic energy. These are called the *matching conditions.* They are specified in Theorem 1 below.

Put

$$F^\delta_\alpha = G_{\alpha\beta}\widetilde{G}^{\beta\delta},$$

where

$$G_{\alpha\beta} = g_{\alpha\beta} - \delta^{ab} g_{\alpha a}\widetilde{g}_{\beta b}, \qquad \widetilde{G}_{\alpha\beta} = \widetilde{g}_{\alpha\beta} - \delta^{ab}\widetilde{g}_{\alpha a}\widetilde{g}_{\beta b}.$$

The matrix $\widetilde{G}_{\alpha\beta}$ is invertible as it represents the controlled Lagrangian's reduced kinetic energy metric, which is assumed to be non-degenerate.

Theorem 1: Equations (10) and (11) are equivalent if and only if

$$F^{\delta}_{\alpha}\left[\frac{1}{2}\frac{\partial\widetilde{g}_{\beta\gamma}}{\partial r^{\delta}}-\frac{\partial\widetilde{g}_{\delta\beta}}{\partial r^{\gamma}}+\delta^{ab}\widetilde{g}_{\delta a}\frac{\partial\widetilde{g}_{\beta b}}{\partial r^{\gamma}}\right] = \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial r^{\alpha}}-\frac{\partial g_{\alpha\beta}}{\partial r^{\gamma}}+\delta^{ab}g_{\alpha a}\frac{\partial\widetilde{g}_{\beta b}}{\partial r^{\gamma}},$$

$$F^{\delta}_{\alpha}\left[\frac{\partial\widetilde{g}_{\beta a}}{\partial r^{\delta}}-\frac{\partial\widetilde{g}_{\delta a}}{\partial r^{\beta}}\right]=\frac{\partial g_{\beta a}}{\partial r^{\alpha}}-\frac{\partial g_{\alpha a}}{\partial r^{\beta}},$$

$$F^{\delta}_{\alpha}\left[\frac{\partial\widetilde{a}_{\beta}}{\partial r^{\delta}}-\frac{\partial\widetilde{a}_{\delta}}{\partial r^{\beta}}\right]=\frac{\partial a_{\beta}}{\partial r^{\alpha}}-\frac{\partial a_{\alpha}}{\partial r^{\beta}},$$

$$F^{\delta}_{\alpha}\left[\widetilde{a}_{\delta}+\delta^{ab}k_b\widetilde{g}_{\delta a}+\frac{\partial U}{\partial r^{\delta}}\right]=a_{\alpha}+\alpha^{ab}k_b g_{\alpha a}+\frac{\partial U}{\partial r^{\alpha}}$$

and the control inputs are

$$u_a=(g_{\alpha a}-\widetilde{g}_{\alpha a})\ddot{r}^{\alpha}+\Big(\frac{\partial g_{\alpha a}}{\partial r^{\beta}}-\frac{\partial\widetilde{g}_{\alpha a}}{\partial r^{\beta}}\Big)\dot{r}^{\alpha}\dot{r}^{\beta}. \qquad (12)$$

Using equations (11), one can eliminate the accelerations $\ddot{r}^{\alpha}$ from the control law (12).

The controlled dynamics is s-invariant, and thus one can use the modified energy (9) for stability analysis of its relative equilibria. These relative equilibria represent the accelerated motions (3). Below we demonstrate this approach using our mechanical example.

B. Stabilization of the Accelerating Wheel-Pendulum System

1) The Stability Condition: Recall that the Lagrangian has the following structure:

$$L=\frac{1}{2}\left(\alpha(\theta)\dot{\theta}^2+2\beta(r)\dot{\theta}\dot{\phi}+\dot{\phi}^2\right)-k\dot{\phi}t-U(\theta).$$

The controlled dynamics is governed by the equations

$$\alpha\ddot{\theta}+\beta\ddot{\phi}+\frac{1}{2}\alpha'\dot{\theta}^2=-U',$$
$$\beta\ddot{\theta}+\ddot{\phi}+\beta'\dot{\theta}^2=k+u. \qquad (13)$$

Consider the controlled Lagrangian

$$\widetilde{L}=\frac{1}{2}\left(\widetilde{\alpha}(\theta)\dot{\theta}^2+2\widetilde{\beta}(\theta)\dot{\theta}\dot{\phi}+\dot{\phi}^2\right)+\widetilde{a}(\theta)\dot{\theta}t-k\dot{\phi}t-U(\theta).$$

The equations of motion associated with $\widetilde{L}$ become

$$\widetilde{\alpha}\ddot{\theta}+\widetilde{\beta}\ddot{\phi}+\frac{1}{2}\widetilde{\alpha}'\dot{\theta}^2+\widetilde{a}=-U',$$
$$\widetilde{\beta}\ddot{\theta}+\ddot{\phi}+\widetilde{\beta}'\dot{\theta}^2=k. \qquad (14)$$

We require that these equations are equivalent to (13). This equivalence implies the following matching conditions (see Theorem 1):

$$\widetilde{\alpha}'(\alpha-\beta\widetilde{\beta})+2\widetilde{\beta}'(-\alpha\widetilde{\beta}+\beta\widetilde{\alpha})=\alpha'(\widetilde{\alpha}-\widetilde{\beta}^2), \qquad (15)$$
$$(\widetilde{a}+U')(\alpha-\beta\widetilde{\beta})-k(-\alpha\widetilde{\beta}+\beta\widetilde{\alpha})=U'(\widetilde{\alpha}-\widetilde{\beta}^2). \qquad (16)$$

After $\widetilde{\beta}$ has been chosen, (15) becomes a linear first order differential equation for $\widetilde{\alpha}$. After solving (15), one finds $\widetilde{a}$ that satisfies (16). The equivalence requirement also determines the feedback control input u.

We now discuss the conditions for stability of the accelerated motion (1). As before, we discuss stability with respect to θ.

Using the Routh reduction, one finds the reduced dynamics

$$\frac{d}{dt}\frac{\partial R}{\partial\dot{\theta}}=\frac{\partial R}{\partial\theta},$$

or, explicitly,

$$(\widetilde{\alpha}-\widetilde{\beta}^2)\ddot{\theta}+(\widetilde{\alpha}'/2-\widetilde{\beta}\widetilde{\beta}')\dot{\theta}^2+\widetilde{a}+\widetilde{\beta}k=-U'.$$

The (time-dependent) Routhian is

$$R=\frac{1}{2}(\widetilde{\alpha}-\widetilde{\beta}^2)\dot{\theta}^2+(\widetilde{\beta}kt+\widetilde{a}t+\widetilde{\beta}p)\dot{\theta}-U.$$

According to (9),

$$\mathcal{E}(\theta,\dot{\theta})=\frac{1}{2}(\widetilde{\alpha}-\widetilde{\beta}^2)\dot{\theta}^2+U+\int(\widetilde{a}+\widetilde{\beta}k)\,d\theta. \qquad (17)$$

The relative equilibrium $\theta=\theta_0$ is stable if $\mathcal{E}$ is definite at $(\theta_0,0)$.

We now discuss how one achieves stability. First, we obtain a new representation of the second matching condition (16). We have

$$\begin{aligned}(\widetilde{a}+U')(\alpha-\beta\widetilde{\beta})-k(-\alpha\widetilde{\beta}+\beta\widetilde{\alpha}) &=(\widetilde{a}+U'+\widetilde{\beta}k)(\alpha-\beta\widetilde{\beta})\\ &\quad-\widetilde{\beta}k(\alpha-\beta\widetilde{\beta})-k(-\alpha\widetilde{\beta}+\beta\widetilde{\alpha})\\ &=(\widetilde{a}+U'+\widetilde{\beta}k)(\alpha-\beta\widetilde{\beta})-\beta k(\widetilde{\alpha}-\widetilde{\beta}^2)\end{aligned}$$

and thus (16) becomes

$$(\widetilde{a}+\widetilde{U}'+\widetilde{\beta}k)(\alpha-\beta\widetilde{\beta})=(U'+\beta k)(\widetilde{\alpha}-\widetilde{\beta}^2). \qquad (18)$$

At $\theta=\theta_0$, both $U'+k\beta$ and $\widetilde{a}+U'+k\widetilde{\beta}$ vanish, and therefore

$$(\widetilde{a}'+U''+\widetilde{\beta}'k)(\alpha-\beta\widetilde{\beta})=(U''+\beta'k)(\widetilde{\alpha}-\widetilde{\beta}^2).$$

Recall that $U''+\beta'k$ is negative at $\theta=\theta_0$ and the stability condition requires that $\widetilde{a}'+U''+\widetilde{\beta}'k$ and $\widetilde{\alpha}-\widetilde{\beta}^2$ are of the same sign. The stability condition thus becomes

$$\alpha(\theta_0)-\beta(\theta_0)\widetilde{\beta}(\theta_0)<0. \qquad (19)$$

After choosing $\widetilde{\beta}$ that satisfies (19) one can assign a suitable initial condition $\widetilde{\alpha}(\theta_0)$, find $\widetilde{\alpha}(\theta)$ from (15), and find $\widetilde{a}(\theta)$ from (18). The above procedure determines the controlled Lagrangian $\widetilde{L}$.

2) The Control Input: The equivalence of (13) and (14) implies that the control input is given by

$$u = \Big[\big((\widetilde{\alpha} - \widetilde{\beta}^2)(\beta' - \widetilde{\beta}') - (\beta - \widetilde{\beta})(\widetilde{\alpha}'/2 - \widetilde{\beta}\widetilde{\beta}')\big)\dot{\theta}^2 - (\beta - \widetilde{\beta})(\widetilde{a} + \widetilde{\beta}k + U')\Big] \Big/ (\widetilde{\alpha} - \widetilde{\beta}^2). \quad (20)$$

Summarizing, we have:

Theorem 2: If (19) holds for the system (13) and the control is defined by (20), then the accelerated motion (1) of the wheel-pendulum system is stable in the orbital sense, i.e., it is stable with respect to the variables $(\theta, \dot{\theta}, \ddot{\phi})$.

3) The Stability Region: We now demonstrate that the controller proposed in this paper is capable of producing a large region of stability. Choose the numerical values of the parameters of the wheel-pendulum system to be such that

$$\frac{mRl}{A + MR^2 + mR^2} = \frac{1}{8}, \quad \frac{ml^2}{A + MR^2 + mR^2} = \frac{1}{16}, \quad \frac{mgl}{A + MR^2 + mR^2} = 1.$$

For the motion with the pendulum tilt of $\theta = \pi/4$ one computes

$$\ddot{\phi} = \frac{8}{8\sqrt{2} + 1}.$$

The controlled Lagrangian can be evaluated explicitly for this problem (details are omitted here and will appear in a future publication). The stability region for the relative equilibrium $\theta = \pi/4$, which is the region inside the critical level of the modified energy (17) is shown in Fig. 2.

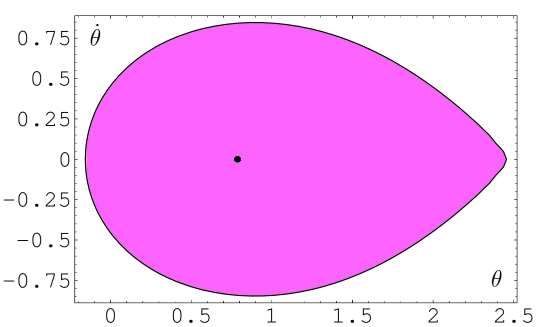

Fig. 2. The stability region for $\theta = \pi/4$.

V. Conclusions

This paper has extended matching techniques to tracking of relative equilibria of a class of time-dependent Lagrangian dynamical systems as well as developed an energy-based procedure for stability analysis of these equilibria. Although the stability analysis proposed here relies on the time-independence of the modified energy, we expect our approach to be applicable to more general tracking problems and we intend to address this issue in a future publication.

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VI. References

[1] Auckly, D., L. Kapitanski, & W. White, Control of Nonlinear Underactuated Systems, *Commun. Pure Appl. Math.* **53**, 2000, 354–369.

[2] Blankenstein, G., R. Ortega & A. van der Schaft, The Matching Conditions of Controlled Lagrangians and Interconnection and Damping Assignment Passivity Based Control, *Internat. J. Control* **75**, 2002, 645–665.

[3] Bloch, A.M., *Nonholonomic Mechanics and Control*, Interdisciplinary Appl. Math. **24**, Springer-Verlag, 2003.

[4] Bloch, A.M., N. Leonard, & J.E. Marsden, Stabilization of Mechanical Systems Using Controlled Lagrangians, *Proc. CDC* **36**, 1997, 2356–2361.

[5] Bloch, A.M., N. Leonard, & J.E. Marsden, Matching and Stabilization by the Method of Controlled Lagrangians, *Proc. CDC* **37**, 1998, 1446–1451.

[6] Bloch, A.M., N. Leonard, & J.E. Marsden, Potential Shaping and the Method of Controlled Lagrangians, *Proc. CDC* **38**, 1999, 1652–1657.

[7] Bloch, A.M., N.E. Leonard, & J.E. Marsden, Controlled Lagrangians and the Stabilization of Mechanical Systems I: The First Matching Theorem, *IEEE Trans. on Systems and Control* **45**, 2000, 2253–2270.

[8] Bloch, A.M., D-E. Chang, N.E. Leonard, & J.E. Marsden, Controlled Lagrangians and the Stabilization of Mechanical Systems II: Potential Shaping, *Trans. IEEE on Autom. Contr.* **46**, 2001, 1556–1571.

[9] Chang, D-E., A.M. Bloch, N.E. Leonard, J.E. Marsden, & C. Woolsey, The Equivalence of Controlled Lagrangian and Controlled Hamiltonian Systems, *Control and the Calculus of Variations (special issue dedicated to J.L. Lions)* **8**, 2002, 393–422.

[10] Hamberg, J., General Matching Conditions in the Theory of Controlled Lagrangians, *Proc. CDC* **38**, 1999, 2519–2523.

[11] Hamberg, J., Controlled Lagrangians, Symmetries and Conditions for Strong Matching. *In: Lagrangian and Hamiltonian Methods for Nonlinear Control,* Elsevier, 2000.

[12] Marsden, J.E., *Lectures on Mechanics,* London Mathematical Society Lecture Note Series **174**, Cambridge University Press, 1992.

[13] Marsden, J.E. & T.S. Ratiu, *An Introduction to Mechanics and Symmetry.* Texts in Appl. Math. **17**, Springer-Verlag, 1999.

[14] Maschke, B., R. Ortega, & A. van der Schaft, Energy-Based Lyapunov Functions for Forced Hamiltonian Systems with Dissipation, *IEEE Trans. Automat. Control* **45**, 2001, 1498–1502.
[15] Ortega, R., A. Loria, P.J. Nicklasson, & H. Sira-Ramirez, *Passivity-based Control of Euler–Lagrange Systems*, Springer-Verlag, Communication & Control Engineering Series, 1998.
[16] Ortega, R., M.W. Spong, F. Gómez-Estern, & G. Blankenstein, Stabilization of a Class of Underactuated Mechanical Systems via Interconnection and Damping Assignment, *IEEE Trans. Aut. Control* **47**, 2002, 1218–1233.
[17] Zenkov, D.V., A.M. Bloch, N.E. Leonard, & J.E. Marsden, Matching and Stabilization of Low-Dimensional Nonholonomic Systems. *Proc. CDC* **39**, 2000, 1289–1295.
[18] Zenkov, D.V., A.M. Bloch, & J.E. Marsden, Flat Nonholonomic Matching. *Proc. ACC*, 2002, 2812–2817.