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The Complex Geometry of Weak Piecewise Smooth Solutions of Integrable Nonlinear PDE's of Shallow Water and Dym Type

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To the 70th birthday of Solomon Alber

Abstract: An extension of the algebraic-geometric method for nonlinear integrable PDE's is shown to lead to new piecewise smooth weak solutions of a class of N-component systems of nonlinear evolution equations. This class includes, among others, equations from the Dym and shallow water equation hierarchies. The main goal of the paper is to give explicit theta-functional expressions for piecewise smooth weak solutions of these nonlinear PDE's, which are associated to nonlinear subvarieties of hyperelliptic Jacobians.

The main results of the present paper are twofold. First, we exhibit some of the special features of integrable PDE's that admit piecewise smooth weak solutions, which make them different from equations whose solutions are globally meromorphic, such as the KdV equation. Second, we blend the techniques of algebraic geometry and weak solutions of PDE's to gain further insight into, and explicit formulas for, piecewise-smooth finite-gap solutions.

The basic technique used to achieve these aims is rather different from earlier papers dealing with peaked solutions. First, profiles of the finite-gap piecewise smooth solutions are linked to certain finite dimensional billiard dynamical systems and ellipsoidal billiards. Second, after reducing the solution of certain finite dimensional Hamiltonian

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systems on Riemann surfaces to the solution of a nonstandard Jacobi inversion problem, this is resolved by introducing new parametrizations.

Amongst other natural consequences of the algebraic-geometric approach, we find finite dimensional integrable Hamiltonian dynamical systems describing the motion of peaks in the finite-gap as well as the limiting (soliton) cases, and solve them exactly. The dynamics of the peaks is also obtained by using Jacobi inversion problems. Finally, we relate our method to the shock wave approach for weak solutions of wave equations by determining jump conditions at the peak location.

Contents

1. Introduction

An important feature of many integrable nonlinear evolution equations is the nature of their soliton solutions. There are many examples of such solutions found in a variety of physical applications, such as nonlinear optics and water wave equations. Nonsmooth soliton solutions of integrable equations are now well known, and include solutions of the shallow water equation (SW) with peaks, the points at which their spatial derivative changes sign (see Camassa and Holm [1993] and Camassa, Holm and Hyman [1994]). It was noted in Alber et al. [1994, 1995, 1999] that the spatial structure of these "peakon" and finite-gap piecewise smooth weak solutions are closely related to finite dimensional integrable billiard systems.

Some history. Camassa and Holm [1993] described classes of n-peakon solutions for an integrable equation in the context of a model for shallow water theory. This work (see also Camassa, Holm and Hyman [1994]) contains many other facts about these equations as well, such as a Hamiltonian derivation of the equation, the associated linear isospectral eigenvalue problem and its discrete spectrum corresponding to the peakons, a steepening lemma important for understanding how solutions lose regularity, numerical stability, etc. Of particular interest to us is their description of the dynamics of the peakons in terms of a finite-dimensional completely integrable Hamiltonian system. In other words, each peakon solution can be associated with a mechanical system of moving particles. Calogero [1995] and Calogero and Francoise [1996] further extended the class of mechanical systems of this type.

It is well-known (see, for example, Ablowitz and Segur [1981]), that solitons and quasi-periodic solutions of most classical integrable equations can be obtained by using the inverse scattering transform (IST) method. This is done by establishing a connection with an isospectral eigenvalue problem for an associated operator that is often a Schrödinger operator. In some cases it involves a potential in the form of an entire function of the spectral parameter. Such an operator is called an *energy-dependent*

Schrödinger operator. The scattering problem for the operators of this type was studied by Jaulent [1972] and Jaulent and Jean [1976].

On the other hand, in connection with certain N-component systems of integrable evolution equations, Antonowicz and Fordy [1989] investigated certain energy dependent *scalar* Schrödinger operators. Using this formalism, they obtained multi-Hamiltonian structures for this class of systems.

Later, Alber et al. [1994, 1995, 1999] showed that in case of certain potentials, a limiting procedure can be applied to generic solutions, which results in solutions with peaks. The latter were related to finite dimensional integrable dynamical systems with reflections and were termed *piecewise-smooth solutions*, a terminology that hereafter we will adopt. This relation provides an efficient route to the study of finite-gap and piecewise soliton solutions of nonlinear PDE's. The approach is based on studying finite dimensional Hamiltonian systems on certain Riemann surfaces and can be used for a number of equations including the shallow water equation, the Dym type equation, as well as certain N-component systems and equations in their hierarchies.

Finite-gap solutions of the Dym equation were studied in Dmitrieva [1993a] and Novikov [1999] by making use of a connection with the KdV equation and with the aid of additional phase functions. Soliton solutions of Dym type equations were studied in Dmitrieva [1993b]. Periodic solutions of the shallow water equation were discussed in McKean and Constantin [1999]. The papers by Beals et al. [1998, 1999, 2000] used Stieltjes' theorem on continued fractions and the classical moment problem for studying multi-peakon solutions of the (SW) equation. Multi-peakon solutions have also been derived in Camassa [2000] by Gram–Schmidt orthogonalization.

The main results of this paper. While our techniques are rather general and can be applied to large classes of N-component systems, we shall illustrate them in detail for two specific integrable PDE's. One of these equations is a member of the Dym hierarchy that has been studied by, amongst others, Kruskal [1975], Cewen [1990], Hunter and Zheng [1994] and Alber et al. [1995, 1999]. Using subscript notation for partial derivatives, this equation is

$$U_{xxt} + 2U_x U_{xx} + U U_{xxx} - 2\kappa U_x = 0. \tag{HD}$$

The other equation, derived from the Euler equations of hydrodynamics in a shallow water framework by Camassa and Holm [1993], is

$$U_t + 3UU_x = U_{xxt} + 2U_x U_{xx} + U U_{xxx} - 2\kappa U_x. \tag{SW}$$

In both equations, the dependent variable $U(x, t)$ may be interpreted as a horizontal fluid velocity and κ is a parameter.

Under appropriate boundary conditions, applying the limit $\kappa \to 0$ to (SW) leads to an equation that has peaked solutions. For equation (HD), such solutions exist also for $\kappa \neq 0$ (for example periodic and finite-gap peaked solutions).

By using the method of generating equations for nonlinear integrable PDE's, we reduce the equations to a Jacobi inversion problem associated with hyperelliptic curves. The solutions $U(x, t)$ themselves are given by trace formulae, i.e., sums of coordinates of points on such curves.

An important feature is that the corresponding Abel–Jacobi mapping is not a standard one. First of all, the holomorphic differentials that are involved do not form a complete set of such differentials on a hyperelliptic curve. Second, it involves a meromorphic differential. As a result, the image of the mapping turns out to be a non-Abelian subvariety (a stratum) of a *generalized* Jacobian. This also implies that the x- and t-flows of (HD) and (SW) are essentially nonlinear, i.e., they are not translationally invariant. Seen from the viewpoint of algebraic geometry, these nonstandard aspects constitute the main difference between shallow water and Dym type equations, and equations of KdV type and more generally equations from the whole KP hierarchy which lead to standard Abel–Jacobi mappings.

The basic technique of the present paper is rather different from earlier papers dealing with peaked solutions. First, profiles of the finite-gap piecewise-smooth solutions are linked to certain finite dimensional billiard dynamical systems and ellipsoidal billiards in the field of Hooke potentials. Second, after reducing the solution of the finite dimensional Hamiltonian systems on Riemann surfaces to the solution of a nonstandard Jacobi inversion problem, it is resolved by introducing new parametrizations.

The philosophy that "justifies" procedures of this sort is that, in the end, by using the trace formulae, *we obtain weak solutions of the PDE's (HD) and (SW) in the spacetime sense*. This is regarded as equivalent to the validity of Hamilton's principle for these PDE's and is taken as a fundamental criterion for the definition of their solutions. It is worth emphasizing that Hamilton's principle naturally leads to weak solutions in the *spacetime* sense (and not in the spatial sense alone). We might also remark that even for billiards, one has to be careful about the sense in which solutions are interpreted. In the case of a point particle bouncing off a wall, for example, the equations of motion themselves do not rigorously make sense at the collision; what does make sense is the fundamental principle of Hamilton. This point of view of course is not new – see, e.g., Young [1969] and Kane et al. [1999].

The contents of the paper. In Sect. 2, basic trace formulae and μ-variable representations are used to establish a connection between solutions of the nonlinear equations and finite dimensional Hamiltonian systems on Riemann surfaces. These representations describe finite-gap and soliton type solutions, as well as mixed soliton–finite-gap solutions. Then, solving the Hamiltonian systems is reduced to Jacobi inversion problems with meromorphic differentials. These inversion problems are solved by introducing a new parameterization that yields a Hamiltonian flow on a nonlinear subvariety of the Jacobi variety. The approach of recurrence chains used in this section is demonstrated in detail in the case of Dym-type equations.

In Sect. 3 the geodesic motion and motion in the field of a Hooke potential on an ellipsoid are linked, at any fixed time t, to finite-gap solutions of (HD) and (SW) equations respectively through trace formulae. In Sect. 4 it is shown how peaked finite-gap solutions of (HD) and (SW) equations arise in the particular limit of smooth solutions. Based on this, a connection to ellipsoidal and hyperbolic billiards is used to construct the *peak* solutions of equations (HD) and (SW) in the form of an infinite sequence of pieces, corresponding to the segments between impacts, glued together along peaks. The motion between impacts in the billiard problems is made linear on generalized Jacobians of hyperelliptic curves.

By solving the corresponding generalized Jacobi inversion problem, we find theta-function solutions to the billiards, which thereby enables us to describe explicit peak solutions for the above PDE. We then extend the analysis from fixed-time peak solutions to time-dependent ones and show that the latter are described by an infinite number of meromorphic pieces in x and t that are glued along peak lines (surfaces) where the solution has discontinuous derivatives in the dependent variables. We give theta-function expressions for the pieces and the peak surfaces. These formulae may be useful for stability analysis as well as for numerical investigations of the perturbed nonlinear PDE's.

In Sect. 5 the Hamiltonian structure for the motion of the peaks of the finite-gap piecewise-solutions is obtained by using algebraic-geometric methods. Lastly, in Sect. 6 we relate our method to the shock wave approach for weak solutions of wave equations by determining jump conditions at the shock location.

2. Finite-Gap Solutions

In this section we will show that even on the level of finite-gap solutions, there are crucial differences between the KdV equation case and equations (HD) or (SW). The same method can be applied to other equations forming the HD and SW hierarchy as well as to N-component systems of nonlinear evolution equations which have associated with them energy dependent Schrödinger operators (see Alber et al. [1997]).

We will start by describing the algebraic geometrical structure of finite-gap solutions of equations (HD) and (SW) related to a hyperelliptic curve of genus n, also called n-gap solutions. The same method can be applied also to the other equations forming the HD and SW hierarchy.

For the HD equation such solutions were obtained in terms of theta-functions by Dmitrieva [1993a] (see also Dmitrieva [1993b]) and Novikov [1999]. For equation (SW) on a circle, the problem was discussed in Constantin and McKean [1999].

Lax pairs and recurrence chains. We now use the recurrence chain approach to develop a basic trace formula which establishes a connection between solutions of equation (HD) and finite dimensional Hamiltonian systems on Riemann surfaces, written in the so-called μ-variables representation. This representation describes finite-gap solutions, as well as their limiting forms of soliton-type. This representation also yields the existence of peakons in a special limiting case. For definiteness, we concentrate here on equation (HD). Analogous results are available in the case of equation (SW) (for details see Alber et al. [1994, 1995]).

The hierarchy of Dym equations is obtained from the Lax equations

$$\frac{\partial}{\partial t_n} L = [L, A_n], \quad n \in \mathbb{N}, \quad L = -\frac{\partial^2}{\partial x^2} + V(E, x, t_n),$$

where the potential $V(E, x, t_n)$ is written in terms of a complex parameter E in the form

$$V(x, t_n, E) = \frac{M(x, t_n)}{2E}, \tag{2.1}$$

for a function $M(x, t_n)$ to be determined below. Assuming $[L, A_n]$ to be a scalar operator, we choose $A_n = B_n \partial_x - \frac{1}{2} B_n'$ for some function $B_n(E, x, t_n)$ and obtain the following sequence of equations for V,

$$\frac{\partial V}{\partial t_n} = -\frac{1}{2} \frac{\partial^3 B_n}{\partial x^3} + 2 \frac{\partial B_n}{\partial x} V + B_n \frac{\partial V}{\partial x}. \tag{2.2}$$

Now we choose B_n to be a polynomial in E of degree n:

$$B_n(x, t, E) = b_0 \prod_{k=1}^{n} (E - \mu_k(x, t)) = \sum_{k=0}^{n} b_{n-k}(x, t)\, E^k. \tag{2.3}$$

Substituting the expressions (2.1) and (2.3) into the generating equation (2.2) and equating like powers of E, we obtain a recurrence chain for coefficients of $B(x,t)$ which yields the n^{th} equation of the Dym hierarchy. For example, putting $t_1 = t$ and choosing $n = 1$, $B_1(x,t,E) = b_0(x,t)E + b_1(x,t)$ yields the following chain

$$\begin{aligned} E^1 &: -b_0^{'''} = 0, \\ E^0 &: -b_1^{'''} + 2b_0' M + b_0 M' = 0, \\ E^{-1} &: 2b_1' M + b_1 M' = \frac{\partial M}{\partial t}. \end{aligned} \tag{2.4}$$

After setting $b_0 = 1$ and using (2.1), we get

$$\begin{aligned} M' &= b_1^{'''}, \\ 2b_1' M + b_1 M' &= \frac{\partial M}{\partial t}. \end{aligned} \tag{2.5}$$

The first equation defines b_1 in terms of M, $M = b_1^{''} + \kappa$, with κ a constant. Renaming $b_1 = -U$, so that

$$M = -U'' + \kappa, \tag{2.6}$$

and putting this into the second equation of the set (2.5) results in equation (HD). (For further details about the hierarchies of (HD) and (SW), see for example Alber et al. [1994, 1995, 1999].) The method of generating equations is due to S. Alber [1979] and another exposition of it can be found in Alber et al. [1985, 1997].

We call (2.2) the “dynamical generating equations”, because it generates a hierarchy of equations governing the dynamics of the dependent variable $M(x,t)$.

Remark. The flows where B_n is a polynomial E, as in the definition (2.3) and in the example above, will in general lead to nonlocal equations, i.e., the evolution equation for M involves terms that depend on nonlocal operators acting on combinations of M and its derivatives. This can be seen, for instance, in Eq. (2.5) where both b_1 and b_1' require inverting (2.6) to write U in terms of M. Thus, flows generated by polynomials B_n in E should be properly classified as integro-differential evolution equations, rather than PDE’s. In contrast, the choice of polynomials in $1/E$ for B_n leads to flows that are local, i.e., M_t only depends on combinations M and its (spatial) derivatives, and these flows are proper PDE’s. This feature of equations of Dym (HD) or shallow water (SW) type is somewhat different from other completely integrable PDE’s like the KdV or Sine-Gordon equation. Equations (HD) and (SW) possess “open ended” hierarchies: the recurrence chain can be extended from negative to positive powers of E, by choosing B_n in (2.2) to be a rational function of the parameter E. The case when the chain includes only negative powers of E is in fact the one most studied in the literature (see, e.g., Dimitrieva [1993a], Novikov [1999] for the case of Dym equation).

Now let us consider the stationary flow for the n^{th} equation of the hierarchy, which is obtained by dropping the time derivative of V in the left-hand side of (2.2). By definition a stationary equation describes a finite-dimensional system for the coefficients of B_n and is equivalent to the 2×2 Lax pair

$$\frac{\partial}{\partial x} W_n(E) = -[W_n(E), \mathcal{L}(E)], \quad \text{or} \quad \left[\frac{\partial}{\partial x} + \mathcal{L}(E), W_n(E)\right] = 0,$$
$$W_n(E) = \begin{pmatrix} -\frac{1}{2}B_n' & B_n \\ -\frac{1}{2}B_n'' + B_n \frac{M}{E} & \frac{1}{2}B_n' \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & 1 \\ \frac{M}{E} & 0 \end{pmatrix}. \tag{2.7}$$

The matrix $W_n(E)$ undergoes an isospectral deformation. Hence the spectral curve

$$\Gamma = \{|W_n(E) - zI| = 0\}$$

is an invariant of the stationary flow. The curve is hyperelliptic and can be represented in the form

$$\Gamma = \{w^2 = \mu C(\mu)\}, \tag{2.8}$$

where $z = wE$ and

$$C(E) = E\left(-B_n'' B_n + \frac{1}{2}B_n'^2\right) + B_n^2 M. \tag{2.9}$$

Since B_n is a polynomial of degree n, $C(E)$ becomes a polynomial of degree (at most) $2n$:

$$C(E) = \sum_{j=0}^{2n} C_j\, E^j = C_{2n} \prod_{k=0}^{2n} (E - m_k), \tag{2.10}$$

for some constants m_k, $k = 1, \ldots, 2n$. In this case the curve Γ has genus n and we set the coefficient C_{2n} to be a negative number: $C_{2n} \equiv -L_0^2$. We shall refer to (2.9) as the *stationary generating equation*.

Equating like-powers of E in both sides of the stationary generating equation yields first integrals

$$\begin{aligned} E^{2n} &: C_{2n} = -b_1'' + M, \\ E^{2n-1} &: C_{2n-1} = -b_1 b_1'' - b_2'' + \frac{1}{2}(b_1')^2 + 2b_1 M, \\ E^j &: \qquad \cdots \\ E^0 &: C_0 = 2b_n^2 M. \end{aligned} \tag{2.11}$$

Let us consider the divisor of points $P_1 = (\mu_1, w_1), \ldots, P_n = (\mu_n, w_n)$ on Γ. Substituting (2.3) into (2.9) and setting $E = \mu_1, \ldots, \mu_n$ successively, one gets the following system of equations describing evolution of the points under the stationary flow:

$$\mu_i' \equiv \frac{\partial \mu_i}{\partial x} = \frac{\sqrt{R(\mu_i)}}{\mu_i \prod_{j\neq i}^{n} (\mu_i - \mu_j)}, \tag{2.12}$$

where

$$R(\mu) = \mu C(\mu) = -L_0^2 \mu \prod_{r=1}^{2n} (\mu - m_r). \tag{2.13}$$

In the case of equation (SW), this should be replaced by

$$R(\mu) = \mu \prod_{r=1}^{2n+1} (\mu - m_r). \tag{2.14}$$

We now proceed to describe finite-gap solutions of equation (HD) and the other equations from its hierarchy. According to a general theory (see, e.g., Dubrovin [1981], Belokolos et al. [1994], for any fixed t, the x-profile of an n-gap solution of an integrable PDE satisfies the n^{th} stationary equation of the hierarchy. Hence, n-gap solutions $M(x, t_k)$ of k^{th} equation of HD hierarchy must satisfy the stationary generating equation (2.9) represented by the Lax pair (2.7), as well as the dynamical generating equation

$$\frac{\partial V}{\partial t_k} = -\frac{1}{2}\frac{\partial^3 B_k}{\partial x^3} + 2\frac{\partial B_k}{\partial x} V + B_k \frac{\partial V}{\partial x}, \quad V = \frac{M(x, t_k)}{2E}, \tag{2.15}$$

where the coefficients of $B_k(E)$ are found recursively. Notice that the latter equation is equivalent to the matrix commutativity relation

$$\left[\frac{\partial}{\partial x} + \mathcal{L}, \frac{\partial}{\partial t_k} + W_k\right] = 0, \tag{2.16}$$

where

$$W_k(E) = \begin{pmatrix} -\frac{1}{2}B_k' & B_k \\ -\frac{1}{2}B_k'' + B_k \frac{M}{E} & \frac{1}{2}B_k' \end{pmatrix}, \tag{2.17}$$

and $\mathcal{L}$ is defined in (2.7). The compatibility of conditions (2.16), and (2.7) leads to the following Lax pair:

$$\frac{\partial}{\partial t_k} W_n(E) = -[W_n(E), W_k(E)], \quad k \in \mathbb{N}, \quad k \neq n. \tag{2.18}$$

For $k = n$, we replace (2.18) with the Lax pair (2.7) thus identifying t_n with x.

The (1,2)-entry of the matrix equation (2.18) implies the following t_k-evolution of the polynomial $B_n(E)$:

$$\frac{\partial B_n}{\partial t_k} = \frac{\partial B_n}{\partial x} B_k - B_n \frac{\partial B_k}{\partial x}, \quad k \neq n. \tag{2.19}$$

In case $k = n$ this relation is replaced by

$$\frac{\partial B}{\partial t_n} = v b_0 \frac{\partial B}{\partial x},$$

where v is a constant, which can always be eliminated by rescaling t_n.

Expanding the right-hand side of (2.19) in E and using the condition that it must be a polynomial of degree $n-1$, we find

$$B_k(E) = \left[\frac{1}{E^{n-k}} B_n(E) \right]_+ , \tag{2.20}$$

where $[\ ]_+$ denotes the polynomial part of the expression.

As follows from the first equation in (2.11), $M = C_{2n} + b_1''$. On the other hand, according to formula (2.3), $b_1 = -\sum_{i=1}^n \mu_i$. Finally, using the definition (2.6) of M in terms of the solution U and integrating twice with respect to x, we obtain

$$U = \sum_{i=1}^{n} \mu_i + \frac{1}{2}\Big(\kappa - C_{2n}\Big)\, x^2 + K_1 x + K_2, \tag{2.21}$$

where K_1 and K_2 are constants of integration. If we assume that all the variables μ_i are bounded, which is related to the choice of sign of the leading order coefficient C_{2n}, then b_1 is a bounded function of x. To find bounded solutions $U(x,t)$ of the PDE, we set

$$C_{2n} = \kappa, \quad \text{and} \quad K_1 = 0.$$

Hence, when the above requirements are imposed, we see that the leading order coefficient of the polynomial $C(E)$ must coincide with the parameter κ of the PDE.

The Dym equation (HD) is invariant under the Galilean transformation

$$\hat{x} = x + K_2 t, \quad \hat{t} = t, \quad \hat{U} = U - K_2,$$

so that the constant K_2 can always be eliminated from expression (2.21). Therefore, under the boundedness conditions above, and up to a Galilean transformation, we assume that the finite-gap and soliton solutions of the Dym equation (HD) is reconstructed in terms of the root variables $\mu' s$ by the "trace" formula which in case of equations (HD) and (SW) have the form

$$U(x,t) = \sum_{i=1}^{n} \mu_i - \mathfrak{m}. \tag{2.22}$$

Here $\mathfrak{m}$ is a constant, which equals zero in the case of equation (HD).

Through (2.22) a solution of the system (2.12) allows to construct the instantaneous profile of $U(x,\cdot)$ from a set of initial conditions $\mu_i(x,\cdot) = \mu_i(0,\cdot) \in [m_{2i}, m_{2i+1}]$, $i = 1, \ldots, n$. Here the "dot" notation stresses the fact that time t is just a parameter in this system.

On the other hand, substitution of (2.3) into (2.19), setting $E = \mu_1, \ldots, \mu_n$ successively, and taking into account expressions (2.12) results in the following t_k-evolution equations for μ_i,

$$\frac{\partial \mu_i}{\partial t_k} = B_k(\mu_i) \frac{\partial \mu_i}{\partial t_x} = B_k(\mu_i) \frac{\sqrt{R(\mu_i)}}{\mu_i \prod_{j\neq i}^{n} (\mu_i - \mu_j)}, \quad i = 1, \ldots, n, \tag{2.23}$$

where, in view of (2.3) and (2.20), for $k = 1, \ldots, n-1$,

$$B_k(\mu_i) = \operatorname{Res}_{s=0} \frac{1}{s^{n-k}} \frac{(s-\mu_1)\cdots(s-\mu_n)}{s-\mu_i},$$

i.e., up to the sign, the k^{th} elementary symmetric function of $\{\mu_1, \dots \mu_n\} \setminus \mu_i$. In the case $k = 1$,

$$\dot{\mu}_i \equiv \frac{\partial \mu_i}{\partial t_1} = \frac{(\mu_i - \Sigma)\sqrt{R(\mu_i)}}{\mu_i \prod_{j \neq i}^{n} (\mu_i - \mu_j)}, \quad i = 1, \dots, n, \quad \Sigma = \mu_1 + \dots + \mu_n, \tag{2.24}$$

the solution of which produces the μ's, and hence the PDE's solution U, at any (later) time t. We notice that for $k > n$, the derivatives $\partial/\partial t_k$ are linear combinations of $\partial/\partial t_1, \dots, \partial/\partial t_n$.

Expressions (2.12), (2.23), and (2.12) provide the so-called *μ-variables representation* for the finite-gap solutions of an evolution equation. They are the analogs of Drach–Dubrovin equations which describe evolution of points on the spectral hyperelliptic curve in the case of the KdV equation. (For further details see Dubrovin [1975], Drach [1919], Alber et al. [1994, 1995, 1999], Gesztesy et al. [1996], and Alber and Fedorov [2001].)

With the initial conditions chosen, the right-hand-side of system (2.12) is real, and the derivative of μ_i changes sign when μ_i reaches the end points of its gap, $\mu_i = m_{2i}$ or $\mu_i = m_{2i+1}$, corresponding to a change of the sheet of the spectral curve Γ. Thus each variable μ undergoes (real) oscillations between the end points of a gap (so that the resulting PDE solution $U(x, t)$ remains real).

Remark. The condition that the root variables μ's are real (or, equivalently, their initial conditions are chosen as described above), while certainly sufficient to assure reality of the PDE's solution U resulting from (2.21), is clearly not necessary (namely, some of the μ's could occur in conjugate pairs). A wider class of real solutions U could be constructed by relaxing the reality assumption on the μ-variables. However, a thorough discussion of the reality condition for U and its implications for the root variables, while certainly desirable, lies beyond the scopes of the present paper, and it will be addressed in future work.

By rearranging and summing up Eqs. (2.12) and (2.24), (2.23), one obtains the following nonstandard Abel–Jacobi equations

$$\sum_{i=1}^{n} \frac{\mu_i^k \, d\mu_i}{\sqrt{R(\mu_i)}} = \begin{cases} dt_k & k = 1, \dots, n-1, \\ x & k = n, \end{cases} \tag{2.25}$$

which contain $(n-1)$ holomorphic differentials and one *meromorphic* differential on Γ. Thus, the number of holomorphic differentials is less than genus of the Riemann surface, which implies that the corresponding inversion problem cannot be solved in terms of meromorphic functions of x and $t_1, \dots, t_{n-1}$ (see e.g., Markushevich [1977]).

Finite-gap stationary flows in x. Let us first consider the x-flow by fixing time variables in (2.25): $t_k = t_k^0$ =const, $k = 1, \dots, n-1$, so that $dt_k = 0$. Now introduce a new spatial variable x_1 defined as follows:

$$x = \int_0^{x_1} \frac{1}{L_0} \, \mu_1 \cdots \mu_n \, dx_1. \tag{2.26}$$

In view of the well-known Jacobi identities

$$\frac{\mu_i^k}{\prod_{j\neq i}^n(\mu_i-\mu_j)} = \begin{cases} 1/(\mu_1\cdots\mu_n) & k=-1, \\ 0 & k=0,\ldots,n-2, \\ 1 & k=n-1, \\ \Sigma & k=n, \end{cases} \tag{2.27}$$

Eqs. (2.12) give rise to the following system:

$$\sum_{i=1}^n \int_{\mu_0}^{\mu_i} \frac{\mu^{k-1}\,d\mu}{\sqrt{R(\mu)}} = \begin{cases} x_1+\phi_1 & k=1, \\ \phi_k & k=2,\ldots,n, \end{cases} \tag{2.28}$$

where $\phi_1,\ldots,\phi_n$ are constant phases which depend on t_k^0 as on parameters.

Equations (2.28) include n holomorphic differentials on Γ and determine the standard Abel–Jacobi map of the symmetric product $\Gamma^{(n)}$ of n copies of Γ to the Jacobi variety (Jacobian) $\mathrm{Jac}(\Gamma)$. Thus, the flow generated by the system (2.12) is made linear on $\mathrm{Jac}(\Gamma)$ after introducing the reparametrization (2.26). By using standard methods (see e.g., Dubrovin [1981] or Mumford [1983]), the map can be inverted, resulting in expressions for algebraic symmetric functions of μ-variables in terms of theta-functions of n arguments which depend linearly on x_1 and, in a transcendental way, on t_k^0 as parameters. Then, by using the trace formula (2.22), one obtains a theta-functional expression for U as a function of x_1, t_k^0, $U=\tilde U(x_1,t_k^0)$.

On the other hand, substituting the theta-functional expression for the product $\mu_1\cdots\mu_n$ into (2.26) yields a quadrature. By solving it, one finds x as a meromorphic function of x_1 which depends on t_0 as a parameter. However, the inverse function $x_1(x,t_0)$ is no longer meromorphic in x.

Finally, the composition function $U(x,t_0)=\tilde U(x_1(x,t_0),t_k^0)$ gives a profile of the finite-gap solutions of the (HD) or (SW) equation (for explicit theta-functional expressions $\tilde U(x_1,t_0)$, $x(x_1,t_0)$ see Alber and Fedorov [2001]). Notice that as seen from (2.26) and (2.28), the original x-flow is also made linear on $\mathrm{Jac}(\Gamma)$. However the straight line motion is not uniform.

The transformation (2.26) involving x and x_1 coincides with a change of variable in the well-known Liouville transformation (see, e.g., Verhulst [1996]).

Finite-gap flows in t_k. Now let us fix the coordinate $x=x_0$ as well as all the times $t_1,\ldots,t_{n-1}$ but t_k. Then introduce a new time variable $\tilde t$ defined by

$$dt_k = \frac{\mu_1\cdots\mu_n}{L_0(\Sigma_{k-1})}d\tilde t, \tag{2.29}$$

where Σ_{k-1} are the elementary symmetric functions of $\mu_1,\ldots,\mu_n$ such that

$$(s-\mu_1)\cdots(s-\mu_n) = s^n + s^{n-1}\Sigma_1+\cdots+s^0\Sigma_n.$$

Applying again the identities (2.27), from (2.24) and (2.23) we arrive at the following canonical Abel–Jacobi mapping

$$\sum_{i=1}^n \int_{\mu_0}^{\mu_i} \frac{\mu^{s-1}\,d\mu}{2\sqrt{R(\mu)}} = \begin{cases} \psi_1=\tilde t+\phi_1 & s=1, \\ \psi_s=\delta_{s,k}t_k+\phi_s & s=2,\ldots,n, \end{cases} \tag{2.30}$$

where $\phi_1, \ldots, \phi_n$ are constant phases which depend on x_0 and the rest of times t_l as on parameters, and δ_{ij} is the Kronecker delta.

As a result of inversion of (2.30), elementary symmetric functions of μ's and therefore the solution of equations (HD) and (SW) can be found in terms of theta-functions of n arguments which depend linearly on ψ_s. This means that the arguments depend linearly on $\tilde{t}$, as well as on the original time t_k. However, $\tilde{t}$ itself depends on t_k in a nonlinear way.

Indeed, to describe the relation between $\tilde{t}$ and t_k, we substitute the theta-functional expressions for the symmetric functions $\Sigma_n = \mu_1 \cdots \mu_n$ and Σ_{k-1} into (2.29). As a result, in contrast to the quadrature (2.26) relating x and x_1, we now get *a differential equation* of the form

$$\frac{dt_k}{d\tilde{t}} = F(t_k, \tilde{t}|x_0),$$

where F is a transcendental function of $t, \tilde{t}$ and the parameter x_0. It can be shown that the equation involves a transcendental integral.

Remarks. 1. In contrast to the x_1- and x-flows considered above, the flows generated by (2.23) (t_k-flows) including (2.24), are *nonlinear* flows on the Jacobi variety $\mathrm{Jac}(\Gamma)$. From the point of view of algebraic geometry, this phenomenon constitutes the main difference between solutions of such well known equations as KdV and sine Gordon equations and equations of (HD) or (SW) type.

2. The problem of inversion of the full nonstandard Abel mapping defined by (2.25) can be also studied by using a generalized Jacobian of the curve Γ. Namely, one has to extend the mapping by including an extra holomorphic differential on Γ to get a complete set of such differentials. As a result of this procedure, one gets a flow on nonlinear subvarieties (strata) of generalized Jacobians. The complete algebraic geometrical description and explicit formulae are presented in Alber and Fedorov [2001].

3. Flows on *n*-Dimensional Quadrics and Stationary *n*-Gap Solutions of the (HD) and (SW) Equations

Consider a family of confocal quadrics in $\mathbb{R}^{n+1} = (X_1, \ldots, X_{n+1})$

$$\tilde{Q}(s) = \left\{ \frac{X_1^2}{a_1 - s} + \cdots + \frac{X_{n+1}^2}{a_{n+1} - s} = 1 \right\}, \quad s \in \mathbb{R}, \quad 0 < a_{n+1} < a_1 < \cdots < a_n. \tag{3.1}$$

The elliptic coordinates $\mu_1, \ldots, \mu_{n+1}$ can be defined in $\mathbb{R}^{n+1}$ in a standard way (see, e.g., Jacobi [1884a]) as follows. The condition $s = c$ determines the quadric $\tilde{Q}(c)$ on which one of the coordinates, say μ_{n+1}, equals c, and the other coordinates $\mu_1, \ldots, \mu_n$ are elliptic coordinates on $\tilde{Q}(c)$ defined by relations

$$X_j^2 = (a_j - c)\frac{\prod_{l=1}^{n}(a_j - \mu_l)}{\prod_{k=1, k\neq j}^{n+1}(a_j - a_k)}, \quad j = 1, \ldots, n+1. \tag{3.2}$$

In the sequel without loss of generality we assume $c = 0$.

It is well-known that the problem of geodesics on the ellipsoid $\tilde{Q} = \tilde{Q}(0)$ is completely integrable (Jacobi [1884 a,b]). Moreover, as noticed by Jacobi himself and later by many other authors (see e.g. Rauch-Wojciechowski [1995]), there exists an infinite sequence of integrable generalizations of the problem describing a motion on $\tilde{Q}$ in the force field of certain polynomial potentials $\mathcal{V}_p(X_1, \ldots, X_{n+1})$, $p \in \mathbb{N}$ of degree $2p$. The simplest integrable potential is the quadratic Hooke potential or the potential of an elastic string joining the center of the ellipsoid $\tilde{Q}$ to the point mass on it:

$$\mathcal{V}_1 = \frac{\sigma}{2}(X_1^2 + \cdots + X_{n+1}^2), \quad \sigma = \text{const.}$$

In this case in terms of the ellipsoidal coordinates, the total energy (Hamiltonian) takes the Stäckel form:

$$H = \frac{1}{8}\sum_{i=1}^{n} \frac{\prod_{j\neq i}(\mu_i - \mu_j)\mu_i}{\Phi(\mu_i)} \left(\frac{d\mu_i}{dx}\right)^2 + \frac{\sigma}{2}\sum_{i=1}^{n} \mu_i + \text{const},$$

where

$$\Phi(\mu) = (\mu - a_1)\cdots(\mu - a_{n+1})$$

and x denotes time. After fixing constants of motion, the system is reduced to the Abel–Jacobi equations

$$\sum_{k=1}^{n} \int_{\mu_0}^{\mu_k} \frac{\mu^i \, d\mu}{2\sqrt{\mathcal{R}(\mu_k)}} = \delta_{in}\, x + \phi_i, \qquad i = 1, \ldots, n, \tag{3.3}$$

$$\mathcal{R}(\mu) = -\mu\Phi(\mu)[c_0(\mu - c_1)\cdots(\mu - c_{n-1}) - \sigma\mu^n], \qquad c_0, \ldots, c_{n-1} = \text{const},$$

where $\phi_1, \ldots, \phi_n$ are constant phases and $c_1, \ldots, c_{n-1}$ are constants of motion.

Notice that for $\sigma = 0$ the order of the polynomial $\mathcal{R}(\mu)$ is $2n+1$, whereas for $\sigma \neq 0$ it is $2n + 2$. The case $\sigma = 0$ corresponds to the free (geodesic) motion on $\tilde{Q}$. c_0 is the constant in the first integral $(\dot{X}, \dot{X})$ and the remaining constants admit a clear geometric interpretation: the tangent line to a geodesic is also tangent to the fixed confocal quadrics $\tilde{Q}(c_1), \ldots, \tilde{Q}(c_{n-1})$ (Chasles theorem).

Now notice that Eqs. (3.3) are equivalent to the system (2.25) with $dt = 0$ describing stationary (HD) and (SW) equations, provided we identify the roots of the polynomial $\mathcal{R}(\mu)$ with those of the odd order polynomial (2.13) (for $\sigma = 0$ and $L_0 = 1$) and of the even order polynomial (2.14) (for $\sigma = 1$) respectively. The equivalence also holds when some of the parameters a_i in (3.3) are negative, which correspond to the motion on a hyperboloid. For concreteness, we shall consider only the case of ellipsoids. Taking into account the trace formula (2.22), we arrive at the following theorem:

Theorem 3.1. *The geodesic motion and motion in the field of a Hooke potential on the ellipsoid $\tilde{Q}$ are linked, at any fixed time t, to the n-gap solutions of (HD) and (SW) equations respectively through the trace formula (2.22). Namely, if the roots of the polynomials $R(\mu)$ in (2.13) or (2.14) coincide with the roots of $\mathcal{R}(\mu)$ in (3.3), the profiles of such solutions are given by the sum of the elliptic coordinates of the moving point on $\tilde{Q}$ with addition of $(-\mathfrak{m})$ in case of equation (SW).*

For the geodesic flow on $\tilde{Q}$ ($\sigma = 0$) and equation (HD), this result was obtained in Alber and Alber [1985], Cewen [1990], and Alber et al. [1995]).

As with Eq. (2.25), under the change of parameter (2.26), Eqs. (3.3) reduce to those containing holomorphic differentials only and having the same structure as (2.28). By inverting the corresponding Abel–Jacobi mapping, one obtains explicit expressions for elementary symmetric functions of μ_i and, in view of (3.2), for the Cartesian coordinates $X_1, \ldots, X_{n+1}$ in terms of theta-functions of the new parameter x_1 (for the case of the geodesic flow, see Weierstrass [1844], Moser[1978], and Knörrer [1982]). In the case $n = 2$, the change of parameter (2.26) was first applied by Weierstrass [1844] to solve the classical Jacobi geodesic problem on a triaxial ellipsoid (Jacobi [1884a,1884b]).

4. Billiard Dynamical Systems and Piecewise-Smooth Weak Solutions of PDE's

In this section it is first shown how peaked finite-gap solutions of (HD) and (SW) equations arise in the limit $m_1 \to 0$, where m_1 is the smallest root of the polynomial $R(E)$ in Eqs. (2.12)–(2.24). Then a connection to ellipsoidal and hyperbolic billiards is established.

Ellipsoidal billiards and generalized Jacobians. Suppose that one of the semi-axes of the ellipsoid $\tilde{Q}$ tends to zero, namely, $a_{n+1} \to 0$. In the limit, $\tilde{Q}$ passes into the interior of $(n-1)$-dimensional ellipsoid

$$Q = \{X_1^2/a_1 + \cdots + X_n^2/a_n = 1\} \in \mathbb{R}^n, \quad \mathbb{R}^n = (X_1, \ldots, X_n).$$

The elliptic coordinates $\mu_1, \ldots, \mu_n$ on $\tilde{Q}$ transform to elliptic coordinates *in* $\mathbb{R}^n$ giving

$$X_j^2 = \frac{\prod_{l=1}^{n}(a_j - \mu_l)}{\prod_{k=1,k\neq j}^{n}(a_j - a_k)}, \quad j = 1, \ldots, n, \tag{4.1}$$

which appear as the corresponding limits of (3.2).

Then the motion on $\tilde{Q}$ gets transformed into *billiard* motion inside the ellipsoid Q. Geodesics on $\tilde{Q}$ pass into straight line segments inside Q, whereas the points of intersection of the geodesics with the plane $\{X_{n+1} = 0\}$ are mapped into *impact points* on Q with elastic reflection. Also, the motion on $\tilde{Q}$ under the Hooke force passes to the motion inside Q under the action of the Hooke force with the potential $\mathcal{V} = \sigma(X_1^2 + \cdots + X_n^2)/2$. However, in contrast to cases $\sigma = 0$ or $\sigma < 0$, for $\sigma > 0$ (an attracting Hooke potential), for the trajectory to reach Q the total energy h must be sufficiently large. Namely, there ought to exist a positive ε such that inside Q the following double inequality holds:

$$h + \sigma(X_1^2 + \cdots + X_n^2)/2 > \varepsilon > 0.$$

Under this condition, the motion on $\tilde{Q}$ transforms to *billiard* motion inside the ellipsoid Q again having impacts and elastic reflections along Q. Thus, we have "an ellipsoidal billiard with the Hooke potential" which is described by the mapping $\mathcal{B} : (\mathbf{x}, \mathbf{v}) \to (\tilde{\mathbf{x}}, \tilde{\mathbf{v}})$, where $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$ are the Cartesian coordinates of a point on Q and the *starting* velocity vector respectively, while $(\tilde{\mathbf{x}}, \tilde{\mathbf{v}})$ are the coordinates and the starting velocity at the next impact point. Following Fedorov [2001], the mapping has the form

$$\begin{aligned} \tilde{\mathbf{x}} &= \frac{-1}{\nu}\,[(\sigma - (\mathbf{v}, a^{-1}\mathbf{v}))\mathbf{x} + 2(\mathbf{x}, a^{-1}\mathbf{v})\mathbf{v}], \\ \tilde{\mathbf{v}} &= \frac{-1}{\nu}\,[(\sigma - (\mathbf{v}, a^{-1}\mathbf{v}))\mathbf{v} - 2\sigma(\mathbf{x}, a^{-1}\mathbf{v})\mathbf{x}] + \varrho a^{-1}\tilde{\mathbf{x}} \\ &= \frac{-1}{\nu}\,[(\sigma - (\mathbf{v}, a^{-1}\mathbf{v}))(\mathbf{v} + \varrho a^{-1}\mathbf{x}) + 2(\mathbf{x}, a^{-1}\mathbf{v})(\varrho a^{-1}\mathbf{v} - \sigma\mathbf{x})], \end{aligned} \tag{4.2}$$

$$\nu = \sqrt{4\sigma(\mathbf{x}, a^{-1}\mathbf{v})^2 + (\sigma - (\mathbf{v}, a^{-1}\mathbf{v}))^2}, \quad \varrho = \frac{2(\tilde{\mathbf{v}}, a^{-1}\tilde{\mathbf{x}})}{(\tilde{\mathbf{x}}, a^{-2}\tilde{\mathbf{x}})}.$$

Notice that in the limit $\sigma \to 0$ this reduces to a standard billiard mapping given in Veselov [1988]

$$\tilde{\mathbf{x}} = \mathbf{x} - \frac{2(\mathbf{x}, a^{-1}\mathbf{v})}{(\mathbf{v}, a^{-1}\mathbf{v})}\,\mathbf{v}, \quad \tilde{\mathbf{v}} = \mathbf{v} + \frac{2(\tilde{\mathbf{v}}, a^{-1}\tilde{\mathbf{x}})}{(\tilde{\mathbf{x}}, a^{-2}\tilde{\mathbf{x}})}\,a^{-1}\tilde{\mathbf{x}}.$$

The mapping (4.2), as well as the billiard limits of the motion on $\tilde{Q}$ with the higher order potentials $\mathcal{V}_p(X_1, \ldots, X_n, X_{n+1})$ $(X_{n+1} = 0)$ are completely integrable.

In the limit $a_{n+1} \to 0$ and after using the change of variable (2.26), the Abel–Jacobi equations (3.3) are transformed as follows:

$$\begin{aligned} \sum_{k=1}^{n} \int_{\mu_0}^{\mu_k} \frac{\mu^{i-1}\,d\mu}{2\sqrt{\rho(\mu)}} &= \phi_i = \text{const}, \quad i = 1, \ldots, n-1, \\ \sum_{k=1}^{n} \int_{\mu_0}^{\mu_k} \frac{d\mu}{2\mu\sqrt{\rho(\mu)}} &= x_1 + \phi_n, \end{aligned} \tag{4.3}$$

$$\rho(\mu) = -(\mu - a_1)\cdots(\mu - a_n)\,[c_0(\mu - c_1)\cdots(\mu - c_{n-1}) - \sigma\mu^n].$$

This system contains $n-1$ holomorphic differentials on the Riemann surface $\mathcal{C} = \{w^2 = \rho(\mu)\}$ of genus $g = n-1$ and one differential of the third kind having a pair of simple poles $\mathcal{Q}_-, \mathcal{Q}_+$ on $\mathcal{C}$ with $\mu(\mathcal{Q}_\pm) = 0$. Here again $\phi_1, \ldots, \phi_n$ are constant phases and $c_0, \ldots, c_{n-1}$ are constants of motion. The elliptic coordinates $\mu_1, \ldots, \mu_n$ represent the divisor of n points $P_i = (\mu_i, w_i)$ on $\mathcal{C}$.

Equations (4.3) describe a well defined mapping of the symmetric product $\mathcal{C}^{(g+1)}$ to $\mathrm{Jac}(\mathcal{C}, \mathcal{Q}_-, \mathcal{Q}_+)$, the $(g+1)$-dimensional *generalized Jacobian* of the curve $\mathcal{C}$ with two distinguished points $\mathcal{Q}_\pm$. The later is obtained from the genus n curve $w^2 = \mathcal{R}(\mu)$ in (3.3) as a result of confluence of two Weierstrass points $(a_{n+1} \to 0)$ and regularization: cutting out the double point and gluing $\mathcal{Q}_-, \mathcal{Q}_+$.

The generalized Jacobian is a noncompact algebraic variety which is topologically equivalent to the product of the customary g-dimensional Jacobian variety $\mathrm{Jac}(\mathcal{C})$ with complex angle coordinates $\phi_1, \ldots, \phi_g$ and the cylinder $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (for the definition and description of generalized Jacobians see, among others, Serre [1959], Previato [1985], Gavrilov [1999], and Fedorov [1999]).

As follows from (4.3), the geodesic and the potential billiard motion parameterized by x_1 is represented by a straight line flow on $\mathrm{Jac}(\mathcal{C}, \mathcal{Q}_-, \mathcal{Q}_+)$, which is directed along the real section of $\mathbb{C}^*$ and leaves the coordinates ϕ on $\mathrm{Jac}(\mathcal{C})$ invariant.

As we shall see below, the solutions to the generalized inversion problem (4.3) have different structures, depending on whether $R(\mu)$ is an even or an odd order polynomial.

Solutions in terms of generalized theta-functions. First we concentrate on straight line billiards corresponding to the case $\sigma = 0$ when the curve $\mathcal{C}$ has one infinite point ∞. Fix a canonical basis of cycles $A_1, \dots, A_g, B_1, \dots, B_g$ on $\mathcal{C}$ and let $\bar{\omega}_1, \dots, \bar{\omega}_g$ be the dual basis of *normalized* holomorphic differentials on $\mathcal{C}$ and $z_1, \dots, z_g$ be corresponding coordinates on the universal covering of $\mathrm{Jac}(\mathcal{C})$. There exists a unique $g \times g$ constant normalizing matrix D such that

$$\bar{\omega}_k = \sum_{j=1}^{g} \frac{D_{kj}\, \mu^{j-1}\, d\mu}{\sqrt{\rho(\mu)}}, \quad z_k = \sum_{j=1}^{g} D_{kj}\phi_j, \quad k = 1, \dots, g = n-1. \tag{4.4}$$

Let us also introduce a *normalized* differential of the third kind Ω_0 having simple poles at $\mathcal{Q}_\pm$ with residues ± 1 respectively:

$$\Omega_0 = \frac{\sqrt{\rho(0)}\, d\mu}{\mu\sqrt{\rho(\mu)}} + \sum_{k=1}^{g} \beta_k \bar{\omega}_k, \quad \sqrt{\rho(0)} = \sqrt{a_1 \cdots a_n \cdot c_1 \cdots c_{n-1}}, \tag{4.5}$$

where β_k are unique constants such that Ω_0 has zero A-periods on $\mathcal{C}$. Then the last equation in (4.3) can be represented in the following form:

$$\sum_{k=1}^{n} \int_{\mu_0}^{\mu_k} \Omega_0 = Z, \quad Z = 2\sqrt{\rho(0)}x_1 + \text{const}. \tag{4.6}$$

Notice that in case of the ellipsoidal billiards $\sqrt{R(0)}$ is always real and hence Z is also real. Let us also choose the base point (μ_0, w_0) of the mapping (4.3) to be an infinite point $\infty \in \mathcal{C}$. According to Fedorov [1999], the solution of the problem of inversion (4.3) together with (4.1) yields the following expressions for the Cartesian coordinates X_i of the point moving inside the ellipsoid Q:

$$X_i(x_1, z) = \kappa_i \frac{e^{-Z/2}\theta[\Delta + \eta_{(i)}](z - q/2) + e^{Z/2}\theta[\Delta + \eta_{(i)}](z + q/2)}{e^{-Z/2}\theta[\Delta](z - q/2) + e^{Z/2}\theta[\Delta](z + q/2)}, \tag{4.7}$$

$$i = 1, \dots, n, \quad z = (z_1, \dots, z_{n-1})^T, \quad Z = 2\sqrt{R(0)}x_1 + Z_0,$$

$$z, Z_0 = \text{const}, \quad q = 2\left(\int_\infty^{\mathcal{Q}_+} \bar{\omega}_1, \dots, \int_\infty^{\mathcal{Q}_+} \bar{\omega}_g\right)^T \in \mathbb{C}^g,$$

$$\kappa_i = \text{const}.$$

These expressions involve quotients of generalized theta-functions, where $\theta[\Delta+\eta_{(i)}](z)$ and $\theta[\Delta](z)$ are customary theta-functions associated with the Riemann surface $\mathcal{C}$ with appropriately chosen half-integer theta-characteristics $\eta_{(i)}$ (Δ is the half-integer theta-characteristic corresponding to the vector of Riemann's constants). The vector q coincides with the vector of B-periods of the meromorphic differential Ω_0. The constant factors κ_i depend on the parameters of the curve $\mathcal{C}$ only. (For the definition and properties of the generalized theta-functions see e.g., Belokolos et al. [1994], Gagnon et al. [1992], Ercolani [1987], and Fedorov [1999].)

The expressions (4.7) describe a straight line segment in $\mathbb{R}^n$ ($\mathbb{C}^n$) with z playing a role of a constant phase vector which defines the position of the segment. When one of the μ-variables, say μ_1, equals zero, the corresponding point $P_1 = (\mu_1, \sqrt{R(\mu_1)})$ on the curve $\mathcal{C}$ coincides with one of the poles $\mathcal{Q}_-$, $\mathcal{Q}_+$ of the differential Ω_0. Then, as follows from the mapping (4.3) and (4.6), x_1 and Z become infinite. On the other hand, in view of (4.1), at this moment the moving point in $\mathbb{R}^n$ meets an ellipsoid Q.

It follows that as x_1 and Z change from $-\infty$ to ∞ along the real axis, the expressions (4.7) have finite limits, giving the coordinates of two subsequent impact points on Q. Notice that $X_i(\infty, z)$ have the same values as $X_i(-\infty, z+q)$. Hence the next segment of the billiard trajectory is given by (4.7) with z being replaced by $z+q$. This yields the following algebraic-geometrical description of the billiard motion (see also Fedorov [1999]).

Theorem 4.1. *As the point mass inside Q approaches the ellipsoid, the point P_1 on $\mathcal{C}$ tends to the pole $\mathcal{Q}_+$. At the moment of impact, P_1 jumps from $\mathcal{Q}_+$ back to $\mathcal{Q}_-$, whereas the phase vector z is increased by q defined in formulas (4.7). The process repeats itself for each impact.*

Using this property and by applying induction, from (4.7) the coordinates of the whole sequence of impact points are found in the form

$$\mathbf{x}_i(N) = \kappa_i \frac{\theta[\Delta + \eta_{(i)}](z_0 + Nq)}{\theta[\Delta](z_0 + Nq)}, \quad i = 1, \ldots, n, \tag{4.8}$$

where $N \in \mathbb{N}$ is the number of impacts and the phase vector $z_0 = (z_{10}, \ldots, z_{g0})^T$ is the same for all the segments of the billiard trajectory.

These expressions depend on customary theta-functions only and, as functions of z_0, are meromorphic on a covering of the Jacobian variety of $\mathcal{C}$. They have also been obtained by Veselov [1988] by using a factorization of matrix polynomials (see also Moser and Veselov [1991]). The work of Veselov is closely related to the discretization of mechanics that preserves the integrable structure. The numerical implementation of Veselov's procedures was given in Wendlandt and Marsden [1997], a discrete reduction procedure in Marsden, Pekarsky and Shkoller [1999], Bobenko and Suris [1999] and an extension to PDE's in Marsden, Patrick and Shkoller [1999].

The generalized Abel map (4.3) yields expressions in terms of generalized theta-functions for the elementary symmetric functions of the variables μ. In particular, following Fedorov [1999], one obtains

$$\begin{aligned} \mu_1 \cdots \mu_n &= \partial_{x_1} \partial_V \log \tilde{\theta}[\Delta](z, Z) \\ &= 2\sqrt{\rho(0)}\, \partial_Z \frac{e^{-Z/2} \partial_V \theta[\Delta](z - q/2) + e^{Z/2} \partial_V \theta[\Delta](z + q/2)}{e^{-Z/2} \theta[\Delta](z - q/2) + e^{Z/2} \theta[\Delta](z + q/2)}, \end{aligned} \tag{4.9}$$

where

$$\tilde{\theta}[\Delta](z, Z) = e^{-Z/2} \theta[\Delta](z - q/2) + e^{Z/2} \theta[\Delta](z + q/2), \tag{4.10}$$

$$Z = 2\sqrt{\rho(0)} x_1 + Z_0, \quad \partial_V = V_1 \frac{\partial}{\partial z_1} + \cdots + V_n \frac{\partial}{\partial z_n},$$

and where V is the last column of the normalizing matrix D defined in (4.4): $V = (D_{1g}, \ldots, D_{gg})^T$. The phases z and Z_0 are the same as in (4.7). As follows from (4.9), for $x_1, Z \to \pm\infty$, the product $\mu_1 \cdots \mu_n$ tends to zero, as expected. Taking the integral (2.26) with $L_0 = 1$ yields

$$\begin{aligned} x(x_1, z) &= \int \mu_1 \cdots \mu_n \, dx_1 = \partial_V \log \tilde{\theta}(z, Z) + \text{const} \\ &= \frac{e^{-Z/2} \partial_V \theta[\Delta](z - q/2) + e^{Z/2} \partial_V \theta[\Delta](z + q/2)}{e^{-Z/2} \theta[\Delta](z - q/2) + e^{Z/2} \theta[\Delta](z + q/2)} + \text{const}. \end{aligned} \tag{4.11}$$

It follows from this expression that the original parameter x has finite values as $x_1 \to \pm\infty$ and $x(\infty, z)$ has the same value as $x(-\infty, z + q)$. Now, substituting in (4.11) $Z = -\infty$, $Z = \infty$, by induction, we find the length of the N^{th} segment of the billiard trajectory in the form

$$x(N) - x(N-1) = \frac{\partial_V \theta[\Delta](z_0 + Nq)}{\theta[\Delta](z_0 + Nq)} - \frac{\partial_V \theta[\Delta](z_0 + Nq - q)}{\theta[\Delta](z_0 + Nq - q)}, \quad N \in \mathbb{N} \tag{4.12}$$

z_0 being the same as in (4.8).

As a result, the solution $X_i(x)$, $x \in \mathbb{R}$, of the *continuous* geodesic billiard problem should be viewed as consisting of an infinite number of pieces each parameterized by $x_1 \in (-\infty, \infty)$ and given by (4.7) and (4.11). These pieces are obtained by iteratively adding vector q to the phase z in (4.7) and (4.11) and they are glued together at the impact points corresponding to $x_1 = \pm\infty$.

Now we turn to the ellipsoidal billiard with the Hooke potential ($\sigma = 1$). In this case the curve $\mathcal{C}$ appearing in (4.3) has 2 infinite points at $\pm\infty$. We again introduce normalized differentials $\bar{\omega}_k$, Ω_0, and coordinates z_k, Z according to (4.4) and (4.6). Let the base point of the mapping (4.3) be one of the Weierstrass points of $\mathcal{C}$, say $\mu_0 = a_n$. Then, instead of (4.7), the inversion of the generalized mapping (4.3) yields the following expressions for the *squares* of the Cartesian coordinates of the mass point moving inside an ellipsoid Q:

$$X_i^2(x_1, z) = \kappa_i' \frac{\tilde{\theta}^2[\Delta + \eta_{(i)}](z, Z)}{\tilde{\theta}[\Delta](z - \hat{q}/2, Z - \hat{S}/2)\, \tilde{\theta}[\Delta](z + \hat{q}/2, Z + \hat{S}/2)}, \tag{4.13}$$

$$\begin{aligned} & i = 1, \ldots, n, \\ & z = (z_1, \ldots, z_{n-1})^T, \quad Z = 2\sqrt{\rho(0)} x_1 + Z_0, \quad z, Z_0 = \text{const}, \\ & q = \int_{Q_-}^{Q_+} (\bar{\omega}_1, \ldots, \bar{\omega}_g)^T, \quad \hat{q} = 2 \int_{a_n}^{\infty_+} (\bar{\omega}_1, \ldots, \bar{\omega}_g)^T, \\ & \hat{S} = \int_{\infty_-}^{\infty_+} \Omega_0, \end{aligned}$$

where $\tilde{\theta}[\Delta](z, Z)$ is defined in (4.10) and

$$\tilde{\theta}[\Delta + \eta_{(i)}](z, Z) = e^{-Z/2} \theta[\Delta + \eta_{(i)}](z - q/2) + e^{Z/2} \theta[\Delta + \eta_{(i)}](z + q/2) \tag{4.14}$$

Here κ_i' are constants, and $\sqrt{\rho(0)}$ is the same as in (4.5). Similarly to (4.7), as x_1 and Z pass from $-\infty$ to ∞, $X_i^2(x_1, z)$ tend to finite values resulting in the squares of the coordinates of subsequent impact points on Q. Thus, expressions (4.13) describe a segment of trajectory of the billiard in the field of the Hooke potential between two impacts. After each impact the phase vector z changes according to Theorem 4.1. Then, by using induction, the sequence of impact points is described as follows:

$$\mathbf{x}_i^2(N) = \kappa_i' \frac{\theta^2[\Delta + \eta_{(i)}](z_0 + Nq)}{\theta[\Delta](z_0 - \hat{q} + Nq)\, \theta[\Delta](z_0 + \hat{q} + Nq)}, \quad i = 1, \dots, n, \qquad N \in \mathbb{N}, \quad z_0 = (z_{10}, \dots, z_{g0})^T = \text{const}. \tag{4.15}$$

Apparently, this theta-functional solution for the billiard with the Hooke potential was not previously known. Lastly, we find the following expression for x

$$x(x_1, z) = \text{const} + \log \frac{\tilde{\theta}[\Delta](z - \hat{q}/2, Z - \hat{S}/2)}{\tilde{\theta}[\Delta](z + \hat{q}/2, Z + \hat{S}/2)}, \quad Z = 2\sqrt{\rho(0)}x_1 + \text{const}, \tag{4.16}$$

which, for $x_1 \to \pm\infty$ and $Z \to \pm\infty$, has finite limits determining x for two subsequent impacts. Then, using the expression (4.10), by induction, we express a x-interval between the impacts in terms of the customary theta-function:

$$x(N) - x(N-1) = \log \frac{\theta[\Delta](z_0 - \hat{q}/2 + Nq)}{\theta[\Delta](z_0 + \hat{q}/2 + Nq)} - \log \frac{\theta[\Delta](z_0 - \hat{q}/2 + Nq - q)}{\theta[\Delta](z_0 + \hat{q}/2 + Nq - q)} - \log \hat{S}. \tag{4.17}$$

We emphasize that, in contrast to the geodesic billiard, for the billiard in the potential field the "time" x is not proportional to the length of a trajectory.

Stationary finite-gap peaked solutions. Now we return to the finite-gap solutions of equations (HD) and (SW). Notice that under the limit $m_1 \to 0$ the mapping (2.28) takes the form (4.3) with $\rho(\mu)$ being a polynomial of degree $2n-1$ and $2n$ respectively.

The trace formula (2.22) and relations (4.1) yield

$$U = \sum_{j=1}^{n} X_j^2 + \sum_{i=1}^{n} a_i + \mathfrak{m}.$$

Then solution to the billiard problems (4.7)–(4.17) provide solutions $U(x, t_0)$ for the above equations which consist of infinite sequences of smooth pieces each one corresponding to a segment between two impacts. The impacts themselves give peaks of $U(x, t_0)$. This leads to the following theorem.

Theorem 4.2. 1) *At any fixed time $t = t_0$, finite-gap peaked solution of the equation (HD) consists of an infinite number of pieces $U_N(x, t_0)$, $N \in \mathbb{Z}$ glued at peak points. Let $\rho(\mu)$ be any polynomial with distinct roots $a_1, \dots, a_n$. Then, for any N, every piece is given by the following pair of theta-functional expressions parameterized by $x_1 \in \mathbb{R}$,*

$$U_N = \sum_{j=1}^{n} X_j^2(x_1, z_N) + \sum_{i=1}^{n} a_i, \tag{4.18}$$

$$x(x_1, z) = \frac{e^{-Z/2}\partial_V \theta[\Delta](z_N - q/2) + e^{Z/2}\partial_V \theta[\Delta](z_N + q/2)}{e^{-Z/2}\theta[\Delta](z_N - q/2) + e^{Z/2}\theta[\Delta](z_N + q/2)} + x_0, \tag{4.19}$$

$$z_N = z_0 + Nq \in \mathbb{C}^{n-1}, \quad Z = 2\sqrt{\rho(0)}x_1 + Z_0,$$

where $X_j^2(x_1, z)$ and q are given by (4.7) and z_0, Z_0, x_0 are constant phases of the solution depending on t_0, which are the same for any piece. The length of the N^{th} piece equals

$$\frac{\partial_V \theta[\Delta](z_0 + Nq)}{\theta[\Delta](z_0 + Nq)} - \frac{\partial_V \theta[\Delta](z_0 + Nq - q)}{\theta[\Delta](z_0 + Nq - q)}. \tag{4.20}$$

2) *At any fixed time $t = t_0$ finite-gap peaked solution to equation (SW) consists of an infinite number of pieces $U_N(x, t_0)$, $N \in \mathbb{Z}$ which are glued at peak points. The pieces are given in the following parametric form*

$$U_N = \sum_{j=1}^{n} X_j^2(x_1, z_N) + \sum_{i=1}^{n} a_i + \mathfrak{m}, \quad z_N = z_0 + Nq \in \mathbb{C}^{n-1}, \tag{4.21}$$

$$x(x_1, t_0) = \log \frac{\tilde{\theta}[\Delta](z_N - \hat{q}/2, Z - \hat{S}/2)}{\tilde{\theta}[\Delta](z_N + \hat{q}/2, Z + \hat{S}/2)} + x_0, \quad Z = 2\sqrt{\rho(0)}x_1 + Z_0, \tag{4.22}$$

where $X_j^2(x_1, z)$ are given by (4.13) and z_0, Z_0, x_0 are constant phases which depend on t_0. The x-length of N^{th} piece equals

$$\log \frac{\theta[\Delta](z_0 - \hat{q}/2 + Nq)}{\theta[\Delta](z_0 + \hat{q}/2 + Nq)} - \log \frac{\theta[\Delta](z_0 - \hat{q}/2 + Nq - q)}{\theta[\Delta](z_0 + \hat{q}/2 + Nq - q)} - \log \hat{S}. \tag{4.23}$$

When in the polynomials (2.13) or (2.14) $m_1 = 0$ and m_2 tends to zero, the distance between subsequent peaks of a profile tends to zero and in the limit the peaks coalesce. (Notice that this is done for a fixed t.) The solution $U(x, t_0)$ for this limiting case is smooth.

Remark. It is known (see, for instance, Fedorov [1999]) that there are special degenerate umbilic billiard solutions of the classical billiard problem (without a potential) that have straight line segments meeting $n - 1$ fixed focal conics of Q between any subsequent impacts and, as $x \to \pm\infty$, the billiard motion converges to simple oscillations along the largest axis of the ellipsoid. This corresponds to the confluence of the roots of the polynomial $\rho(\mu)$ in (4.3),

$$c_1 = a_1, \quad \ldots, \quad c_{n-1} = a_{n-1}.$$

As a result, the hyperelliptic curve $\mathcal{C}$ becomes singular of arithmetic genus zero and the asymptotic billiard motion is described in terms of tau-functions. The corresponding asymptotic peaked solutions of equations (HD) and (SW) are given in Alber and Fedorov [2001].

Time-dependent piecewise-meromorphic solutions. Now we pass to global algebraic geometrical description of the finite-gap peaked solutions. After setting $m_1 \to 0$, the system (2.25) is formally reduced to the following Abel–Jacobi mapping:

$$\int_{\mu_0}^{\mu_1} \frac{\mu^{k-1}\, d\mu}{2\sqrt{\rho(\mu)}} + \cdots + \int_{\mu_0}^{\mu_n} \frac{\mu^{k-1}\, d\mu}{2\sqrt{\rho(\mu)}} = \begin{cases} t_k + \phi_k & k = 1, \ldots, n-1, \\ x + \phi_n & k = n, \end{cases} \tag{4.24}$$

where

$$\rho(\mu) = -L_0^2 \prod_{r=2}^{2n} (\mu - m_r) \quad \text{and} \quad \rho(\mu) = \prod_{r=2}^{2n+1} (\mu - m_r)$$

in the case of equations (HD) or (SW) respectively. Here $\phi_1, \dots, \phi_n$ are constant phases. This system contains $n-1$ independent holomorphic differentials defined on the genus $g = n - 1$ Riemann surface $\{w^2 = \rho(\mu)\}$, which can be identified with the curve $\mathcal{C}$ described above. However, in contrast to the system (4.3), in the case of a polynomial $\rho(\mu)$ of odd order which corresponds to equation (HD), the last equation in (4.24) contains a meromorphic differential of the *second* kind having a double pole at the infinite point ∞ on $\mathcal{C}$. In case of a polynomial $\rho(\mu)$ of even order corresponding to equation (SW), the last equation includes a meromorphic differential of the *third* kind with a pair of simple poles at the infinite points ∞_-, ∞_+ on $\mathcal{C}$.

According to Clebsch and Gordon [1866] and Gavrilov [1999], in the odd order case, such a system describes a well defined and invertible mapping of the symmetric product $\mathcal{C}^{(g+1)}$ to $\mathrm{Jac}(\mathcal{C}, \infty)$, the generalized Jacobian of the curve $\mathcal{C}$ with *one* distinguished point at ∞. The set $\mathrm{Jac}(\mathcal{C}, \infty)$ is a noncompact algebraic variety which is topologically equivalent to the product $\mathrm{Jac}(\mathcal{C}) \times \mathbb{C}$. To describe this case we introduce a normalized differential of second kind having a double pole at ∞,

$$\Omega_\infty^{(1)} = \frac{\sqrt{-1} L_0 \mu^g \, d\mu}{2\sqrt{\rho(\mu)}} + \sum_{k=1}^{g} d_k \bar{\omega}_k, \quad g = n - 1, \tag{4.25}$$

where $\bar{\omega}_k$ are the normalized holomorphic differentials specified in (4.4), d_k are normalizing constants such that all A-periods of $\Omega_\infty^{(1)}$ on $\mathcal{C}$ are zeros. Then the last equation in (4.24) implies that

$$\sum_{i=1}^{n} \int_{\mu_0}^{\mu_i} \Omega_\infty^{(1)} = Z, \quad Z = \sqrt{-1} L_0 x + (d, Dt) + \text{const} \tag{4.26}$$

$$d = (d_1, \dots, d_{n-1})^T, \quad t = (t_n, \dots, t_2)^T,$$

where D is an $(n-1) \times (n-1)$ normalizing matrix defined in (4.11).

Since ∞ now is a pole of $\Omega_\infty^{(1)}$, we choose the basepoint $P_0 = (\mu_0, w_0)$ to be a *finite* Weierstrass point on $\mathcal{C}$. For concreteness we choose $P_0 = (m_{2n}, 0)$. Applying the residue theorem to the generalized theta-function associated with $\mathrm{Jac}(\mathcal{C}, \infty)$ we solve the inversion problem (4.24) and find the following expression:

$$\sum_{i=1}^{n} \mu_i = C_1 - Z^2 + \frac{2Z\partial_V \theta[\Delta + \eta_{2n}](z) - \partial_V^2 \theta[\Delta + \eta_{2n}](z)}{\theta[\Delta + \eta_{2n}](z)}, \tag{4.27}$$

$$Z = \sqrt{-1} L_0 x + (d, Dt) + Z_0, \quad z = Dt + z_0 \in \mathbb{C}^{n-1},$$

$$Z_0, z_0 = \text{const}, \quad C_1 = \sum_{k=1}^{g} \oint_{A_k} \mu \, \bar{\omega}_k + m_{2n},$$

where the half-integer characteristic η_{2n} labels the point $(m_{2n}, 0)$, the vector $V = (D_{1g}, \dots, D_{gg})^T$ is specified in (4.4), and the constant C_1 contains the sum of integrals along the canonical cycles $A_1, \dots, A_g$ on $\mathcal{C}$. Notice that in the above formula

$\partial_V = \partial_{t_2}$. Expression (4.27) is meromorphic in x and $t_1, \dots, t_{n-1}$ and can be regarded as a generalization of the Matveev–Its formula to the case of the noncompact variety $\mathrm{Jac}(\mathcal{C}, \infty)$.

In the case of an even order curve $\mathcal{C}$, corresponding to finite-gap peaked solutions of equation (SW), system (4.24) defines a mapping of the symmetric product $\mathcal{C}^{(g+1)}$ to the generalized Jacobian $\mathrm{Jac}(\mathcal{C}, \infty_\pm)$ which is topologically equivalent to the product $\mathrm{Jac}(\mathcal{C}) \times \mathbb{C}^*$. As above, we set P_0 to be the *last* Weierstrass point $(m_{2n+1}, 0)$ and introduce the normalized differential of the third kind having a pair of simple poles at $\infty_-, \infty_+ \in \mathcal{C}$, as well as the corresponding variable Z:

$$\Omega_{\infty_\pm} = \frac{\mu^g \, d\mu}{2\sqrt{\rho(\mu)}} + \sum_{k=1}^{g} \bar{d}_k \bar{\omega}_k, \quad Z = \sum_{i=1}^{n} \int_{m_{2n+1}}^{\mu_i} \Omega_{\infty_\pm}, \tag{4.28}$$

where $(\bar{d}_1, \dots, \bar{d}_g) = \bar{d}$ are chosen such that all the A-periods of $\Omega_{\infty_\pm}$ are zeros. Then, applying the residue theorem to the generalized theta-function associated with the $\mathrm{Jac}(\mathcal{C}, \infty_\pm)$ yields

$$\sum_{i=1}^{n} \mu_i + \mathfrak{m} = \text{const} - \frac{e^{-Z}\theta[\Delta](z - \hat{q}) + e^{Z}\theta[\Delta](z + \hat{q})}{\theta[\Delta](z)}, \tag{4.29}$$

where, in view of (4.28),

$$\begin{gathered} Z = x + (\bar{d}, Dt) + Z_0, \quad z = Dt + z_0 \in \mathbb{C}^g, \\ \hat{q} = \left(\int_{\infty_-}^{\infty_+} \bar{\omega}_1, \dots, \int_{\infty_-}^{\infty_+} \bar{\omega}_g \right)^T \in \mathbb{C}^g, \quad Z_0, z_0 = \text{const}. \end{gathered} \tag{4.30}$$

Remark. According to the formula (2.22), expressions (4.27) and (4.29) describe formal solutions to equations (HD) and (SW) respectively. However, while treating these solutions, one needs to take into account the reflection phenomenon described in Theorem 4.1. Namely, when a certain variable μ_i passes zero, the point $P_i = (\mu_i, \sqrt{\rho(\mu_i)})$ jumps from one sheet of the Riemann surface $\mathcal{C}$ to another or, in other words, from the pole $\mathcal{Q}_+$ of the differential of the third kind Ω_0 to another pole $\mathcal{Q}_-$. Therefore, the above expressions do not provide global solutions to the equations. Instead, the following theorem holds.

Theorem 4.3. 1) *The time-dependent finite-gap peaked solution $U(x,t)$ of (HD) consists of an infinite number of pieces in $\mathbb{R}^n = (t_1, \dots, t_{n-1}, x)$ described by meromorphic functions*

$$\begin{aligned} U_N(x,t) &= C_1 - Z_N^2 + \frac{2Z_N \partial_V \theta[\Delta + \eta_{2n}](z_N) - \partial_V^2 \theta[\Delta + \eta_{2n}](z_N)}{\theta[\Delta + \eta_{2n}](z_N)}, \quad N \in \mathbb{Z}, \\ & z_N = D\mathbf{t} + Nq + z_0, \quad Z_N = \sqrt{-1} L_0 x + (d, z_N) + Nh + Z_0, \\ & Z_0, z_0 = \text{const}, \quad \mathbf{t} = (t_1, \dots, t_{n-1}), \\ & h = \int_{\mathcal{Q}_-}^{\mathcal{Q}_+} \Omega_\infty^{(1)}, \quad q = \left(\int_{\mathcal{Q}_-}^{\mathcal{Q}_+} \bar{\omega}_1, \dots, \int_{\mathcal{Q}_-}^{\mathcal{Q}_+} \bar{\omega}_g \right)^T, \end{aligned} \tag{4.31}$$

where C_1 is the constant specified in (4.27).

For a fixed N the corresponding piece $U_N(x,t)$ is bounded by nonintersecting surfaces $\mathcal{S}_{N-1}$ and $\mathcal{S}_N$ in $\mathbb{R}^n$ given by equations

$$\begin{aligned} &\mathcal{S}_N = \{x = p_N(t)\}, \\ &p_N(t) = \frac{1}{\sqrt{-1}L_0}\left(\partial_V \log\theta[\Delta+\eta_{2n}](z_N+q/2) - (d, z_N) - Nh\right). \end{aligned} \tag{4.32}$$

The adjacent pieces $U_N(x,t)$ and $U_{N+1}(x,t)$ are thus glued to each other along $\mathcal{S}_N$, where

$$U(p_N(t),t) = C_1 - \partial_V^2 \log\theta[\Delta+\eta_{2n}](z_N+q/2). \tag{4.33}$$

2) *The finite-gap peaked solution $U(x,t)$ of (SW) consists of an infinite number of pieces given by meromorphic functions*

$$\begin{aligned} U_N(x,t) &= \text{const} - \frac{e^{-Z_N}\theta[\Delta](z_N-\hat{q}) + e^{Z_N}\theta[\Delta](z_N+\hat{q})}{\theta[\Delta](z_N)}, \quad N\in\mathbb{Z}, \\ &z_N = Dt + qN + z_0, \quad Z_N = x + (\bar{d}, z_N) + N\bar{h} + Z_0, \\ &t = (t_n,\dots,t_2), \quad Z_0, z_0 = \text{const}, \quad \bar{h} = \int_{\mathcal{Q}_-}^{\mathcal{Q}_+} \Omega_{\infty_\pm}, \end{aligned} \tag{4.34}$$

where the vector $\hat{q}$ is described in (4.30). The piece $U_N(x,t)$ is bounded by peak surfaces $\bar{\mathcal{S}}_{N-1}$ and $\bar{\mathcal{S}}_N$ defined as follows:

$$\bar{\mathcal{S}}_N = \{x = \bar{p}_N(t)\}, \quad \bar{p}_N(t) = \text{const} - \log\frac{\theta[\Delta](z_N-\hat{q}+q/2)}{\theta[\Delta](z_N+\hat{q}+q/2)}. \tag{4.35}$$

The adjacent pieces $U_N(x,t)$ and $U_{N+1}(x,t)$ are glued together along $\bar{\mathcal{S}}_N$, where

$$U(p_N(t),t) = \text{const} - \partial_V \log\frac{\theta[\Delta](z_N-\hat{q})}{\theta[\Delta](z_N+\hat{q})}. \tag{4.36}$$

Notice that along the peak surfaces, the solutions described in 1) and 2) have discontinuous partial derivatives with respect to x and $t_1,\dots,t_{n-1}$.

Remark. By fixing all the times but t_k in the above expressions, one obtains 2-dimensional piecewise solutions $U_N(x,t_k)$, whereas the corresponding sections of $\mathcal{S}_N, \bar{\mathcal{S}}_N \subset (x,t) = \mathbb{R}^n$ describe peak lines in (x,t_k)-plane. As follows from (4.32) and (4.35), the motion of the N^{th} peak $p_N(t_k)$ along the x-axis is described by a sum of a linear function in t_k and a quasi-periodic one. The latter function becomes periodic in the case $g=1$.

Finally, after fixing all the times without exception, expressions (4.31) and (4.34) provide pieces of the stationary finite-gap peaked solution already described in Theorem 4.2.

Proof of Theorem 4.3. According to Theorem 4.2, the profiles of finite-gap peaked solutions are associated with geodesic ellipsoidal billiards and billiards in the field of a Hooke potential. An impact point on the boundary of a billiard trajectory corresponds to a peak of the profile $U(x,t_0)$, and this happens when one of the μ_i passes zero. Hence, the solution (4.27) is valid until one of the points $P_1,\dots,P_n$ on $\mathcal{C}$ coincides with $\mathcal{Q}_-$ or $\mathcal{Q}_+$, the poles of the differential Ω_0 in (4.5). Putting, for example, $P_n \equiv \mathcal{Q}_+$ ($\mu_n \equiv 0$)

in (4.24), one arrives at the following relations involving the normalized differentials defined in (4.4) and (4.25):

$$\sum_{i=1}^{g} \int_{P_0}^{\mu_i} \bar{\omega}_k = z_k - q_k/2, \quad k = 1, \dots, g, \tag{4.37}$$

$$\sum_{i=1}^{g} \int_{P_0}^{\mu_i} \left(\Omega_\infty^{(1)} - \sum_{k=1}^{g} d_k \bar{\omega}_k \right) = \sqrt{-1} L_0 x - \int_{P_0}^{\mathcal{Q}_+} \left(\Omega_\infty^{(1)} - \sum_{k=1}^{g} d_k \bar{\omega}_k \right), \tag{4.38}$$

where $P_0 = (m_{2n}, 0)$. Notice that Eqs. (4.37) form a closed system for the variables $\mu_1, \dots, \mu_{n-1}$ and describe the standard Abel–Jacobi mapping $\mathcal{C}^{(g)} \to \mathrm{Jac}(\mathcal{C})$. Hence, the first symmetric polynomial has the following standard form in terms of theta-functions in the odd order case:

$$\mu_1 + \cdots + \mu_{n-1} = c_1 - \partial_V^2 \log \theta[\Delta + \eta_{2n}](z - q/2), \quad c_1 = \sum_{k=1}^{g} \oint_{A_k} \bar{\omega}_k. \tag{4.39}$$

On the other hand, Eq. (4.38) implies that at a peak point the coordinate x becomes a function of z and therefore of t: $x = p_0(t)$. In the odd order case, this equation contains a sum of Abelian integrals of the second kind, the so-called *Abelian transcendent*. By making use of the following standard expression for the normalized transcendent (Clebsch and Gordon [1866])

$$\sum_{i=1}^{g} \int_{\mu_0}^{\mu_i} \Omega_\infty^{(1)} = -\partial_V \log \theta[\Delta + \eta_{2n}] \left(\sum_{i=1}^{g} \int_{P_0}^{\mu_i} \bar{\omega}_k \right),$$

from (4.37) and (4.38) we find

$$p_0(t) = \frac{1}{\sqrt{-1} L_0} \left(\partial_V \log \theta[\Delta + \eta_{2n}](z - q/2) - (d, z) + h/2 \right), \quad h = \int_{\mathcal{Q}_-}^{\mathcal{Q}_+} \Omega_\infty^{(1)}. \tag{4.40}$$

Using the trace formula for the solution $U(p_0(t), t) = \mu_1 + \cdots + \mu_{n-1}$ and expression (4.39) it follows that the equation $x = p_0(t)$ determines a surface $\mathcal{S}_0$ in $\mathbb{C}^n$ along which the solution U has a peak.

Now setting in (4.24) $P_n \equiv \mathcal{Q}_-$ and taking into account (4.4), (4.25) and

$$\int_{P_0}^{\mathcal{Q}_-} \Omega_\infty^{(1)} = -\int_{P_0}^{\mathcal{Q}_+} \Omega_\infty^{(1)}, \quad \int_{P_0}^{\mathcal{Q}_-} \bar{\omega} = -\int_{P_0}^{\mathcal{Q}_+} \bar{\omega}$$

we obtain an expression for another peak surface $\mathcal{S}_1$ determined by the equation $\{x = p_1(t)\}$ with

$$p_1(t) = \frac{1}{\sqrt{-1} L_0} \left(\partial_V \log \theta[\Delta + \eta_{2n}](z + q/2) - (d, z) - h/2 \right), \tag{4.41}$$

along which

$$U(p_1(t), t) = \mu_1 + \cdots + \mu_{n-1} = C_1 - \partial_V^2 \log \theta[\Delta + \eta_{2n}](z + q/2). \tag{4.42}$$

Under the reality condition, the surfaces $\mathcal{S}_0$ and $\mathcal{S}_1$ do not intersect and therefore determine a connected domain in $\mathbb{C}^n = (x, t)$ where the solution (4.27) is applicable. We denote this piece of solution as $U_1(x, t)$. As follows from (4.40) and (4.41) $\mathcal{S}_1$ is obtained from $\mathcal{S}_0$ by changing the phase as follows:

$$Z \to Z + h, \quad z \to z + q \quad \text{that is} \quad \begin{aligned} & x \to x + \frac{1}{\sqrt{-1}L_0}(h - (d, Dt)), \\ & t \to t + D^{-1}q. \end{aligned} \tag{4.43}$$

In addition, according to (4.42) and (4.39) at any two points on $\mathcal{S}_0$ and $\mathcal{S}_1$ which are equivalent modulo the shift, $U_1(x, t)$ takes the same values:

$$U_1(q_1(t), t) = U_1\left(q_0(t) + \frac{1}{\sqrt{-1}L_0}(h - (d, Dt)), t + D^{-1}q\right). \tag{4.44}$$

Now let us define the function $U_2(x, t) = U_1(x + (h - (d, Dt))/(\sqrt{-1}L_0), t + D^{-1}q)$, which is also a local solution to (HD). In view of (4.44), U_1 and U_2 take the same values along $\mathcal{S}_1$, which ensures a correct gluing of two pieces together. By using iteration with respect to both positive and negative $N's$, we construct a complete sequence of peak surfaces and obtain formulae given in part 1) of the theorem.

Similarly, solution (4.29) of (SW) is valid until one of the points $P_1, \dots, P_n$ on $\mathcal{C}$ coincides with $\mathcal{Q}_-$ or $\mathcal{Q}_+$, the poles of Ω_0. Setting $P_n \equiv \mathcal{Q}_+$ in (4.24) for the case of an even order curve $\mathcal{C}$, and using (4.4) and (4.28) yields

$$\sum_{i=1}^{n-1} \int_{P_0}^{\mu_i} \bar{\omega}_k = z_k - q_k/2, \quad k = 1, \dots, n-1, \tag{4.45}$$

$$\sum_{i=1}^{n-1} \int_{P_0}^{\mu_i} \left(\Omega_{\infty_\pm} - \sum_{k=1}^{n-1} \bar{d}_k \bar{\omega}_k\right) = x - \int_{P_0}^{\mathcal{Q}_+} \left(\Omega_{\infty_\pm} - \sum_{k=1}^{n-1} \bar{d}_k \bar{\omega}_k\right), \tag{4.46}$$

where $P_0 = (m_{2n+1}, 0)$. Inverting (4.45) results in the following expression for a symmetric polynomial (see e.g., Clebsch and Gordon [1866])

$$\mu_1 + \dots + \mu_{n-1} = \text{const} - \partial_V \log \frac{\theta[\Delta](z - \hat{q} - q/2)}{\theta[\Delta](z + \hat{q} - q/2)}. \tag{4.47}$$

After applying the theta-functional formula for the normalized transcendent of the third kind (Clebsch and Gordon [1866]),

$$\sum_{i=1}^{g} \int_{P_0}^{\mu_i} \Omega_{\infty_\pm} = \text{const} - \log \frac{\theta[\Delta](s - \hat{q})}{\theta[\Delta](s + \hat{q})}, \quad s = \sum_{i=1}^{g} \int_{P_0}^{\mu_i} \bar{\omega}, \quad g = n - 1,$$

from (4.46) and (4.45) we obtain

$$x = p_0(t) = \text{const} - \log \frac{\theta[\Delta](z - \hat{q} + q/2)}{\theta[\Delta](z + \hat{q} + q/2)} - (z, \bar{d}) + \bar{h}/2. \tag{4.48}$$

By choosing $P_n \equiv \mathcal{Q}_+$ in (4.24), one arrives at the expressions (4.47) and (4.48) with $q/2, \bar{h}/2$ replaced by $-q/2, -\bar{h}/2$. Then, following similar arguments and applying induction, the piecewise solution of part 2) is constructed.

We emphasize that although the different pieces $U_N(x,t)$ of the solution are obtained by iterative shifting the phases z, Z by the same vector, the pieces $U_N(x,t_0)$ of the solution (t_0 being fixed) are all distinct because the shift occurs in both x- and t-directions.

Remark. If we omit the reality condition above, then the hypersurfaces $\mathcal{S}_\mathcal{N}$, $\bar{\mathcal{S}}_N$ in $\mathbb{C}^n$ intersect, bounding a set of n-dimensional domains adjacent to each other in a rather complicated manner. Then the procedure of gluing different pieces of the functions $U_N(x,t)$ meromorphic inside each domain cannot be defined uniquely. As a result, the generic complex solution $U(x,t)$ branches along the peak surfaces.

5. Kinematics of Peaks

Now we obtain expressions for the velocity of the N^{th} peak $p_N(t)$ of the piecewise solution of (HD) with respect to time t_k. As was shown above, the solution has a peak when one of the μ-variables passes zero implying that $P_n = \mathcal{Q}_-$ or $P_n = \mathcal{Q}_+$.

Theorem 5.1. *Let $y_1, \ldots, y_{n-1}$ denote the μ-coordinates of the points $P_1, \ldots, P_{n-1}$ at the moment in time when one of the μ-variables passes zero. The following system of equations holds:*

$$\frac{\partial p_N(t)}{\partial t_k} = -\Sigma_{k-1}(y_1, \ldots, y_{n-1}), \tag{5.1}$$

where Σ_k is k^{th} the symmetric function of $y_1, \ldots, y_{n-1}$. In particular, we have

$$\frac{\partial p_N(t)}{\partial t_2} = y_1 + \cdots + y_{n-1} = U(p_N(t), t), \tag{5.2}$$

i.e., the t_2-velocity of the peak coincides with its height.

Proof. After applying limit $m_1 \to 0$, Eqs. (2.12) and (2.23) for the derivatives of μ_n take the form

$$\frac{\partial \mu_n}{\partial x} = \frac{\sqrt{\rho(\mu_n)}}{(\mu_n - \mu_1)\cdots(\mu_n - \mu_{n-1})}, \tag{5.3}$$

$$\frac{\partial \mu_n}{\partial t_k} = \Sigma_{k-1}(\mu_1, \ldots, \mu_{n-1}) \frac{\sqrt{\rho(\mu_n)}}{(\mu_n - \mu_1)\cdots(\mu_n - \mu_{n-1})}. \tag{5.4}$$

On the other hand, along the peak line $\{x = p_N(t_k)\}$, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\mu_n(p_N(t_k), t_k) \equiv \frac{\partial \mu_n}{\partial x}\frac{\mathrm{d}\, p_N(t_k)}{\mathrm{d}\, t_k} + \frac{\partial \mu_n}{\partial t_k} = 0,$$

which, in view of (5.3) and (5.4) and after setting $\mu_n \equiv 0$, yields

$$\frac{\sqrt{\rho(0)}}{\mu_1 \cdots \mu_{n-1}} \left(\frac{\partial \mu_n}{\partial t_k} + \Sigma_{k-1}(y_1, \ldots, y_{n-1}) \right) = 0.$$

Since $\rho(0) \neq 0$ and $\mu_1 = y_1, \ldots, \mu_{n-1} = y_{n-1}$ are finite, the latter relation gives (5.1).

Remark. The relations (5.1) can be also found by using direct differentiation of the expression for the N^{th} peak surface (4.35) with respect to t_k. Namely, putting without loss of generality $N = 0$, and taking into account $\partial_V = \partial_{t_2}$, we write

$$\frac{\partial p_0(t)}{\partial t_k} = \partial_{t_k}\partial_{t_2} \log\theta[\Delta + \eta_{2n}]\left(\sum_{i=1}^{n-1}\int_{P_0}^{\mu_i} \bar{\omega}\right). \tag{5.5}$$

According to Mumford [1983], in case of odd order hyperelliptic curves, this gives a theta-functional expression for the coefficient in front of λ^{n-k} in the polynomial $(\lambda - \mu_1)\cdots(\lambda - \mu_g)$ which coincides with $\Sigma_{k-1}(y_1, \ldots, y_{n-1})$.

6. The Dynamics of Peaks and Weak Solutions

Expression (5.2) states that for equations in the hierarchies of (HD) or (SW), every peak in the solution profile moves with velocity determined by the local value of the solution. In this section, we derive this property without recourse to tools related to the complete integrability of the evolution equation. Thus, this property of peak motion can hold in general for equations that admit piecewise-smooth weak solutions, with jumps in the first spatial derivative at isolated points in the solution's support. In this case, the derivative discontinuity can be viewed as a "shock" in the appropriate weak form of the evolution equation.

We will take the weak form of the equation (HD) or (SW) to be

$$\int_{\Omega} \nabla\phi(x,t)\cdot \mathbf{V}(x,t)\; dx\; dt = 0, \tag{6.1}$$

where the equality is satisfied for all test functions $\phi(x,t)$ is C^∞ with compact support in a domain Ω in the (x,t) plane. Here $\nabla\phi = (\phi_t, \phi_x)$, the dot denotes the $\mathbb{R}^2$ inner product, and the vector function $\mathbf{V}(x,t) = (V_1, V_2)$ is defined by

$$\begin{aligned} V_1 &= U_x, \\ V_2 &= \partial_x\left[\frac{1}{2}U^2 - \frac{1}{4}\int_{-\infty}^{\infty} |x-y|\;(U_y^2 - 2\kappa U)\; dy\right], \end{aligned} \tag{6.2}$$

for equation (HD) and

$$\begin{aligned} V_1 &= U_x, \\ V_2 &= \partial_x\left[\frac{1}{2}U^2 + \frac{1}{4}\int_{-\infty}^{\infty} e^{-|x-y|}\;\left(2U^2 + U_y^2 - 2\kappa U\right)\; dy\right], \end{aligned} \tag{6.3}$$

for equation (SW), respectively. We will look for jump conditions satisfied by the solutions of Eq. (6.1). If the jump discontinuities are isolated, by adjusting the support of the test functions $\phi(x,t)$ we only need to consider the case of a single discontinuity.

Let us suppose that the function $U(x,t)$ is infinitely differentiable almost everywhere in Ω, except along the curve $x = q(t)$ where the first derivative U_x has a discontinuity. If we partition the domain Ω into $\Omega = \Omega_1 \cup \Omega_2$ by cutting along the portion of the discontinuity curve $x - q(t) = 0$ in Ω, the divergence theorem and the choice of test functions $\phi(x,t)$ vanishing on the boundary $\partial\Omega$ allow us to write Eq. (6.1) as

$$
\begin{aligned}
0 = \int\limits_{\Omega} dx\; dt\; \nabla\phi\; \cdot \mathbf{V} = \int\limits_{\Omega_1} dx\; dt\; \phi\, \nabla \cdot \mathbf{V} \\
+ \int\limits_{\Omega_2} dx\; dt\; \phi\, \nabla \cdot \mathbf{V} + \int\limits_{\partial\Omega_1 \cap \partial\Omega_2} dl\, \phi\, \mathbf{n} \cdot [\mathbf{V}]_-^+ .
\end{aligned}
\tag{6.4}
$$

Here the unit vector $\mathbf{n}$ is directed along the normal $[-\dot{q}, 1]$ to the discontinuity curve $\partial\Omega_1 \cap \partial\Omega_2$ in Ω, and $[\mathbf{V}]_-^+$ denotes the jump of the vector $\mathbf{V}$ across this curve,

$$
[\mathbf{V}]_-^+ \equiv \lim_{x \to q(t)^+} \mathbf{V}(x,t) - \lim_{x \to q(t)^-} \mathbf{V}(x,t).
$$

By the arbitrariness of $\phi(x,t)$, each integrand term on the right-hand side of (6.4) has to vanish separately. Thus, from the first two terms,

$$
\nabla \cdot \mathbf{V} = 0, \quad \text{or} \quad \frac{\partial V_1}{\partial t} + \frac{\partial V_2}{\partial x} = 0, \tag{6.5}
$$

in Ω_1 or Ω_2, where $U(x.t)$ is smooth. This smoothness and zero divergence condition, by the definition (6.2) or (6.3) for (HD) or (SW) respectively, imply that $U(x,t)$ is a solution of these equations in Ω_1 or Ω_2. For instance, (6.5) becomes

$$
U_{xt} + \partial_{xx} \left[\frac{1}{2} U^2 - \frac{1}{4} \int_{-\infty}^{\infty} |x-y|\; (U_y^2 - 2\kappa U)\; dy \right] = 0,
$$

which is the integrated form of the Harry-Dym equation (HD).

The last (jump) condition in (6.4), $\mathbf{n} \cdot [\mathbf{V}]_-^+ = 0$ along $\partial\Omega$, implies

$$
\dot{q}[V_1]_-^+ = [V_2]_-^+ . \tag{6.6}
$$

The left-hand side of this expression is simply

$$
\dot{q}[V_1]_-^+ = \dot{q}[U_x]_-^+ . \tag{6.7}
$$

As to the right-hand side, the second (integral) term in the definitions (6.2) or (6.3) of $V_2(x,t)$ is a continuous function of x, as the integral wipes out the discontinuity $\operatorname{sgn}(x-y)$ as well as additional ones that U_y^2 might have. Hence the integral terms do not contribute to the right hand side of (6.6). The jump of $V_2(x,t)$ across the discontinuity curve $x = q(t)$ then reduces to

$$
[V_2]_-^+ = \frac{1}{2} \left[\left(U^2 \right)_x \right]_-^+ = U(q,t)\, [U_x]_-^+ . \tag{6.8}
$$

If

$$
[U_x]_-^+ \equiv U_x(q^+, t) - U_x(q^-, t) \neq 0,
$$

Eqs. (6.7) and (6.8) yield

$$
\dot{q} = U(q,t), \tag{6.9}
$$

i.e., the location of the discontinuity (shock) in the U_x moves at the local speed $U(q,t)$. We have then proved the following

Theorem 6.1. *Let $U(x,t)$ be a solution of Eq. (6.1), with the vector $\mathbf{V}(x,t)$ defined in terms of $U(x,t)$ by the nonlinear, nonlocal operators (6.2) and (6.3) respectively for equations (HD) and (SW). Let $U(x,t)$ be a smooth function of (x,t) in the domain $\Omega \subseteq \mathbb{R}^2$, except along the curve $x = q(t)$, where U is continuous while the first derivative U_x has a jump discontinuity (peak) $U(q^+,t) \neq U(q^-,t)$. Then $U(x,t)$ is a solution of equations (HD) and (SW) in each domain Ω_1 and Ω_2 in which the curve $x = q(t)$ partitions Ω, and the location of the peak $q(t)$ moves with velocity equal to its height, $\dot{q} = U(q,t)$.*

Conclusions. In this paper, profiles of the weak finite-gap piecewise-smooth solutions of the integrable nonlinear equations of shallow water and Dym type are linked to billiard dynamical systems and geodesic flows with reflections described in terms of finite dimensional Hamiltonian systems on Riemann surfaces. After reducing the solution of these systems to that of a nonstandard Jacobi inversion problem, solutions are found by introducing new parametrizations. The extension of the algebraic-geometric method for nonlinear integrable PDE's given in this paper leads to a description of piecewise-smooth weak solutions of a class of N-component systems of nonlinear evolution equations and its associated energy dependent Schrödinger operators.

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