

Stabilization of the Unicycle with Rider

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Abstract

In this paper we discuss stabilization of a nonholonomic system consisting of a unicycle with rider. We show in particular that one can achieve stability of slow steady vertical motions by imposing a feedback control force on the rider's limb.

1 Introduction

In this paper we study the stabilization problem for a model of a rider on a unicycle. We use some of the ideas discussed in [10], where various techniques for studying the stability of motion of nonholonomic mechanical systems are analyzed. In particular, energy-based methods are considered as well as use of the so-called Lyapunov-Malkin theorem.

We model the rider on a unicycle by a double pendulum on a wheel, the two pendula representing the body and the limb of the rider. This leads to complicated but tractable equations. See [11] for details. To achieve stabilization of slow vertical steady state motions of the unicycle we then apply linear control to the pendulum representing the limb of the rider. The stability of the overall system is tested using the Lyapunov-Malkin theorem. This theorem, which enables us to conclude overall nonlinear stability using partial spectral information about the system, turns out to be particularly useful for the analysis of nonholonomic systems (see

[5, 8, 10]). Fast motions may also be stabilized and are in fact easier to handle because of the stabilizing effect of the wheel velocity.

While the analysis here is quite nontrivial in itself we intend to extend it both to more complex nonholonomic/robotic systems and to more complicated nonlinear control techniques; for example, the matching control technique discussed in [2, 3].

2 Modeling the Unicycle with Rider

We discuss here the dynamics of a homogeneous disk on a horizontal plane with a double pendulum attached. The upper pendulum is free to move in the plane orthogonal to the disk while the lower pendulum stays “vertical” in the disk's plane. We view this as a simple model of a rider on a unicycle. The configuration space with coordinates $(\theta, \varkappa, \psi, \phi, x, y)$ is $S^1 \times S^1 \times S^1 \times SE(2)$. See Figure 1 for details. This mechanical system is

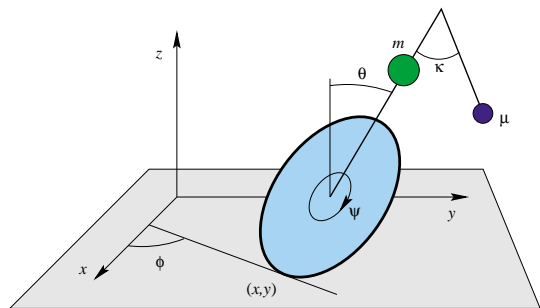


Figure 1: The disk on the horizontal plane.

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$SO(2) \times SE(2)$ -invariant. The action by the group element (α, β, a, b) on the configuration space is given by $(\theta, \varkappa, \psi, \phi, x, y) \mapsto (\theta, \varkappa, \psi + \alpha, \phi + \beta, x \cos \beta - y \sin \beta + a, x \sin \beta + y \cos \beta + b)$.

To stabilize slow vertical steady state motions of the unicycle with rider moving along a straight line, we introduce a single linear control

$$u = k_1\theta + k_2\kappa + k_3\dot{\theta} + k_4\dot{\kappa}$$

with gain parameters (k_1, k_2, k_3, k_4) into the upper pendulum. One can think of this as a controlled limb of the rider. We remark that fast steady state motions of the unicycle without rider do not require stabilization (see [10]). It is this fact that makes fast motions of the unicycle with rider easier to stabilize than slow motions.

3 Feedback Stabilization

Using results of [10], we find the special local coordinates (r, s) in the neighborhood of an upright steady state motion of the unicycle in which the dynamics of the reduced system is governed by the equations

$$\dot{r} = A_k r + R(r, s), \quad \dot{s} = S(r, s), \quad (1)$$

where A_k is a 4×4 matrix, s is a two dimensional vector, and $R(r, s)$, $S(r, s)$ represent nonlinear terms vanishing when $r = 0$. The entries of the matrix A_k depend on the gain parameters. The steady state motions are the equilibria

$$r = 0, \quad s = c \quad (2)$$

of (1). The equilibrium $r = 0$, $s = 0$ represents the steady state motion under consideration while equilibria (2) with $c \neq 0$ represent nearby steady state motions.

According to the Lyapunov-Malkin theorem [6, 7], the equilibria (2) are stable for small values of c if all eigenvalues of the matrix A_k have negative real parts. We emphasize the importance of the fact that $R(0, s) = 0$, $S(0, s) = 0$. This theorem has been used by a number of authors in analyzing stability of nonholonomic systems. See [5, 8, 10] and references therein. In particular, we stress that the conditions $R(0, s) = 0$ and $S(0, s) = 0$ are valid for all systems considered in [1, 10].

In the next theorem we prove existence of the linear feedback control that forces the spectrum of the matrix A_k to belong to the left half plane.

Theorem. *There exists an open non-empty stability region \mathcal{S} in the space of the gain parameters. For any $(k_1, k_2, k_3, k_4) \in \mathcal{S}$ the spectrum of A_k belongs to the left half plane and therefore by the Lyapunov-Malkin theorem the steady state motion $r = 0$, $s = 0$ is stable.*

The proof of this theorem is based on the Routh-Hurwitz analysis of the characteristic polynomial of the matrix A_k . For details see [11].

4 Final Remarks

We emphasize that the Lyapunov-Malkin theorem can be used for nonlinear feedback stabilization. It extends a spectral stability condition to a nonlinear setting. We expect that the domain of the local coordinates (r, s) can be expanded by an appropriate choice of nonlinear control terms. The basin of attraction therefore may be enlarged. We intend to address this issue in a future publication.

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