The Euler–Poincaré Equations in Geophysical Fluid Dynamics

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Abstract

Recent theoretical work has developed the Hamilton's-principle analog of Lie-Poisson Hamiltonian systems defined on semidirect products. The main theoretical results are twofold:

1. Euler–Poincaré equations (the Lagrangian analog of Lie-Poisson Hamiltonian equations) are derived for a parameter dependent Lagrangian from a general variational principle of Lagrange d'Alembert type in which variations are constrained;

2. an abstract Kelvin–Noether theorem is derived for such systems.

By imposing suitable constraints on the variations and by using invariance properties of the Lagrangian, as one does for the Euler equations for the rigid body and ideal fluids, we cast several standard Eulerian models of geophysical fluid dynamics (GFD) at various levels of approximation into Euler-Poincaré form and discuss their corresponding Kelvin–Noether theorems and potential vorticity conservation laws. The various levels of GFD approximation are related by substituting a sequence of velocity decompositions and asymptotic

expansions into Hamilton's principle for the Euler equations of a rotating stratified ideal incompressible fluid. We emphasize that the shared properties of this sequence of approximate ideal GFD models follow directly from their Euler-Poincaré formulations. New modifications of the Euler-Boussinesq equations and primitive equations are also proposed in which nonlinear dispersion adaptively filters high wavenumbers and thereby enhances stability and regularity without compromising either low wavenumber behavior or geophysical balances.

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1 Introduction

The Eulerian formulation of the action principle for an ideal fluid casts it into a form that is amenable to asymptotic expansions and thereby facilitates the creation of approximate theories. This Eulerian action principle is part of the general procedure of the reduction theory of Lagrangian systems, including the theory of the Euler–Poincaré equations (the Lagrangian analog of Lie-Poisson Hamiltonian equations). This setting provides a shared structure for many problems in GFD, with several benefits, both immediate (such as a systematic approach to hierarchical modeling and versions of Kelvin's theorem for these models) and longer term (e.g., structured multisymplectic integration algorithms).

This paper will be concerned with Euler–Poincaré equations arising from a family of action principles for a sequence of standard GFD models in purely Eulerian variables at various levels of approximation. We use the method of *Hamilton's principle asymptotics* in this setting. In particular, the action principles of these models are related by different levels of truncation of asymptotic expansions and velocity-pressure decompositions in Hamilton's principle for the unapproximated Euler equations of rotating stratified ideal incompressible fluid dynamics. This sequence of GFD models includes the Euler equations themselves, followed by their approximations, namely: Euler-Boussinesq equations (EB), primitive equations (PE), Hamiltonian balance equations (HBE), and generalized Lagrangian mean (GLM) equations. We also relate our approach to the rotating shallow water equations (RSW), semigeostrophic equations (SG), and quasigeostrophic equations (QG). Thus, asymptotic expansions and velocity-pressure decompositions of Hamilton's principle for the Euler equations describing the motion of a rotating stratified ideal incompressible fluid will be used to cast the standard EB, PE, HBE and GLM models of GFD into Euler-Poincaré form and thereby unify these descriptions and their properties at various levels of approximation. See Tables 4.1 and 4.2 for summaries.

These GFD models have a long history dating back at least to Rossby [1940], Charney [1948] and Eliassen [1949], who used them, in their simplest forms (particularly the quasigeostrophic and semigeostrophic approximations), to study structure formation on oceanic and atmospheric mesoscales. The history of the efforts to establish the proper equations for synoptic motions is summarized by Pedlosky [1987] and Cushman-Roisin [1994]; see also Phillips [1963]. One may consult, for example, Salmon [1983, 1985, 1988], Holm, Marsden, Ratiu and Weinstein [1985], Abarbanel, Holm, Marsden, and Ratiu [1986], and Holm [1996] for recent applications of the approach of Hamilton's principle asymptotics to derive approximate equations in GFD.

Well before Rossby, Charney, and Eliassen, at the end of the 19th century, Poincaré [1901] investigated the formulation of the Euler equations for the dynamics of a rigid body in Lie algebraic form. Poincaré's formulation of

the Euler equations for a rigid body carries over naturally to the dynamics of ideal continua, as shown by Holm, Marsden and Ratiu [1998a], and Poincaré's ideas will form the basis of the present study. Some of Poincaré's other key papers in this area are listed in the bibliography.

Starting from the action principle for the Euler equations, the present work first expresses the various GFD equations in the Euler-Poincaré form for continua due to Holm, Marsden and Ratiu [1998a] and discusses the properties acquired by casting the GFD equations into this form. The main property so obtained is the Kelvin–Noether theorem for the theory. This, in turn, leads to conservation of potential vorticity on fluid parcels. Domain-integrated energy is also conserved and the relation of the Euler–Poincaré equations to the Lie-Poisson Hamiltonian formulation of the dynamics is given by a Legendre transformation at the level of the Lie algebra of divergenceless vector fields.

The methods of this paper are based on reduction of variational principles; that is, on Lagrangian reduction (see Cendra et al. [1986, 1987] and Marsden and Scheurle [1993a,b]), which is also useful for systems with nonholonomic constraints. This has been demonstrated in the work of Bloch, Krishnaprasad, Marsden and Murray [1996], who derived the reduced Lagrange d'Alembert equations for such nonholonomic systems. Coupled with the methods of the present paper, these techniques for handling nonholonomic constraints should also be useful for continuum systems. In addition, it seems likely that the techniques of multisymplectic geometry, associated variational integrators, and the multisymplectic reduction will be exciting developments for the present setting; see Marsden, Patrick and Shkoller [1997] for the beginnings of such a theory.

Organization of the Paper. In §2 we recall from Holm, Marsden and Ratiu [1998a] the abstract Euler-Poincaré theorem for Lagrangians depending on parameters along with the associated Kelvin–Noether theorem. These theorems play a key role in the rest of our analysis. In §3 we discuss their implications for continuum mechanics and then in §4 we apply them to a sequence of models in geophysical fluid dynamics. We begin in §4.1 and §4.2 by recalling the action principles in the Eulerian description for the Euler equations and their Euler-Boussinesq approximation, respectively. Then we show how these standard GFD models satisfy the Euler-Poincaré theorem. These sections also introduce the scaling regime and small parameters we use in making asymptotic expansions and velocity-pressure decompositions that are used in the remaining sections. Next, §4.3 introduces the hydrostatic approximation into the Euler-Poincaré formulation of the Euler-Boussinesq equations to yield the corresponding formulation of the primitive equations. Later sections cast further approximations of the Euler-Boussinesq equations into the Euler-Poincaré formulation, starting in §4.4 with the Hamiltonian balance equations and proceeding to the generalized Lagrangian mean (GLM) theory for wave, mean flow interaction (WMFI), due to Andrews and McIntyre

[1978a,b] in §5. In §6 we use the Euler–Poincaré theorem, including advected parameters, to formulate a new model of ideal GFD called the EBα model that includes *nonlinear dispersion* along with stratification and rotation. The EBα equations modify the usual Euler-Boussinesq equations by introducing a length scale, α. The length scale α is interpreted physically in the GLM setting as the amplitude of the rapidly fluctuating component of the flow. We derive the Euler–Poincaré equations for the EBα model by making an asymptotic expansion of the GLM Lagrangian for WMFI in powers of α and the Rossby number. Thus, the EBα model is a WMFI turbulence closure model for a rotating stratified incompressible fluid. In this model, nonlinear dispersion (parameterized by α) acts to filter the high wavenumbers ($k > 1/\alpha$) and thereby enhances solution stability and regularity without compromising either low wavenumber behavior ($k < 1/\alpha$), or geophysical balances. We also present the corresponding nonlinear dispersive modification of the primitive equations, called the PEα model. The nonlinear dispersive filtering of high wavenumber activity in the EBα and PEα models regularizes these equations and thereby makes them good candidates for long term numerical integration.

2 The Euler–Poincaré Equations, Semidirect Products, and Kelvin's Theorem

Here we recall from Holm, Marsden and Ratiu [1998a] the statements of the Euler–Poincaré equations and their associated Kelvin–Noether theorem. In the next section, we will discuss these statements in the context of continuum mechanics and then in the following section apply them to a sequence of models in geophysical fluid dynamics. Although there are several possible permutations of the conventions, we shall state the Euler–Poincaré theorem for the case of *right actions* and *right invariant Lagrangians*, which is appropriate for fluids and, in particular, for the GFD situation.

2.1 The Euler–Poincaré Equations and Semidirect Products

Assumptions and Notation. We shall begin with the abstract framework which will be a convenient setting for the several special cases of GFD to follow.

- Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. We consider a vector space V and assume we have a *right* representation of G on V. The group G then acts in a natural way on the *right* on the dual space V^* (the action by $g \in G$ on V^* is the dual of the action by g^{-1} on V). We denote the action of g on an element $v \in V$ by vg and on an element

$a \in V^*$ by ag. In general we use this *concatenation notation* for group actions. Then G also acts by right translation on TG and hence it acts on $TG \times V^*$. We denote the action of a group element g on a point (v_h, a) by $(v_g, a)g = (v_h g, ag)$.

- Assume we have a Lagrangian $L : TG \times V^* \to \mathbb{R}$ that is right G–invariant.
- For each $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) = L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0 (that is, the subgroup of elements of G that leave the element $a_0 \in V^*$ invariant).
- Right G–invariance of L permits us to define the ***reduced Lagrangian*** through the equation $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(v_g g^{-1}, ag^{-1}) = L(v_g, a).$$

  Conversely, this relation defines for any $l : \mathfrak{g} \times V^* \to \mathbb{R}$ a right G–invariant function $L : TG \times V^* \to \mathbb{R}$.
- For a curve $g(t) \in G$, let $\xi(t) := \dot{g}(t)g(t)^{-1}$ and define the curve $a(t) = a_0 g(t)^{-1}$, which is the unique solution of the linear differential equation with time dependent coefficients $\dot{a}(t) = -a(t)\xi(t)$ with initial condition $a(0) = a_0$.
- Let $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ be the *infinitesimal adjoint operator*; that is, the linear map given by the Lie algebra bracket: $\mathrm{ad}_\xi(\eta) = [\xi, \eta]$. Let $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ be the dual of the linear transformation ad_ξ.

Theorem 2.1 (Euler–Poincaré reduction.) *The following are equivalent:*

i *Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0 \tag{2.1}$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii *$g(t)$ satisfies the Euler–Lagrange equations for L_{a_0} on G.*

iii *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0 \tag{2.2}$$

holds on $\mathfrak{g} \times V^*$, using variations of the form

$$\delta\xi = \dot{\eta} - \mathrm{ad}_\xi \eta = \dot{\eta} - [\xi, \eta], \quad \delta a = -a\eta, \tag{2.3}$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

iv *The* ***Euler–Poincaré equations*** *hold on* $\mathfrak{g} \times V^*$

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = -\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a. \tag{2.4}$$

We refer to Holm, Marsden and Ratiu [1998a] for the proof of this in the abstract setting. We shall see some of the features of this result in the concrete setting of continuum mechanics shortly.

Important Notation. Following the notational conventions of Holm, Marsden and Ratiu [1998a], we let $\rho_v : \mathfrak{g} \to V$ be the linear map given by $\rho_v(\xi) = v\xi$ (the right action of ξ on $v \in V$), and let $\rho_v^* : V^* \to \mathfrak{g}^*$ be its dual. For $a \in V^*$, we write

$$\rho_v^* a = v \diamond a \in \mathfrak{g}^* \,,$$

which is a bilinear operation in v and a. Continuing to use the concatenation notation for Lie algebra actions, the $\mathfrak{g}$–action on $\mathfrak{g}^*$ and V^* is defined to be *minus* the dual map of the $\mathfrak{g}$–action on $\mathfrak{g}$ and V respectively and is denoted by $\mu\xi$ and $a\xi$ for $\xi \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$, and $a \in V^*$. The following is a useful way to write the definition of $v \diamond a \in \mathfrak{g}^*$: for all $v \in V$, $a \in V^*$ and $\xi \in \mathfrak{g}$, we have (note minus sign)

$$\langle v \diamond a \,, \xi \rangle = \langle a \,, v\xi \rangle = - \langle a\xi, v \rangle \ . \tag{2.5}$$

The Legendre Transformation. As explained in Marsden and Ratiu [1994], one normally thinks of passing from Euler–Poincaré equations on a Lie algebra $\mathfrak{g}$ to Lie–Poisson equations on the dual $\mathfrak{g}^*$ by means of the Legendre transformation. In our case, we start with a Lagrangian on $\mathfrak{g} \times V^*$ and perform a *partial* Legendre transformation in the variable ξ only, by writing

$$\mu = \frac{\delta l}{\delta \xi} \,, \quad h(\mu, a) = \langle \mu, \xi \rangle - l(\xi, a). \tag{2.6}$$

Therefore, we have the formulae

$$\frac{\delta h}{\delta \mu} = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \frac{\delta l}{\delta \xi} \,, \frac{\delta \xi}{\delta \mu} \right\rangle = \xi \quad \text{and} \quad \frac{\delta h}{\delta a} = - \frac{\delta l}{\delta a} \,. \tag{2.7}$$

One of the points is that, consistent with the examples, *we do not attempt* to use the full Legendre transformation to make Euler–Poincaré equations on $\mathfrak{g} \times V$ correspond to Lie–Poisson equations on the dual space $\mathfrak{g}^* \times V^*$. In fact, such attempts will fail because in most interesting examples, the full Legendre transform will be degenerate (the heavy top, compressible fluids, etc). It is for this reason that we take a partial Legendre transformation. In this case, our Euler–Poincaré equations on $\mathfrak{g} \times V^*$ will correspond to the Lie–Poisson equations on $\mathfrak{g}^* \times V^*$. We next briefly recall the Hamiltonian setting on $\mathfrak{g}^* \times V^*$.

Lie–Poisson Systems on Semidirect Products. Let $S = G \circledS V$ be the semidirect product Lie group for right actions. Explicitly, the conventions for S are the following: the multiplication has the expression

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + v_1 g_2), \tag{2.8}$$

the identity element is $(e, 0)$, and the inverse is given by $(g, v)^{-1} = (g^{-1}, -vg^{-1})$. The Lie algebra of S is denoted $\mathfrak{s} = \mathfrak{g} \circledS V$ and it has the bracket operation given by

$$[(\xi_1, v_1),\, (\xi_2, v_2)] = ([\xi_1, \xi_2],\, v_1\xi_2 - v_2\xi_1). \tag{2.9}$$

Let $\mathfrak{s}^*$ denote the dual of $\mathfrak{s}$. For a *right* representation of G on V the (+) Lie-Poisson bracket of two functions $f, k : \mathfrak{s}^* \to \mathbb{R}$ has the expression

$$\{f, k\}_+(\mu, a) \;=\; \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\right] \right\rangle - \left\langle a, \frac{\delta k}{\delta a}\frac{\delta f}{\delta \mu} - \frac{\delta f}{\delta a}\frac{\delta k}{\delta \mu} \right\rangle , \tag{2.10}$$

where $\delta f/\delta \mu \in \mathfrak{g}$, and $\delta f/\delta a \in V$ are the functional derivatives of f. Using the diamond notation (2.5), the corresponding Hamiltonian vector field for $h : \mathfrak{s}^* \to \mathbb{R}$ is easily seen to have the expression

$$X_h(\mu, a) = -\left(\mathrm{ad}^*_{\delta h/\delta \mu}\mu + \frac{\delta h}{\delta a} \diamond a,\; a\,\frac{\delta h}{\delta \mu} \right) . \tag{2.11}$$

Thus, Hamilton's equations on the dual of a semidirect product are given by

$$\dot{\mu} \;=\; \{\mu, h\} = -\,\mathrm{ad}^*_{\delta h/\delta \mu}\mu - \frac{\delta h}{\delta a} \diamond a\,, \tag{2.12}$$

$$\dot{a} \;=\; \{a, h\} = -\,a\,\frac{\delta h}{\delta \mu}\,. \tag{2.13}$$

where overdot denotes time derivative. Thus, the partial Legendre transformation (2.6) maps the Euler–Poincaré equations (2.4), together with the equations $\dot{a} = -a(t)\xi(t)$ for a to the Lie–Poisson equations (2.12) and (2.13).

Cautionary Remark. If the vector space V is absent and one has just the equations

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = -\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \tag{2.14}$$

for $\xi \in \mathfrak{g}$ on a Lie algebra, one speaks of them as the ***basic Euler–Poincaré equations***. As explained in Holm, Marsden and Ratiu [1998a], the Euler–Poincaré equations (2.4) *are not* the basic Euler–Poincaré equations on the larger semidirect product Lie algebra $\mathfrak{g} \circledS V^*$. This is a critical difference between the Lie–Poisson and the Euler–Poincaré cases.

Advected Parameters. As we shall see in the examples, and as indicated by the above Euler–Poincaré reduction theorem, the parameters $a \in V^*$ acquire dynamical meaning under Lagrangian reduction. For the heavy top, the parameter is the unit vector in the direction of gravity, which becomes a dynamical variable in the body representation. For stratified incompressible fluids, the parameters are the buoyancy b and volume D of a fluid element in the reference configuration, which in the spatial representation become dynamical variables satisfying the passive scalar advection equation and continuity equation, respectively.

2.2 The Kelvin–Noether Theorem

In this section, we explain a version of the Noether theorem that holds for solutions of the Euler–Poincaré equations. Our formulation is motivated and designed for continuum theories (and hence the name Kelvin–Noether), but it may be also of interest for finite dimensional mechanical systems. Of course it is well known that the Kelvin circulation theorem for ideal flow is closely related to the Noether theorem applied to continua using the particle relabelling symmetry group (see, for example, Arnold [1966]).

The Kelvin–Noether Quantity. We start with a Lagrangian L_{a_0} depending on a parameter $a_0 \in V^*$ as above. We introduce a manifold $\mathcal{C}$ on which G acts (on the right, as above) and suppose we have an equivariant map $\mathcal{K} : \mathcal{C} \times V^* \to \mathfrak{g}^{**}$.

In the case of continuum theories, the space $\mathcal{C}$ will be a loop space and $\langle \mathcal{K}(c, a), \mu \rangle$ for $c \in \mathcal{C}$ and $\mu \in \mathfrak{g}^*$ will be a circulation. This class of examples also shows why we *do not* want to identify the double dual $\mathfrak{g}^{**}$ with $\mathfrak{g}$.

Define the ***Kelvin–Noether quantity*** $I : \mathcal{C} \times \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$I(c, \xi, a) = \left\langle \mathcal{K}(c, a), \frac{\delta l}{\delta \xi}(\xi, a) \right\rangle. \tag{2.15}$$

Theorem 2.2 (Kelvin–Noether) *Fixing $c_0 \in \mathcal{C}$, let $\xi(t), a(t)$ satisfy the Euler–Poincaré equations and define $g(t)$ to be the solution of $\dot{g}(t) = \xi(t)g(t)$ and, say, $g(0) = e$. Let $c(t) = c_0 g(t)^{-1}$ and $I(t) = I(c(t), \xi(t), a(t))$. Then*

$$\frac{d}{dt} I(t) = \left\langle \mathcal{K}(c(t), a(t)), \frac{\delta l}{\delta a} \diamond a \right\rangle. \tag{2.16}$$

Proof. First of all, write $a(t) = a_0 g(t)^{-1}$ and use equivariance to write $I(t)$ as follows:

$$\left\langle \mathcal{K}(c(t), a(t)), \frac{\delta l}{\delta \xi}(\xi(t), a(t)) \right\rangle = \left\langle \mathcal{K}(c_0, a_0), \left[\frac{\delta l}{\delta \xi}(\xi(t), a(t)) \right] g(t) \right\rangle$$

The g^{-1} pulls over to the right side as g (and not g^{-1}) because of our conventions of always using right representations. We now differentiate the right hand side of this equation. To do so, we use the following well known formula for differentiating the coadjoint action (see Marsden and Ratiu [1994], §9.3):

$$\frac{d}{dt}[\mu(t)g(t)] = \left[\mathrm{ad}^*_{\xi(t)}\mu(t) + \frac{d}{dt}\mu(t)\right] g(t),$$

where $\mu \in \mathfrak{g}^*$, and $\xi \in \mathfrak{g}$ is given by

$$\xi(t) = \dot{g}(t)g(t)^{-1}.$$

Using this and the Euler–Poincaré equations, we get

$$\begin{aligned}
\frac{d}{dt}I &= \frac{d}{dt}\left\langle \mathcal{K}(c_0,a_0), \left[\frac{\delta l}{\delta \xi}(\xi(t),a(t))\right] g(t)\right\rangle \\
&= \left\langle \mathcal{K}(c_0,a_0), \frac{d}{dt}\left\{\left[\frac{\delta l}{\delta \xi}(\xi(t),a(t))\right] g(t)\right\}\right\rangle \\
&= \left\langle \mathcal{K}(c_0,a_0), \left[\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} - \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a}\diamond a\right] g(t)\right\rangle \\
&= \left\langle \mathcal{K}(c_0,a_0), \left[\frac{\delta l}{\delta a}\diamond a\right] g(t)\right\rangle \\
&= \left\langle \mathcal{K}(c_0,a_0)g(t)^{-1}, \left[\frac{\delta l}{\delta a}\diamond a\right]\right\rangle \\
&= \left\langle \mathcal{K}(c(t),a(t)), \left[\frac{\delta l}{\delta a}\diamond a\right]\right\rangle
\end{aligned}$$

where, in the last steps we used the definitions of the coadjoint action, the Euler–Poincaré equation (2.4) and the equivariance of the map $\mathcal{K}$.

Because the advected terms are absent for the basic Euler–Poincaré equations, we obtain the following.

Corollary 2.3 *For the basic Euler–Poincaré equations, the Kelvin quantity $I(t)$, defined the same way as above but with $I : \mathcal{C} \times \mathfrak{g} \to \mathbb{R}$, is conserved.*

3 The Euler–Poincaré Equations in Continuum Mechanics

In this section we will apply the Euler–Poincaré equations to the case of continuum mechanical systems.

Vector Fields and Densities. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{D}$ (or, more generally, a smooth compact manifold with boundary and given volume form or density). We let $\mathrm{Diff}(\mathcal{D})$ denote the diffeomorphism group of $\mathcal{D}$ of an appropriate Sobolev class (for example, as in Ebin and Marsden [1970]). If the domain $\mathcal{D}$ is not compact, then various decay hypotheses at infinity need to be imposed. Under such conditions, $\mathrm{Diff}(\mathcal{D})$ is a smooth infinite dimensional manifold and a topological group relative to the induced manifold topology. Right translation is smooth but left translation and inversion are only continuous. Thus, $\mathrm{Diff}(\mathcal{D})$ is not literally a Lie group in the naive sense and so the previous theory must be applied with care. Nevertheless, if one uses right translations and right representations, the Euler–Poincaré equations of Theorem 2.1 do make sense, as a direct verification shows. We shall illustrate such computations, by verifying several key facts in the proof as we proceed.

Let $\mathfrak{X}(\mathcal{D})$ denote the space of vector fields on $\mathcal{D}$ of the same differentiability class as $\mathrm{Diff}(\mathcal{D})$. Formally, this is the *right* Lie algebra of $\mathrm{Diff}(\mathcal{D})$, that is, its standard *left* Lie algebra bracket is *minus* the usual Lie bracket for vector fields. To distinguish between these brackets, we shall reserve in what follows the notation $[\mathbf{u}, \mathbf{w}]$ for the standard Jacobi-Lie bracket of the vector fields $\mathbf{u}, \mathbf{w} \in \mathfrak{X}(\mathcal{D})$ whereas the notation $\mathrm{ad}_{\mathbf{u}}\mathbf{w} := -[\mathbf{u}, \mathbf{w}]$ denotes the adjoint action of the *left* Lie algebra on itself. (The sign conventions will also be clear in the coordinate expressions.)

We also let $\mathfrak{X}(\mathcal{D})^*$ denote the geometric dual space of $\mathfrak{X}(\mathcal{D})$, that is, $\mathfrak{X}(\mathcal{D})^* := \Omega^1(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$, the space of one–form densities on $\mathcal{D}$. If $\alpha \otimes m \in \Omega^1(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$, the pairing of $\alpha \otimes m$ with $\mathbf{w} \in \mathfrak{X}(\mathcal{D})$ is given by

$$\langle \alpha \otimes m, \mathbf{w} \rangle = \int_{\mathcal{D}} \alpha \cdot \mathbf{w}\, m \tag{3.1}$$

where $\alpha \cdot \mathbf{w}$ denotes the contraction of a one–form with a vector field. For $\mathbf{w} \in \mathfrak{X}(\mathcal{D})$ and $\alpha \otimes m \in \mathfrak{X}(\mathcal{D})^*$, the dual of the adjoint representation is defined by (note the minus sign)

$$\langle \mathrm{ad}^*_{\mathbf{w}}(\alpha \otimes m), \mathbf{u} \rangle = -\int_{\mathcal{D}} \alpha \cdot [\mathbf{w}, \mathbf{u}]\, m$$

and its expression is

$$\mathrm{ad}^*_{\mathbf{w}}(\alpha \otimes m) = (\pounds_{\mathbf{w}}\alpha + (\mathrm{div}_m \mathbf{w})\alpha) \otimes m = \pounds_{\mathbf{w}}(\alpha \otimes m)\,, \tag{3.2}$$

where $\mathrm{div}_m \mathbf{w}$ is the divergence of $\mathbf{w}$ relative to the measure m,, which is related to the Lie derivative by $\pounds_{\mathbf{w}} m = (\mathrm{div}_m \mathbf{w})m$. Hence, if $\mathbf{w} = w^i \partial/\partial x^i$, and $\alpha = \alpha_i dx^i$, the one–form factor in the preceding formula for $\mathrm{ad}^*_{\mathbf{w}}(\alpha \otimes m)$ has the coordinate expression

$$\left(w^j \frac{\partial \alpha_i}{\partial x^j} + \alpha_j \frac{\partial w^j}{\partial x^i} + (\mathrm{div}_m \mathbf{w})\alpha_i \right) dx^i \;=\; \left(\frac{\partial}{\partial x^j}(w^j \alpha_i) + \alpha_j \frac{\partial w^j}{\partial x^i} \right) dx^i \;,$$

the last equality assuming that the divergence is taken relative to the standard measure $m = d^n\mathbf{x}$ in $\mathbb{R}^n$.

Configurations, Motions and Material Velocities. Throughout the rest of the paper we shall use the following conventions and terminology for the standard quantities in continuum mechanics. Elements of $\mathcal{D}$ representing the material particles of the system are denoted by X; their coordinates $X^A, A = 1, ..., n$ may thus be regarded as the particle labels. A ***configuration***, which we typically denote by η, is an element of $\mathrm{Diff}(\mathcal{D})$. A ***motion*** η_t is a path in $\mathrm{Diff}(\mathcal{D})$. The ***Lagrangian*** or ***material velocity*** $\mathbf{U}(X,t)$ of the continuum along the motion η_t is defined by taking the time derivative of the motion keeping the particle labels (the reference particles) X fixed:

$$\mathbf{U}(X,t) := \frac{d\eta_t(X)}{dt} := \left.\frac{\partial}{\partial t}\right|_X \eta_t(X),$$

the second equality being a convenient shorthand notation of the time derivative holding X fixed.

Consistent with this definition of velocity, the tangent space to $\mathrm{Diff}(\mathcal{D})$ at $\eta \in \mathrm{Diff}(\mathcal{D})$ is given by

$$T_\eta\mathrm{Diff}(\mathcal{D}) = \{\mathbf{U}_\eta : \mathcal{D} \to T\mathcal{D} \mid \mathbf{U}_\eta(X) \in T_{\eta(X)}\mathcal{D}\}.$$

Elements of $T_\eta\mathrm{Diff}(\mathcal{D})$ are usually thought of as vector fields on $\mathcal{D}$ covering η. The tangent lift of right and left translations on $T\mathrm{Diff}(\mathcal{D})$ by $\varphi \in \mathrm{Diff}(\mathcal{D})$ have the expressions

$$\mathbf{U}_\eta\varphi := T_\eta R_\varphi(\mathbf{U}_\eta) = \mathbf{U}_\eta \circ \varphi \qquad \text{and} \qquad \varphi\mathbf{U}_\eta := T_\eta L_\varphi(\mathbf{U}_\eta) = T\varphi \circ \mathbf{U}_\eta\,.$$

Eulerian Velocities. During a motion η_t, the particle labeled by X describes a path in $\mathcal{D}$ whose points $x(X,t) := \eta_t(X)$ are called the ***Eulerian*** or ***spatial points*** of this path. The derivative $\mathbf{u}(x,t)$ of this path, keeping the Eulerian point x fixed, is called the ***Eulerian*** or ***spatial velocity*** of the system:

$$\mathbf{u}(x,t) := \mathbf{U}(X,t) := \left.\frac{\partial}{\partial t}\right|_x \eta_t(X).$$

Thus, the Eulerian velocity $\mathbf{u}$ is a time dependent vector field on $\mathcal{D}$: $\mathbf{u}_t \in \mathfrak{X}(\mathcal{D})$, where $\mathbf{u}_t(x) := \mathbf{u}(x,t)$. We also have the fundamental relationship

$$\mathbf{U}_t = \mathbf{u}_t \circ \eta_t\,,$$

where $\mathbf{U}_t(X) := \mathbf{U}(X,t)$.

The representation space V^* of $\mathrm{Diff}(\mathcal{D})$ in continuum mechanics is often some subspace of $\mathfrak{T}(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$, the tensor field densities on $\mathcal{D}$ and the

representation is given by pull back. It is thus a *right* representation of $\mathrm{Diff}(\mathcal{D})$ on $\mathfrak{T}(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$. The right action of the Lie algebra $\mathfrak{X}(\mathcal{D})$ on V^* is given by $a\mathbf{u} := \pounds_{\mathbf{u}} a$, the Lie derivative of the tensor field density a along the vector field $\mathbf{u}$.

The Lagrangian. The Lagrangian of a continuum mechanical system is a function $L : T\mathrm{Diff}(\mathcal{D}) \times V^* \to \mathbb{R}$ which is right invariant relative to the tangent lift of right translation of $\mathrm{Diff}(\mathcal{D})$ on itself and pull back on the tensor field densities.

Thus, the Lagrangian L induces a *reduced Lagrangian* $l : \mathfrak{X}(\mathcal{D}) \times V^* \to \mathbb{R}$ defined by

$$l(\mathbf{u}, a) = L(\mathbf{u} \circ \eta, \eta^* a),$$

where $\mathbf{u} \in \mathfrak{X}(\mathcal{D})$ and $a \in V^* \subset \mathfrak{T}(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$, and where $\eta^* a$ denotes the pull back of a by the diffeomorphism η and $\mathbf{u}$ is the Eulerian velocity. The evolution of a is given by solving the equation

$$\dot{a} = -\pounds_{\mathbf{u}}\, a.$$

The solution of this equation, given the initial condition a_0, is $a(t) = \varphi_{t*} a_0$, where the lower star denotes the push forward operation and φ_t is the flow of $\mathbf{u}$.

Advected Eulerian Quantities. These are defined in continuum mechanics to be those variables which are Lie transported by the flow of the Eulerian velocity field. Using this standard terminology, the above equation states that the tensor field density a (which may include mass density and other Eulerian quantities) is advected.

As we have discussed, in many examples, $V^* \subset \mathfrak{T}(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$. On a general manifold, tensors of a given type have natural duals. For example, symmetric covariant tensors are dual to symmetric contravariant tensor densities, the pairing being given by the integration of the natural contraction of these tensors. Likewise, k–forms are naturally dual to $(n-k)$–forms, the pairing being given by taking the integral of their wedge product.

The operation $\diamond$ between elements of V and V^* producing an element of $\mathfrak{X}(\mathcal{D})^*$ introduced in equation (2.5) becomes

$$\langle v \diamond a, \mathbf{w} \rangle = \int_{\mathcal{D}} a \cdot \pounds_{\mathbf{w}}\, v = -\int_{\mathcal{D}} v \cdot \pounds_{\mathbf{w}}\, a\ , \tag{3.3}$$

where $v \cdot \pounds_{\mathbf{w}}\, a$ denotes the contraction, as described above, of elements of V and elements of V^*. (These operations do *not* depend on a Riemannian structure.)

For a path $\eta_t \in \mathrm{Diff}(\mathcal{D})$ let $\mathbf{u}(x,t)$ be its Eulerian velocity and consider, as in the hypotheses of Theorem 2.1 the curve $a(t)$ with initial condition a_0 given by the equation

$$\dot{a} + \pounds_{\mathbf{u}} a = 0. \tag{3.4}$$

Let $L_{a_0}(\mathbf{U}) := L(\mathbf{U}, a_0)$. We can now state Theorem 2.1 in this particular, but very useful, setting.

Theorem 3.1 (Euler–Poincaré reduction for continua.) *For a path η_t in $\mathrm{Diff}(\mathcal{D})$ with Lagrangian velocity $\mathbf{U}$ and Eulerian velocity $\mathbf{u}$, the following are equivalent:*

i *Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L\left(X, \mathbf{U}_t(X), a_0(X)\right)\, dt = 0 \tag{3.5}$$

holds, for variations $\delta\eta_t$ vanishing at the endpoints.

ii *η_t satisfies the Euler–Lagrange equations for L_{a_0} on $\mathrm{Diff}(\mathcal{D})$.*

iii *The constrained variational principle in Eulerian coordinates*

$$\delta \int_{t_1}^{t_2} l(\mathbf{u}, a)\, dt = 0 \tag{3.6}$$

holds on $\mathfrak{X}(\mathcal{D}) \times V^*$, using variations of the form

$$\delta\mathbf{u} = \frac{\partial \mathbf{w}}{\partial t} - \mathrm{ad}_{\mathbf{u}}\mathbf{w} = \frac{\partial \mathbf{w}}{\partial t} + [\mathbf{u}, \mathbf{w}], \quad \delta a = -\pounds_{\mathbf{w}}\, a, \tag{3.7}$$

where $\mathbf{w}_t = \delta\eta_t \circ \eta_t^{-1}$ vanishes at the endpoints.

iv *The* ***Euler–Poincaré equations for continua***

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \mathbf{u}} = -\,\mathrm{ad}^*_{\mathbf{u}}\frac{\delta l}{\delta \mathbf{u}} + \frac{\delta l}{\delta a} \diamond a = -\pounds_{\mathbf{u}}\frac{\delta l}{\delta \mathbf{u}} + \frac{\delta l}{\delta a} \diamond a\,, \tag{3.8}$$

hold, where the $\diamond$ operation given by (3.2) needs to be determined on a case by case basis, depending on the nature of the tensor a. (Remember that $\delta l/\delta\mathbf{u}$ is a one–form density.)

Remarks.

1. The following string of equalities shows *directly* that **iii** is equivalent to **iv**:

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} l(\mathbf{u}, a) dt = \int_{t_1}^{t_2} \left(\frac{\delta l}{\delta \mathbf{u}} \cdot \delta \mathbf{u} + \frac{\delta l}{\delta a} \cdot \delta a \right) dt \\ &= \int_{t_1}^{t_2} \left[\frac{\delta l}{\delta \mathbf{u}} \cdot \left(\frac{\partial \mathbf{w}}{\partial t} - \mathrm{ad}_{\mathbf{u}} \mathbf{w} \right) - \frac{\delta l}{\delta a} \cdot \pounds_{\mathbf{w}} a \right] dt \\ &= \int_{t_1}^{t_2} \mathbf{w} \cdot \left[-\frac{\partial}{\partial t} \frac{\delta l}{\delta \mathbf{u}} - \mathrm{ad}^*_{\mathbf{u}} \frac{\delta l}{\delta \mathbf{u}} + \frac{\delta l}{\delta a} \diamond a \right] dt \,. \end{aligned} \tag{3.9}$$

2. Similarly, one can deduce the form (3.7) of the variations in the constrained variational principle (3.6) by a direct calculation, as follows. One writes the basic relation between the spatial and material velocities, namely $\mathbf{u}(x,t) = \dot{\eta}(\eta_t^{-1}(x), t)$. One then takes the variation of this equation with respect to η and uses the definition $\mathbf{w}(x,t) = \delta\eta((\eta_t^{-1}(x), t)$ together with a calculation of its time derivative. Of course, one can also do this calculation using the inverse map η_t^{-1} instead of η as the basic variable, see Holm, Marsden, and Ratiu [1986], Holm [1996].

3. The preceding sort of calculation for $\delta\mathbf{u}$ in fluid mechanics and the interpretation of this restriction on the form of the variations as the so-called Lin constraints is due to Bretherton [1970]. The variational form (3.7) for the 'basic' Euler–Poincaré equations (i.e., without the advected parameters a) is due to Marsden and Scheurle [1993a] and Bloch, Krishnaprasad, Marsden and Ratiu [1996]. For the full Euler–Poincaré case, this form is due to Holm, Marsden and Ratiu [1998a] and for the general Lagrangian reduction case, to Cendra, Holm, Marsden and Ratiu [1997] and Cendra, Marsden and Ratiu [1997]. These ideas were used for Maxwell-Vlasov plasmas by Cendra, Holm, Hoyle and Marsden [1997].

4. The coordinate expressions for $(\delta l/\delta a) \diamond a$ required to complete the equations of motion for GFD models are given in the next section for $a_0(X)$ in three dimensions.

5. As with the general theory, variations of the action in the advected tensor quantities contribute to the equations of motion which follow from Hamilton's principle. At the level of the action l for the Euler-Poincaré equations, the Legendre transform in the variable $\mathbf{u}$ alone is often nonsingular, and when it is, it produces the Hamiltonian formulation of Eulerian fluid motions with a Lie-Poisson bracket defined on the dual of the semidirect product algebra of vector fields acting amongst themselves by Lie bracket and on tensor fields and differential forms by the

Lie derivative. This is a special instance of the more general facts for Lie algebras that was discussed earlier.

The Inverse Map and the Tensor Fields a for Fluids. In the case of fluids in the Lagrangian picture, the flow of the fluid is a diffeomorphism which takes a fluid parcel along a path from its initial position X, in a "reference configuration" to its current position x in the "container", i.e., in the Eulerian domain of flow. As we have described, this ***forward map*** is denoted by $\eta : X \mapsto x$. The inverse map $\eta^{-1} : x \mapsto X$ provides the map assigning the Lagrangian labels to a given spatial point. Interpreted as passive scalars, these Lagrangian labels are simply advected with the fluid flow, $\dot{X} = 0$. In the Lagrangian picture, a tensor density in the reference configuration $a_0(X)$ (satisfying $\dot{a}_0(X) = 0$) consists of *invariant* tensor functions of the initial reference positions and their differentials. These tensor functions are parameters of the initial fluid reference configuration (e.g., the initial density distribution, which is an invariant n-form).

When viewed in the Eulerian picture as

$$a_t(x) := (\eta_{t*}a_0)(x) = (\eta_t^{-1*}a_0)(x),$$

i.e.,

$$a_0(X) := (\eta_t^* a_t)(X) = (\eta_{t*}^{-1}a_0)(X),$$

the time invariant tensor density $a_0(X)$ in the reference configuration acquires advective dynamics in the Eulerian picture, namely

$$\dot{a}_0(X) = \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) a(x,t) = 0,$$

where $\pounds_{\mathbf{u}}$ denotes Lie derivative with respect to the Eulerian velocity field $\mathbf{u}(x,t)$. This relation results directly from the well known Lie derivative formula for tensor fields. (See, for example, Abraham, Marsden and Ratiu [1988].)

Mapping the time invariant quantity a_0 (a tensor density function of X) to the time varying quantity a_t (a tensor density function of x) as explained above is a special case of the way we advect quantities in V^* in the general theory. Specifically, we can view this advection of a_t as being the fluid analogue of the advection of the unit vector along the direction of gravity (a spatially fixed quantity) by means of the body rotation vector in the heavy top example.

Consistent with the fact that the heavy top is a *left invariant* system while continuum theories are *right invariant*, the advected tensor density a_t is a spatial quantity, while the advected direction of gravity is a body quantity. If we were to take the inverse map η^{-1} as the basic group variable, rather than the map η, then continuum theories would also become left invariant.

The Continuity Equation for the Mass Density. We will need to make an additional assumption on our continuum theory. Namely, we assume that amongst the tensor densities being advected, there is a special one, namely the mass density. This of course is a tensor density that occurs in all continuum theories. We denote this density by $\rho d^n x$ and it is advected according to the standard principles discussed above. Thus, ρ satisfies the usual continuity equation:

$$\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{u}) = 0.$$

In the Lagrangian picture we have $\rho d^n x = \rho_0(X) d^n X$, where $\rho_0(X)$ is the mass density in the reference configuration. It will also be convenient in the continuum examples below to define Lagrangian *mass* coordinates $\ell(X)$ satisfying $\rho d^n x = d^n \ell$ with $\dot{\ell} = 0$. (When using Lagrangian mass coordinates, we shall denote the density ρ as D.)

The Kelvin–Noether Theorem. Let

$$\mathcal{C} := \left\{ \gamma : S^1 \to \mathcal{D} \;\middle|\; \gamma \text{ continuous} \right\}$$

be the space of continuous loops in $\mathcal{D}$ and let the group $\mathrm{Diff}(\mathcal{D})$ act on $\mathcal{C}$ on the left by $(\eta, \gamma) \in \mathrm{Diff}(\mathcal{D}) \times \mathcal{C} \mapsto \eta\gamma \in \mathcal{C}$, where $\eta\gamma = \eta \circ \gamma$.

Next we shall define the ***circulation map*** $\mathcal{K} : \mathcal{C} \times V^* \to \mathfrak{X}(\mathcal{D})^{**}$. Given a one form density $\alpha \in \mathfrak{X}^*$ we can form a one form (no longer a density) by dividing it by the mass density ρ; we denote the result just by α/ρ. We let $\mathcal{K}$ then be defined by

$$\langle \mathcal{K}(\gamma, a), \alpha \rangle = \oint_\gamma \frac{\alpha}{\rho}\,. \tag{3.10}$$

The expression in this definition is called the ***circulation*** of the one–form α/ρ around the loop γ.

This map is equivariant in the sense that

$$\langle \mathcal{K}(\eta \circ \gamma, \eta_* a), \eta_* \alpha \rangle = \langle \mathcal{K}(\gamma, a), \alpha \rangle$$

for any $\eta \in \mathrm{Diff}(\mathcal{D})$. This is proved using the definitions and the change of variables formula.

Given the Lagrangian $l : \mathfrak{X}(\mathcal{D}) \times V^* \to \mathbb{R}$, the Kelvin–Noether quantity is given by (2.15) which in this case becomes

$$I(\gamma_t, \mathbf{u}, a) = \oint_{\gamma_t} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}}\,. \tag{3.11}$$

With these definitions of $\mathcal{K}$ and I, the statement of Theorem 2.2 becomes the classical Kelvin circulation theorem.

Theorem 3.2 (Kelvin Circulation Theorem.) *Assume that* $\mathbf{u}(x,t)$ *satisfies the Euler–Poincaré equations for continua (3.8):*

$$\frac{\partial}{\partial t}\left(\frac{\delta l}{\delta \mathbf{u}}\right) = -\pounds_{\mathbf{u}}\left(\frac{\delta l}{\delta \mathbf{u}}\right) + \frac{\delta l}{\delta a}\diamond a$$

and a *satisfies*

$$\frac{\partial a}{\partial t} + \pounds_{\mathbf{u}} a = 0.$$

Let η_t *be the flow of the Eulerian velocity field* $\mathbf{u}$*, that is,* $\mathbf{u}_t = (d\eta_t/dt)\circ\eta_t^{-1}$*. Define* $\gamma_t := \eta_t\circ\gamma_0$ *and* $I(t) := I(\gamma_t, \mathbf{u}_t, a_t)$*. Then*

$$\frac{d}{dt}I(t) = \oint_{\gamma_t}\frac{1}{\rho}\frac{\delta l}{\delta a}\diamond a\,. \tag{3.12}$$

In this statement, we use a subscript t to emphasise that the operations are done at a particular t and to avoid having to write the other arguments, as in $a_t(x) = a(x,t)$; we omit the arguments from the notation when convenient. Due to the importance of this theorem we shall give here a separate proof tailored for the case of continuum mechanical systems.

Proof. First we change variables in the expression for $I(t)$:

$$I(t) = \oint_{\gamma_t}\frac{1}{\rho_t}\frac{\delta l}{\delta \mathbf{u}} = \oint_{\gamma_0}\eta_t^*\left[\frac{1}{\rho_t}\frac{\delta l}{\delta \mathbf{u}}\right] = \oint_{\gamma_0}\frac{1}{\rho_0}\eta_t^*\left[\frac{\delta l}{\delta \mathbf{u}}\right].$$

Next, we use the Lie derivative formula, namely

$$\frac{d}{dt}\left(\eta_t^*\alpha_t\right) = \eta_t^*\left(\frac{\partial}{\partial t}\alpha_t + \pounds_{\mathbf{u}}\alpha_t\right)\,,$$

applied to a one–form density α_t. This formula gives

$$\begin{aligned}
\frac{d}{dt}I(t) &= \frac{d}{dt}\oint_{\gamma_0}\frac{1}{\rho_0}\eta_t^*\left[\frac{\delta l}{\delta \mathbf{u}}\right] \\
&= \oint_{\gamma_0}\frac{1}{\rho_0}\frac{d}{dt}\left(\eta_t^*\left[\frac{\delta l}{\delta \mathbf{u}}\right]\right) \\
&= \oint_{\gamma_0}\frac{1}{\rho_0}\eta_t^*\left[\frac{\partial}{\partial t}\left(\frac{\delta l}{\delta \mathbf{u}}\right) + \pounds_{\mathbf{u}}\left(\frac{\delta l}{\delta \mathbf{u}}\right)\right].
\end{aligned}$$

By the Euler–Poincaré equations (3.8), this becomes

$$\frac{d}{dt}I(t) = \oint_{\gamma_0}\frac{1}{\rho_0}\eta_t^*\left[\frac{\delta l}{\delta a}\diamond a\right] = \oint_{\gamma_t}\frac{1}{\rho_t}\left[\frac{\delta l}{\delta a}\diamond a\right],$$

again by the change of variables formula.

Corollary 3.3 *Since the last expression holds for every loop* γ_t*, we may write it as*

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\frac{1}{\rho}\frac{\delta l}{\delta \mathbf{u}} = \frac{1}{\rho}\frac{\delta l}{\delta a}\diamond a\,. \tag{3.13}$$

4 Applications of the Euler–Poincaré Theorem in GFD

Variational Formulae in Three Dimensions. We compute explicit formulae for the variations δa in the cases that the set of tensors a is drawn from a set of scalar fields and densities on $\mathbb{R}^3$. We shall denote this symbolically by writing

$$a \in \{b, D\, d^3x\}\,. \tag{4.1}$$

We have seen that invariance of the set a in the Lagrangian picture under the dynamics of $\mathbf{u}$ implies in the Eulerian picture that

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) a = 0\,,$$

where $\pounds_{\mathbf{u}}$ denotes Lie derivative with respect to the velocity vector field $\mathbf{u}$. Hence, for a fluid dynamical action $\mathfrak{S} = \int dt\, l(\mathbf{u}; b, D)$, the advected variables $\{b, D\}$ satisfy the following Lie-derivative relations,

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) b = 0, \quad \text{or} \quad \frac{\partial b}{\partial t} = -\,\mathbf{u}\cdot\nabla\, b\,, \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) D\, d^3x = 0, \quad \text{or} \quad \frac{\partial D}{\partial t} = -\,\nabla\cdot(D\mathbf{u})\,. \tag{4.3}$$

In fluid dynamical applications, the advected Eulerian variables b and D represent the buoyancy b (or specific entropy, for the compressible case) and volume element (or mass density) D, respectively. According to Theorem 3.1, equation (3.6), the variations of the tensor functions a at fixed $\mathbf{x}$ and t are also given by Lie derivatives, namely $\delta a = -\pounds_{\mathbf{w}}\, a$, or

$$\begin{aligned} \delta b &= -\pounds_{\mathbf{w}}\, b = -\mathbf{w}\cdot\nabla\, b\,, \\ \delta D\; d^3x &= -\pounds_{\mathbf{w}}\,(D\, d^3x) = -\nabla\cdot(D\mathbf{w})\; d^3x\,. \end{aligned} \tag{4.4}$$

Hence, Hamilton's principle with this dependence yields

$$\begin{aligned} 0 &= \delta \int dt\; l(\mathbf{u}; b, D) \\ &= \int dt\; \left[\frac{\delta l}{\delta \mathbf{u}}\cdot\delta\mathbf{u} + \frac{\delta l}{\delta b}\;\delta b + \frac{\delta l}{\delta D}\;\delta D\right] \\ &= \int dt\; \left[\frac{\delta l}{\delta \mathbf{u}}\cdot\Big(\frac{\partial \mathbf{w}}{\partial t} - \mathrm{ad}_{\mathbf{u}}\,\mathbf{w}\Big) - \frac{\delta l}{\delta b}\;\mathbf{w}\cdot\nabla\, b - \frac{\delta l}{\delta D}\;\Big(\nabla\cdot(D\mathbf{w})\Big)\right] \\ &= \int dt\; \mathbf{w}\cdot\left[-\frac{\partial}{\partial t}\frac{\delta l}{\delta \mathbf{u}} - \mathrm{ad}^*_{\mathbf{u}}\;\frac{\delta l}{\delta \mathbf{u}} - \frac{\delta l}{\delta b}\;\nabla\, b + D\;\nabla\frac{\delta l}{\delta D}\right] \\ &= -\int dt\; \mathbf{w}\cdot\left[\Big(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\Big)\frac{\delta l}{\delta \mathbf{u}} + \frac{\delta l}{\delta b}\;\nabla\, b - D\;\nabla\frac{\delta l}{\delta D}\right], \end{aligned} \tag{4.5}$$

where we have consistently dropped boundary terms arising from integrations by parts, by invoking natural boundary conditions. Specifically, we impose $\hat{\mathbf{n}} \cdot \mathbf{w} = 0$ on the boundary, where $\hat{\mathbf{n}}$ is the boundary's outward unit normal vector.

The Euler–Poincaré equations for continua (3.8) may now be summarized for advected Eulerian variables a in the set (4.1). We adopt the notational convention of the circulation map $\mathcal{K}$ in equation (3.10) that a one form density can be made into a one form (no longer a density) by dividing it by the mass density D and use the Lie-derivative relation for the continuity equation $(\partial/\partial t + \pounds_{\mathbf{u}})D d^3x = 0$. Then, the Euclidean components of the Euler–Poincaré equations for continua in equation (4.5) are expressed in Kelvin theorem form (3.13) with a slight abuse of notation as

$$\Big(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\Big)\Big(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x}\Big) \,+\, \frac{1}{D}\frac{\delta l}{\delta b}\nabla b \cdot d\mathbf{x} \,-\, \nabla\Big(\frac{\delta l}{\delta D}\Big) \cdot d\mathbf{x} = 0\,, \tag{4.6}$$

in which the variational derivatives of the Lagrangian l are to be computed according to the usual physical conventions, i.e., as Fréchet derivatives. Formula (4.6) is the Kelvin–Noether form of the equation of motion for ideal continua. Hence, we have the explicit Kelvin theorem expression, cf. equations (3.11) and (3.12),

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u})} \frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} = -\oint_{\gamma_t(\mathbf{u})} \frac{1}{D}\frac{\delta l}{\delta b}\nabla b \cdot d\mathbf{x}\,, \tag{4.7}$$

where the curve $\gamma_t(\mathbf{u})$ moves with the fluid velocity $\mathbf{u}$. Then, by Stokes' theorem, the Euler equations generate circulation of $(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}})$ whenever ∇b and $\nabla(\frac{1}{D}\frac{\delta l}{\delta b})$ are not collinear. The corresponding conservation of potential vorticity q on fluid parcels is given by

$$\frac{\partial q}{\partial t} + \mathbf{u}\cdot\nabla q = 0\,, \quad \text{where} \quad q = \frac{1}{D}\nabla b \cdot \operatorname{curl}\left(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}\right). \tag{4.8}$$

Remark on Lagrangian Mass Coordinates. An alternative way to treat Hamilton's principle for an action $\mathfrak{S} = \int dt\; l(\mathbf{u}; b, D)$ is to perform variations at fixed $\mathbf{x}$ and t of the *inverse* maps $\mathbf{x} \to \boldsymbol{\ell}$, described by the Lagrangian mass coordinate functions $\ell^A(\mathbf{x}, t)$, $A = 1, 2, \ldots, n$, which determine $\mathbf{u}$, b and D by the formulae (in which one sums on repeated indices)

$$\frac{\partial \ell^A}{\partial t} = -u^i D_i^A\,, \quad b = b(\ell^A)\,, \quad D_i^A = \frac{\partial \ell^A}{\partial x^i}\,, \quad D = \det(D_i^A)\,. \tag{4.9}$$

As discussed above, the relation of mass coordinates ℓ to the usual Lagrangian coordinates X is given by a simple change of variables in the fluid reference configuration to make $\rho_0(X)d^nX = d^n\ell$. Variation of an action of the form

$\mathfrak{S} = \int dt\ l(\mathbf{u}, b, D)$ with respect to ℓ^A with p imposing volume preservation then yields (Holm, Marsden, and Ratiu [1986], Holm [1996]),

$$\delta\mathfrak{S} = \int dt \int d^n x \left\{ D(D^{-1})^i_A \delta\ell^A \Big[\frac{d}{dt}\frac{1}{D}\frac{\delta l}{\delta u^i} + \frac{1}{D}\frac{\delta l}{\delta u^j} u^j_{,i} + \frac{1}{D}\frac{\delta l}{\delta b} b_{,i} - \Big(\frac{\delta l}{\delta D}\Big)_{,i} \Big] - \delta p (D-1) \right\}, \quad (4.10)$$

where $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$ is the material derivative and we again invoke natural boundary conditions ($\hat{\mathbf{n}}\cdot\mathbf{u} = 0$ on the boundary) when integrating by parts. Hence, the vanishing of the coefficient of $\delta\ell^A$ in the variational formula (4.10) recovers the Euler–Poincaré equations for continua (3.8) expressed in Kelvin theorem form (3.13) or (4.7), by stationarity of the action $\mathfrak{S} = \int dt\ l(\mathbf{u}; b, D)$ with respect to variations of the Lagrangian mass coordinates $\ell^A(\mathbf{x}, t)$. In vector notation, these equations are

$$\frac{d}{dt}\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} + \frac{1}{D}\frac{\delta l}{\delta u^j}\nabla u^j + \frac{1}{D}\frac{\delta l}{\delta b}\nabla b - \nabla\frac{\delta l}{\delta D} = 0, \quad (4.11)$$

or, in three dimensions,

$$\frac{\partial}{\partial t}\Big(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}\Big) - \mathbf{u}\times \mathrm{curl}\Big(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}\Big) + \nabla\Big(\mathbf{u}\cdot\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} - \frac{\delta l}{\delta D}\Big) + \frac{1}{D}\frac{\delta l}{\delta b}\nabla b = 0\,. \quad (4.12)$$

In writing the last equation, we have used the fundamental vector identity of fluid dynamics,

$$(\mathbf{b}\cdot\nabla)\mathbf{a} + a_j\nabla b^j = -\,\mathbf{b}\times(\nabla\times\mathbf{a}) + \nabla(\mathbf{b}\cdot\mathbf{a})\,, \quad (4.13)$$

for any three dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ with, in this case, $\mathbf{a} = (\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}})$ and $\mathbf{b} = \mathbf{u}$. Taking the curl of equation (4.12) and using advection of the buoyancy b yields conservation of potential vorticity on fluid parcels as given in equation (4.8). For incompressible flows $D = 1$ in equation (4.8). The Euclidean component formulae (4.11), (4.12) and (4.8) are especially convenient for direct calculations of motion equations in geophysical fluid dynamics, to which we turn our attention next.

4.1 Euler's Equations for a Rotating Stratified Ideal Incompressible Fluid

The Lagrangian. In the Eulerian velocity representation, we consider Hamilton's principle for fluid motion in an three dimensional domain with action functional $\mathfrak{S} = \int dt\, l$ and Lagrangian $l(\mathbf{u}, b, D)$ given by

$$l = \int d^3x\ \rho_0 D(1+b)\Big(\frac{1}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - gz\Big) - p(D-1)\,, \quad (4.14)$$

where $\rho_{tot} = \rho_0 D(1+b)$ is the total mass density, ρ_0 is a dimensional constant and $\mathbf{R}$ is a given function of $\mathbf{x}$. This Lagrangian produces the following variations at fixed $\mathbf{x}$ and t

$$\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} = \rho_0(1+b)(\mathbf{u}+\mathbf{R})\,, \quad \frac{\delta l}{\delta b} = \rho_0 D\Big(\frac{1}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - gz\Big)\,,$$
$$\frac{\delta l}{\delta D} = \rho_0(1+b)\Big(\frac{1}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - gz\Big) - p\,, \quad \frac{\delta l}{\delta p} = -\,(D-1)\,. \quad (4.15)$$

Hence, from the Euclidean component formula (4.11) for Hamilton principles of this type and the fundamental vector identity (4.13), we find the motion equation for an Euler fluid in three dimensions,

$$\frac{d\mathbf{u}}{dt} - \mathbf{u}\times \operatorname{curl}\mathbf{R} + g\hat{\mathbf{z}} + \frac{1}{\rho_0(1+b)}\nabla p = 0\,, \quad (4.16)$$

where $\operatorname{curl}\mathbf{R} = 2\mathbf{\Omega}(\mathbf{x})$ is the Coriolis parameter (i.e., twice the local angular rotation frequency). In writing this equation, we have used advection of buoyancy,

$$\frac{\partial b}{\partial t} + \mathbf{u}\cdot\nabla b = 0,$$

from equation (4.2).

The Kelvin–Noether Theorem. From equation (4.7), the Kelvin–Noether circulation theorem corresponding to the motion equation (4.16) for an ideal incompressible stratified fluid in three dimensions is,

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u})}(\mathbf{u}+\mathbf{R})\cdot d\mathbf{x} = -\oint_{\gamma_t(\mathbf{u})}\frac{1}{\rho_0(1+b)}\nabla p\cdot d\mathbf{x}\,, \quad (4.17)$$

where the curve $\gamma_t(\mathbf{u})$ moves with the fluid velocity $\mathbf{u}$. By Stokes' theorem, the Euler equations generate circulation of $(\mathbf{u}+\mathbf{R})$ around $\gamma_t(\mathbf{u})$ whenever the gradients of bouyancy and pressure are not collinear. Using advection of buoyancy b, one finds conservation of potential vorticity $q_{\rm Eul}$ on fluid parcels, cf. equation (4.8),

$$\frac{\partial q_{\rm Eul}}{\partial t} + \mathbf{u}\cdot\nabla q_{\rm Eul} = 0\,, \quad \text{where} \quad q_{\rm Eul} = \nabla b\cdot\operatorname{curl}(\mathbf{u}+\mathbf{R})\,. \quad (4.18)$$

The constraint $D = 1$ (volume preservation) is imposed by varying p in Hamilton's principle, according to equation (4.15). Incompressibility then follows from substituting $D = 1$ into the Lie-derivative relation (4.3) for D, which gives $\nabla\cdot\mathbf{u} = 0$. Upon taking the divergence of the motion equation (4.16) and requiring incompressibility to be preserved in time, one finds an elliptic equation for the pressure p with a Neumann boundary condition obtained from the normal component of the motion equation (4.16) evaluated on the boundary.

4.2 Euler-Boussinesq Equations

The Lagrangian. The Lagrangian (4.14) for the Euler fluid motion nondimensionalizes as follows, in terms of units of L for horizontal distance, B_0 for vertical depth, $\mathcal{U}_0$ for horizontal velocity, $B_0\mathcal{U}_0/L$ for vertical velocity, f_0 for Coriolis parameter, ρ_0 for density and $\rho_0 f_0 L\mathcal{U}_0$ for pressure:

$$l = \int d^3x\ D(1+b)\Big(\frac{\epsilon}{2}\mathbf{u}_3\cdot\mathbf{v}_3 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - \frac{z}{\epsilon\mathcal{F}}\Big) - p(D-1)\,. \tag{4.19}$$

Here we take $\mathbf{R}(\mathbf{x})$ to be horizontal and independent of the vertical coordinate, so $\operatorname{curl}\mathbf{R} = f(\mathbf{x})\hat{\boldsymbol{z}}$. In this nondimensional notation, three-dimensional vectors and gradient operators have subscript 3, while horizontal vectors and gradient operators are left unadorned. Thus, we denote, in three dimensional Euclidean space,

$$\begin{aligned}
\mathbf{x}_3 &= (x,y,z), \quad \mathbf{x} = (x,y,0),\\
\mathbf{u}_3 &= (u,v,w), \quad \mathbf{u} = (u,v,0),\\
\nabla_3 &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right), \quad \nabla = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},0\right),\\
\frac{d}{dt} &= \frac{\partial}{\partial t} + \mathbf{u}_3\cdot\nabla_3 = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla + w\frac{\partial}{\partial z}\ ,
\end{aligned} \tag{4.20}$$

and $\hat{\boldsymbol{z}}$ is the unit vector in the vertical z direction. For notational convenience, also we define *two* nondimensional velocities

$$\mathbf{u}_3 = (u,v,w), \quad \text{and} \quad \mathbf{v}_3 = (u,v,\sigma^2 w)\,, \tag{4.21}$$

as well as three nondimensional parameters,

$$\epsilon = \frac{\mathcal{U}_0}{f_0 L}\,, \quad \sigma = \frac{B_0}{L}\,, \quad \mathcal{F} = \frac{f_0^2 L^2}{gB_0}\,, \tag{4.22}$$

corresponding respectively to ***Rossby number***, ***aspect ratio*** and ***(squared) rotational Froude number***. Typically, the Rossby number and the aspect ratio are small, $\epsilon, \sigma \ll 1$, while the rotational Froude number is of order unity in atmospheric and oceanic dynamics for L at synoptic scales and larger. The ***nondimensional Euler fluid equations*** corresponding to the Lagrangian l in equation (4.19) are obtained from the Euler–Poincaré equations (4.11) or (4.12) with $\mathbf{u} \to \mathbf{u}_3$ and $\nabla \to \nabla_3$. Namely,

$$\epsilon\frac{d\mathbf{v}_3}{dt} - \mathbf{u}\times\operatorname{curl}\mathbf{R} + \frac{1}{\epsilon\mathcal{F}}\hat{\mathbf{z}} + \frac{1}{(1+b)}\nabla_3 p = 0\,. \tag{4.23}$$

Clearly, the leading order balances in these equations are hydrostatic in the vertical and geostrophic in the horizontal direction. In this notation, the Kelvin–Noether circulation theorem (4.7) for the Euler fluid becomes

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u}_3)}(\epsilon\mathbf{v}_3 + \mathbf{R})\cdot d\mathbf{x}_3 = -\oint_{\gamma_t(\mathbf{u}_3)}\frac{1}{\rho_0(1+b)}\nabla_3 p\cdot d\mathbf{x}_3\ , \tag{4.24}$$

Likewise, conservation of nondimensional potential vorticity $q_{\rm Eul}$ on fluid parcels is given by, cf. equation (4.18),

$$\frac{\partial q_{\rm Eul}}{\partial t} + \mathbf{u}_3 \cdot \nabla_3 q_{\rm Eul} = 0\,, \quad \text{where} \quad q_{\rm Eul} = \nabla_3 b \cdot \nabla_3 \times (\epsilon \mathbf{v}_3 + \mathbf{R})\,. \tag{4.25}$$

Hamilton's Principle Asymptotics. For sufficiently small buoyancy, $b = o(\epsilon)$, we define

$$p' = p + \frac{z}{\epsilon \mathcal{F}} \quad \text{and} \quad b' = \frac{b}{\epsilon \mathcal{F}}\,,$$

and expand the Lagrangian (4.19) in powers of ϵ as

$$l_{\rm EB} = \int dt \int d^3x \; D\Big(\frac{\epsilon}{2}\mathbf{u}_3 \cdot \mathbf{v}_3 + \mathbf{u} \cdot \mathbf{R}(\mathbf{x}) - b'z\Big) - p'(D-1) + o(\epsilon)\,. \tag{4.26}$$

Upon dropping the order $o(\epsilon)$ term in the Lagrangian $l_{\rm EB}$ the corresponding Euler–Poincaré equation gives the ***Euler-Boussinesq equation*** for fluid motion in three dimensions, namely,

$$\epsilon \frac{d\mathbf{v}_3}{dt} - \mathbf{u} \times \operatorname{curl} \mathbf{R} + b'\hat{\mathbf{z}} + \nabla_3 p' = 0\,, \tag{4.27}$$

or, in horizontal and vertical components, with $\operatorname{curl} \mathbf{R} = f(\mathbf{x})\hat{\boldsymbol{z}}$,

$$\epsilon \, \frac{d\mathbf{u}}{dt} + f\hat{\boldsymbol{z}} \times \mathbf{u} + \nabla p' = 0\,, \qquad \epsilon \, \sigma^2 \, \frac{dw}{dt} + b' + \frac{\partial p'}{\partial z} = 0\,, \tag{4.28}$$

where

$$\frac{db'}{dt} = 0 \quad \text{and} \quad \nabla_3 \cdot \mathbf{u}_3 = \nabla \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0\,.$$

Even for order $O(\epsilon)$ buoyancy, the leading order balances are still hydrostatic in the vertical, and geostrophic in the horizontal. Equations (4.28) describe the motion of an ideal incompressible stratified fluid relative to a stable hydrostatic equlibrium in which the density is taken to be constant except in the buoyant force. See, for example, Phillips [1969] for a derivation of this approximate system via direct asymptotic expansions of the Euler equations.

The Kelvin–Noether Theorem. The Kelvin–Noether circulation theorem (4.7) for the Euler-Boussinesq motion equation (4.27) for an ideal incompressible stratified fluid in three dimensions is,

$$\frac{d}{dt} \oint_{\gamma_t(\mathbf{u}_3)} (\epsilon \mathbf{v}_3 + \mathbf{R}) \cdot d\mathbf{x} = - \oint_{\gamma_t(\mathbf{u}_3)} b' dz\,, \tag{4.29}$$

where the curve $\gamma_t(\mathbf{u}_3)$ moves with the fluid velocity $\mathbf{u}_3$. (The two Kelvin theorems in equations (4.24) and (4.29) differ in their right hand sides.)

By Stokes' theorem, the Euler-Boussinesq equations generate circulation of $\epsilon\mathbf{v}_3+\mathbf{R}$ around $\gamma_t(\mathbf{u}_3)$ whenever the gradient of bouyancy is not vertical. Conservation of potential vorticity $q_{\rm EB}$ on fluid parcels for the Euler-Boussinesq equations is given by

$$\frac{\partial q_{\rm EB}}{\partial t} + \mathbf{u}_3\cdot\nabla_3\, q_{\rm EB} = 0\,, \quad \text{where} \quad q_{\rm EB} = \nabla_3 b'\cdot\nabla_3\times(\epsilon\mathbf{v}_3+\mathbf{R})\,. \tag{4.30}$$

4.3 Primitive Equations

The Lagrangian. The primitive equations (PE) arise from the Euler Boussinesq equations, upon imposing the approximation of hydrostatic pressure balance. Setting the aspect ratio parameter σ to zero in the Lagrangian $l_{\rm EB}$ in equation (4.26) (see equations (4.21) and (4.22)), provides the Lagrangian for the nondimensional primitive equations (PE),

$$l_{\rm PE} = \int dt \int d^3x \left[\frac{\epsilon}{2}D|\mathbf{u}|^2 + D\mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - Db'z - p'(D-1)\right]. \tag{4.31}$$

The Euler–Poincaré equations for $l_{\rm PE}$ now produce the PE; namely, equations (4.28) with $\sigma = 0$,

$$\epsilon\,\frac{d\mathbf{u}}{dt} + f\hat{\boldsymbol{z}}\times\mathbf{u} + \nabla p' = 0\,, b' + \frac{\partial p'}{\partial z} = 0\,, \tag{4.32}$$

where

$$\frac{db'}{dt} = 0 \quad \text{and} \quad \nabla_3\cdot\mathbf{u}_3 = \nabla\cdot\mathbf{u} + \frac{\partial w}{\partial z} = 0\,.$$

Thus, from the viewpoint of Hamilton's principle, imposition of hydrostatic balance corresponds to ignoring the kinetic energy of vertical motion by setting $\sigma = 0$ in the nondimensional EB Lagrangian (4.26).

The Kelvin–Noether Theorem. The Kelvin–Noether circulation theorem for the primitive equations is obtained from equation (4.29) for the Euler-Boussinesq equations simply by setting $\sigma = 0$. Namely,

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u}_3)}(\epsilon\mathbf{u}+\mathbf{R})\cdot d\mathbf{x} = -\oint_{\gamma_t(\mathbf{u}_3)} b'dz\,, \tag{4.33}$$

where the curve $\gamma_t(\mathbf{u}_3)$ moves with the fluid velocity $\mathbf{u}_3$. By Stokes' theorem, the primitive equations generate circulation of $\epsilon\mathbf{u}+\mathbf{R}$ around $\gamma_t(\mathbf{u}_3)$ whenever the gradient of bouyancy is not vertical. The conservation of potential vorticity on fluid parcels for the primitive equations is given by, cf. equation (4.8),

$$\frac{\partial q_{\rm PE}}{\partial t} + \mathbf{u}_3\cdot\nabla_3\, q_{\rm PE} = 0\,, \quad \text{where} \quad q_{\rm PE} = \nabla_3 b'\cdot\nabla_3\times(\epsilon\mathbf{u}+\mathbf{R})\,. \tag{4.34}$$

Remark. In the limit, $\epsilon \to 0$, Hamilton's principle for either $l_{\rm EB}$, or $l_{\rm PE}$ gives,

$$f\hat{\boldsymbol{z}} \times \mathbf{u} + b'\,\hat{\boldsymbol{z}} + \nabla_3\, p' = 0, \tag{4.35}$$

which encodes the leading order hydrostatic and geostrophic equilibrium balances. These balances form the basis for further approximations for near-geostrophic, hydrostatic flow.

4.4 Hamiltonian Balance Equations

Balanced Fluid Motions. A fluid motion equation is said to be ***balanced***, if specification of the fluid's stratified buoyancy and divergenceless velocity determines its pressure through the solution of an equation which does not contain partial time-derivatives among its highest derivatives. This definition of balance makes pressure a diagnostic variable (as opposed to the dynamic, or prognostic variables such as the horizontal velocity components). The Euler equations (4.23) and the Euler-Boussinesq equations (4.27) for the incompressible motion of a rotating continuously stratified fluid are balanced in this sense, because the pressure in these cases is determined diagnostically from the buoyancy and velocity of the fluid by solving a Neumann problem. However, the hydrostatic approximation of this motion by the primitive equations (PE) is not balanced, because the Poisson equation for the pressure in PE involves the time-derivative of the horizontal velocity divergence, which alters the character of the Euler system from which PE is derived and may lead to rapid time dependence, as discussed in Browning et al. [1990]. Balanced approximations which eliminate this potentially rapid time dependence have been sought and found, usually by using asymptotic expansions of the solutions of the PE in powers of the small Rossby number, $\epsilon \ll 1$, after decomposing the horizontal velocity $\mathbf{u}$ into order $O(1)$ rotational and order $O(\epsilon)$ divergent components, as $\mathbf{u} = \hat{\boldsymbol{z}} \times \nabla\psi + \epsilon\nabla\chi$, where ψ and χ are the stream function and velocity potential, respectively, for the horizontal motion. (This is just the Helmholtz decomposition with relative weight ϵ.)

Balance equations (BE) are reviewed in the classic paper of McWilliams and Gent [1980]. Succeeding investigations have concerned the well-posedness and other features of various BE models describing continuously stratified oceanic and atmospheric motions. For example, consistent initial boundary value problems and regimes of validity for BE are determined in Gent and McWilliams [1983a,b]. In other papers by these authors and their collaborators listed in the bibliography, balanced models in isentropic coordinates are derived, methods for the numerical solution of BE are developed, and the applications of BE to problems of vortex motion on a β-plane and wind-driven ocean circulation are discussed. In studies of continuously stratified incompressible fluids, solutions of balance equations that retain terms of order $O(1)$ and order $O(\epsilon)$ in a Rossby number expansion of the PE solutions have been

found to compare remarkably well with numerical simulations of the PE; see Allen, Barth, and Newberger [1990a,b] and Barth et al. [1990]. Discussions of the relation between BE and semigeostrophy have also recently appeared, see, e.g., Gent, McWilliams and Snyder [1994] and Xu [1994].

Conservation of Energy and Potential Vorticity. One recurring issue in the literature is that, when truncated at order $O(\epsilon)$ in the Rossby number expansion, the BE for continuously stratified fluids conserve energy (Lorenz [1960]), but do not conserve potential vorticity on fluid parcels. Recently, Allen [1991] found a set of BE for continuously stratified fluids that retains additional terms of order $O(\epsilon^2)$ and *does conserve potential vorticity* on fluid parcels. Allen calls these balance equations "BEM equations", because they are based on momentum equations, rather than on the equation for vertical vorticity, as for the standard BE. An advantage of the momentum formulation of BEM over the vorticity formulation of the original BE is that boundary conditions are more naturally imposed on the fluid's velocity than on its vorticity. Holm [1996] derives Hamiltonian balance equations (HBE) in the momentum formulation by using the ϵ-weighted Helmholtz decomposition for $\mathbf{u}$ and expanding Hamilton's principle (HP) for the PE in powers of the Rossby number, $\epsilon \ll 1$. This expansion is truncated at order $O(\epsilon)$, then all terms are retained that result from taking variations. As we have seen, an asymptotic expansion of HP for the Euler-Boussinesq (EB) equations which govern rotating stratified incompressible inviscid fluid flow has two small dimensionless parameters: the aspect ratio of the shallow domain, σ, and the Rossby number, ϵ. Setting σ equal to zero in this expansion yields HP for PE. Setting ϵ also equal to zero yields HP for equilibrium solutions in both geostrophic and hydrostatic balance. Setting $\sigma = 0$, substituting the ϵ-weighted Helmholtz decomposition for $\mathbf{u}$ and truncating the resulting asymptotic expansion in ϵ of the HP for the EB equations, yields HP for a set of nearly-geostrophic Hamiltonian balance equations (HBE). The resulting HBE are equivalent to the BEM equations in Allen [1991].

The Lagrangian. The Lagrangian for the HBE model is given in Holm [1996], cf. equation (4.31) for the PE action,

$$\mathfrak{S}_{\rm HBE} = \int dt \int d^3x \left[D\mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - Dbz - p(D-1) + \epsilon\frac{D}{2}|\mathbf{u}-\epsilon\mathbf{u}_D|^2 \right], \tag{4.36}$$

where the horizontal fluid velocity is taken in balance equation form as $\mathbf{u} = \mathbf{u}_R + \epsilon\mathbf{u}_D = \hat{\boldsymbol{z}}\times\nabla\psi + \epsilon\nabla\chi$. The corresponding Euler–Poincaré equations give

the dynamics of the HBE model

$$\begin{aligned}
\epsilon\frac{d}{dt}\mathbf{u}_R + \epsilon^2 u_{Rj}\nabla u_D^j + f\hat{\boldsymbol{z}}\times\mathbf{u} + \nabla p &= 0,\\
b + \frac{\partial p}{\partial z} + \epsilon^2\mathbf{u}_R\cdot\frac{\partial \mathbf{u}_D}{\partial z} &= 0,\\
\text{with}\quad \frac{db}{dt} = \frac{\partial}{\partial t}\,b + \mathbf{u}\cdot\nabla b + \epsilon\, w\frac{\partial b}{\partial z} &= 0,\\
\text{and}\quad \nabla\cdot\mathbf{u} + \epsilon\frac{\partial w}{\partial z} &= 0.
\end{aligned} \tag{4.37}$$

Here the notation is the same as for the PE, except that $w \to \epsilon\, w$ for HBE.

Dropping all terms of order $O(\epsilon^2)$ from the HBE model equations (4.37) recovers the BE discussed in Gent and McWilliams [1983a,b]. Retaining these order $O(\epsilon^2)$ terms restores the conservation laws due to symmetries of HP at the truncation order $O(\epsilon)$. As explained in Holm [1996], the resulting HBE model has the same order $O(\epsilon)$ accuracy as the BE, since not *all* of the possible order $O(\epsilon^2)$ terms are retained. Since the HBE model shares the same conservation laws and Euler–Poincaré structure as EB and PE, and differs from them only at order $O(\epsilon^2)$, it may be a valid approximation for times longer than the expected order $O(1/\epsilon)$ for BE.

The Kelvin-Noether Theorem. The HBE model (4.37) possesses the following Kelvin–Noether circulation theorem,

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u}_3)}(\mathbf{R}+\epsilon\mathbf{u}_R)\cdot d\mathbf{x}_3 = -\oint_{\gamma_t(\mathbf{u}_3)} b dz, \tag{4.38}$$

for any closed curve $\gamma_t(\mathbf{u}_3)$ moving with the fluid velocity $\mathbf{u}_3$. We compare this result with the Kelvin–Noether circulation theorem for PE in equation (4.33), rewritten as

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u}_3)}\underbrace{(\mathbf{R}+\epsilon\mathbf{u})}_{\text{PE}}\cdot d\mathbf{x}_3 = \frac{d}{dt}\oint_{\gamma_t(\mathbf{u}_3)}\underbrace{(\mathbf{R}+\epsilon\mathbf{u}_R}_{\text{HBE}} + \underbrace{\epsilon^2\mathbf{u}_D)\cdot d\mathbf{x}_3}_{\text{ZERO}} = -\oint_{\gamma_t(\mathbf{u}_3)} b dz. \tag{4.39}$$

Because

$$\oint \mathbf{u}_D\cdot d\mathbf{x}_3 = \oint d\chi = 0,$$

the ϵ^2 term vanishes, and so the HBE circulation integral differs from that of PE only through the differences in buoyancy between the two theories.

The conservation of potential vorticity on fluid parcels for the HBE model is given by, cf. equation (4.8),

$$\frac{\partial q_{\text{HBE}}}{\partial t} + \mathbf{u}_3\cdot\nabla_3\, q_{\text{HBE}} = 0\,, \quad \text{where} \quad q_{\text{HBE}} = \nabla_3\, b\cdot\nabla_3\times(\epsilon\mathbf{u}_R + \mathbf{R})\,. \tag{4.40}$$

Combining this with advection of b and the tangential boundary coundition on $\mathbf{u}_3$ yields an infinity of conserved quantities,

$$C_\Phi = \int d^3x \ \Phi(q_{\rm HBE}, b), \tag{4.41}$$

for any function Φ. These are the Casimir functions for the Lie-Poisson Hamiltonian formulation of the HBE given in Holm [1996].

HBE Discussion. By their construction as Euler–Poincaré equations from a Lagrangian which possesses the classic fluid symmetries, the HBE conserve integrated energy and conserve potential vorticity on fluid parcels. Their Lie-Poisson Hamiltonian structure endows the HBE with the same type of self-consistency that the PE possess (for the same Hamiltonian reason). After all, the conservation laws in both HBE and PE are not accidental. They correspond to symmetries of the Hamiltonian or Lagrangian for the fluid motion under continuous group transformations, in accordance with Noether's theorem. In particular, energy is conserved because the Hamiltonian in both theories does not depend on time explicitly, and potential vorticity is conserved on fluid parcels because the corresponding Hamiltonian or Lagrangian is right invariant under the infinite set of transformations that relabel the fluid parcels without changing the Eulerian velocity and buoyancy. See, e.g., Salmon [1988] for a review of these ideas in the GFD context, as well as Holm, Marsden and Ratiu [1998a,b] and the earlier sections of the present paper for the general context for such results.

The vector fields which generate these relabeling transformations turn out to be the *steady flows* of the HBE and PE models. By definition, these steady flows leave invariant the Eulerian velocity and buoyancy as they move the Lagrangian fluid parcels along the flow lines. Hence, as a direct consequence of their shared Hamiltonian structure, the steady flows of both HBE and PE are relative equilibria. That is, steady HBE and PE flows are critical points of a sum of conserved quantities, including the (constrained) Hamiltonian. This shared critical-point property enables one, for example, to use the Lyapunov method to investigate stability of relative equilibrium solutions of HBE and PE. See Holm and Long [1989] for an application of the Lyapunov method in the Hamiltonian framework to the stability of PE relative equilibria. According to the Lyapunov method, convexity of the constrained Hamiltonian at its critical point (the relative equilibrium) is sufficient to provide a norm that limits the departure of the solution from equilibrium under perturbations. See, e.g., Abarbanel et al. [1986] for applications of this method to the Euler equations for incompressible fluid dynamics and Holm et al. [1985] for a range of other applications in fluid and plasma theories.

Thus, the HBE arise as Euler–Poincaré equations and possess the same Lie-Poisson Hamiltonian structure as EB and PE, and differ in their Hamiltonian

and conservation laws by small terms of order $O(\epsilon^2)$. Moreover, the HBE conservation laws are fundamentally of the same nature as those of the EB equations and the PE from which they descend. These conserved quantities — particularly the quadratic conserved quantities — may eventually be useful measures of the deviations of the HBE solutions from EB and PE solutions under time evolution starting from identical initial conditions.

4.5 Remarks on Two-dimensional Fluid Models in GFD

The search for simpler dynamics than those of the primitive equations natually leads to considerations of two-dimensional fluid models. This certainly holds for applications in GFD, because the aspect ratio of the domain (σ) and the Rossby number (ϵ) of the rotating flow are often small in these applications. Many treatments of two-dimensional GFD models have been given using asymptotic expansion methods in Hamilton's principle, see, e.g., Salmon [1983, 1985, 1988]. These treatments tend to focus especially on the rotating shallow water (RSW) equations, their quasigeostrophic (QG) approximation, and certain intermediate approximations, such as the semigeostrophic (SG) equations (Eliassen [1949], Hoskins [1975], Cullen and Purser [1989], Holm, Lifschitz and Norbury [1998]) and the Salmon [1985] L_1 model. A unified derivation of the RSW, L_1, QG and SG equations using Hamilton's principle asymptotics is given in Allen and Holm [1996]. This paper also derives as Euler–Poincaré equations a new class of "extended geostrophic" (EG) models. The EG models include nonlocal corrections to the ageostrophic velocity which could produce more accurate models than the L_1, QG and SG approximations of the RSW equations.

There are also three dimensional versions of the QG and SG equations, and recently a continuously stratified L_1 model was derived in Allen, Holm and Newberger [1998] through the use of Hamilton's principle asymptotics and the Euler–Poincaré theory. For the suite of idealized, oceanographic, moderate Rossby number, mesoscale flow test problems in Allen and Newberger [1993], this continuously stratified L_1 model produces generally accurate approximate solutions. These solutions are not quite as accurate as those from the BEM/HBE or BE models, but are substantially more accurate than those from three dimensional SG or QG.

Due to their wide applicability in GFD, the properties of the two dimensional QG equations have been studied extensively. Weinstein [1983] wrote down a Lie-Poisson bracket for them in preparation for studying stability of quasigeostrophic equilibria. The Hamiltonian structure and nonlinear stability of the equilibrium solutions for the QG system and its variants has been thoroughly explored. For references, see Marsden and Weinstein [1982], Weinstein [1983] and Holm et al. [1985]. See also the introduction and bibliography of Marsden et al. [1983] for a guide to some of the history and literature of this subject. A discussion of the geodesic properties of the QG equations in the

framework of Euler–Poincaré theory is given in Holm, Marsden and Ratiu [1998a,b]. A related discussion of QG in both two and three dimensions is given from the viewpoint of Hamilton's principle asymptotics in Holm and Zeitlin [1997].

Formulae showing the asymptotic expansion relationships among the Lagrangians for the various GFD models are summarized in Tables (4.1) and (4.2). In the next two sections, we turn our attention to dealing with the mean effects of rapid fluctuations in GFD.

Table 1. Successive GFD approximations in Hamilton's principle.

$$l_{\rm Euler} \;=\; \int d^3x \left[D(1+b)\Bigg(\underbrace{\mathbf{R}(\mathbf{x})\cdot\mathbf{u}}_{\rm Rotation} + \underbrace{\frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{\epsilon}{2}\sigma^2 w^2}_{\rm Kinetic\ Energy} \Bigg) \right.$$
$$\left. - \underbrace{D(1+b)\left(\frac{z}{\epsilon\mathcal{F}}\right)}_{\rm Potential\ Energy} - \underbrace{p(D-1)}_{\rm Constraint} \right]$$

- $l_{\rm Euler} \to l_{\rm EB}$, for small buoyancy, $b = O(\epsilon)$.
- $l_{\rm EB} \to l_{\rm PE}$, for small aspect ratio, $\sigma^2 = O(\epsilon)$.
- $l_{\rm PE} \to l_{\frac{\rm HBE}{\rm BEM}}$, for horizontal velocity decomposition, $\mathbf{u} = \hat{\boldsymbol{z}}\times\nabla\psi + \epsilon\nabla\chi = \mathbf{u}_R + \epsilon\mathbf{u}_D$, and $|\mathbf{u}|^2 \to |\mathbf{u}_R|^2$ in $l_{\rm PE}$.
- $l_{\frac{\rm HBE}{\rm BEM}} \to l_{\rm EG}$, upon rearranging KE in $l_{\frac{\rm HBE}{\rm BEM}}$ and decomposing horizontal velocity as $\mathbf{u} = \mathbf{u}_1 + \epsilon\mathbf{u}_2$, where $\mathbf{u}_1 = \hat{\boldsymbol{z}}\times\nabla\tilde{\phi}$ with

$$\tilde{\phi}(\mathbf{x}_3, t) = \phi_S(x, y, t) + \int_z^0 dz'\; b\,,$$

  i.e., $\partial\tilde{\phi}/\partial z = -\,b$ and where $\mathbf{u}_2$ is the prescribed function,

$$\mathbf{u}_2 = \tau(\mathbf{u}_1\cdot\nabla)\hat{\boldsymbol{z}}\times\mathbf{u}_1 - \,\tilde{\alpha}\tau\nabla\left((\mathcal{F}-\nabla^2)^{-1}J(\tilde{\phi},\psi)\right) - \beta\tau f_1\mathbf{u}_1,$$

  with $\psi = f_1 - b_1 + \nabla^2\tilde{\phi}$. The constants τ, $\tilde{\alpha}$, β, and γ are free parameters and the functions f_1 and b_1 denote prescribed order $O(\epsilon)$ Coriolis and topography variations.
- $l_{\rm EG} \to l_1$, for horizontal velocity decomposition, $\mathbf{u} = \mathbf{u}_1 = \hat{\boldsymbol{z}}\times\nabla\tilde{\phi}$ and dropping terms of order $O(\epsilon^2)$ in $l_{\rm EG}$.
- $l_1 \to l_{QG}$, on dropping terms of order $O(\epsilon^2)$ in the Euler–Poincaré equations for l_1.

Table 2. Nondimensional Euler–Poincaré Lagrangians at successive levels of approximation via asymptotic expansions.

$$l_{\rm Euler} = \int d^3x \left[D(1+b)\left(\mathbf{R}(\mathbf{x})\cdot\mathbf{u} + \frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{\epsilon}{2}\sigma^2 w^2 - \frac{z}{\epsilon\mathcal{F}} \right) - p(D-1) \right]$$

$$l_{\rm EB} = \int d^3x \left[D\left(\mathbf{R}\cdot\mathbf{u} + \frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{\epsilon}{2}\sigma^2 w^2 - bz \right) - p(D-1) \right]$$

$$l_{\rm PE} = \int d^3x \left[D\left(\mathbf{R}\cdot\mathbf{u} + \frac{\epsilon}{2}|\mathbf{u}|^2 - bz \right) - p(D-1) \right]$$

$$l_{\frac{\rm HBE}{\rm BEM}} = \int d^3x \left[D\left(\mathbf{R}\cdot\mathbf{u} + \frac{\epsilon}{2}|\mathbf{u} - \epsilon\mathbf{u}_D|^2 - bz \right) - p(D-1) \right]$$

$$l_{\rm EG} = \int d^3x \left[D\left((\mathbf{R} + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2)\cdot\mathbf{u} - \frac{\epsilon}{2}|\mathbf{u}_1 + \gamma\epsilon\mathbf{u}_2|^2 - bz \right) - p(D-1) \right]$$

$$l_1 = \int d^3x \left[D\left((\mathbf{R} + \epsilon\mathbf{u}_1)\cdot\mathbf{u} - \frac{\epsilon}{2}|\mathbf{u}_1|^2 - bz \right) - p(D-1) \right]$$

$$l_{\rm QG/AW} = \int_{\mathcal{D}} d^2x \int_{z_0}^{z_1} dz \left[D\left(\mathbf{R}\cdot\mathbf{u} + \frac{\epsilon}{2}\mathbf{u}\cdot(1 - \mathcal{L}(z)\Delta^{-1})\mathbf{u} \right) - p(D-1) \right],$$

where

$$\mathcal{L}(z) = \Big(\frac{\partial}{\partial z} + B\Big)\frac{1}{\mathcal{S}(z)}\Big(\frac{\partial}{\partial z} - B\Big) - \mathcal{F}$$

and $B = 0$ for QG.

5 Generalized Lagrangian Mean (GLM) Equations

The GLM theory of Andrews and McIntyre [1978a] is a hybrid Eulerian-Lagrangian description in which Langrangian-mean flow quantities satisfy equations expressed in Eulerian form. A related set of equations was introduced by Craik and Leibovich [1976] in their study of Langmuir vortices driven by a prescribed surface wave field. The GLM equations are extended from prescribed fluctuation properties to a theory of self-consistent Hamiltonian dynamics of wave, mean-flow interaction for a rotating stratified incompressible fluid in Gjaja and Holm [1996].

GLM Approximations. In GLM theory, one decomposes the fluid trajectory at fixed Lagrangian label ℓ^A, $A = 1, 2, 3$, as follows,

$$\mathbf{x}^\xi(\ell^A, t) = \mathbf{x}(\epsilon\ell^A, \epsilon t) + \alpha\boldsymbol{\xi}(\mathbf{x}, t), \quad \text{with} \quad \overline{\boldsymbol{\xi}} = 0 \quad \text{and} \quad \overline{\mathbf{x}\cdot\boldsymbol{\xi}} = 0\,, \tag{5.1}$$

where scaling with ϵ denotes slow Lagrangian space and time dependence, and an overbar denotes an appropriate time average at fixed Eulerian position. For example, overbar may denote the average over the rapid oscillation phase of a single-frequency wave displacement $\boldsymbol{\xi}$ of amplitude α relative to its wavelength. Thus, the displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ associated with such waves has zero Eulerian mean, $\overline{\boldsymbol{\xi}(\mathbf{x}, t)} = 0$. Superscript ξ on a function denotes its evaluation at the displaced Eulerian position associated with the rapidly fluctuating component of the fluid parcel displacement. Thus, e.g., $\mathbf{x}^\xi = \mathbf{x} + \alpha\boldsymbol{\xi}$ and $\mathbf{u}^\xi(\mathbf{x}) = \mathbf{u}(\mathbf{x}^\xi)$, for a function $\mathbf{u}$.

The GLM operator $\overline{(\ )}^L$ averages over parcels at the displaced positions $\mathbf{x}^\xi = \mathbf{x} + \alpha\boldsymbol{\xi}$ and produces slow Eulerian space and time dependence. This defines the Lagrangian mean velocity:

$$\overline{\mathbf{u}}^L(\epsilon\mathbf{x}, \epsilon t) \equiv \overline{\mathbf{u}(\mathbf{x}, t)}^L = \overline{\mathbf{u}^\xi(\mathbf{x}, t)} = \overline{\mathbf{u}(\mathbf{x}^\xi, t)}\,, \tag{5.2}$$

where unadorned overbar denotes the Eulerian average and scaling with ϵ denotes slow dependence. Thus, the GLM description associates to an instantaneous Eulerian velocity field $\mathbf{u}(\mathbf{x}^\xi, t)$ a unique Lagrangian mean velocity, written (with a slight abuse of notation) as $\overline{\mathbf{u}}^L(\epsilon\mathbf{x}, \epsilon t)$, such that when a fluid parcel at $\mathbf{x}^\xi = \mathbf{x} + \alpha\boldsymbol{\xi}$ moves at its velocity $\mathbf{u}(\mathbf{x}^\xi, t)$, a fictional parcel at $\mathbf{x}$ is moving at velocity $\overline{\mathbf{u}}^L(\epsilon\mathbf{x}, \epsilon t)$. Hence,

$$\mathbf{u}(\mathbf{x} + \alpha\boldsymbol{\xi}, t) = \Big(\frac{\partial}{\partial t}\Big|_{\mathbf{x}} + \overline{\mathbf{u}}^L \cdot \boldsymbol{\nabla}\Big)\Big[\mathbf{x} + \alpha\boldsymbol{\xi}(\mathbf{x}, t)\Big] \;=\; \overline{\mathbf{u}}^L + \alpha\overline{D}^L\boldsymbol{\xi}\,, \tag{5.3}$$

where $\overline{D}^L \equiv \partial/\partial t|_{\mathbf{x}} + \overline{\mathbf{u}}^L \cdot \boldsymbol{\nabla}$ is the material derivative with respect to the slowly varying Lagrangian mean velocity, $\overline{\mathbf{u}}^L(\epsilon\mathbf{x}, \epsilon t)$.

In GLM theory one finds the following basic Eulerian relations and definitions,

$$\begin{aligned}
\mathbf{u}^\xi(\mathbf{x},t) \equiv \mathbf{u}(\mathbf{x}^\xi,t) &= \overline{\mathbf{u}}^L(\epsilon\mathbf{x},\epsilon t) + \alpha\mathbf{u}^l(\mathbf{x},t),\\
\overline{\mathbf{u}}^L(\epsilon\mathbf{x},\epsilon t) &\equiv \overline{\mathbf{u}(\mathbf{x}+\alpha\boldsymbol{\xi},t)},\\
\mathbf{u}^l \equiv \overline{D}^L\boldsymbol{\xi}\,,\ \ \overline{D}^L &\equiv \frac{\partial}{\partial t}\Big|_{\mathbf{x}} + \overline{\mathbf{u}}^L\cdot\boldsymbol{\nabla}\,, \qquad (5.4)\\
\mathbf{R}^\xi(\mathbf{x}) \equiv \mathbf{R}(\mathbf{x}^\xi) &= \overline{\mathbf{R}}^L(\epsilon\mathbf{x}) + \alpha\mathbf{R}^l(\boldsymbol{\xi})\,,\\
\overline{\mathbf{R}}^L(\epsilon\mathbf{x}) &\equiv \overline{\mathbf{R}(\mathbf{x}+\alpha\boldsymbol{\xi})},\\
\overline{\mathbf{u}^l} = 0 &= \overline{\mathbf{R}^l}\,,
\end{aligned}$$

where $\mathbf{R}$ denotes the vector potential for the Coriolis parameter, as before. The basic identity used in deriving these formulae is

$$\Big(\frac{\partial}{\partial t}\Big|_{\mathbf{x}^\xi} + \mathbf{u}^\xi\cdot\frac{\partial}{\partial\mathbf{x}^\xi}\Big)f(\mathbf{x}^\xi,t) = \Big(\frac{\partial}{\partial t}\Big|_{\mathbf{x}} + \overline{\mathbf{u}}^L\cdot\frac{\partial}{\partial\mathbf{x}}\Big)f(\mathbf{x}+\alpha\boldsymbol{\xi},t), \qquad (5.5)$$

which holds for any differentiable function f. This identity may be shown by taking the partial time derivative at constant ℓ^A of the decomposition (5.1) and using the chain rule, cf. Andrews and McIntyre [1978a].

The Lagrangian. We return to Hamilton's principle with Lagrangian (4.26) for the Euler-Boussinesq equations. This is expressed in terms of unscaled (i.e., dimensional) Eulerian instantaneous quantities in the GLM notation as

$$\begin{aligned}
\mathfrak{S}_{\rm EB} = \int dt\int d^3x\Bigg\{ D^\xi\left[\frac{1}{2}|\mathbf{u}^\xi(\mathbf{x},t)|^2 - b^\xi gz + \ \mathbf{u}^\xi(\mathbf{x},t)\cdot\mathbf{R}^\xi(\mathbf{x})\right]\\
-p^\xi\left(D^\xi-1\right)\Bigg\}, \qquad (5.6)
\end{aligned}$$

where g is the constant acceleration of gravity and

$$D^\xi(\mathbf{x},t) = D(\mathbf{x}+\alpha\boldsymbol{\xi},t) = \det\Big(\delta^i_j + \alpha\frac{\partial\xi^i}{\partial x^j}\Big)\,, \qquad (5.7)$$

which is a cubic expression in α. We denote the corresponding pressure decomposition as,

$$p^\xi(\mathbf{x},t) = p(\mathbf{x}+\alpha\boldsymbol{\xi},t) = \overline{p}^L(\epsilon\mathbf{x},\epsilon t) + \sum_{j=1}^{3}\alpha^j h_j(\epsilon\mathbf{x},\epsilon t)p_j(\mathbf{x},t)\,. \qquad (5.8)$$

After expanding D^ξ in powers of α, Gjaja and Holm [1996] average over the rapid space and time scales in the action (5.6) for the Euler-Boussinesq equations while keeping the Lagrangian coordinates ℓ^A and ϵt fixed. (This is the Lagrangian mean of the action.)

Remark. Note that averaging in this setting is a formal operation associated with the addition of a new degree of freedom describing the rapid fluctuations. Thus, averaging in itself does not entail any approximations. The approximations occur next, in the truncations of the expansions of the averaged action in powers of the small parameters ϵ and α.

The GLM dynamics follows upon making the decompositions in (5.1)-(5.3) and (5.7)-(5.8), averaging the action (5.6) and assuming that the rapidly fluctuating displacement $\boldsymbol{\xi}$ is a *prescribed* function of $\mathbf{x}$ and t, which satisfies the transversality condition given in (5.10) below.

The averaged, truncated action is (taking $\mathbf{R}(\mathbf{x}) = \boldsymbol{\Omega} \times \mathbf{x}$ with constant rotation frequency $\boldsymbol{\Omega}$ for simplicity)

$$\begin{aligned}
\overline{\mathfrak{S}}_{GLM} \;=\; & \int dt \int d^3x \Bigg\{ D \Bigg[\frac{1}{2}|\overline{\mathbf{u}}^L|^2 + \frac{\alpha^2}{2} \overline{\left| \frac{\partial \boldsymbol{\xi}}{\partial t} + (\overline{\mathbf{u}}^L \cdot \boldsymbol{\nabla})\boldsymbol{\xi} \right|^2} - b\Big(\ell^A(\mathbf{x},t)\Big) gz \\
& + \overline{\mathbf{u}}^L \cdot (\boldsymbol{\Omega} \times \mathbf{x}) + \alpha^2 \overline{\Big(\frac{\partial \boldsymbol{\xi}}{\partial t} + (\overline{\mathbf{u}}^L \cdot \boldsymbol{\nabla})\boldsymbol{\xi} \Big) \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi})} \Bigg] \\
& + \overline{p}^L \Bigg[1 - D + \frac{\alpha^2}{2} \frac{\partial}{\partial x^i} \overline{\Big(\xi^i \frac{\partial \xi^j}{\partial x^j} - \xi^j \frac{\partial \xi^i}{\partial x^j} \Big)} \Bigg] + \alpha^2 h_1 \overline{\left(p_1 \frac{\partial \xi^j}{\partial x^j} \right)} \\
& + O(\alpha^4). \Bigg\}
\end{aligned} \tag{5.9}$$

Note that the buoyancy b is a function of the Lagrangian labels ℓ^A which is held fixed during the averaging.

The variation of $\overline{\mathfrak{S}}_{GLM} = \int dt \overline{L}_{GLM}$ in equation (5.9) with respect to h_1 at fixed $\mathbf{x}$ and t yields the transversality condition,

$$\overline{\left(p_1 \frac{\partial \xi^j}{\partial x^j} \right)} = O(\alpha^2 \epsilon). \tag{5.10}$$

When $\boldsymbol{\xi}$ and p_1 are single-frequency wave oscillations, this condition implies that the wave amplitude is transverse to the wave vector; hence the name "transversality condition." Gjaja and Holm [1996] show that this condition is required for the Euler–Poincaré equations resulting from averaging in Hamilton's principle to be consistent with applying the method of averages directly to the Euler-Boussinesq equations.

Next, the variation of $\overline{\mathfrak{S}}_{GLM}$ with respect to $\overline{p}^L$ at fixed $\mathbf{x}$ and t gives

$$D = 1 + \frac{\alpha^2}{2} \frac{\partial}{\partial x^i} \overline{\Big(\xi^i \frac{\partial \xi^j}{\partial x^j} - \xi^j \frac{\partial \xi^i}{\partial x^j} \Big)}, \tag{5.11}$$

where the second term on the right side is order $O(\alpha^2\epsilon)$, and thus is negligible at order $O(\alpha^2)$. This follows because mean quantities only depend on slow space and slow time, for which $\partial/\partial x^j = \epsilon \partial/\partial(\epsilon x^j)$.

The Euler–Poincaré Equations. The Euler–Poincaré equation for the action $\overline{\mathfrak{S}}_{GLM}$ results in the GLM motion equation,

$$\frac{\partial(\mathbf{m}/D)}{\partial\epsilon t} - \overline{\mathbf{u}}^L \times \left(\frac{\partial}{\partial\epsilon\mathbf{x}} \times \frac{\mathbf{m}}{D}\right) + \frac{1}{\epsilon} bg\hat{\boldsymbol{z}} \tag{5.12}$$
$$+ \frac{\partial}{\partial\epsilon\mathbf{x}}\left[\overline{p}^L + |\overline{\mathbf{u}}^L|^2 - \frac{1}{2}\overline{\mathbf{u}^\xi \cdot \mathbf{u}^\xi} - \overline{\mathbf{u}^\xi \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi})} - \alpha^2 \mathbf{p} \cdot \overline{\mathbf{u}}^L\right] = 0,$$

where the Lagrangian mean momentum $\mathbf{m}$ is defined as

$$\mathbf{m} \equiv \frac{\delta \overline{L}_{GLM}}{\delta \overline{\mathbf{u}}^L} = D\big[\overline{\mathbf{u}}^L + (\boldsymbol{\Omega} \times \mathbf{x})\big] - \alpha^2 \mathbf{p}, \tag{5.13}$$

and (leaving α^2 explicit) $\mathbf{p}$ is the "pseudomomentum density," defined by, cf. Andrews and McIntyre [1978a],

$$\mathbf{p} \equiv -\, D\, \overline{(u^l_j + R^l_j)\boldsymbol{\nabla}\xi^j} \quad \text{and} \quad D \equiv \det\left(\frac{\partial \ell^A}{\partial x^j}\right). \tag{5.14}$$

Equations (5.11) and (5.12) (and implicitly (5.10)) are the equations of the GLM theory for incompressible flow discussed in Andrews and McIntyre [1978a].

The Kelvin–Noether Theorem. The Kelvin–Noether circulation theorem for the GLM motion equation (5.12) states that

$$\frac{d}{d\epsilon t}\oint_{\overline{\gamma}(\epsilon t)} \mathbf{m}\cdot d\mathbf{x} = -\,\frac{g}{\epsilon}\oint_{\overline{\gamma}(\epsilon t)} b\, dz\,, \tag{5.15}$$

where the curve $\overline{\gamma}(\epsilon t)$ moves with the Lagrangian mean fluid velocity $\overline{\mathbf{u}}^L$. By Stokes' theorem, the GLM equations generate circulation of $\mathbf{m}$ around $\overline{\gamma}(\epsilon t)$ whenever the gradient of bouyancy is not vertical. The conservation of potential vorticity on fluid parcels for the GLM equations is given by

$$\overline{D}^L q_{GLM} = 0\,, \quad \text{where} \quad q_{GLM} = \frac{1}{D}\frac{\partial b}{\partial \epsilon\mathbf{x}} \cdot \left(\frac{\partial}{\partial\epsilon\mathbf{x}} \times \frac{\mathbf{m}}{D}\right). \tag{5.16}$$

The constraint $D = 1$ is imposed at order $O(\alpha^2)$ in Hamilton's principle for the action (5.9) by the slow component of the pressure. This condition holds at order $O(\alpha^2)$ when $\boldsymbol{\nabla}_{\epsilon\mathbf{x}} \cdot \overline{\mathbf{u}}^L = 0$, since D satisfies $\partial D/\partial\epsilon t + \boldsymbol{\nabla}_{\epsilon\mathbf{x}} \cdot D\overline{\mathbf{u}}^L = 0$.

Alternative Derivation of the Kelvin–Noether Circulation Theorem for GLM. The *unapproximated* Euler-Boussinesq equations (4.27) for a rotating stratified incompressible fluid are, in the GLM notation,

$$\left(\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}^\xi} + \mathbf{u}^\xi \cdot \frac{\partial}{\partial \mathbf{x}^\xi}\right)\mathbf{u}^\xi \;-\; \mathbf{u}^\xi \times 2\boldsymbol{\Omega}(\mathbf{x}^\xi) \;=\; -\,\frac{\partial p^\xi}{\partial \mathbf{x}^\xi} - b^\xi g\hat{\boldsymbol{z}}^\xi, \tag{5.17}$$
$$\left(\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}^\xi} + \mathbf{u}^\xi \cdot \frac{\partial}{\partial \mathbf{x}^\xi}\right) b^\xi = 0, \quad \frac{\partial}{\partial \mathbf{x}^\xi} \cdot \mathbf{u}^\xi = 0, \quad 2\boldsymbol{\Omega}(\mathbf{x}^\xi) = \frac{\partial}{\partial \mathbf{x}^\xi} \times \mathbf{R}(\mathbf{x}^\xi)\,.$$

Being Euler–Poincaré, these equations admit the following Kelvin–Noether circulation theorem, cf. equation (4.29),

$$\frac{d}{dt}\oint_{\gamma(t)}\Big(\mathbf{u}(\mathbf{x}^\xi,t)+\mathbf{R}(\mathbf{x}^\xi)\Big)\cdot d\mathbf{x}^\xi = -\,g\oint_{\gamma(t)} b(\mathbf{x}^\xi,t)\;dz^\xi, \tag{5.18}$$

for a contour $\gamma(t)$ which moves with the fluid.

The time average of the Kelvin circulation integral is expressed as

$$\begin{aligned}
\overline{I}(\epsilon t) &= \overline{\oint_{\gamma(t)}\Big(\mathbf{u}(\mathbf{x}^\xi,t)+\mathbf{R}(\mathbf{x}^\xi)\Big)\cdot d\mathbf{x}^\xi} \\
&= \overline{\oint_{\gamma(t)}(\overline{\mathbf{u}}^L+\overline{\mathbf{R}}^L+\alpha\mathbf{u}^l+\alpha\mathbf{R}^l)\cdot(d\mathbf{x}+\alpha\,d\boldsymbol{\xi})} \\
&= \oint_{\overline{\gamma}(\epsilon t)}\left[(\overline{\mathbf{u}}^L+\overline{\mathbf{R}}^L)\cdot d\mathbf{x}+\alpha^2\overline{(\mathbf{u}^l+\mathbf{R}^l)\cdot d\boldsymbol{\xi}}\right],
\end{aligned} \tag{5.19}$$

where the contour $\overline{\gamma}(\epsilon t)$ moves with velocity $\overline{\mathbf{u}}^L$, since it follows the fluid parcels as the average is taken. Using equations (5.13) and (5.14), we rewrite $\overline{I}(\epsilon t)$ as

$$\overline{I}(\epsilon t)=\oint_{\overline{\gamma}(\epsilon t)}\Big(\overline{\mathbf{u}}^L+\overline{\mathbf{R}}^L-\alpha^2\mathbf{p}/D\Big)\cdot d\mathbf{x}=\oint_{\overline{\gamma}(\epsilon t)}\mathbf{m}\cdot d\mathbf{x}\,, \tag{5.20}$$

which re-introduces Lagrangian mean momentum $\mathbf{m}(\epsilon\mathbf{x},\epsilon t)$ and the pseudo-momentum density of the rapid motion, $\mathbf{p}(\epsilon\mathbf{x},\epsilon t)$. In terms of these quantities, we may write the Lagrangian mean of the Kelvin–Noether circulation theorem for the Euler-Boussinesq equations as

$$\frac{d}{dt}\overline{I}(\epsilon t)=\epsilon\frac{d}{d\epsilon t}\oint_{\overline{\gamma}(\epsilon t)}\mathbf{m}\cdot d\mathbf{x}=-\,g\,\overline{\oint_{\gamma(t)} b(\mathbf{x}^\xi,t)\;dz^\xi}=-\,g\oint_{\overline{\gamma}(\epsilon t)} b\,dz\,. \tag{5.21}$$

This result recovers the Kelvin–Noether circulation theorem (5.15) for the GLM equations (5.12) which were derived above as Euler–Poincaré equations for the averaged Lagrangian $\overline{\mathfrak{S}}_{GLM}$ in equation (5.9).

Geometry of the Stokes Mean Drift Velocity and the Pseudomomentum. The quantity $\mathbf{p}$ in equation (5.14) is an additional slowly varying fluid degree of freedom which emerges in the process of averaging to describe the rectified effects of the rapidly varying component of the fluid motion acting on the slowly varying component. The rectified effects of wave fluctuations in fluids are traditionally discussed in terms of another quantity, namely, the Stokes mean drift velocity, defined as

$$\overline{\mathbf{u}}^S=\overline{\mathbf{u}}^L-\overline{\mathbf{u}}=\alpha^2\overline{(\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\mathbf{u}^l}+O(\alpha^4)\,, \tag{5.22}$$

We define the analogous Stokes mean drift quantity corresponding to the rotation vector potential, $\mathbf{R}$, as

$$\overline{\mathbf{R}}^S = \overline{\mathbf{R}}^L - \overline{\mathbf{R}} = \alpha^2\overline{(\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\mathbf{R}^l} + O(\alpha^4)\,, \tag{5.23}$$

The time-averaged contour integral $\overline{I}(\epsilon t)$ in equation (5.19) may be rewritten as

$$\begin{aligned}
\overline{I}(\epsilon t) &= \oint_{\overline{\gamma}(\epsilon t)} \left[(\overline{\mathbf{u}} + \overline{\mathbf{R}})\cdot d\mathbf{x} + (\overline{\mathbf{u}}^S + \overline{\mathbf{R}}^S)\cdot d\mathbf{x} - \mathbf{p}\cdot d\mathbf{x} \right] \\
&= \oint_{\overline{\gamma}(\epsilon t)} \left[(\overline{\mathbf{u}} + \overline{\mathbf{R}})\cdot d\mathbf{x} + \alpha^2\overline{\left(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}(\mathbf{u}^l + \mathbf{R}^l) \right)}\cdot d\mathbf{x} + \alpha^2\overline{(\mathbf{u}^l + \mathbf{R}^l)\cdot d\boldsymbol{\xi}} \right] \\
&= \oint_{\overline{\gamma}(\epsilon t)} \left[(\overline{\mathbf{u}} + \overline{\mathbf{R}})\cdot d\mathbf{x} + \alpha^2\overline{\left(\pounds_\xi\Big((\mathbf{u}^l + \mathbf{R}^l)\cdot d\mathbf{x} \Big) \right)} \right]. \qquad (5.24)
\end{aligned}$$

Thus, the last two terms combine into the time-averaged Lie derivative with respect to $\boldsymbol{\xi}$ of the total fluctuation circulation $(\mathbf{u}^l + \mathbf{R}^l)\cdot d\mathbf{x}$. This is a remarkable formula which shows the geometric roles of the Stokes mean drift velocity and the pseudomomentum: The Stokes mean drift velocity derives from *transport*, while the pseudomomentum is caused by *line-element stretching* by the fluctuations. For *divergence-free* fluctuations, we have $\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0$, and the Stokes mean drift velocity in the rotating frame is a higher order term. In particular,

$$\alpha^2\bar{\mathbf{u}}^S = \alpha^2\overline{\left((\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\frac{d\boldsymbol{\xi}}{dt} \right)} = \alpha^2\frac{\partial}{\partial x^i}\overline{\left(\xi^i\,\frac{d\boldsymbol{\xi}}{dt} \right)} = \epsilon\;\alpha^2\frac{\partial}{\partial \epsilon x^i}\overline{\left(\xi^i\,\frac{d\boldsymbol{\xi}}{dt} \right)}\,. \tag{5.25}$$

The last step in this calculation follows because mean quantities only depend on slow space and slow time. Thus, the Stokes mean drift velocity (i.e., the difference between the Lagrangian and Eulerian mean velocities) is order $O(\epsilon)$ smaller than the pseudomomentum, for divergence-free fluctuations.

6 Nonlinear Dispersive Modifications of the EB Equations and PE

In generalizing earlier work by Camassa and Holm [1993] from one dimension to n dimensions, Holm, Marsden and Ratiu [1998a,b] used the Euler–Poincaré framework to formulate a modified set of Euler equations for ideal homogeneous incompressible fluids. This modification introduces nonlinear dispersion into Euler's equations, which is designed physically to model *nondissipative* unresolved rapid fluctuations. This nonlinear dispersion is founded mathematically on the geometrical property that solutions of the basic Euler–Poincaré equations are geodesics on an underlying group when their Lagrangian is a

metric. Here we use the Euler–Poincaré reduction theorems (2.1) and (3.1) including advected parameters and Hamilton's principle asymptotics in the EB Lagrangian in GLM notation (5.6) to formulate a modified set of Euler-Boussinesq equations that includes nonlinear dispersion along with stratification and rotation. In this new modification of the Euler-Boussinesq equations, nonlinear dispersion adaptively filters high wavenumbers and thereby enhances stability and regularity without compromising either low wavenumber behavior, or geophysical balances. Here, 'high' and 'low' refer respectively to wavenumbers greater, or less than the inverse of a fundamental length scale α, which parameterizes the nonlinearly dispersion in the model. We also present the corresponding nonlinear dispersive modification of the primitive equations. We leave it as an open question, whether our nonlinearly dispersive primitive equation model will have a slow manifold when dissipation and forcing are included.

6.1 Higher Dimensional Camassa–Holm Equation.

The Lagrangian and Action Functionals. As shown in Holm, Marsden and Ratiu [1998a,b], the Camassa-Holm (CH) equation (Camassa and Holm [1993]) in n dimensions describes geodesic motion on the diffeomorphism group of $\mathbb{R}^n$ with respect to the metric given by the H^1 norm of the Eulerian fluid velocity. This Euler–Poincaré equation follows from the Lagrangian $l_{\rm CH}$ given by the H^1 norm of the fluid velocity $\mathbf{u}$ in n dimensions, subject to volume preservation (for $n \neq 1$), namely,

$$\mathfrak{S}_{\rm CH} = \int dt\, l_{\rm CH} = \int dt \int_{\mathcal{M}} d^n x \, \left\{ \frac{D}{2}\Big(|\mathbf{u}|^2 + \alpha^2 |\nabla \mathbf{u}|^2 \Big) - p(D-1) \right\}, \tag{6.1}$$

in which the parameter α has dimensions of length. We denote $(\nabla \mathbf{u})^i_j = u^i_{,j} \equiv \partial u^i / \partial x^j$, $|\nabla \mathbf{u}|^2 \equiv u^i_{,j} u^j_i = {\rm tr}(\nabla \mathbf{u} \cdot \nabla \mathbf{u}^T)$, tr is the trace operation for matrices and the superscript $(\cdot)^T$ denotes transpose.

The action $\mathfrak{S}_{\rm CH}$ in (6.1) is the order $O(\alpha^2)$ expression for the mean of the EB action in equation (5.6) in the absence of stratification and rotation, namely

$$\mathfrak{S}_{\rm CH} \approx \int dt \int d^n x \, \left\{ D \frac{1}{2} \overline{|\mathbf{u}^\xi(\mathbf{x},t)|^2} - p\,(D-1) \right\}, \tag{6.2}$$

in which we neglect the order $O(\alpha^2 \epsilon)$ pressure and density corrections discussed in Gjaja and Holm [1996] and approximate the mean kinetic energy in a Taylor expansion as follows,

$$\begin{aligned} \frac{1}{2}\overline{|\mathbf{u}^\xi(\mathbf{x},t)|^2} &= \frac{1}{2}\overline{|\mathbf{u}(\mathbf{x}+\alpha\boldsymbol{\xi},t)|^2} = \frac{1}{2}\overline{|\mathbf{u}(\mathbf{x},t) + \alpha\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\mathbf{u}|^2} + O(\alpha^4) \\ &= \frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}\overline{|\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\mathbf{u}|^2} + O(\alpha^4) \approx \frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\boldsymbol{\nabla}\mathbf{u}|^2 . \end{aligned} \tag{6.3}$$

In the last step, we have also dropped terms of order $O(\alpha^4)$ and assumed isotropy of the rapid fluctuations, so that $\overline{\xi^i\xi^j} \approx \delta^{ij}$. As in GLM theory, the length-scale parameter α represents the amplitude of the rapidly fluctuating component of the fluid parcel trajectory (or its amplitude to length-scale ratio in a nondimensional formulation).

The Euler–Poincaré Equations. Varying the action $\mathfrak{S}_{\rm CH}$ in (6.1) at fixed $\mathbf{x}$ and t gives

$$\begin{aligned}\delta\mathfrak{S}_{\rm CH} \;=\; \int dt \int_{\mathcal{M}} d^n x \Big[\Big(\frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2 - p\Big)\,\delta D \\ + \big(D\mathbf{u} - \alpha^2(\mathrm{div} D\mathrm{grad})\mathbf{u}\big)\cdot\delta\mathbf{u} - (D-1)\delta p\Big] \\ + \alpha^2 \int dt \oint_{\partial\mathcal{M}} d^{n-1}x\; (D\hat{\boldsymbol{n}}\cdot\nabla\mathbf{u}\cdot\delta\mathbf{u})\,,\end{aligned} \tag{6.4}$$

whose natural boundary conditions on $\partial\mathcal{M}$ are

$$\mathbf{u}\cdot\hat{\mathbf{n}} = 0 \quad \text{and} \quad (\hat{\mathbf{n}}\cdot\nabla)\mathbf{u} \parallel \hat{\mathbf{n}}, \tag{6.5}$$

where $\parallel$ denotes "parallel to" in the second boundary condition, which of course is not imposed when α^2 is absent. (Recall that $\delta\mathbf{u}$ in equation (6.4) is arbitrary except for being tangent on the boundary. This tangency, along with the second condition in equation (6.5) is sufficient for the boundary integral in equation (6.4) to vanish.) By equation (4.11), the Euler–Poincaré equation for the action $\mathfrak{S}_{\rm CH}$ in equation (6.1) is

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)\mathbf{v} + v_j\nabla u^j + \nabla\Big(p - \frac{1}{2}|\mathbf{u}|^2 - \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2\Big) = 0\,, \tag{6.6}$$

where

$$\mathbf{v} \equiv \mathbf{u} - \alpha^2\Delta\mathbf{u} = \frac{1}{D}\frac{\delta l_{\rm CH}}{\delta\mathbf{u}}\bigg|_{D=1}.$$

In writing this equation, we have substituted the constraint $D = 1$, which as before implies incompressibility via the continuity equation for D. Requiring the motion equation (6.6) to preserve $\mathrm{div}\,\mathbf{u} = 0 = \mathrm{div}\,\mathbf{v}$ implies a Poisson equation for the pressure p with a Neumann boundary condition, which is obtained just as usual in the case of incompressible ideal fluid dynamics, by taking the normal component of the motion equation evaluated at the boundary. Of course, the n-dimensional extension of the CH equation (6.6) reduces to Euler's equation when we set $\alpha = 0$. The properties of the n-dimensional CH equation (6.6) are summarized in Holm, Marsden and Ratiu [1998a,b] and Holm, Kouranbaeva, Marsden, Ratiu, and Shkoller [1998] for the ideal case. In particular, equation (6.6) is shown to be the geodesic spray

equation for geodesic motion on the group Diff($\mathcal{D}$) with respect to the metric given by the H^1 norm of the fluid velocity $\mathbf{u}$ in n dimensions. See also Chen, Foias, Holm, Olson, Titi and Wynne [1998] and Foias, Holm and Titi [1998] for discussions of the corresponding viscous, forced case and its connection to mean turbulence closure models.

Discussion of the CH Equation. The essential idea of the CH equation is that its specific momentum (i.e., its momentum per unit mass) is transported by a velocity which is smoothed by inverting the elliptic Helmholtz operator $(1 - \alpha^2\Delta)$, where α corresponds to the length scale at which this smoothing becomes important, i.e., when it becomes of order $O(1)$. When the smoothing operator $(1-\alpha^2\Delta)^{-1}$ is applied to the transport velocity in Euler's equation to produce the CH equation, its effect on length scales smaller than α is that steep gradients of the specific momentum $\mathbf{v}$ tend not to steepen much further, and thin vortex tubes tend not to get much thinner as they are transported. And, its effect on length scales that are considerably larger than α is negligible. Hence, the transport of vorticity in the CH equation is intermediate between that for the Euler equations in two and three dimensions. As for Euler vorticity, the curl of the CH specific momentum is *convected* as an *active* two form, but its transport velocity is the *smoothed, or filtered* CH specific momentum.

The effects of this smoothing or filtering of the transport velocity in the CH equation can be seen quite clearly from its Fourier spectral representation in the periodic case. In this case, we define $\mathbf{v_k}$ as the $\mathbf{k}$-th Fourier mode of the specific momentum $\mathbf{v} \equiv (1 - \alpha^2\Delta)\mathbf{u}$ for the CH equation; so that $\mathbf{v_k} \equiv (1 + \alpha^2|\mathbf{k}|^2)\mathbf{u_k}$. Then the Fourier spectral representation of the CH equation for a periodic three-dimensional domain is expressed as

$$\Pi\left(\frac{d}{dt}\mathbf{v_k} - i\sum_{\mathbf{p}+\mathbf{n}=\mathbf{k}} \frac{\mathbf{v_p}}{1+\alpha^2|\mathbf{p}|^2} \times (\mathbf{n}\times\mathbf{v_n})\right) = 0, \tag{6.7}$$

where Π is the Leray projection onto Fourier modes transverse to $\mathbf{k}$. (As usual, the Leray projection ensures incompressibility.) In this Fourier spectral representation of the CH equation, one sees that the coupling to high modes is suppressed by the denominator when $1 + \alpha^2|\mathbf{p}|^2 \gg 1$. Consequently, when $|\mathbf{p}| \geq O(1/\alpha)$, the smoothing of the transport velocity suppresses the Fourier-mode interaction coefficients. In fact, the CH smoothing of the transport velocity suppresses *both* the forward and backward cascades for wave numbers $|\mathbf{p}| \geq O(1/\alpha)$, but leaves the Euler dynamics essentially unchanged for smaller wave numbers. The result is that the vortex stretching term in the dynamics of $\mathbf{q} = \operatorname{curl}\mathbf{v}$ is mollified in the CH model and so the vortices at high wave numbers will tend to be "shorter and fatter" than in the corresponding Euler case for the same initial conditions.

The Kelvin–Noether Theorem. Since the n-dimensional CH equation (6.6) is Euler–Poincaré, it also has a corresponding ***Kelvin–Noether circulation theorem***. Namely,

$$\frac{dI}{dt} = 0 \quad \text{where} \quad I(t) = \oint_{\gamma_t} (\mathbf{u} - \alpha^2 \Delta \mathbf{u}) \cdot d\mathbf{x} = \oint_{\gamma_t} \mathbf{v} \cdot d\mathbf{x} \,, \tag{6.8}$$

for any closed curve γ_t that moves with the fluid velocity $\mathbf{u}$. This expression for the Kelvin–Noether circulation of the CH equation in three dimensions is reminiscent of the corresponding expression (5.20) in GLM theory involving the pseudomomentum for wave, mean-flow interaction theory. This correspondence confirms the physical interpretation of the α^2 term in the Kelvin–Noether circulation integral as a *Lagrangian mean closure relation* for the pseudomomentum of the high frequency (i.e., rapidly fluctuating, turbulent) components of the flow. In this interpretation, α for the CH equation corresponds to both the amplitude of these high frequency components and the length scale at which they become important.

Energy Conservation for the CH Equation. Legendre transforming the action (6.1) gives the following conserved ***Hamiltonian*** (still expressed in terms of the velocity, instead of the momentum density $\mathbf{m} = \delta l / \delta \mathbf{u}$),

$$H = \int_{\mathcal{M}} d^n x \, \Big[\frac{D}{2} \left(|\mathbf{u}|^2 + \alpha^2 |\nabla \mathbf{u}|^2 \right) + p(D-1) \Big] \,. \tag{6.9}$$

Thus, when evaluated on the constraint manifold $D = 1$, the Lagrangian and the Hamiltonian for the CH equation coincide in n dimensions. (This, of course, is not unexpected for a stationary principle giving rise to geodesic motion.)

The curl of the 3D Camassa-Holm motion equation (6.6) yields

$$\frac{\partial}{\partial t} \mathbf{q} = \mathbf{q} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{q} \equiv [\mathbf{u}, \mathbf{q}], \quad \text{where} \quad \mathbf{q} \equiv \operatorname{curl}(\mathbf{u} - \alpha^2 \Delta \mathbf{u}) \,, \tag{6.10}$$

and we have used incompressibility and commutativity of the divergence and Laplacian operators. Thus, $\mathbf{u}$ is the transport velocity for the generalized vorticity $\mathbf{q}$ and the "vortex stretching" term $\mathbf{q} \cdot \nabla \mathbf{u}$ involves $\nabla \mathbf{u}$, whose L^2 norm is *controlled* by the conservation of energy in equation (6.9). Boundedness of this norm will be useful in future analytical studies of the 3D Camassa-Holm equation; for example, in the investigation of the Liapunov stability properties of its equilibrium solutions.

The Riemannian CH Equation. One can formulate the CH equation on a general Riemannian manifold, possibly with boundary. Although this formulation will be the subject of future papers, we comment on it here because of the importance of spherical geometry, in particular, for GFD models.

6.2 The Euler-Boussinesq α Model (EBα)

We introduce nonlinear dispersion into the Euler-Boussinesq equations using the Euler–Poincaré framework and following the example of the CH equations.

The Lagrangian. To carry this out, we modify the EB Lagrangian (4.26) by simply adding the α^2 term (while dropping other scale factors ϵ, σ, as well as primes and subscripts)

$$\begin{aligned}\mathfrak{S}_{\mathrm{EB}\alpha} &= \int dt\; l_{\mathrm{EB}\alpha} \qquad (6.11)\\ &= \int dt \int_{\mathcal{M}} d^n x \Big[D\Big(\frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\nabla \mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - gbz\Big) - p(D-1)\Big]\,,\end{aligned}$$

This action is the order $O(\alpha^2)$ approximation of the mean of the EB action in equation (5.6), using the Taylor expansion (6.3) and neglecting order $O(\alpha^2\epsilon)$ pressure and density corrections. (An order $O(\alpha^2)$ term involving $\overline{(\boldsymbol{\xi}\cdot\nabla\mathbf{u})(\boldsymbol{\xi}\cdot\nabla\mathbf{R})}$ is also dropped, since it makes only a negligible contribution in the resulting motion equation.)

The Euler–Poincaré Equations. Varying this action at fixed $\mathbf{x}$ and t gives

$$\begin{aligned}\delta\mathfrak{S}_{\mathrm{EB}\alpha} \;=\; &\int dt \int_{\mathcal{M}} d^n x \Big[\Big(\frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - gbz - p\Big)\,\delta D\\ &\quad -Dgz\delta b - (D-1)\delta p + \big(D\mathbf{u} - \alpha^2(\mathrm{div}D\mathrm{grad})\mathbf{u}\big)\cdot\delta\mathbf{u}\Big]\\ &\quad + \alpha^2 \int dt \oint_{\partial\mathcal{M}} d^{n-1}x\; (D\hat{\boldsymbol{n}}\cdot\nabla\mathbf{u}\cdot\delta\mathbf{u})\,, \qquad (6.12)\end{aligned}$$

where the natural boundary conditions are again given in (6.5). The corresponding Euler–Poincaré equation for the action $\mathfrak{S}_{\mathrm{EB}\alpha}$ in equation (6.11) is,

$$\begin{aligned}&\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)\mathbf{v} - \mathbf{u}\times\mathrm{curl}\mathbf{R} + v_j\nabla u^j + gb\hat{\boldsymbol{z}}\\ &\qquad +\nabla\Big(p - \frac{1}{2}|\mathbf{u}|^2 - \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2\Big) = 0\,, \qquad (6.13)\end{aligned}$$

where

$$\begin{aligned}&\mathbf{v} \equiv \mathbf{u} - \alpha^2\Delta\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0,\\ &\frac{db}{dt} = 0, \quad \frac{d}{dt} = \Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)\,. \qquad (6.14)\end{aligned}$$

Again, we have substituted the constraint $D = 1$, which implies incompressibility via the continuity equation for D. Relative to the usual EB equations (4.27), the EBα equation (6.13) has a smoothed, or filtered transport velocity, since $\mathbf{u} = (1-\alpha^2\Delta)^{-1}\mathbf{v}$.

The Kelvin–Noether Theorem. The Kelvin–Noether circulation theorem for the EBα equation (6.13) is,

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u})}(\mathbf{v}+\mathbf{R})\cdot d\mathbf{x} = -\oint_{\gamma_t(\mathbf{u})} b dz\,, \tag{6.15}$$

where the curve $\gamma_t(\mathbf{u})$ moves with the fluid velocity $\mathbf{u}$. (The two Kelvin theorems in equations (6.15) and (4.29) differ in their definitions of $\mathbf{v}$.) By Stokes' theorem, the EBα equations generate circulation of $\mathbf{v}+\mathbf{R}$ around $\gamma_t(\mathbf{u})$ whenever the gradient of bouyancy is not vertical. The conservation of potential vorticity on fluid parcels for the EBα equations is given by

$$\frac{\partial q_{\mathrm{EB}\alpha}}{\partial t} + \mathbf{u}\cdot\nabla\, q_{\mathrm{EB}\alpha} = 0\,, \quad \text{where} \quad q_{\mathrm{EB}\alpha} = \nabla b\cdot\nabla\times(\mathbf{v}+\mathbf{R})\,. \tag{6.16}$$

The curl of the EBα motion equation gives,

$$\frac{\partial}{\partial t}\mathbf{q} = -\mathbf{u}\cdot\nabla\mathbf{q} + \mathbf{q}\cdot\nabla\mathbf{u} + g\nabla b\times\hat{\boldsymbol{z}}, \quad \text{where} \quad \mathbf{q}\equiv \mathrm{curl}(\mathbf{u}-\alpha^2\Delta\mathbf{u}+\mathbf{R})\,. \tag{6.17}$$

This the usual expression for transport, stretching and creation of vorticity in a buoyant flow, except that here the vortex stretching coefficient is $\nabla\mathbf{u}$, which is *moderated* relative to the usual EB equations.

Energy Conservation. The EBα equations (6.13) conserve the following Hamiltonian (found, e.g., by Legendre transforming the Lagrangian $l_{\mathrm{EB}\alpha}$ in equation (6.11)). Namely,

$$H_{\mathrm{EB}\alpha} = \int_{\mathcal{M}} d^n x\Big[\, D\Big(\frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2\Big) + Dgbz + p(D-1)\Big]\,. \tag{6.18}$$

The corresponding conserved energy is

$$E_{\mathrm{EB}\alpha} = H_{\mathrm{EB}\alpha}\Big|_{D=1} = \int_{\mathcal{M}} d^n x\Big[\frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2 + gbz\Big]\,. \tag{6.19}$$

Since the (finite) value of this conserved energy for the EBα model is determined by its initial conditions and b is advected (so it has a maximum value in L^∞) there is H^1 control of the velocity $\mathbf{u}$. This is the effect of the "filtering" of the solution produced by the nonlinear dispersion for $\alpha\neq 0$. This filtering moderates the growth of instabilities at wavenumbers $|\mathbf{k}|\geq 1/\alpha$. So if the EB$\alpha$ model is used as a large eddy simulation (LES) model, one would choose the value of α to determine the size of the minimum resolved length scale. The filtering by the α term also allows nonlinear Liapunov stability conditions to be formulated for equilibrium solutions of the EBα model. This stability result is clear from the work of Abarbanel et al. [1986], who introduced the notion of "conditional" Liapunov stability for the EB model, using

wavenumber conditions that now turn out to be satisfied for the EBα model. The Euler fluid equations (4.23) may also be modified analogously to include nonlinear dispersion. However, this case is ignored for now, as we proceed to discuss the modified primitive equations.

6.3 Primitive Equation α Model (PEα)

In horizontal and vertical components, with $\mathbf{v}^\perp \equiv \mathbf{v} - v^\parallel \hat{\boldsymbol{z}}$, $v^\parallel \equiv (\mathbf{v}\cdot\hat{\boldsymbol{z}})$ and $\operatorname{curl}\mathbf{R} = f(\mathbf{x})\hat{\boldsymbol{z}}$, the the EB$\alpha$ equations are expressed in nondimensional form as, cf. equations (4.28) and (6.13),

$$\epsilon\,\frac{d\mathbf{v}^\perp}{dt} + \epsilon\, v_j \nabla^\perp u^j + f\hat{\boldsymbol{z}}\times\mathbf{u} + \nabla^\perp \pi = 0\,, \qquad \epsilon\,\sigma^2\,\frac{dv^\parallel}{dt} + b + \frac{\partial\pi}{\partial z} = 0\,, \tag{6.20}$$

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{u}^\perp\cdot\nabla^\perp + u^\parallel\frac{\partial}{\partial z}\right), \quad \frac{db}{dt} = 0\,, \qquad \nabla\cdot\mathbf{u} = \nabla^\perp\cdot\mathbf{u}^\perp + \frac{\partial u^\parallel}{\partial z} = 0\,,$$

$$\text{where} \quad \pi \equiv \left(p - \frac{1}{2}|\mathbf{u}|^2 - \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2\right), \qquad \mathbf{v} \equiv \mathbf{u} - \alpha^2\Delta\mathbf{u}\,.$$

Here, ϵ and σ are the Rossby number and aspect ratio, respectively, and α has been scaled in units of horizontal length scale, L. The leading order balances are still hydrostatic in the vertical, and geostrophic in the horizontal. Setting $\sigma = 0$ in equation (6.20) removes the vertical acceleration and thereby produces the primitive equation α model (PEα). We expect that the filtering property for $\alpha \neq 0$ discussed above for the CH and EBα equations should make the PEα model much more regular than the ordinary PE, from the viewpoint of gravity wave oscillations. The problematic asspects of gravity waves in atmospheric and oceanic numerical simulations and data assimilation using the PE model have often been addressed by invoking the concept of an idealized "slow manifold" on which gravity waves are absent, see, e.g., Lorenz [1992]. However, the existence of a slow manifold has never been proven for the PE model. It is an open question, whether the new PEα model will have a slow manifold when dissipation and forcing are included. We shall report on this matter elsewhere.

Final Remarks

In this paper we have shown how asymptotic expansions in Hamilton's principle for the Euler–Poincaré equations of geophysical fluid dynamics provide an organizing principle for many GFD systems and produce a clear, unified understanding of their Kelvin theorems. Hamilton's principle asymptotics in the Euler–Poincaré setting thus explains the shared properties of these GFD models and provides a unified approach to making additional approximations and creating new models. In this setting, we have introduced a new class of fluid models, called α models, that possess nonlinear dispersion which smooths the

transport velocity relative to the circulation velocity in the Euler–Poincaré equations. The effect of this smoothing is to moderate the vortex stretching process while preserving the Kelvin circulation theorem for these equations. The efficacy and utility of the α models are yet to be determined, but initial studies of them are promising. We expect that one can also perform the sort of asymptotics done here on the group level first (to get an approximating group) and then perform Euler–Poincaré reduction and arrive at the same conclusions. In other words, there should be a general principle for asymptotics in Hamilton's principle, which allows passage from one group and its corresponding advected quantities to an approximating one. Moreover, this process should commute with Lagrangian and Hamiltonian reduction.

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New progress in Lagrangian averaged models

Since this paper was originally submitted in 1997, the Euler-Poincaré framework has inspired considerable progress in formulating and analyzing closed systems of Lagrangian averaged Euler (LAE) equations. The closed model LAE equations were first obtained in Holm, Marsden and Ratiu [1998a,1998b] by using Taylor's hypothesis as a closure step as we did in section 6 for the EB equations and PE. The equations including the effects of stratification and rotation introduced in Gjaja and Holm [1996] in the Lagrangian fluid specification by using a WKB approximation for the fluctuating vector field $\xi = \mathbf{x}^\xi - \mathbf{x}$ were later recognized as a self-consistent variant of the LAE closure. See Holm [1999a,1999b,2001,2002] and Marsden and Shkoller [2001a,2001b] for further discussions of that approach, which uses asymptotic expansions of Hamilton's principle for GLM to order $O(|\xi|^2)$ in combination with Taylor's hypothesis in developing the closure equations.

This type of closure method has recently been developed to the point of applications as the basis of a turbulence model (after properly including viscous dissipation) in Chen *et al.* [1998,1999a,1999b,1999]. This LANS$-\alpha$ model – the Lagrangian averaged Navier-Stokes$-\alpha$ equations – was compared to Large Eddy Simulation (LES) methods in Domaradzki and Holm [2001], Mohseni *et al.* [2000], Holm and Kerr [2001] and in Geurts and Holm [2001]. See Shkoller [1998,2000], Holm [1999a,1999b], Marsden, Ratiu and Shkoller [2000], Marsden and Shkoller [2001a,2001b] and Foias, Holm and Titi [2001,2002] for additional studies and discussions of the mathematical properties of the LANS$-\alpha$ and Euler$-\alpha$ equations.

Of course, the Lagrangian averaged Euler-Poincaré (LAEP) approach is also versatile enough to derive LA equations for *compressible* fluid motion. This was already shown in the original GLM theory in Andrews and McIntyre [1978a]. For brevity now, we only remark that the LAEP approach preserves

helicity conservation for barotropic compressible flows. It also preserves magnetic helicity and cross-helicity conservation when applied to magnetohydrodynamics (MHD). For more details in this regard, see Holm[2001,2002].

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