Variational Multisymplectic Formulations of Nonsmooth Continuum Mechanics

R.C. Fetecau
J.E. Marsden
M. West

To Larry Sirovich on the occasion of his 70th birthday

ABSTRACT This paper develops the foundations of the multisymplectic formulation of *nonsmooth* continuum mechanics. It may be regarded as a PDE generalization of previous techniques that developed a variational approach to collision problems. These methods have already proved of value in computational mechanics, particularly in the development of asynchronous integrators and efficient collision methods. The present formulation also includes solid-fluid interactions and material interfaces and, in addition, lays the groundwork for a treatment of shocks.

Contents

1 Introduction

There has been much interest lately in using variational methods for computational mechanics, such as Kane, Repetto, Ortiz, and Marsden [1999], Kane, Marsden, Ortiz, and West [2000] and Pandolfi, Kane, Marsden, and Ortiz [2002]. These variational methods have the attractive property that one can give the precise sense in which the algorithms used preserve the mechanical structure. This sense is a natural consequence of the variational structure of the algorithms and involves the symplectic and multisymplectic character of the algorithm. We refer the reader to the survey Marsden and West [2001], references therein as well as the references in the following paragraphs for additional details about the general setting and properties of variational integrators.

There have been two developments in this area that bear directly on the present work. First of all, variational collision algorithms for finite dimensional problems (such as the collision of rigid bodies) have been developed that also share the conservation properties of smooth variational integrators; for example, the paper of Fetecau, Marsden, Ortiz, and West [2002] shows that the collision algorithms are symplectic, including the dynamics through the collision.

Second, in the area of variational algorithms for PDE's, a basic work was that of Marsden, Patrick, and Shkoller [1998] which laid the foundations of the method. The continuous part of these techniques were applied to the context of continuum mechanics in Marsden, Pekarsky, Shkoller, and West [2001]. This approach to continuum mechanics and discrete multisymplectic mechanics was developed further in Lew, Marsden, Ortiz, and West [2002], which introduced the notion of AVI's (asynchronous variational integrators). AVI's allow one to spatially and temporally adapt the algorithm and still retain its variational and multisymplectic character. This technique was also shown to be efficient computationally for two- and three-dimensional elasticity.

The purpose of the present paper is to combine the ideas from the continuous variational collision theory with the continuous part of multisymplectic theory of continuum mechanics. The result is a variational theory for PDE's of the sort arising in continuum mechanics that allow for material interfaces, elastic collisions, shocks and fluid-solid interactions. We shall not attempt to give a full account of all the literature in this area as it is extensive and complex, but we do mention the important works of Moreau [1982, 1986, 1988] that bear on the topics treated here.

In §5.2 we classify a collection of nonsmooth dynamic models that we will study. In this paper we do not address the discrete and algorithmic aspects of this theory. In fact, we plan to merge this work with variational algorithms and discrete mechanics (such as the work on AVI's) in future publications.

2 Multisymplectic Geometry

In this section we will review some aspects of basic covariant field theory in the framework of multisymplectic geometry. The multisymplectic framework is a PDE generalization of classical non-relativistic mechanics (or particle mechanics) and has diverse applications, including to electromagnetism, continuum mechanics, gravity, bosonic strings, etc.

The traditional approach to the multisymplectic geometric structure closely follows the derivation of the canonical symplectic structure in particle mechanics. The derivation first defines the field theoretic analogues of the tangent and cotangent bundles (called the first jet bundle and the dual jet bundle, respectively). It then introduces a canonical multisymplectic form on the dual jet bundle and pulls it back to the Lagrangian side using the covariant Legendre transform. As an alternative, Marsden, Patrick, and Shkoller [1998] gave a very elegant approach of deriving the multisymplectic structure by staying entirely on the Lagrangian side, which we will use here.

We start by reviewing the main concepts of the multisymplectic field-theoretic setting.

Let X be an oriented manifold, which in many examples is spacetime, and let $\pi_{XY} : Y \to X$ be a finite-dimensional fiber bundle called the ***covariant configuration bundle***. The physical fields will be sections of this bundle, which is the covariant analogue of the configuration space in classical mechanics.

The role of the tangent bundle is played by J^1Y (or $J^1(Y)$), the ***first jet bundle*** of Y. We identify J^1Y with the *affine* bundle over Y whose fiber over $y \in Y_x = \pi_{XY}^{-1}(x)$ consists of those linear maps $\gamma : T_xX \to T_yY$ satisfying

$$T\pi_{XY} \circ \gamma = \mathrm{Id}_{T_xX}.$$

We let $\dim X = n+1$ and the fiber dimension of Y be N. Coordinates on X are denoted $x^\mu, \mu = 1, 2, \ldots, n, 0$, and fiber coordinates on Y are denoted by $y^A, A = 1, \ldots, N$. These induce coordinates $v^A\mu$ on the fibers of J^1Y. For a section $\varphi : X \to Y$, its tangent map at $x \in X$, denoted $T_x\varphi$, is an element of $J^1Y_{\varphi(x)}$. Thus, the map $x \mapsto T_x\varphi$ is a local section of J^1Y regarded as a bundle over X. This section is denoted $j^1(\varphi)$ or $j^1\varphi$ and is called the first jet of φ. In coordinates, $j^1(\varphi)$ is given by

$$x^\mu \mapsto (x^\mu, \varphi^A(x^\mu), \partial_\nu\varphi^A(x^\mu)), \tag{2.1}$$

where $\partial_\nu = \frac{\partial}{\partial x^\nu}$.

We will study Lagrangians defined on J^1Y and derive the Euler-Lagrange equations by a procedure similar to that used in Lagrangian mechanics on the tangent bundle of a configuration manifold (see Marsden and Ratiu [1999]). We thus consider theories for which Lagrangians depend at most on the fields and their *first* derivatives (first order field theories). For a geometric-variational approach to second-order field theories we refer the reader to Kouranbaeva and Shkoller [2000].

Higher order jet bundles of Y, J^mY, can be defined as $J^1(\cdots(J^1(Y)))$ and are used in the higher order field theories. In this paper we will use only J^1Y and a specific subbundle Y'' of J^2Y which we will define below.

Let $\gamma \in J^1Y$ so that $\pi_{X,J^1Y}(\gamma) = x$. Analogous to the tangent map of the projection π_{Y,J^1Y}, $T\pi_{Y,J^1Y} : TJ^1Y \to TY$, we may define the jet map of this projection which takes J^2Y onto J^1Y:

$$J\pi_{Y,J^1Y} : \mathrm{Aff}(T_xX, T_\gamma J^1Y) \to \mathrm{Aff}(T_xX, T\pi_{Y,J^1Y} \cdot T_\gamma J^1Y).$$

We define the subbundle Y'' of J^2Y over X which consists of second-order jets so that on each fiber

$$Y''_x = \{s \in J^2Y_\gamma \mid J\pi_{Y,J^1Y}(s) = \gamma\}. \tag{2.2}$$

In coordinates, if $\gamma \in J^1Y$ is given by $(x^\mu, y^A, v^A{}_\mu)$, and $s \in J^2Y_\gamma$ is given by $(x^\mu, y^A, v^A{}_\mu, w^A{}_\mu, k^A{}_{\mu\nu})$, then s is a second order jet if $v^A{}_\mu = w^A{}_\mu$. Thus, the second jet of a section φ, $j^2(\varphi)$, given in coordinates by the map $x^\mu \mapsto (x^\mu, \varphi^A, \partial_\nu\varphi^A, \partial_\mu\partial_\nu\varphi^A)$, is an example of a second-order jet.

Next we introduce the field theoretic analogue of the cotangent bundle. We define the ***dual jet bundle*** J^1Y^* to be the *vector* bundle over Y whose fiber at $y \in Y_x$ is the set of affine maps from J^1Y_y to $\Lambda^{n+1}(X)_x$, the bundle of $(n+1)$-forms on X. A smooth section of J^1Y^* is therefore an affine bundle map of J^1Y to $\Lambda^{n+1}(X)$ covering π_{XY}.

Fiber coordinates on J^1Y^* are $(p, p_A{}^\mu)$, which correspond to the affine map given in coordinates by

$$v^A{}_\mu \mapsto (p + p_A{}^\mu v^A{}_\mu) d^{n+1}x, \tag{2.3}$$

where

$$d^{n+1}x = dx^1 \wedge \cdots \wedge dx^n \wedge dx^0.$$

Analogous to the canonical one- and two-forms on a cotangent bundle, there are canonical $(n+1)$- and $(n+2)$-forms on the dual jet bundle J^1Y^*. We will omit here the intrinsic definitions of these canonical forms (see Gotay, Isenberg, and Marsden [1997] for details). In coordinates, with $d^nx_\mu = \partial_\mu \lrcorner\, d^{n+1}x$, these forms are given by

$$\Theta = p_A{}^\mu dy^A \wedge d^nx_\mu + p d^{n+1}x \tag{2.4}$$

and

$$\Omega = dy^A \wedge dp_A{}^\mu \wedge d^nx_\mu - dp \wedge d^{n+1}x. \tag{2.5}$$

A ***Lagrangian density*** $\mathcal{L} : J^1Y \to \Lambda^{n+1}(X)$ is a smooth bundle map over X. In coordinates, we write

$$\mathcal{L}(\gamma) = L(x^\mu, y^A, v^A_\mu) d^{n+1}x. \tag{2.6}$$

The covariant Legendre transform for $\mathcal{L}$ is a fiber preserving map over Y, $\mathbb{F}\mathcal{L} : J^1Y \to J^1Y^*$, expressed intrinsically as the first order vertical Taylor approximation to $\mathcal{L}$:

$$\mathbb{F}\mathcal{L}(\gamma) \cdot \gamma' = \mathcal{L}(\gamma) + \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{L}(\gamma + \epsilon(\gamma' - \gamma)), \tag{2.7}$$

where $\gamma, \gamma' \in J^1Y_y$.

The coordinate expression of $\mathbb{F}\mathcal{L}$ is given by

$$p^\mu_A = \frac{\partial L}{\partial v^A{}_\mu}, \quad \text{and} \quad p = L - \frac{\partial L}{\partial v^A{}_\mu} v^A{}_\mu \tag{2.8}$$

for the multimomenta p^μ_A and the covariant Hamiltonian p.

Now we can use the covariant Legendre transform to pull back to the Lagrangian side the multisymplectic canonical structure on the dual jet bundle. We define the ***Cartan form*** as the $(n+1)$-form $\Theta_\mathcal{L}$ on J^1Y given by

$$\Theta_\mathcal{L} = (\mathbb{F}\mathcal{L})^*\Theta \tag{2.9}$$

and the $(n+2)$-form $\Omega_\mathcal{L}$ by

$$\Omega_\mathcal{L} = -d\Theta_\mathcal{L} = (\mathbb{F}\mathcal{L})^*\Omega, \tag{2.10}$$

with local coordinate expressions

$$\Theta_\mathcal{L} = \frac{\partial L}{\partial v^A_\mu} dy^A \wedge d^n x_\mu + \left(L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu \right) d^{n+1}x \tag{2.11}$$

and

$$\Omega_\mathcal{L} = dy^A \wedge d\left(\frac{\partial L}{\partial v^A_\mu} \right) \wedge d^n x_\mu - d\left(L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu \right) \wedge d^{n+1}x. \tag{2.12}$$

To lay the groundwork for the following sections we introduce the concept of jet prolongations. We will show how automorphisms of Y lift naturally to automorphisms of J^1Y and we will construct the covariant analogue of the tangent map.

Let $\eta_Y : Y \to Y$ be a π_{XY}-bundle automorphism covering a diffeomorphism $\eta_X : X \to X$. If $\gamma : T_xX \to T_yY$ is an element of J^1Y, let $\eta_{J^1Y}(\gamma) : T_{\eta_X(x)}X \to T_{\eta_Y(y)}Y$ be defined by

$$\eta_{J^1Y}(\gamma) = T\eta_Y \circ \gamma \circ T\eta_X^{-1}. \tag{2.13}$$

The π_{Y,J^1Y}-bundle automorphism $j^1(\eta_Y)$, also denoted η_{J^1Y}, is called the ***first jet extension*** or ***prolongation*** of η_Y to J^1Y and has the coordinate expression

$$\eta_{J^1Y}(\gamma) = \left(\eta_X^\mu(x), \eta_Y^A(x,y), \left[\partial_\nu \eta_Y^A + \left(\partial_B \eta_Y^A\right) v^B{}_\nu\right] \partial_\mu \left(\eta_X^{-1}\right)^\nu\right), \qquad (2.14)$$

where $\gamma = (x^\mu, y^A, v^A{}_\mu)$.

If V is a vector field on Y whose flow is η_λ, so that

$$V \circ \eta_\lambda = \frac{d\eta_\lambda}{d\lambda},$$

then its ***first jet extension*** or ***prolongation***, denoted $j^1(V)$ or V_{J^1Y}, is the vector field on J^1Y whose flow is $j^1(\eta_\lambda)$; that is

$$j^1(V) \circ j^1(\eta_\lambda) = \frac{d}{d\lambda} j^1(\eta_\lambda). \qquad (2.15)$$

In coordinates, $j^1(V)$ has the expression

$$j^1(V) = \left(V^\mu, V^A, \frac{\partial V^A}{\partial x^\mu} + \frac{\partial V^A}{\partial y^B} v^B_\mu - v^A_\nu \frac{\partial V^\nu}{\partial x^\mu}\right). \qquad (2.16)$$

We note that one can also view V as a section of the bundle $TY \mapsto Y$ and take its first jet in the sense of (2.1). Then one obtains a section of $J^1(TY) \mapsto Y$ which is not to be confused with $j^1(V)$ as defined by (2.15) and (2.16); they are two different objects.

This is the differential-geometric formulation of the multisymplectic structure. However, as we mentioned before, there is a very elegant and interesting way to construct $\Theta_{\mathcal{L}}$ directly from the variational principle, staying entirely on the Lagrangian side. It is this variational approach that we will use in the next sections to extend the multisymplectic formalism to the nonsmooth context.

3 Variational Multisymplectic Geometry in a Nonsmooth Setting

We now consider the variational approach to multisymplectic field theory of Marsden, Patrick, and Shkoller [1998] and formulate it a nonsmooth setting. A novelty of this variational approach is that it considers *arbitrary* and not only *vertical* variations of sections. The motivation for such a generalization is that, even though both the vertical and arbitrary variations result in the same Euler-Lagrange equations, the Cartan form obtained from the vertical variations is missing one term (corresponding to the $d^{n+1}x$ form). However, the horizontal variations account precisely for this extra term and make the Cartan form complete.

We reconsider the need for horizontal variations in the nonsmooth context and adapt the formalism developed in Marsden, Patrick, and Shkoller [1998] to give a rigorous derivation of the jump conditions when fields are allowed to be nonsmooth.

3.1 Nonsmooth Multisymplectic Geometry

Let U be a manifold with smooth closed boundary. In the smooth context, the configuration space is the infinite-dimensional manifold defined by an appropriate closure of the set of smooth maps

$$\mathcal{C}^\infty = \{\phi : U \to Y \mid \pi_{XY} \circ \phi : U \to X \text{ is an embedding}\}.$$

In the the nonsmooth setting, we must also introduce a codimension 1 submanifold $D \subset U$, called the ***singularity submanifolds*** across which the fields ϕ may have singularities. For example, the submanifold D may be the spacetime surface separating two regions of a continuous medium or, in the case of two elastic bodies colliding, D may be the spacetime contact set.

For the smooth case, the observation that the configuration space is a smooth manifold enables the use of differential calculus on the manifold of mappings as required by variational principles (see Marsden and Ratiu [1999],Marsden, Patrick, and Shkoller [1998]). In this subsection we will present various types of configuration spaces that one must consider in the nonsmooth context and discuss their manifold structure.

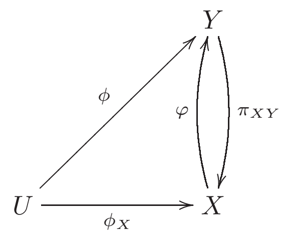
Configuration spaces. The applications that we present in the next sections require different configuration spaces, according to the type of singularities that we allow across the singularity submanifold D.

Case (a). Continuous but nonsmooth. For the first examples presented in this paper (such as rigid-body dynamics with impact and propagating singular surfaces within a continuum medium), the configuration space is the set of continuous maps

$$\begin{aligned}\mathcal{C}^a = \{\phi : U \to Y \mid\ & \pi_{XY} \circ \phi : U \to X \text{ is an embedding}, \\ & \phi \text{ is } C^0 \text{ in } U \text{ and of class } C^2 \text{ in } U \setminus D\}\end{aligned} \tag{3.1}$$

For each $\phi \in \mathcal{C}^a$, we set $\phi_X = \pi_{XY} \circ \phi$ and $U_X = \pi_{XY} \circ \phi(U)$, $D_X = \pi_{XY} \circ \phi(D)$, so that $\phi_X : U \to U_X$ and its restriction to D are diffeomorphisms.

We also denote the section $\phi \circ {\phi_X}^{-1}$ by φ, as in



The submanifold D separates the interior of U into two disjoint open subsets U^+ and U^-, that is $\text{int}(U) = U^+ \cup U^- \cup (U \cap D)$ and we let $U_X^+ = \phi_X(U^+)$ and $U_X^- = \phi_X(U^-)$ be their corresponding images in X. It follows that $\text{int}(U_X) = U_X^+ \cup U_X^- \cup (U_X \cap D_X)$.

Remark. For particle mechanics, the formalism reduces to the spacetime formulation that we developed in Fetecau, Marsden, Ortiz, and West [2002] to study nonsmooth rigid-body dynamics. We will discuss this example in detail in §4. ♦

Case (b). Discontinuous without separation (slip). For problems such as propagation of free surfaces in fluids or interaction of an elastic body and a fluid, the configuration map ϕ is no longer continuous. We must therefore choose a new configuration space to include these cases. Observe that in such problems the fluid-fluid and the solid-fluid boundaries are material surfaces, in the sense that particles which are on the separating surface at a given time remain on the surface at later times.

Let U_S and U_F two open subsets of U such that $\partial U_S = \partial U_F = D$ is a codimension 1 submanifold in U. We adopt the subscripts S and F because one of the applications of this general setting will be solid-fluid interactions; U_S, U_F and D will be interpreted as the spacetime regions of the solid, the fluid and of the surface separating the two materials, respectively. For fluid-fluid boundaries, D is the spacetime free surface.

The requirement that there be no flow across the material interface is expressed by considering the configuration space

$$\begin{aligned} \mathcal{C}^b = \{ & \phi : U \to Y \mid \pi_{XY} \circ \phi : U \to X \text{ is an embedding}, \\ & \phi \text{ is of class } C^2 \text{ in } U_S \cup U_F \text{ and } \overline{\phi_S}(D) = \overline{\phi_F}(D) = \phi(D) \}, \end{aligned} \tag{3.2}$$

where ϕ_S, ϕ_F are the restrictions of the map ϕ on U_S, U_F, respectively, and the notation $\overline{f}$ represents the continuous extension of the map f to the closure of its domain.

Remark. One may alternatively denote U_S and U_F by U^+ and U^-, respectively. This will be particularly useful in §3.2, where we retain only these notations for the domains where there are no singularities. As in case (a), we denote by U_X^+, U_X^- and D_X the images in X of U^+, U^- and D, under ϕ_X,. ◆

Case (c). Discontinuous with separation (collisions). Collisions of elastic bodies may exhibit both of the features of the two classes of configuration maps presented so far. The mechanical impact of two solids generates stress waves that propagate through their bodies, reflect on the boundaries and then return to the contact interface. At the instant when a returning wave first reaches the contact surface, the release begins and separation will eventually occur. Because of the complicated, non-linear structure of the governing equations, little of a general nature may be presented for impact problems. We refer to Graff [1991] for a detailed discussion on the longitudinal impact of two elastic rods and on the impact of an elastic sphere with a rod, to demonstrate some of the complexities encountered in such problems.

We will consider the frictionless impact with slipping of two elastic bodies. The analog of D from the previous paragraphs will be the spacetime contact set. However, to the contact set in the spatial configuration, there correspond two distinct surfaces in the reference configuration. In the multisymplectic formalism we consider two disjoint open sets U_1 and U_2 in U and $D_1 \subset \partial U_1$, $D_2 \subset \partial U_2$ two codimension 1 submanifolds. We consider the following set as the configuration space for collision problems

$$\begin{aligned}\mathcal{C}^c = \{\phi : U \to Y \mid \pi_{XY} \circ \phi : U \to X \text{ is an embedding},\\ \phi \text{ is of class } C^2 \text{ in } U \text{ and } \overline{\phi}(D_1) = \overline{\phi}(D_2)\},\end{aligned} \tag{3.3}$$

where the notation $\overline{f}$ represents, as before, the continuous extension of the map f.

Remark. The set $\overline{\phi}(D_1)$ (or equivalently, $\overline{\phi}(D_2)$) must be interpreted as a *subset* of the spacetime contact set. The subset does not contain points which belong to other types of discontinuity surfaces (such as, for example, the points on the interface at the very moment of impact, which also belong to $\mathcal{C}^a$ type waves that are generated by the mechanical impact and propagate through the bodies). Intersections of different types of discontinuity surfaces are extremely important and we intend to treat this subject in our future work on this topic. We will thus not discuss this point further here. ◆

Remark. For a more unified presentation and to include all three cases a-c in the general result from §3.2, we will refer later to U_1 and U_2 as U^+ and U^-. ♦

Remark. For purposes of connecting this work with PDE methods, one can consider the closure of either of $\mathcal{C}^{a-c}$ in the topology of a larger space such as $H^s(U,Y)$ or $\mathcal{C}^\infty(U^+,Y)\times\mathcal{C}^\infty(U^-,Y)$. This enables one to regard $\mathcal{C}^{a-c}$ as subsets in a manifold of mappings of the appropriate Sobolev class, as in Palais [1968] and Ebin and Marsden [1970]. ♦

Remark. In the remainder of the paper we will write $\mathcal{C}^{a-c}$ to indicate the appropriate configuration space with the manifold structure obtained by the procedure explained above. However, whenever we state a general result which applies to all configuration manifolds a-c, we will write $\mathcal{C}$ to mean any of the three. ♦

Variations and tangent spaces. We will account for general variations of maps $\phi \in \mathcal{C}$ induced by a family of maps ϕ^λ defined by the action of some Lie group. More precisely, let $\mathcal{G}$ be a Lie group of π_{XY}-bundle automorphism η_Y covering diffeomorphisms η_X, with Lie algebra $\mathfrak{g}$, acting on $\mathcal{C}$ by $\Phi : \mathcal{G}\times\mathcal{C}\to\mathcal{C}$, where

$$\Phi(\eta_Y,\phi) = \eta_Y\circ\phi. \tag{3.4}$$

Now let $\lambda\mapsto\eta_Y^\lambda$ be an arbitrary smooth path in $\mathcal{G}$ such that $\eta_Y^0 = \mathrm{Id}_Y$, and let $V\in T_\phi\mathcal{C}$ be given by

$$V = \left.\frac{d}{d\lambda}\right|_{\lambda=0}\Phi(\eta_Y^\lambda,\phi), \quad \text{and} \quad V_X = \left.\frac{d}{d\lambda}\right|_{\lambda=0}\eta_X^\lambda\circ\phi_X. \tag{3.5}$$

We define the vertical component V_yY of the tangent space at y to be

$$V_yY = \{\mathcal{V}\in T_yY \mid T\pi_{XY}\cdot\mathcal{V} = 0\}. \tag{3.6}$$

Using this, we can naturally split the tangent space at each point $y=\phi(u)$ in the image of ϕ into vertical and horizontal components, $T_yY = V_yY\oplus H_yY$, where

$$H_yY = T_u\phi\cdot T_uU.$$

This decomposition of T_yY induces a decomposition of $T_\phi\mathcal{C}$, so that any vector $V\in T_\phi\mathcal{C}$ may be decomposed as $V = V^h + V^v$, where

$$V^h = T(\phi\circ\phi_X^{-1})\cdot V_X, \tag{3.7}$$

and by (3.5), $V_X = T\pi_{XY}\cdot V$.

Case (a). For $\phi \in \mathcal{C}^a$, it is easy to show that the tangent space $T_\phi \mathcal{C}^a$ is given by

$$T_\phi \mathcal{C}^a = \{V : U \to TY \mid V \text{ is } C^0 \text{ in } U \text{ and of class } C^2 \text{ in } U \setminus D, \\ \pi_{Y,TY} \circ V = \phi \text{ and } T_{\pi_{XY}} \circ V = V_X \text{ is a vector field on } X\}. \tag{3.8}$$

Case (b). In the multisymplectic description of a continuum medium, the bundle Y over X is trivial. It consists of a fiber manifold M (also called the ambient space) attached to each point of the spacetime $X = B \times \mathbb{R}$ (B is called the reference configuration). The fiber components of the points $\phi(u)$ with $u \in D$ (the interface in the reference configuration) constitute the image of the interface in the spatial configuration.

By constructing variations of maps $\phi \in \mathcal{C}^b$ as in (3.4), we can prove the following lemma

3.1 Lemma. *Let N_A be the outward unit normal of the current configuration interface and $V = (V^\mu, V^A)$ be a tangent vector obtained by (3.5). Then, $[\![V^A(u)]\!] N_A = 0$, for all $u \in D$.*

Proof. Let us first explain the sense in which the jump condition of this lemma must be interpreted. A point $y \in D$ is mapped by ϕ to a point $\phi(y) \in Y$ and denote by y the fiber component of $\phi(y)$ (y is a point on the current interface). By definition (3.2), there exist two points $y_S, y_F \in D$ such that $\overline{\phi_S}(y_S) = \overline{\phi_F}(y_F) = \phi(y)$.

Consider now a variation of a map $\phi \in \mathcal{C}^b$ given by $\phi^\lambda = \eta_Y^\lambda \circ \phi$, where η_Y^λ are π_{XY}-bundle automorphisms covering diffeomorphisms η_X and $\eta_Y^0 = \mathrm{Id}_Y$. By $[\![V(x)]\!]$ we mean

$$[\![V(u)]\!] = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \overline{\phi_S^\lambda}(u_S) - \left.\frac{d}{d\lambda}\right|_{\lambda=0} \overline{\phi_F^\lambda}(u_F), \tag{3.9}$$

where ϕ_S^λ, ϕ_F^λ are the restrictions of the map ϕ^λ on U_S, U_F, respectively. The two terms in the right-hand-side of (3.9) are vectors in $T_{\phi(u)}Y$, so the addition operation makes sense.

Since $\phi^\lambda \in \mathcal{C}^b$, there exists a point $u_F^\lambda \in D$ such that

$$\overline{\phi_S^\lambda}(u_S) = \overline{\phi_F^\lambda}(u_F^\lambda).$$

We note that $u_F^0 = u_F$. Then, we can derive

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \overline{\phi_S^\lambda}(u_S) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \overline{\phi_F^\lambda}(u_F) + T\overline{\phi_F}(u_F) \cdot v, \tag{3.10}$$

where

$$v = \left.\frac{d}{d\lambda}\right|_{\lambda=0} u_F^\lambda \in T_{u_F} D. \tag{3.11}$$

By using (3.10), (3.9) becomes

$$[\![V(u)]\!] = T\overline{\phi_F}(u_F) \cdot v. \tag{3.12}$$

Now, (3.11) and (3.12) prove the lemma. ■

Using the same notation as before, we thus proved

$$\begin{aligned} T_\phi \mathcal{C}^b = \{V : U \to TY \mid & V \text{ is of class } C^2 \text{ in } U_S \cup U_F, \pi_{Y,TY} \circ V = \phi, \\ & T_{\pi_{XY}} \circ V = V_X \text{ is a vector field on } X, \text{ and } [\![V^A]\!] N_A = 0 \text{ on } D\}. \end{aligned} \tag{3.13}$$

c. A result very similar to Lemma 3.1 will hold for case (c) as well. More precisely, consider two points $u_1 \in D_1$ and $u_2 \in D_2$ such that $\overline{\phi}(u_1) = \overline{\phi}(u_2) = y$. By an argument similar to the one used in the proof of Lemma 3.1, we can prove

$$V(u_2) - V(u_1) = T\overline{\phi}(u_1) \cdot v, \tag{3.14}$$

where $v \in T_{u_1} D_1$. Then, $V(u_2) - V(u_1) \in T_y Y$ and, if we denote by N_A the components of the outward unit normal of the contact set in the current configuration, we abuse the notation and write

$$[\![V^A]\!] N_A = 0.$$

Hence, the tangent space $T_\phi \mathcal{C}^c$ is given by

$$\begin{aligned} T_\phi \mathcal{C}^c = \{V : U \to TY \mid & V \text{ is of class } C^2 \text{ in } U_1 \cup U_2, \pi_{Y,TY} \circ V = \phi, \\ & T_{\pi_{XY}} \circ V = V_X \text{ is a vector field on } X \text{ and } [\![V^A]\!] N_A = 0 \text{ on } D\}, \end{aligned} \tag{3.15}$$

where the jump relation has the interpretation explained before.

3.2 Variational Approach

We will show next how to derive the equations of motion and the jump conditions directly from the variational principle, staying entirely on the Lagrangian side.

The ***action function*** $S : \mathcal{C} \to \mathbb{R}$ is defined by

$$S(\phi) = \int_{U_X} \mathcal{L}(j^1(\phi \circ \phi_X^{-1})). \tag{3.16}$$

We say that $\phi \in \mathcal{C}$ is a ***stationary point*** or ***critical point*** of S if

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} S(\Phi(\eta_Y^\lambda, \phi)) = 0 \tag{3.17}$$

for all curves η_Y^λ with $\eta_Y^0 = \mathrm{Id}_Y$.

Using the infinitesimal generators defined in (3.5), we compute:

$$\begin{aligned} dS_\phi \cdot V &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} S(\Phi(\eta_Y^\lambda, \phi)) \\ &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} \int_{\eta_X^\lambda(U_X)} \mathcal{L}(j^1(\Phi(\eta_Y^\lambda, \phi))) \\ &= \int_{U_X} \left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathcal{L}(j^1(\Phi(\eta_Y^\lambda, \phi))) + \int_{U_X} \mathfrak{L}_{V_X}\left[\mathcal{L}(j^1(\phi \circ \phi_X^{-1}))\right]. \end{aligned}$$

In Marsden, Patrick, and Shkoller [1998] the following lemma is proved.

3.2 Lemma. *For any $V \in T_\phi \mathcal{C}$,*

$$dS_\phi \cdot V^h = \int_{\partial U_X} V_X \,\lrcorner\, \left[\mathcal{L}(j^1(\phi \circ \phi_X^{-1}))\right], \tag{3.18}$$

and

$$dS_\phi \cdot V^v = \int_{U_X} \left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathcal{L}(j^1(\Phi(\eta_Y^\lambda, \phi))). \tag{3.19}$$

The previous lemma leads to the following fundamental theorem.

3.3 Theorem. *Given a Lagrangian density $\mathcal{L} : J^1Y \to \Lambda^{n+1}(X)$, which is smooth away from the discontinuity, there exists a unique smooth section $D_{EL}\mathcal{L} \in C^\infty(Y'', \Lambda^{n+1}(X) \otimes T^*Y))$ and a unique differential form $\Theta_\mathcal{L} \in \Lambda^{n+1}(J^1Y)$ such that for any $V \in T_\phi\mathcal{C}$ which is compactly supported in U and any open subset U_X such that $\overline{U_X} \cap \partial X = \emptyset$,*

$$\begin{aligned} dS_\phi \cdot V = \int_{U_X^+} & D_{EL}\mathcal{L}(j^2(\phi \circ \phi_X^{-1})) \cdot V + \int_{U_X^-} D_{EL}\mathcal{L}(j^2(\phi \circ \phi_X^{-1})) \cdot V \\ &+ \int_{U_X \cap D_X} [\![j^1(\phi \circ \phi_X^{-1})^* \left(j^1(V) \,\lrcorner\, \Theta_\mathcal{L}\right)]\!], \end{aligned} \tag{3.20}$$

where $[\![\cdot]\!]$ denotes the jump.

Furthermore,

$$D_{EL}\mathcal{L}(j^2(\phi \circ \phi_X^{-1})) \cdot V = j^1(\phi \circ \phi_X^{-1})^*[j^1(V) \,\lrcorner\, \Omega_\mathcal{L}]. \tag{3.21}$$

In coordinates, the action of the ***Euler-Lagrange derivative*** *$D_{EL}\mathcal{L}$ on Y'' is given by*

$$\begin{aligned} D_{EL}\mathcal{L}(j^2(\phi \circ \phi_X^{-1})) = \Bigg[& \frac{\partial L}{\partial y^A}(j^1(\phi \circ \phi_X^{-1})) - \frac{\partial^2 L}{\partial x^\mu \partial {v^A}_\mu}(j^1(\phi \circ \phi_X^{-1})) \\ & - \frac{\partial^2 L}{\partial y^B \partial {v^A}_\mu}(j^1(\phi \circ \phi_X^{-1})) \cdot (\phi \circ \phi_X^{-1})^B_{,\mu} \\ & - \frac{\partial^2 L}{\partial {v^B}_\mu \partial {v^A}_\nu}(j^1(\phi \circ \phi_X^{-1})) \cdot (\phi \circ \phi_X^{-1})^B_{,\mu\nu} \Bigg] d^{n+1}x \otimes dy^A, \end{aligned} \tag{3.22}$$

while the form $\Theta_{\mathcal{L}}$ matches the definition of the ***Cartan form*** *obtained via the Legendre transform and has the coordinate expression*

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial v^A{}_\mu} dy^A \wedge d^n x_\mu + \left(L - \frac{\partial L}{\partial v^A{}_\mu} v^A{}_\mu \right) d^{n+1}x. \tag{3.23}$$

Proof. We choose $U_X = \phi_X(U)$ small enough so that it is contained in a coordinate chart O. In the coordinates on O, let $V = (V^\mu, V^A)$ so that along $\phi \circ \phi_X^{-1}$, the decomposition (3.7) can be written as

$$V_X = V^\mu \frac{\partial}{\partial x^\mu} \text{ and } V^v = (V^v)^A \frac{\partial}{\partial y^A} = \left(V^A - V^\mu \frac{\partial(\phi \circ \phi_X^{-1})^A}{\partial x^\mu} \right) \frac{\partial}{\partial y^A}.$$

Now, we use (3.19) to obtain

$$dS_\phi \cdot V^v = \int_{U_X} \left[\frac{\partial L}{\partial y^A} (j^1(\phi \circ \phi_X^{-1})) \cdot (V^v)^A + \frac{\partial L}{\partial v^A{}_\mu} (j^1(\phi \circ \phi_X^{-1})) \cdot \frac{\partial (V^v)^A}{\partial x^\mu} \right] d^{n+1}x. \tag{3.24}$$

We split the integral $\int_{U_X}$ into $\int_{U_X^+} + \int_{U_X^-}$ and integrate by parts to obtain

$$\begin{aligned} dS_\phi \cdot V^v = & \int_{U_X^+} \left[\frac{\partial L}{\partial y^A} (j^1(\phi \circ \phi_X^{-1})) - \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial v^A{}_\mu} (j^1(\phi \circ \phi_X^{-1})) \right] \\ & \cdot (V^v)^A d^{n+1}x \\ & + \int_{U_X^-} \left[\frac{\partial L}{\partial y^A} (j^1(\phi \circ \phi_X^{-1})) - \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial v^A{}_\mu} (j^1(\phi \circ \phi_X^{-1})) \right] \\ & \cdot (V^v)^A d^{n+1}x \\ & + \int_{U_X \cap D_X} \left[\!\left[\frac{\partial L}{\partial v^A{}_\mu} (j^1(\phi \circ \phi_X^{-1})) \cdot (V^v)^A \right]\!\right] d^n x_\mu. \end{aligned} \tag{3.25}$$

The jump arises from the different orientations of D_X when we use Stokes theorem in U_X^+ and U_X^-. Additionally, from (3.18) we obtain the horizontal contribution

$$dS_\phi \cdot V^h = \int_{U_X \cap D_X} [\![V^\mu L]\!] d^{n+1}x. \tag{3.26}$$

We note that the terms corresponding to $\int_{\partial U_X}$ vanish in both (3.25) and (3.26) since V is compactly supported in U. Now, we can combine (3.25) and (3.26) to obtain

$$dS_\phi \cdot V = \int_{U_X^+ \cup U_X^-} \left\{ \left[\frac{\partial L}{\partial y^A} - \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial v_\mu^A} \right] (j^1(\phi \circ \phi_X^{-1})) \right\} d^{n+1}x \otimes dy^A \cdot V$$
$$+ \int_{U_X \cap D_X} \left[\!\left[V \lrcorner \left\{ \frac{\partial L}{\partial v_\mu^A} (j^1(\phi \circ \phi_X^{-1})) dy^A \wedge d^n x_\mu \right.\right.\right.$$
$$\left.\left.\left. + \left[L - \frac{\partial L}{\partial v_\mu^A} (j^1(\phi \circ \phi_X^{-1})) \frac{\partial (\phi \circ \phi_X^{-1})^A}{\partial x^\mu} \right] d^{n+1}x \right\} \right]\!\right]. \quad (3.27)$$

Let α be the n-form in the jump brackets of the integrand of the boundary integral in (3.27). Then, $\int_{U_X \cap D_X} \alpha = \int_{j^1(\phi \circ \phi_X^{-1})(U_X \cap D_X)} \alpha$, since α is invariant under this lift. Moreover, the vector V in the second term of (3.27) (now written as an integral over $j^1(\phi \circ \phi_X^{-1})(U_X \cap D_X)$) may be replaced by $j^1(V)$ since $\pi_{Y,J^1(Y)}$-vertical vectors are in the kernel of the form that V is acting on. Now we can pull back the integrand with $j^1(\phi \circ \phi_X^{-1})^*$ to get an n-form on $U_X \cap D_X$. To summarize, we proved that the boundary integral in (3.27) can be written as

$$\int_{U_X \cap D_X} \left[\!\left[j^1(\phi \circ \phi_X^{-1})^* \left(j^1(V) \lrcorner \Theta_{\mathcal{L}} \right) \right]\!\right],$$

where $\Theta_{\mathcal{L}} \in \Lambda^{n+1}(J^1(Y))$ has the coordinate expression given by (3.23).

The integrand of the first integral in (3.27) defines the coordinate expression of the Euler-Lagrange derivative $D_{EL}\mathcal{L}$. However, if we choose another coordinate chart O', the coordinate expressions of $D_{EL}\mathcal{L}$ and $\Theta_{\mathcal{L}}$ must agree on the overlap $O \cap O'$ since the left hand side of (3.20) is intrinsically defined. Thus, we have uniquely defined $D_{EL}\mathcal{L}$ and $\Theta_{\mathcal{L}}$.

Now, we can define intrinsically $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}}$ and check that (3.21) holds, as both sides have the same coordinate expressions. ■

We now use Hamilton's principle of critical action and look for those paths $\phi \in \mathcal{C}$ which are critical points of the action function. More precisely, we call a field $\phi \in \mathcal{C}$ a ***solution*** if

$$dS(\phi) \cdot V = 0, \quad (3.28)$$

for all vector fields $V \in T_\phi \mathcal{C}$ which vanish on the boundary ∂U.

From Theorem 3.3 it follows that a field ϕ is a solution if and only if the Euler-Lagrange derivative (evaluated at $j^2(\phi \circ \phi_X^{-1})$) is zero on U_X^+ and U_X^- and the n-form $j^1(\phi \circ \phi_X^{-1})^* \left[j^1(V) \lrcorner \Theta_{\mathcal{L}} \right]$ has a zero jump across $U_X \cap D_X$.

We thus obtain the Euler-Lagrange equations in U_X^+ and U_X^-, away from the singularities. In coordinates, they read

$$\frac{\partial L}{\partial y^A}(j^1(\phi \circ \phi_X^{-1})) - \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial v^A{}_\mu}(j^1(\phi \circ \phi_X^{-1})) = 0 \quad \text{in} \quad U_X^+ \cup U_X^-. \quad (3.29)$$

Finally, the intrinsic jump condition

$$\int_{U_X \cap D_X} [\![j^1(\phi \circ \phi_X^{-1})^* \left(j^1(V) \lrcorner \Theta_{\mathcal{L}} \right)]\!] = 0 \tag{3.30}$$

has the following coordinate expression

$$\begin{aligned} \int_{U_X \cap D_X} \Bigg(& \left[\!\!\left[\frac{\partial L}{\partial v^A{}_\mu}(j^1(\phi \circ \phi_X^{-1})) \cdot V^A \right]\!\!\right] \\ & + \left[\!\!\left[LV^\mu - \frac{\partial L}{\partial v^A{}_\mu}(j^1(\phi \circ \phi_X^{-1})) \frac{\partial(\phi \circ \phi_X^{-1})^A}{\partial x^\nu} V^\nu \right]\!\!\right] \Bigg) d^n x_\mu = 0. \end{aligned} \tag{3.31}$$

In the next section we will write the jump conditions (3.31) for the particle and continuum mechanics multisymplectic models and give their physical interpretations. Here, we simply note that by taking vertical variations only ($V^\mu = 0$) we obtain a jump condition involving only momenta $p_A{}^\mu$; this will represent the jump in linear momentum condition. Horizontal variations will in turn give the correct energy jump and a kinematic compatibility condition.

4 Classical Mechanics

For a classical mechanical system (such as particles or rigid bodies) with configuration space Q, let $X = \mathbb{R}$ (parameter time) and $Y = \mathbb{R} \times Q$, with π_{XY} the projection onto the first factor. The first jet bundle J^1Y is the bundle whose holonomic sections are tangents of sections $\phi : X \to Y$, so we can identify $J^1Y = \mathbb{R} \times TQ$. Using coordinates (t, q^A) on $\mathbb{R} \times Q$, the induced coordinates on J^1Y are the usual tangent coordinates (t, q^A, v^A).

We will apply the multisymplectic formalism described in §2 to nonsmooth rigid-body dynamics. We are particularly interested in the problem of rigid-body collisions, for which the velocity, acceleration and forces are all nonsmooth or even discontinuous. The multisymplectic formalism will elegantly recover the spacetime formulation of nonsmooth Lagrangian mechanics of Fetecau, Marsden, Ortiz, and West [2002].

In Fetecau, Marsden, Ortiz, and West [2002] a mechanical system with configuration manifold Q is considered, but with the dynamics restricted to a submanifold with boundary $C \subset Q$, which represents the subset of admissible configurations. The boundary ∂C is called the contact set; for rigid body collision problems, the submanifold ∂C is obtained from the condition that interpenetration of matter cannot occur. The dynamics is specified by a regular Lagrangian $L : TQ \to \mathbb{R}$. We note that the multisymplectic framework allows us to consider time dependent Lagrangians as well (see the general definition of a Lagrangian density in §2), but we will restrict our discussion here to only autonomous systems.

To apply the multisymplectic formalism for such systems, we choose U to be the interval $[0, 1]$ and D the set containing only one element $\tau_i \in [0, 1]$. The set $\mathcal{C}^a$ from (3.1) becomes:

$$\mathcal{C}' = \{\phi : [0,1] \to \mathbb{R} \times Q \mid \phi \text{ is a } C^0, PW\ C^2 \text{ curve },\\ \phi(\tau) \text{ has only one singularity at } \tau_i\}. \tag{4.1}$$

Now let $U_X = [t_0, t_1]$ be the image in $X = \mathbb{R}$ of the embedding $\phi_X = \pi_{XY} \circ \phi$. The section $\varphi = \phi \circ \phi_X^{-1} : [t_0, t_1] \to \mathbb{R} \times Q$ can be written in coordinates as $t \mapsto (t, q^A(t))$. Let $t_i = \phi_X(\tau_i)$ be the the moment of impact, so that $q(t_i) \in \partial C$. We note that, even though the singularity parameter time τ_i is fixed, it is allowed to vary in the t space according to $t_i = \phi_X(\tau_i)$ and thus, the setting is not restrictive in this sense.

Hence, the map $\phi_X : [0, 1] \to [t_0, t_1]$ is just a time reparametrization. The need for a nonautonomous formulation of an autonomous mechanical system is explained in the following remarks.

Remark. In the smooth context, the dynamics of a mechanical system can be described by sections of smooth fields $\varphi : [t_0, t_1] \to Q$. As we noted in the general setting, the key observation that the set of such smooth fields is a C^∞ infinite-dimensional manifold enables the use of differential calculus on the manifold of mappings (see Marsden and Ratiu [1999]). However, generalization to the nonsmooth setting is not straightforward and this is one of the main issues addressed in Fetecau, Marsden, Ortiz, and West [2002]. ♦

Remark. The approach used in Fetecau, Marsden, Ortiz, and West [2002] is to extend the problem to the nonautonomous case, so that both configuration variables and time are functions of a separate parameter τ. This allows the impact to be fixed in τ space while remaining variable in both configuration and time spaces, and it means that the relevant space of configurations will indeed be a smooth manifold, as proved in that reference. The nonsmooth multisymplectic formalism applied to this problem leads to essentially the same extended formulation. ♦

The Cartan form (3.23) becomes the extended Lagrange 1-form of particle mechanics, with coordinate expression

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial \dot{q}^A} dq^A + \left(L - \frac{\partial L}{\partial \dot{q}^A} \dot{q}^A \right) dt. \tag{4.2}$$

Define the ***energy*** $E : TQ \to \mathbb{R}$ by

$$E(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \cdot \dot{q} - L(q, \dot{q}),$$

which allows us to write the Lagrangian 1-form in the compact notation

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial \dot{q}} dq - E dt. \tag{4.3}$$

Let $V \in T_\phi \mathcal{C}$ be a tangent vector constructed as in (3.5) with coordinates $V = (V^0, V^A)$. As the fiber component of $\varphi(t_i)$ is varied in ∂C, we can write the jump condition (3.30) as

$$\Theta_{\mathcal{L}}|_{t_i^-} = \Theta_{\mathcal{L}}|_{t_i^+} \text{ on } \mathbb{R} \times TQ|\partial C. \tag{4.4}$$

In coordinates, the jump condition (4.4) reads

$$V^A \left.\frac{\partial L}{\partial \dot{q}^A}\right|_{t_i^-}^{t_i^+} + V^0 \left.\left(L - \frac{\partial L}{\partial \dot{q}^A} \dot{q}^A \right)\right|_{t_i^-}^{t_i^+} = 0. \tag{4.5}$$

Splitting this into the two components gives

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{t=t_i^-} \cdot \delta q = \left.\frac{\partial L}{\partial \dot{q}}\right|_{t=t_i^+} \cdot \delta q \tag{4.6}$$

for any $\delta q = V^A \frac{\partial}{\partial q^A} \in T_{q(t_i)} \partial C$ and

$$E(q(t_i^-), \dot{q}(t_i^-)) = E(q(t_i^+), \dot{q}(t_i^+)). \tag{4.7}$$

These equations are the Weierstrass-Erdmann type conditions for impact. That is, equation (4.6) states that the linear momentum must be conserved in the tangent direction to ∂C, while equation (4.7) states that the energy must be conserved during an elastic impact.

Hence, horizontal variations (V^0) give conservation of energy and vertical variations (V^A) give conservation of the Lagrange 1-form on $T\partial C$.

5 Continuum Mechanics

5.1 Multisymplectic Formulation of Continuum Mechanics

Configuration Spaces in the Multisymplectic Formalism. We will use here the formalism constructed in Marsden, Pekarsky, Shkoller, and West [2001] to describe the configurations of a continuous medium. Let (B, G) be a smooth n-dimensional compact oriented Riemannian manifold with smooth boundary and let (M, g) be a smooth N-dimensional compact oriented Riemannian manifold. The space (B, G) will represent what is traditionally called the ***reference configuration***, while (M, g) will denote the ***ambient space***.

We choose $X = B \times \mathbb{R}$; the coordinates on X are $x^\mu = (x^i, x^0) = (x^i, t)$, with $\mu = 0, \ldots, n,\ i = 1, \ldots, n$. Let $Y = X \times M$ be a trivial bundle over X with M being a fiber at each point and let $\pi_{XY} : Y \to X;\ (x, t, y) \mapsto (x, t)$ be the projection on the first factor ($y \in M$ is the fiber coordinate). Let $y^A,\ A = 1, \ldots, N$ be fiber coordinates; they induce the coordinates on J^1Y denoted $\gamma = (x^\mu, y^A, v^A{}_\mu)$. We denote the fiber coordinates on J^1Y^* by $(\Pi, p_A{}^\mu)$; they correspond to the affine map given in coordinates by $v^A{}_\mu \mapsto (\Pi + p_A{}^\mu v^A{}_\mu) d^{n+1}x$.

A section $\varphi : X \to Y$ of π_{XY} has coordinate representation $\varphi(x) = (x^\mu, \varphi^A(x))$, while its first jet $j^1\varphi$ is given by

$$x^\mu \mapsto (x^\mu, \varphi^A(x), \partial_\mu \varphi^A(x)), \tag{5.1}$$

where $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_k = \frac{\partial}{\partial x^k}$.

We note that we introduced two different Riemannian structures on the spatial part of the base manifold X and on the fiber M. Thus, the formalism is general enough to apply for continuum models where the metric spaces (B, G) and (M, g) are essentially different (rods, shells models, fluids with free boundary). However, for classical 2 or 3 dimensional elasticity or for fluid dynamics in a domain with fixed boundaries, the two Riemannian structures may coincide.

Define the function $J : J^1Y \to \mathbb{R}$ with coordinate expression

$$J(x, t, y, v) = \det[v] \sqrt{\frac{\det[g(y)]}{\det[G(x)]}}. \tag{5.2}$$

For a section φ, $J(j^1(\varphi))$ represents the Jacobian of the linear transformation $D\varphi_t$. We note that, even in the cases where the metrics G and g coincide, there is no cancellation in (5.2), as the metric tensors are evaluated at different points ($g(y)$ is different from $G(x)$ unless $y = x$ or both tensors are constant).

Lagrangian dynamics. To describe the dynamics of a particular continuum medium in the variational multisymplectic framework, one needs to specify a Lagrangian density $\mathcal{L}$. The Lagrangian density $\mathcal{L} : J^1Y \to \Lambda^{n+1}X$ is defined as a smooth bundle map

$$\begin{aligned}\mathcal{L}(\gamma) = L(\gamma) d^{n+1}x = \mathbb{K} - \mathbb{P} = \frac{1}{2}\sqrt{\det[G]}\rho(x) g_{AB} v^A{}_0 v^B{}_0 d^{n+1}x \\ - \sqrt{\det[G]}\rho(x) W(x, G(x), g(y), v^A{}_j) d^{n+1}x, \end{aligned} \tag{5.3}$$

where $\gamma \in J^1Y$, $\rho : B \to \mathbb{R}$ is the mass density and W is the ***stored energy function***.

The first term in (5.3), when restricted to first jet extensions, represents the kinetic energy, as $v^A{}_0$ becomes the time derivative $\partial_t \varphi^A$ of the section φ. The second term represents the potential energy and different choices of the function W specify particular models of continuous media. Typically, for elasticity, W depends on the field's partial derivatives through the Green deformation tensor C (see Marsden and Hughes [1983], for example), while for ideal fluid dynamics, W is only a function of the Jacobian J (5.2).

The Lagrangian density (5.3) determines the Legendre transformation $\mathbb{F}\mathcal{L} : J^1Y \to J^1Y^*$. The conjugate momenta are given by

$$p_A{}^0 = \frac{\partial L}{\partial v^A{}_0} = \rho g_{AB} v^B{}_0 \sqrt{\det[G]}, \quad p_A{}^j = \frac{\partial L}{\partial v^A{}_j} = -\rho \frac{\partial W}{\partial v^A{}_j} \sqrt{\det[G]},$$

and

$$\Pi = L - \frac{\partial L}{\partial v^A{}_\mu} v^A{}_\mu = \left[-\frac{1}{2} g_{AB} v^A{}_0 v^B{}_0 - W + \frac{\partial W}{\partial v^A{}_j} v^A{}_j \right] \rho \sqrt{\det[G]}.$$

We define the ***energy density*** e by

$$e = \frac{\partial L}{\partial v^A{}_0} v^A{}_0 - L \quad \text{or, equivalently} \quad e\, d^{n+1}x = \mathbb{K} + \mathbb{P}. \tag{5.4}$$

The Cartan form on J^1Y can be obtained either by using the Legendre transformation and pulling back the canonical $(n+1)$-form on the dual jet bundle as in (2.9), or by a variational route as in Theorem 3.3. The resulting coordinate expression is given by

$$\begin{aligned} \Theta_{\mathcal{L}} = \rho g_{AB} v^B{}_0 \sqrt{\det[G]} dy^A \wedge d^n x_0 - \rho \frac{\partial W}{\partial v^A{}_j} \sqrt{\det[G]} dy^A \wedge d^n x_j \\ + \left[-\frac{1}{2} g_{AB} v^A{}_0 v^B{}_0 - W + \frac{\partial W}{\partial v^A{}_j} v^A{}_j \right] \rho \sqrt{\det[G]} d^{n+1}x. \end{aligned} \tag{5.5}$$

Substituting the Lagrangian density (5.3) into equation (3.29) we obtain the Euler-Lagrange equations for a continuous medium

$$\begin{aligned} \rho g_{AB} \left(\frac{D_g \dot{\varphi}}{Dt} \right)^B - \frac{1}{\sqrt{\det[G]}} \frac{\partial}{\partial x^k} \left(\rho \frac{\partial W}{\partial v^A_k} (j^1\varphi) \sqrt{\det[G]} \right) \\ = -\rho \frac{\partial W}{\partial g_{BC}} \frac{\partial g_{BC}}{\partial y^A} (j^1 \varphi), \end{aligned} \tag{5.6}$$

where

$$\left(\frac{D_g \dot{\varphi}}{Dt} \right)^A = \frac{\partial \dot{\varphi}^A}{\partial t} + \gamma^A_{BC} \dot{\varphi}^B \dot{\varphi}^C \tag{5.7}$$

is the covariant time derivative, and

$$\gamma^A_{BC} = \frac{1}{2} g^{AD} \left(\frac{\partial g_{BD}}{\partial y^C} + \frac{\partial g_{CD}}{\partial y^B} - \frac{\partial g_{BC}}{\partial y^D} \right)$$

are the Christoffel symbols associated with the metric g.

Given a potential energy W which specifies the material, equation (5.6) is a system of PDE's to be solved for a section $\varphi(x,t)$. We remark that all terms in this equation are functions of x and t and hence have the interpretation of material quantities. In particular, (5.7) corresponds to ***material acceleration***.

We define the multisymplectic analogue of the ***Cauchy stress tensor*** σ by

$$\sigma^{AB}(\varphi, x) = \frac{2\rho(x)}{J}\frac{\partial W}{\partial g_{AB}}(j^1\varphi(x)). \tag{5.8}$$

Equation (5.8) is known in the elasticity literature as the Doyle-Ericksen formula. We make the important remark that the balance of angular momentum

$$\sigma^T = \sigma$$

follows from the definition (5.8) and the symmetry of the metric tensor g.

In the case of Euclidean manifolds with constant metrics g and G, equation (5.6) simplifies to the familiar expression

$$\rho\frac{\partial^2 \varphi_A}{\partial t^2} = \frac{\partial}{\partial x^k}\left(\rho\frac{\partial W}{\partial {v^A}_k}(j^1\varphi)\right). \tag{5.9}$$

Next we will describe the multisymplectic formalism for the two main application of the theory we developed: elasticity and ideal fluid dynamics.

Elasticity. As we noted before, for the theory of elasticity the reference configuration (B, G) and the ambient space (M, g) are generally different. The spatial part B of the base manifold X has the interpretation of the reference configuration and the extra dimension of X corresponds to time. Later configurations of the elastic body are captured by a section φ of the bundle Y. For a fixed time t, the sections φ_t play the role of ***deformations***; they map the reference configuration B onto the spatial configuration, which is a subset of the ambient space M.

The fiber coordinates of the first jet $j^1\varphi$ of a section φ, as defined by (5.1), consist of the time derivative of the deformation $\dot\varphi^A$ and the ***deformation gradient*** F^A_i given by

$$F^A_i(x,t) = \frac{\partial \varphi^A}{\partial x^i}. \tag{5.10}$$

Hence, the first jet of a section φ has the local representation

$$j^1\varphi : (x,t) \mapsto ((x,t), \varphi(x,t), \dot\varphi(x,t), F(x,t)).$$

For a given section φ, the ***first Piola-Kirchhoff stress tensor*** $\mathcal{P}_A{}^j$ is defined by

$$\mathcal{P}_A{}^j(\varphi, x) = \rho(x)\frac{\partial W}{\partial v^A{}_j}(j^1\varphi(x)). \tag{5.11}$$

We also define ***the Green deformation tensor*** (also called the right Cauchy-Green tensor) C by $C = \varphi_t^*(g)$; in coordinates we have

$$C_{ij}(x,t) = g_{AB}F_i^A F_j^B. \tag{5.12}$$

Using definitions (5.8) and (5.11), the Euler-Lagrange equations (5.6) become

$$\rho g_{AB}\left(\frac{D_g\dot{\varphi}}{Dt}\right)^B = \mathcal{P}_A{}^i{}_{|i} + \gamma^B_{AC}(\mathcal{P}^j_B F^C_j - J g_{BD}\sigma^{DC}), \tag{5.13}$$

where we have introduced the ***covariant divergence*** defined by

$$\mathcal{P}_A{}^i{}_{|i} = \mathrm{DIV}\mathcal{P} = \frac{\partial \mathcal{P}_A{}^i}{\partial x^i} + \mathcal{P}^j_A\Gamma^k_{jk} - \mathcal{P}^i_B\gamma^B_{AC}F^C_i. \tag{5.14}$$

Here, the Γ^i_{jk} are the Christoffel symbols corresponding to the base metric G.

We note that in (5.13) there is no a-priori relationship between the first Piola-Kirchhoff stress tensor and the Cauchy stress tensor, as W is assumed to have the most general form $W(x, G, g, v)$. However, such a relationship can be derived by imposing material frame indifference on the energy function. This assumption will imply that the energy function W depends on the deformation gradient F (equivalently, on v) and on the field metric g only through the Green deformation tensor given by (5.12) , that is $W = W(C(v, g))$. For this particular form of W, definitions (5.8) and (5.11) lead to

$$\mathcal{P}^i_A = J(\sigma F^{-1})^{\,i}_A. \tag{5.15}$$

Relation (5.15) is known as the Piola transformation law. Substituting it into (5.13), one obtains the Euler-Lagrange equations for the standard elasticity model

$$\rho g_{AB}\left(\frac{D_g\dot{\varphi}}{Dt}\right)^B = \mathcal{P}_A{}^i{}_{|i}. \tag{5.16}$$

For elasticity in a Euclidean space, this equation simplifies to

$$\rho\frac{\partial^2\varphi^A}{\partial t^2} = \frac{\partial \mathcal{P}^{Ai}}{\partial x^i}. \tag{5.17}$$

Barotropic Fluids. For ideal fluid dynamics we have the same multisymplectic bundle picture as that described for elasticity. For fluids moving in a fixed region we set $B = M$ and call it the ***reference fluid container***. However, for fluid dynamics with free boundary, the structures (B, G) and (M, g) are generally different. Configurations of the fluid are captured by a section φ of the bundle Y, which has the interpretation of the particle placement field. In coordinates, the spatial point $y \in Y_{(x,t)}$ corresponds to a position $y = \varphi(x, t)$ of the fluid particle x at time t.

For standard models of barotropic fluids, the potential energy of the fluid depends only on the Jacobian of the deformation, that is $W = W(J(g, G, v))$. The ***pressure*** function is defined to be

$$P(\varphi, x) = -\rho(x)\frac{\partial W}{\partial J}(j^1\varphi(x)). \tag{5.18}$$

For a given section φ, $P(\varphi) : X \to \mathbb{R}$ has the interpretation of the material pressure which is a function of the material density. Using (5.18), the Cauchy stress tensor (5.8) becomes

$$\sigma^{AB}(x) = \frac{2\rho}{J}\frac{\partial W}{\partial J}\frac{\partial J}{\partial g_{AB}}(j^1\varphi) = -P(x)g^{AB}(y(x)). \tag{5.19}$$

We refer to Marsden, Pekarsky, Shkoller, and West [2001] for a discussion on how the pressure function arises in both the compressible and incompressible models. We also remark here that one could also consider (5.19) as a defining equation for pressure, from which (5.18) would follow.

With these notations, the Euler-Lagrange equations (5.6) become

$$\rho g_{AB}\left(\frac{D_g\dot{\varphi}}{Dt}\right)^B = -\frac{\partial P}{\partial x^k}J\left(\left(\frac{\partial \varphi}{\partial x}\right)^{-1}\right)^k_A. \tag{5.20}$$

We introduce the ***spatial density*** $\rho_{\rm sp} = \rho/J$ and define the ***spatial pressure*** $p(y)$ by $p(y(x)) = P(x)$; then (5.20) can be re-written in the familiar form

$$\frac{D_g\dot{\varphi}}{Dt}(x, t) = -\frac{1}{\rho_{\rm sp}}\ \mathrm{grad}\ p \circ \varphi(x, t). \tag{5.21}$$

5.2 Propagating Singular Surfaces Within an Elastic Body

In this subsection we apply the theory developed in §3.1 to investigate the motion of a singular surface of order 1 within a compressible elastic body. The ***order*** of a singular surface is given by the lowest order of the derivatives of the configuration map $\phi(x, t)$ that suffer a non-zero jump across the surface. For a singular surface of order 1, the configuration map

$\phi(x,t)$ is continuous, but its first order derivatives (the velocity $\dot{\phi}$ and the deformation gradient F) may suffer jump discontinuities upon the surface. Thus, the configuration space for this problem belongs to class (a) of the classification considered in §3.1.

The multisymplectic formalism will lead to the derivation of the correct jumps in linear momentum and energy across the discontinuity surface. Moreover, spatial horizontal variations will lead to a kinematic condition known as the Maxwell compatibility condition.

We use the same notation as previously, so let U be diffeomorphic to an open subset of the spacetime X and let D be a codimension 1 submanifold in U representing a discontinuity surface in spacetime, moving within the elastic body. The configuration space $\mathcal{C}$ is given by (3.1) and all the results from §3 apply for this example.

We note first that Theorem 3.3 implies that the Euler-Lagrange equations (5.13) will be satisfied on either side of the discontinuity.

Now let $V \in T_\phi\mathcal{C}$ be a tangent vector with coordinates (V^μ, V^A) and consider initially only *vertical* variations ($V^\mu = 0$). From (3.31) we obtain the jump conditions

$$\int_{D_X} \left[\!\left[\frac{\partial L}{\partial v^A{}_\mu}(j^1(\phi \circ \phi_X^{-1})) \right]\!\right] \cdot V^A d^n x_\mu = 0, \tag{5.22}$$

where we used the continuity of the vector field V.

For simplicity, consider the Euclidean case, where $G_{\mu\nu} = \delta_{\mu\nu}$ and $g_{AB} = \delta_{AB}$. The jump relation (5.22) becomes

$$\int_{D_X} \left[\!\left[\rho(x) \frac{\partial \varphi^A}{\partial t} \right]\!\right] \cdot V^A d^n x_0 + \int_{D_X} [\![\mathcal{P}_A^{\,j}]\!] \cdot V^A d^n x_j = 0, \tag{5.23}$$

where, as before, φ denotes the section $\varphi = \phi \circ \phi_X^{-1}$.

The 1-forms dt and dx^j, $j = 1, \ldots, n$, on D_X are not independent. More precisely, if D_X is given locally by $f(t, x^1, \ldots, x^n) = 0$, then by differentiating we obtain

$$\partial_j f\, dx^j + \partial_t f\, dt = 0. \tag{5.24}$$

Define N_j by $N_j = \partial_j f / |\nabla_x f|$ and define the propagation speed U by

$$U = -\frac{\partial_t f}{|\nabla_x f|}, \tag{5.25}$$

where $|\cdot|$ represents the Euclidean norm. This speed is a measure of the rate at which the moving surface traverses the material; it also gives the excess of the normal speed of the surface over the normal speed of the particles comprising it. Then (5.23) becomes

$$\int_{D_X} \left(\left[\!\left[\rho U \frac{\partial \varphi^A}{\partial t} \right]\!\right] + [\![\mathcal{P}_A^{\,j}]\!]\, N_j \right) \cdot V^A d^n x_0 = 0. \tag{5.26}$$

By a standard argument in the calculus of variations we can pass to the local form and recover the standard jump of linear momentum across a propagating singular surface of order 1, which is

$$\left[\!\!\left[\rho U \frac{\partial \varphi^A}{\partial t}\right]\!\!\right] + [\![\mathcal{P}_A^{\ j}]\!]\, N_j = 0. \tag{5.27}$$

An alternative approach to derive the jump in linear momentum uses the balance of linear momentum for domains traversed by singular surfaces (see Truesdell and Toupin [1960], pg. 545 for example). For such derivations, the jump conditions are usually expressed in spatial coordinates, where the propagation speed U and the Piola-Kirchhoff stress tensor are replaced by a local propagation speed and the Cauchy stress tensor, respectively.

Remark. The conservation of mass implies the following jump relation, known as the *Stokes-Christoffel condition* (see Truesdell and Toupin [1960], pg. 522)

$$[\![\rho U]\!] = 0. \tag{5.28}$$

In Courant and Friedrichs [1948], the continuous quantity ρU is denoted by m and is called the *mass flux* through the surface. Using (5.28), we can re-write (5.27) as

$$\rho U \left[\!\!\left[\frac{\partial \varphi^A}{\partial t}\right]\!\!\right] + [\![\mathcal{P}_A^{\ j}]\!] N_j = 0. \tag{5.29}$$

◆

Remark. In the terminology of Truesdell and Toupin [1960], the singular surfaces that have a non-zero propagation speed are called *propagating* singular surfaces or *waves*. We first note that this definition excludes material surfaces, for which $U = 0$. Moreover, Truesdell and Toupin [1960] classifies the singular surfaces with non-zero jump in velocity into two categories: surfaces with transversal discontinuities ($[\![U]\!] = 0$), called ***vortex sheets***, and surfaces with arbitrary discontinuities in velocity ($[\![U]\!] \neq 0$), called ***shock surfaces***. One can conclude that there are no material shock surfaces, i.e., shock surfaces are always waves. Also, there is a nonzero mass flux ($m \neq 0$) through a shock surface or through a vortex sheet which is not material. ◆

Remark. In the context of gas dynamics, Courant and Friedrichs [1948] defines a ***shock front*** as a discontinuity surface across which there is a non-zero gas flow ($m \neq 0$). Then, a shock surface in the sense of Truesdell and Toupin [1960] is also a shock front. ◆

The linear momentum jump was derived by taking vertical variations of the sections. Next we will focus on *horizontal* variations ($V^\mu \neq 0$) and derive the corresponding jump laws. Using (5.27), the jump conditions (3.31) become

$$\int_{D_X} \left[\!\!\left[LV^\mu - \frac{\partial L}{\partial v^A_\mu}(j^1(\phi \circ \phi_X^{-1})) \frac{\partial(\phi \circ \phi_X^{-1})^A}{\partial x^\nu} V^\nu \right]\!\!\right] d^n x_\mu = 0. \tag{5.30}$$

Consider first only time component variations ($V^0 \neq 0$, $V^j = 0$ for $j = 1, \ldots, n$); then (5.30) gives

$$\begin{aligned} &\int_{D_X} \left[\!\!\left[L - \frac{\partial L}{\partial v^A_0} \frac{\partial(\phi \circ \phi_X^{-1})^A}{\partial t} \right]\!\!\right] V^0 d^n x_0 \\ &\qquad - \int_{D_X} \left[\!\!\left[\frac{\partial L}{\partial v^A_j} \frac{\partial(\phi \circ \phi_X^{-1})^A}{\partial t} \right]\!\!\right] V^0 d^n x_j = 0. \end{aligned} \tag{5.31}$$

Using (5.11), (5.24) and (5.25), (5.31) becomes

$$\int_{D_X} \left([\![Ue]\!] + \left[\!\!\left[\mathcal{P}^j_A \frac{\partial \varphi^A}{\partial t} \right]\!\!\right] N_j \right) V^0 d^n x_0 = 0. \tag{5.32}$$

From (5.32) we recover the standard jump of energy (see Truesdell and Toupin [1960], pg. 610 for example),

$$[\![Ue]\!] + \left[\!\!\left[\mathcal{P}^j_A \frac{\partial \varphi^A}{\partial t} \right]\!\!\right] N_j = 0. \tag{5.33}$$

Finally we consider space component variations ($V^j \neq 0$) in (5.30) and use (5.32) to obtain

$$\begin{aligned} &\int_{D_X} \left[\!\!\left[\frac{\partial L}{\partial v^A_0} \frac{\partial(\phi \circ \phi_X^{-1})^A}{\partial x^j} V^j \right]\!\!\right] d^n x_0 \\ &\qquad - \int_{D_X} \left[\!\!\left[LV^j - \frac{\partial L}{\partial v^A_j} \frac{\partial(\phi \circ \phi_X^{-1})^A}{\partial x^k} V^k \right]\!\!\right] d^n x_j = 0. \end{aligned} \tag{5.34}$$

Then, by using (5.11), (5.24), (5.25) and (5.28), (5.34) becomes

$$\int_{D_X} \left(\rho U \left[\!\!\left[F^A{}_j \frac{\partial \varphi^A}{\partial t} \right]\!\!\right] + [\![L]\!] N_j + [\![F^A{}_j \mathcal{P}_A{}^k]\!] N_k \right) V^j d^n x_0 = 0. \tag{5.35}$$

Since the components V^j are arbitrary we conclude that

$$\rho U \left[\!\!\left[F^A_j \frac{\partial \varphi^A}{\partial t} \right]\!\!\right] + [\![L]\!] N_j + [\![F^A_j \mathcal{P}^k_A]\!] N_k = 0. \tag{5.36}$$

Even though (5.36) does not resemble any standard conservation law, after some algebraic manipulations using (5.27) and (5.33), (5.36) can be rewritten, for continuous U, as

$$U \left[\!\left[F_j^A \right]\!\right] + \left[\!\left[\frac{\partial \varphi^A}{\partial t} \right]\!\right] N_j = 0, \tag{5.37}$$

which is the statement of the Maxwell compatibility condition (see Jaunzemis [1967], Chapter 2 or Truesdell and Toupin [1960], Chapter C.III. for the derivation of the kinematical conditions of compatibility from Hadamard's lemma).

To summarize, the *vertical* variations of the sections led us to derive the jump in linear momentum, while *horizontal* time and space variations accounted for the energy balance and the kinematic compatibility condition, respectively.

5.3 Free Surfaces in Fluids and Solid-fluid Boundaries

Now, we investigate in the multisymplectic framework a different type of discontinuous motion that will illustrate the case (b) of the classification from §3.1. We consider two types of discontinuity surfaces, namely *free surfaces* in fluids and *solid-fluid boundaries*. A free surface or a free boundary is a surface separating two immiscible fluids or two regions of the same fluid in different states of motion (Karamcheti [1966]). The second type of discontinuity considers the interaction of a deformable elastic body with a surrounding barotropic fluid.

As we already noted in §3.1, these types of discontinuous surfaces have one feature in common, namely they are material surfaces (particles which are on the surface at a given time remain on the surface at later times). Equivalently, there is no flow across the discontinuity and the surface is stationary relative to the medium. Hence, in the reference configuration, the surface D_X is given locally by $f(x^1, \ldots, x^n) = 0$ (no dependence of the function f on t). Moreover, from (5.25) we have that the propagation speed U for such surfaces is zero. In the terminology of Truesdell and Toupin [1960], these surfaces are material vortex sheets of order 0.

Free Surfaces in Fluids. Theorem 3.3 implies that Euler-Lagrange equations of type (5.20) will be satisfied on either side of the surface separating the two fluid regions. Next, we will show that Theorem 3.3 gives the correct force balance on the separating surface and the other physical conditions that must be satisfied on such boundaries.

Let $V \in T_\phi \mathcal{C}$ be a tangent vector with coordinates (V^μ, V^A). We consider first only *vertical* variations ($V^\mu = 0$); from (3.31) we obtain the following jump conditions

$$\int_{D_X} \left[\!\left[\frac{\partial L}{\partial v^A_\mu} (j^1(\phi \circ \phi_X^{-1})) \cdot V^A \right]\!\right] d^n x_\mu = 0. \tag{5.38}$$

For simplicity, we will consider Euclidean geometries, that is $G_{\mu\nu} = \delta_{\mu\nu}$ and $g_{AB} = \delta_{AB}$. We recall that for fluids, $W = W(J)$; this relation and the stationarity of the discontinuity surface ($U = 0$ on D_X) simplifies the jump relation (5.38) to

$$\int_{D_X} \left[\!\left[\rho \frac{\partial W}{\partial J} \frac{\partial J}{\partial v_j^A} N_j \cdot V^A \right]\!\right] d^n x_0 = 0, \tag{5.39}$$

where $N_j = \frac{\partial_j f}{|\nabla_x f|}$ is the normal vector to D_X. From the definition of the Jacobian J (5.2), one can derive

$$\frac{\partial J}{\partial v_j^A} = J (v^{-1})^j_A.$$

We use this relation and the definition of the material pressure (5.18) to re-write (5.39) as

$$\int_{D_X} \left[\!\left[PJ \left(\left(\frac{\partial \varphi}{\partial x} \right)^{-1} \right)^j_A N_j V^A \right]\!\right] d^n x_0 = 0. \tag{5.40}$$

We notice that in (5.40), the term $J \left(\left(\frac{\partial \varphi}{\partial x} \right)^{-1} \right)^j_A N_j d^n x_0$ represents the A-th component of the area element in the spatial configuration, as given by the formula of Nanson (see Truesdell and Toupin [1960], pg. 249 or Jaunzemis [1967], pg. 154, for example). Hence, substituting $y = \varphi_t(x)$ in (5.40) and then passing to the local form we can obtain the jump relation

$$[\![p V^A N_A]\!] = 0, \tag{5.41}$$

where p is the spatial pressure defined by $p(y(x)) = P(x)$. Now, we combine (5.41) with the property that the vector field V has a zero normal jump (see (3.13)), to obtain

$$[\![p]\!] = 0, \tag{5.42}$$

which is the standard pressure balance at a free surface.

We take now horizontal variations $V^\mu \neq 0$ such that $V^0 \neq 0$ and $V^j = 0$, for $j = 1, \dots, n$. Then, (3.31) simplifies to

$$\int_{D_X} \left[\!\left[\frac{\partial L}{\partial v_j^A} \frac{\partial (\phi \circ \phi_X^{-1})^A}{\partial t} N_j V^0 \right]\!\right] d^n x_0 = 0. \tag{5.43}$$

Furthermore, using the continuity of V^0 and the particular form of $W = W(J)$ in the Lagrangian (5.3), we can write (5.43) as

$$\int_{D_X} \left[\!\left[PJ \left(\left(\frac{\partial \varphi}{\partial x} \right)^{-1} \right)^j_A \frac{\partial \varphi^A}{\partial t} N_j \right]\!\right] V^0 d^n x_0 = 0. \tag{5.44}$$

As before, we use the formula of Nanson, substitute $y = \varphi_t(x)$ in (5.44), and then pass to the local form to obtain

$$\left[\!\left[p \frac{\partial \varphi^A}{\partial t} N_A \right]\!\right] = 0. \tag{5.45}$$

Using the pressure continuity (5.42), the jump condition (5.45) becomes

$$\left[\!\left[\frac{\partial \varphi^A}{\partial t} N_A \right]\!\right] = 0. \tag{5.46}$$

We can also use the continuity of the normal vector to write (5.46) as

$$\left[\!\left[\frac{\partial \varphi^A}{\partial t} \right]\!\right] N_A = 0. \tag{5.47}$$

The jump condition (5.46) is a kinematic condition which restricts the possible jumps of the fluid velocity only to tangential discontinuities (the normal component is continuous). In the literature, this condition may appear either as a boundary condition (see Karamcheti [1966]) or as a definition for vortex sheets (see Truesdell and Toupin [1960]). However, we recover it through a variational procedure, as a consequence of the general theorem of the §3.2, using the particular form of the space of configurations (3.2) and of its admissible variations (3.13).

Finally, let consider only space component horizontal variations ($V^0 = 0$ and $V^j \neq 0$, for $j = 1, \ldots, n$). The vector field $V^j \frac{\partial}{\partial x^j}$ on X is lifted by $T(\phi \circ \phi_X^{-1})$ to a horizontal vector field on Y (see decomposition (3.7)) with coordinates

$$V^A = \frac{\partial (\phi \circ \phi_X^{-1})^A}{\partial x^j} V^j. \tag{5.48}$$

Then, by using the previous jump conditions (5.42), (3.31) simplifies to

$$\int_{D_X} \left[\!\left[\frac{\partial L}{\partial v_k^A} \frac{\partial (\phi \circ \phi_X^{-1})^A}{\partial x^j} N_k V^j \right]\!\right] d^n x_0 = 0, \tag{5.49}$$

where we also used that $V^j N_j = 0$ for material surfaces. Using (5.48) and the definition of the material pressure (5.18), (5.49) becomes exactly (5.40), so it will provide the already known jump condition (5.42).

Solid-fluid Boundaries. We again apply Theorem 3.3 to find that the Euler-Lagrange equations (5.13) will be satisfied in the domain occupied by the elastic body, while the fluid dynamics in the outer region will be described by (5.20). As for free surfaces, the boundary terms in Theorem 3.3 will give the correct pressure-traction balance on the boundary of the elastic body, as well as restrictions on the jumps in velocity.

For vertical variations only, the jump conditions are those given by (5.38). For Euclidean geometries, these conditions become

$$\int_{D_X}\left[\mathcal{P}^j_A N_j (V^A)^+ - PJ\left(\left(\frac{\partial\varphi}{\partial x}\right)^{-1}\right)^j_A N_j (V^A)^-\right] d^n x_0 = 0, \qquad (5.50)$$

where we adopt the usual notation with superscript + and − for the limit values of a discontinuous function at a point on the singular surface by approaching the point from each side of the discontinuity.

Using the Piola transformation (5.15) and the formula of Nanson we can make the substitution $y = \varphi_t(x)$ in (5.50) and then pass to the local form; we obtain

$$\sigma^{AB} N_B (V^A)^+ - pN_A (V^A)^- = 0. \qquad (5.51)$$

We use the property of the vector field V from (3.13),

$$(V^A)^+ N_A - (V^A)^- N_A = 0,$$

to write (5.51) as

$$(\sigma^{AB} N_B - pN_A) \cdot (V^A)^+ = 0. \qquad (5.52)$$

As there are no restrictions on $(V^A)^+$, we have

$$\sigma^{AB} N_B - pN_A = 0. \qquad (5.53)$$

Moreover, by denoting by $t^A = \sigma^{AB} N_B$ the stress vector, we obtain

$$t^A N_A - p = 0, \qquad (5.54)$$

which is the pressure-traction balance on the boundary of the elastic body.

We now consider horizontal variations $V^\mu \neq 0$ such that $V^0 \neq 0$ and $V^j = 0$, for $j = 1, \ldots, n$. Using the previous result (5.54), the general jump conditions (3.31) reduce to

$$\int_{D_X} \left[\!\left[\frac{\partial L}{\partial v^A_j} \frac{\partial(\phi \circ \phi_X^{-1})^A}{\partial t} N_j V^0 \right]\!\right] d^n x_0 = 0. \qquad (5.55)$$

For solid-fluid interactions, the jump conditions (5.55) become

$$\int_{D_X}\left[\mathcal{P}^j_A N_j \left(\frac{\partial\varphi^A}{\partial t}\right)^+ - PJ\left(\left(\frac{\partial\varphi}{\partial x}\right)^{-1}\right)^j_A N_j \left(\frac{\partial\varphi^A}{\partial t}\right)^-\right] V^0 d^n x_0 = 0, \qquad (5.56)$$

By using the Piola transformation (5.15) and Nanson's formula, we make the substitution $y = \varphi_t(x)$ in (5.56) and then pass to the local form to get

$$\sigma^{AB} N_B \left(\frac{\partial \varphi^A}{\partial t}\right)^+ - pN_A \left(\frac{\partial \varphi^A}{\partial t}\right)^- = 0. \tag{5.57}$$

Now, from (5.53) and (5.57) we can derive

$$\left[\!\left[\frac{\partial \varphi^A}{\partial t} \right]\!\right] N_A = 0, \tag{5.58}$$

which implies the continuity of the normal component of the velocity. Thus, only tangential discontinuities in the velocity are possible. We emphasize again that we obtain this restriction as a consequence of the choice of the configuration space (see (3.2) and (3.13)) and not by prescribing it as a boundary condition.

By an argument similar to the one used for fluid-fluid interfaces, we can show that the space component horizontal variations do not provide new jump conditions; they will lead in fact to the jump condition (5.51), from which the pressure-traction balance (5.54) can be derived.

5.4 Collisions of Elastic Bodies

We now illustrate the last category of the classification of configuration spaces from §3.1. We will apply the general formalism to investigate the collision of two elastic bodies, where the configuration manifold is given by $\mathcal{C}^c$ defined in (3.3) and the analog of the singular surfaces from the previous subsections is the codimension 1 spacetime contact surface. The interface is a material surface, so it has a zero propagation speed $U = 0$. By the choice of the configuration space $\mathcal{C}^c$ we allow the elastic bodies to slip on each other during the collision, but they do so without friction.

If we consider only vertical variations, the jump conditions will be given by (5.38). In the Euclidean case these conditions become

$$\int_{D_X} \left[\!\left[\mathcal{P}_A^{\,j} N_j V^A \right]\!\right] d^n x_0 = 0. \tag{5.59}$$

By making the change of variables $y = \phi_t(x)$ in (5.59) and using the Piola transformation (5.15), we can write the integral in the spatial configuration and then pass to the local form to obtain

$$\left[\!\left[\sigma^{AB} N_B V^A \right]\!\right] = 0, \tag{5.60}$$

where N_A are the components of the outward unit normal to the contact set in the current configuration.

Let $t^A = \sigma^{AB} N_B$ denote the stress vector, as before. By using the jump restriction on V from (3.15) we can derive

$$[\![t^A N_A]\!] = 0, \tag{5.61}$$

which represents the balance of the normal tractions on the contact set during a collision. From the derivation of (5.61) we also obtain that the tangential tractions are zero on the contact surface.

Let us consider now time component horizontal variations ($V^0 \neq 0$ and $V^j = 0$, for $j = 1, \ldots, n$); the general jump conditions (3.31) reduce to (5.55), which in turn become

$$\int_{D_X} \left[\!\!\left[\mathcal{P}_A^{\ j} N_j \left(\frac{\partial \varphi^A}{\partial t} \right) \right]\!\!\right] V^0 d^n x_0 = 0. \tag{5.62}$$

By the same procedure used before, we can pass to the local form in the spatial configuration and obtain

$$\left[\!\!\left[t^A \frac{\partial \varphi^A}{\partial t} \right]\!\!\right] = 0. \tag{5.63}$$

From (5.61) and (5.63) we can derive

$$\left[\!\!\left[\frac{\partial \varphi^A}{\partial t} N_A \right]\!\!\right] = 0, \tag{5.64}$$

which gives the continuity of the normal components of the velocities, once the contact is established. However, the tangential discontinuity in velocities, due to slipping, may be arbitrary.

The space component horizontal variations will not provide new jump conditions; we can show this by the same procedure used in §5.3.

6 Concluding Remarks and Future Directions

There are several directions to pursue in the future to complete the foundations laid in this paper. Perhaps the most important task is to develop algorithms and a discrete mechanics for nonsmooth multisymplectic variational mechanics and to take advantage of the current algorithms (such as that of Pandolfi, Kane, Marsden, and Ortiz [2002]) that are already developing in this direction.

Another task is to further develop the theory of shock waves by combining the geometric approach here with more analytical techniques, such as those used in hyperbolic systems of conservation laws, as well as incorporating appropriate thermodynamic notions.

We also need to extend the basic theory to incorporate constraints, similar to the way that Marsden, Pekarsky, Shkoller, and West [2001] deal with incompressibility constraints.

For some systems, there will be surface tension and other boundary effects; for some of these systems a Hamiltonian structure is already understood (see Lewis, Marsden, Montgomery, and Ratiu [1986] and references therein) but not a multisymplectic structure.

Here we have only considered isolated discontinuities, but there may be degeneracies caused by the intersections of different types and dimensions of discontinuity surfaces that require further attention.

Finally, as in Kane, Marsden, Ortiz, and West [2000] and Pandolfi, Kane, Marsden, and Ortiz [2002], friction (or other dissipative phenomena) and forcing need to be included in the formalism.

Acknowledgments: We are grateful for helpful remarks from Michael Ortiz and Steve Shkoller. This work was partially supported by AFOSR contract F49620-02-1-0176.

7 References

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