Dissipation and Controlled Euler-Poincaré Systems

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Abstract

The method of controlled Lagrangians is a technique for stabilizing underactuated mechanical systems which involves modifying a system's energy and dynamic structure through feedback. These modifications can obscure the effect of physical dissipation in the closed-loop. For example, generic damping can destabilize an equilibrium which is closed-loop stable for a conservative system model. In this paper, we consider the effect of damping on Euler-Poincaré (special reduced Lagrangian) systems which have been stabilized about an equilibrium using the method of controlled Lagrangians. We describe a choice of feedback dissipation which asymptotically stabilizes a sub-class of controlled Euler-Poincaré systems subject to physical damping. As an example, we consider intermediate axis rotation of a damped rigid body with a single internal rotor.

1 Introduction

The method of controlled Lagrangians [6, 3] was specialized to the problem of stabilizing equilibria for Euler-Poincaré systems in [7]. A feedback control law was proposed which preserves the Euler-Poincaré structure but which shapes the kinetic energy of the closed-loop system. More generally, one may choose feedback which also modifies the dynamic structure [4]. Closed-loop stability can be studied using Lyapunov methods.

This paper describes the effect of external damping on an Euler-Poincaré system which has been stabilized using the method of controlled Lagrangians. Left uncompensated, generic damping may destabilize the closed-loop equilibrium. We propose a technique for choosing feedback dissipation which yields asymptotic stability for a sub-class of controlled Euler-Poincaré systems. The approach involves the construction and analysis of a semidefinite Lyapunov function for the closed-loop system. This work builds on previous results for the underwater vehicle with internal rotors [10, 11].

In addition to the previous work of the authors the idea of kinetic energy shaping has been pursued in the Lagrangian setting in [1, 8]. The method of interconnection and damping assignment [2, 9] is an equivalent approach in the Hamiltonian setting [4]. The Hamiltonian approach was also used in the earlier paper [5, 10, 11].

2 Review

The systems considered here have as their configuration space a product of Lie groups $Q = H \times G$ where H is non-abelian and G is Abelian. The dynamics are invariant under the left action of H, the Lagrangian is cyclic in the G-variables, and the control enters in the G-direction [7]. Assuming the control is chosen to preserve the H-symmetry, the dynamics may be described in a reduced velocity phase space isomorphic to $\mathfrak{h} \times G$, where $\mathfrak{h}$ is the Lie algebra of H. Let η^α represent an element of $\mathfrak{h}$ and let θ^a represent an element of G. The reduced Lagrangian has the form

$$l(\eta^\alpha, \dot{\theta}^a) = \frac{1}{2} g_{\alpha\beta}\eta^\alpha\eta^\beta + g_{\alpha b}\eta^\alpha\dot{\theta}^b + \frac{1}{2} g_{ab}\dot{\theta}^a\dot{\theta}^b,$$

where $g_{\alpha\beta}$, $g_{\alpha b}$, and g_{ab} are the (constant) components of the local kinetic energy metric. In the absence of generalized forces other than the control, the open-loop equations are

$$\frac{d}{dt}\frac{\partial l}{\partial \eta^\alpha} = c^\beta_{\alpha\gamma}\eta^\gamma \frac{\partial l}{\partial \eta^\beta} \tag{1}$$

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{\theta}^a} = u_a. \tag{2}$$

The coefficients $c^\beta_{\alpha\gamma}$ are structure constants for $\mathfrak{h}$.

Let M_α represent the momentum conjugate to η^α,

$$M_\alpha = \frac{\partial l}{\partial \eta^\alpha}.$$

In examples, M_α typically represents the total system momentum. In the Lagrangian setting, a Casimir $C^k(M_\alpha)$ satisfies

$$\frac{d}{dt} C^k = \frac{\partial C^k}{\partial M_\alpha}\left(c^\beta_{\alpha\gamma}\eta^\gamma \frac{\partial l}{\partial \eta^\beta}\right) = 0.$$

Physically, Casimirs often correspond to conservation laws for total inertial momentum. For the system described by (1) and (2), the functions C^k are conserved for any choice of control; this observation relates to the fact that internal actuators cannot affect total inertial momentum.

We wish to stabilize an unstable equilibrium

$$(\eta^\alpha, \dot{\theta}^a)|_e = (\eta^\alpha_e, 0) \tag{3}$$

of the uncontrolled dynamics (1) and (2). (This equilibrium corresponds to a *relative* equilibrium for the unreduced system.) The method of controlled Lagrangians provides a control-modified reduced Lagrangian l_c. Under certain conditions on the parameters defining l_c, and for a particular control law u_a, the closed-loop dynamics are

$$\frac{d}{dt}\frac{\partial l_c}{\partial \eta^\alpha} = c^\beta_{\alpha\gamma}\eta^\gamma \frac{\partial l_c}{\partial \eta^\beta} \tag{4}$$

$$\frac{d}{dt}\frac{\partial l_c}{\partial \dot{\theta}^a} = 0. \tag{5}$$

Lyapunov stability analysis gives conditions on control gains for closed-loop stability of (3).

Choosing the modified energy such that

$$\frac{\partial l}{\partial \eta^\alpha} = \frac{\partial l_c}{\partial \eta^\alpha},$$

leads to "matching" of equations (1) and (4). The Euler-Poincaré matching conditions [7] lead to the controlled Lagrangian

$$l_c(\eta^\alpha, \dot{\theta}^a) = \frac{1}{2} g_{\alpha\beta}\eta^\alpha\eta^\beta + g_{\alpha b}\eta^\alpha\dot{\theta}^b + \frac{1}{2}\rho_{ab}\dot{\theta}^a\dot{\theta}^b$$

where ρ_{ab} is a constant control parameter. To state the control law, we first define

$$\begin{aligned} B_{\alpha\beta} &= g_{\alpha\beta} - g_{\alpha a} g^{ab} g_{b\beta}, \\ D^{ab} &= g^{ab} + (g^{ac} - \rho^{ac}) g_{c\alpha} B^{\alpha\beta} g_{\beta e} g^{eb}, \\ k^\beta_a &= D_{ab}(g^{bc} - \rho^{bc}) g_{c\alpha} B^{\alpha\beta}. \end{aligned}$$

By convention, g^{ab} denotes the inverse of g_{ab}. The control law which gives the closed-loop equations (4) and (5) is

$$u_a = k^\alpha_a \left(\frac{d}{dt}\frac{\partial l}{\partial \eta^\alpha}\right) = k^\alpha_a c^\gamma_{\alpha\beta}\eta^\beta \frac{\partial l}{\partial \eta^\gamma}. \tag{6}$$

Note that the coefficient k^α_a depends on the control parameter ρ_{ab}. Choosing u_a as in (6) gives the closed-loop equations (4) and (5).

Define the controlled momenta

$$\begin{aligned} \tilde{J}_a &= \frac{\partial l_c}{\partial \dot{\theta}^a} = g_{a\alpha}\eta^\alpha + \rho_{ab}\dot{\theta}^b \\ \tilde{M}_\alpha &= \frac{\partial l_c}{\partial \eta^\alpha} = A_{\alpha\beta}\eta^\beta + g_{\alpha a}\rho^{ab}\tilde{J}_b \end{aligned}$$

where

$$A_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha a}\rho^{ab} g_{b\beta}.$$

Written in terms of η^α and $\tilde{J}_a$, the controlled energy takes the block diagonal form

$$E_c(\eta^\alpha, \tilde{J}_a) = \frac{1}{2} A_{\alpha\beta}\eta^\alpha\eta^\beta + \frac{1}{2}\rho^{ab}\tilde{J}_a\tilde{J}_b. \tag{7}$$

This form is useful for studying stability of relative equilibria, as described in Section 3.

3 Including Generalized Forces

Assume that the control law u_a has been chosen according to (6). Furthermore, assume that there is a function $E_{\Phi,\Psi}(\eta^\alpha, \tilde{J}_a) =$

$$\frac{1}{2}A_{\alpha\beta}\eta^\alpha\eta^\beta + \frac{1}{2}\rho^{ab}\tilde{J}_a\tilde{J}_b + \Phi(C^k) + \Psi(\tilde{J}_a). \quad (8)$$

which has a minimum or a maximum at the desired equilibrium (3). One method for generating such a function is the energy-Casimir method, which imposes conditions on the control gains and on the equilibrium values of the first and second partial derivatives of Φ and Ψ. The conditions on the first derivatives ensure that the equilibrium is a critical point of $E_{\Phi,\Psi}$. The conditions on the second derivatives ensure that the equilibrium is a strict minimum or maximum. Simple candidates for Φ and Ψ are functions which are linear and quadratic in their arguments. We let

$$\Phi(C^k) = \varphi_k C^k + \frac{\varphi_{kl}}{2}(C^k - C^k_e)(C^l - C^l_e) \quad (9)$$

and

$$\Psi(\tilde{J}_a) = \frac{1}{2\psi}\rho^{ab}\tilde{J}_a\tilde{J}_b, \quad (10)$$

where the scalar constants φ_k, φ_{kl}, and ψ are chosen to satisfy conditions imposed during the energy-Casimir stability analysis. (Note: If $\dot{\theta}|_e \neq 0$, then Ψ must include a linear term.)

In the absence of damping, the controlled energy E_c, the Casimirs C^k, and the controlled momenta $\tilde{J}_a$ are all conserved. Thus $\frac{d}{dt}E_{\Phi,\Psi} = 0$ and Lyapunov stability follows immediately. More generally, suppose the system is subject to generalized forces F_α and F_a, so that

$$\frac{d}{dt}\frac{\partial l}{\partial \eta^\alpha} = c^\beta_{\alpha\gamma}\eta^\gamma\frac{\partial l}{\partial \eta^\beta} + F_\alpha \quad (11)$$

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{\theta}^a} = u_a + F_a. \quad (12)$$

One may consider F_α as "external" forces and F_a as "internal" forces. In general, the internal forces F_a destroy conservation of E_c and $\tilde{J}_a$. These forces do not affect the total momentum $\tilde{M}_\alpha = M_\alpha$, so the Casimirs C^k are unaffected. The external forces F_α typically destroy all conservation laws. One can show that

$$\begin{aligned}
&\frac{d}{dt}E_{\Phi,\Psi} \\
&= A_{\alpha\beta}\eta^\alpha\dot{\eta}^\beta + \rho^{ab}\tilde{J}_a\dot{\tilde{J}}_b + \frac{\partial\Phi}{\partial C^k}\frac{\partial C^k}{\partial \tilde{M}_\alpha}\dot{\tilde{M}}_\alpha + \frac{\partial\Psi}{\partial\tilde{J}_a}\dot{\tilde{J}}_a \\
&= \left(\eta^\alpha A_{\alpha\beta}B^{\beta\gamma} + \frac{\partial\Phi}{\partial C^k}\frac{\partial C^k}{\partial\tilde{M}_\gamma}\right)F_\gamma - \eta^\alpha g_{\alpha a}D^{ab}F_b \\
&\quad + \left(1 + \frac{1}{\psi}\right)\tilde{J}_a D^{ab}\left(F_b - k^\beta_b F_\beta\right). \quad (13)
\end{aligned}$$

If $F_\alpha = 0$, equation (13) suggests a choice of feedback dissipation F_a that will drive the modified energy to its minimum or maximum value [7]. Unfortunately, $E_{\Phi,\Psi}$ is not generally suitable as a Lyapunov function when $F_\alpha \neq 0$. The quadratic terms in (9) can make $\frac{d}{dt}E_{\Phi,\Psi}$ indefinite, regardless of the choice of feedback dissipation. By omitting the quadratic terms from $\Phi(C^k)$, one might obtain a semidefinite function whose rate could be made semidefinite, with opposite sign, by an appropriate choice of feedback dissipation. Stability could then be studied using LaSalle's invariance principle.

Assumption 3.1 *The truncated function*

$$E_{\bar{\Phi},\Psi} = \frac{1}{2}A_{\alpha\beta}\eta^\alpha\eta^\beta + \frac{1}{2}\rho^{ab}\tilde{J}_a\tilde{J}_b + \varphi_k C^k + \Psi(\tilde{J}_a)$$

has a (non-strict) minimum or maximum at the desired equilibrium (3).

Assumption 3.1 is somewhat restrictive, although it holds for many interesting examples.

Suppose the external force F_α represents physical damping, with a thrust term to balance damping at the desired equilibrium. For simplicity, we assume that the damping is linear in velocity and that any internal damping is cancelled through feedback:

$$\frac{d}{dt}\frac{\partial l}{\partial \eta^\alpha} = c^\beta_{\alpha\gamma}\eta^\gamma\frac{\partial l}{\partial \eta^\beta} - d_{\alpha\beta}(\eta^\beta - \eta^\beta_e)$$

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{\theta}^a} = u_a,$$

where $d_{\alpha\beta}$ is a positive definite tensor. Though we assume linear damping, the results should hold for more general drag models, provided these models reflect some basic properties of physical dissipation (e.g., that drag "opposes velocity").

Under the following assumption, $\frac{d}{dt}E_{\bar{\Phi},\Psi}$ given by (13) (with Φ replaced by $\bar{\Phi}$) can be made quadratic in η^α and $\tilde{J}_a$ through dissipative feedback.

Assumption 3.2

$$C^k = \frac{1}{2}h^{k\alpha\beta}\tilde{M}_\alpha\tilde{M}_\beta \tag{14}$$

where $h^{k\alpha\beta}$ is constant and symmetric in α, β.

We note that Casimirs for a number of physically interesting systems, including the free rigid body, the heavy top, and the underwater vehicle, can be expressed as quadratic forms. Substituting the dissipation model and equation (14) into (13) gives

$$\begin{aligned}\frac{d}{dt}E_{\bar{\Phi},\Psi} &= \left(-g_{\alpha a}\eta^\alpha + \left(1+\frac{1}{\psi}\right)\tilde{J}_a\right)D^{ab}F_b \\ &-\tilde{J}_a X^{a\gamma}d_{\gamma\beta}(\eta^\beta - \eta_e^\beta) - \eta^\alpha Y_\alpha^\gamma d_{\gamma\beta}(\eta^\beta - \eta_e^\beta)\end{aligned}$$

where

$$X^{a\alpha} = \left(\rho^{ab}g_{b\beta}\varphi_k h^{k\beta\alpha} - \left(1+\frac{1}{\psi}\right)D^{ab}k_b^\alpha\right) \tag{15}$$

$$Y_\alpha^\beta = A_{\alpha\gamma}\left(B^{\gamma\beta} + \varphi_k h^{k\gamma\beta}\right). \tag{16}$$

Assumption 3.3 *$d_{\alpha\beta}\eta_e^\beta$ is in the null space of $X^{a\alpha}$ and Y_α^β.*

Assumption (3.3) implies that $\frac{d}{dt}E_{\bar{\Phi},\Psi}$ is indefinite in the direction of the propulsive force.

Theorem 3.4 *The feedback dissipation*

$$F_a = D_{ab}\tilde{d}^{bc}\left(-g_{c\beta}\eta^\beta + \left(1+\frac{1}{\psi}\right)\tilde{J}_c\right), \tag{17}$$

with $\tilde{d}_{ab}$ symmetric and negative (positive) definite, makes $\frac{d}{dt}E_{\bar{\Phi},\Psi} \leq (\geq)0$ provided

$$\begin{aligned}&Y_\alpha^\gamma d_{\gamma\beta} + d_{\alpha\gamma}Y_\beta^\gamma + \left(1+\frac{1}{\psi}\right)^{-2} d_{\alpha\gamma}X^{a\gamma}\tilde{d}_{ab}X^{b\delta}d_{\delta\beta} \\ &+ 2\left(1+\frac{1}{\psi}\right)^{-1}\left(d_{\alpha\gamma}X^{a\gamma}g_{a\beta} + g_{\delta b}X^{b\delta}d_{\delta\beta}\right) \\ &+ 2g_{\alpha a}\tilde{d}^{ab}g_{b\beta} \geq (\leq)\ 0.\end{aligned}$$

The proof follows from linear algebra. Having found feedback dissipation which makes $\frac{d}{dt}E_{\bar{\Phi},\Psi} \leq (\geq)0$, one may conclude that the positive (negative) semidefinite function $E_{\bar{\Phi},\Psi}$ approaches a constant value. LaSalle's invariance principle can be used to show asymptotic stability.

4 Example

The problem of stabilizing steady intermediate axis rotation of a rigid body with a single internal rotor using the method of controlled Lagrangians is discussed in [7] and references therein (see also [5]). Let I_i represent the rigid body principal moments of inertia ($i = 1, 2, 3$), let $J_1 = J_2$ and J_3 represent the rotor principal moments of inertia, and define $\lambda_i = I_i + J_i$. We assume that $\lambda_1 > \lambda_2 > \lambda_3$. Let $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)^T$ be the angular velocity of the carrier and let ϕ be the relative angle of the internal rotor about its spin axis, the body 3-axis. The reduced Lagrangian is the total kinetic energy:

$$l(\mathbf{\Omega}, \dot{\phi}) = \frac{1}{2}\left(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + I_3\Omega_3^2 + J_3(\Omega_3 + \dot{\phi})^2\right).$$

With a control torque u acting on the internal rotor, the equations of motion are

$$\begin{aligned}\frac{d}{dt}\frac{\partial l}{\partial \mathbf{\Omega}} &= -\mathbf{\Omega} \times \frac{\partial l}{\partial \mathbf{\Omega}} \\ \frac{d}{dt}\frac{\partial l}{\partial \dot{\phi}} &= u.\end{aligned}$$

The square magnitude of total angular momentum, $C = \frac{1}{2}\|\frac{\partial l}{\partial \mathbf{\Omega}}\|^2$, is a Casimir for the system.

Let $[\rho_{ab}] = \rho J_3$ where ρ is a dimensionless scalar. (Square brackets denote the matrix form of a tensor.) Applying the control law $u = u_{\mathrm{cL}}$ with

$$u_{\mathrm{cL}} = \left(1 + \frac{\rho}{\rho - 1}\frac{I_3}{J_3}\right)^{-1}(\lambda_1 - \lambda_2)\Omega_1\Omega_2 \tag{18}$$

yields the closed-loop equations

$$\begin{aligned}\frac{d}{dt}\frac{\partial l_c}{\partial \mathbf{\Omega}} &= -\mathbf{\Omega} \times \frac{\partial l_c}{\partial \mathbf{\Omega}} \\ \frac{d}{dt}\frac{\partial l_c}{\partial \dot{\phi}} &= 0,\end{aligned}$$

where the controlled Lagrangian is $l_c(\mathbf{\Omega}, \dot{\phi}) =$

$$\frac{1}{2}\left(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + I_3\Omega_3^2\right) + J_3\Omega_3\dot{\phi} + \frac{1}{2}\rho J_3\dot{\phi}^2.$$

The momentum conjugate to ϕ is the controlled conserved quantity $\tilde{l}_3 = J_3(\Omega_3 + \rho\dot{\phi})$. Define the "controlled inertia"

$$I_{C_3} = I_3 + \frac{\rho - 1}{\rho} J_3.$$

For this example, $[A_{\alpha\beta}] = \mathrm{diag}(\lambda_1, \lambda_2, I_{C_3})$ and, referring to equation (7), the controlled energy is

$$E_c(\boldsymbol{\Omega}, \tilde{l}_3) = \frac{1}{2}\left(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + I_{C_3}\Omega_3^2 + \frac{\tilde{l}_3^2}{\rho J_3}\right).$$

The equilibrium

$$\boldsymbol{\Omega}_e = [\,0,\ \bar{\Omega},\ 0\,]^T, \qquad \dot{\phi}_e = 0, \tag{19}$$

with $\bar{\Omega} \neq 0$, corresponds to steady rotation about the intermediate axis with zero relative rotor velocity. This equilibrium is unstable for the uncontrolled system. The control law (18) can be shown to stabilize (19) for appropriate choices of ρ. Conditions on ρ for stability may be found by applying the energy-Casimir method to

$$E_{\Phi,\Psi}(\boldsymbol{\Omega}, \tilde{l}_3) = E_c(\boldsymbol{\Omega}, \tilde{l}_3) + \Phi(C) + \Psi(\tilde{l}_3).$$

In the absence of dissipation, a sufficient condition for nonlinear stability is [7]

$$0 < \rho < \frac{J_3}{J_3 + I_3}. \tag{20}$$

In this case, a negative definite Lyapunov function for (19) is

$$E_{\Phi,\Psi} = E_c - \frac{1}{\lambda_2}C + \frac{\varphi^{11}}{2}(C - C_e)^2 + \frac{\tilde{l}_3^2}{2\psi\rho J_3},$$

where the constant $\varphi^{11} < 0$ and ψ satisfies

$$\left(1 + \frac{1}{\psi}\right) < \frac{J_3}{\rho\,(\lambda_2 - I_{C_3})}.$$

An appropriate choice of feedback dissipation leads to asymptotic stability [7].

Assume that the rigid body is subject to an external torque $-\boldsymbol{D}(\boldsymbol{\Omega} - \boldsymbol{\Omega}_e)$ where $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, d_3) > 0$,

$$\begin{aligned} \frac{d}{dt}\frac{\partial l}{\partial \boldsymbol{\Omega}} &= -\boldsymbol{\Omega} \times \frac{\partial l}{\partial \boldsymbol{\Omega}} - \boldsymbol{D}(\boldsymbol{\Omega} - \boldsymbol{\Omega}_e) \\ \frac{d}{dt}\frac{\partial l}{\partial \dot{\phi}} &= u. \end{aligned}$$

Choose the new control law

$$u = u_{\mathrm{cL}} + u_{\mathrm{diss}} \tag{21}$$

where u_{cL} is given by (18) and u_{diss} is a dissipative feedback term to be chosen.

To illustrate that $\frac{d}{dt}E_{\Phi,\Psi}$ is indefinite, assume that at an instant $\Omega_1 = \Omega_3 = 0$ and $\tilde{l}_3 = 0$. At that instant, regardless of the choice of u_{diss},

$$\frac{d}{dt}E_{\Phi,\Psi} = \varphi^{11}(C - C_e)(\lambda_2\Omega_2)(-d_2(\Omega_2 - \bar{\Omega})).$$

Since $(C - C_e)(\Omega_2 - \bar{\Omega}) > 0$ at this instant, the sign of $\frac{d}{dt}E_{\Phi,\Psi}$ depends on Ω_2. Thus, $E_{\Phi,\Psi}$ cannot be a Lyapunov function.

Instead, we consider the negative semidefinite function

$$E_{\bar{\Phi},\Psi} = E_c - \frac{1}{\lambda_2}C + \frac{\tilde{l}_3^2}{2\psi\rho J_3}$$

and apply the procedure outlined in Section 3. In accordance with Assumption 3.2, we may write C as a quadratic form where $[h^{1\alpha\beta}]$ is the 3×3 identity matrix. One may verify that Assumption 3.3 is satisfied by computing $X^{a\alpha}$ and Y_α^β according to definitions (15) and (16).

Define the dissipative feedback gain $[\tilde{d}_{ab}] = J_3/\tilde{d}$, where $\tilde{d}$ is a scalar parameter, and choose $u_{\mathrm{diss}} = [F_a]$ given in equation (17). Applying Theorem 3.4, one finds that choosing ψ to satisfy

$$0 < 1 + \frac{1}{\psi} < \min\left(-\frac{I_3}{(\rho - 1)\lambda_2},\ \frac{J_3}{\rho\,(\lambda_2 - I_{C_3})}\right) \tag{22}$$

and choosing $\tilde{d} =$

$$\frac{d_3\left(\frac{\psi+1}{\psi\rho\lambda_2} + \frac{\rho-1}{\rho}\frac{1}{I_3}\right)^2}{\left(-\frac{I_{C_3}}{J_3}\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_2}\right) + 2\left(\frac{1}{\left(1+\frac{1}{\psi}\right)\rho\lambda_2} + \frac{\rho-1}{\rho}\frac{1}{I_3}\right)\right)} \tag{23}$$

makes $\frac{d}{dt}E_{\bar{\Phi},\Psi} \geq 0$.

LaSalle's invariance principle applies to systems with semidefinite Lyapunov functions, although the task of finding a trapping region is not trivial. For this example, one may find a trapping region using a physical argument. Since drag increases with angular velocity and the propulsive torque is constant, $\|\mathbf{\Omega}\|$ is bounded; i.e., there is a "maximum sustainable angular rate" above which the body is slowed by drag. Using the Casimir C, one may define a noncompact, positively invariant region whose boundary is determined by the larger of the initial angular rate and the maximum sustainable angular rate. The intersection of this region and the noncompact, positively invariant region obtained by bounding the value of $E_{\bar{\Phi},\Psi}$ is a global trapping region.

Examining the dynamics on the set where $\frac{d}{dt}E_{\bar{\Phi},\Psi} = 0$, one finds that

$$\lambda_2 \dot{\Omega}_2 = -d_2(\Omega_2 - \bar{\Omega}).$$

The largest invariant set within the set where $\frac{d}{dt}E_{\bar{\Phi},\Psi} = 0$ contains only the desired equilibrium (19). By LaSalle's principle, one concludes that the equilibrium is globally asymptotically stable.

Theorem 4.1 *Consider the control law (21) with* u_{cL} *given by (18) and* $u_{\mathrm{diss}} = [F_a]$ *given by (17), where* $[\tilde{d}_{ab}] = J_3/\tilde{d}$. *If* ρ *satisfies (20) and* $\tilde{d}$ *is given by (23), the equilibrium (19) is globally asymptotically stable.*

5 Conclusions

To be of practical value, stabilization techniques which rely on kinetic energy shaping must account for the effect of physical dissipation. Here we have considered the effect of damping on a class of Euler-Poincaré systems which have been stabilized using the method of controlled Lagrangians. For a sub-class of these systems, we have described a choice of feedback dissipation which can yield asymptotic stability in the presence of generic linear damping. The choice of control law and the proof of asymptotic stability rely on the construction and analysis of a semidefinite Lyapunov function for the conservative, closed-loop system.

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