The Euler–Poincaré Equations and Semidirect Products with Applications to Continuum Theories

Darryl D. Holm
Theoretical Division and Center for Nonlinear Studies
Los Alamos National Laboratory, MS B284
Los Alamos, NM 87545
dholm@lanl.gov

Jerrold E. Marsden*
Control and Dynamical Systems
California Institute of Technology 107-81
Pasadena, CA 91125
marsden@cds.caltech.edu

Tudor S. Ratiu †
Department of Mathematics
University of California, Santa Cruz, CA 95064
ratiu@math.ucsc.edu

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Abstract

We study Euler–Poincaré systems (i.e., the Lagrangian analogue of Lie-Poisson Hamiltonian systems) defined on semidirect product Lie algebras. We first give a derivation of the Euler–Poincaré equations for a parameter dependent Lagrangian by using a variational principle of Lagrange d'Alembert type. Then we derive an abstract Kelvin-Noether theorem for these equations. We also explore their relation with the theory of Lie-Poisson Hamiltonian systems defined on the dual of a semidirect product Lie algebra. The Legendre transformation in such cases is often not invertible; thus, it does not produce a corresponding Euler–Poincaré system on that Lie algebra. We avoid this potential difficulty by developing the theory of Euler–Poincaré systems entirely within the Lagrangian framework. We apply the general theory to a number of known examples, including the heavy top, ideal compressible fluids and MHD. We also use this framework to derive higher dimensional Camassa-Holm equations, which have many potentially interesting analytical properties. These equations are Euler-Poincaré equations for geodesics on diffeomorphism groups (in the sense of the Arnold program) but where the metric is H^1 rather than L^2.

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Contents

1 History and Background

To put our paper in context, we shall pick up the thread of the history of mechanics in the later part of the 1800's. By that time, through the work of many people, including Euler, Lagrange, Hamilton, Jacobi, and Routh, it was well understood that the equations of mechanics are expressible in either Hamiltonian or Lagrangian form.

The Lagrangian formulation of mechanics can be based on the variational principles behind Newton's fundamental laws of force balance $\mathbf{F} = m\mathbf{a}$. One chooses a configuration space Q (a manifold, assumed to be of finite dimension n to start the discussion) with coordinates denoted $q^i, i = 1, \ldots, n$, that describe the configuration of the system under study. One then forms the velocity phase space TQ (the tangent bundle of Q). Coordinates on TQ are denoted $(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$, and the Lagrangian is regarded as a function $L : TQ \to \mathbb{R}$. In coordinates, one writes $L(q^i, \dot{q}^i, t)$, which is shorthand notation for $L(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, t)$. Usually, L is the kinetic minus the potential energy of the system and one takes $\dot{q}^i = dq^i/dt$ to be the system velocity. The variational principle of Hamilton states that the variation of the action is stationary at a solution:

$$\delta \mathfrak{S} = \delta \int_a^b L(q^i, \dot{q}^i, t)\, dt = 0. \tag{1.1}$$

In this principle, one chooses curves $q^i(t)$ joining two fixed points in Q over a fixed time interval $[a, b]$, and calculates the action $\mathfrak{S}$, which is the time integral of the Lagrangian, regarded as a function of this curve. Hamilton's principle states that the action $\mathfrak{S}$ has a critical point at a solution in the space of curves. As is well

known, Hamilton's principle is equivalent to the Euler–Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n. \tag{1.2}$$

If the system is subjected to external forces, these are to be added to the right hand side of the Euler-Lagrange equations. For the case in which L comprises kinetic minus potential energy, the Euler-Lagrange equations reduce to a geometric form of Newton's second law. For Lagrangians that are purely kinetic energy, it was already known in Poincaré's time that the corresponding solutions of the Euler-Lagrange equations are geodesics. (This fact was certainly known to Jacobi by 1840, for example.)

To pass to the Hamiltonian formalism, one introduces the conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad i = 1, \dots, n, \tag{1.3}$$

and makes the change of variables $(q^i, \dot{q}^i) \mapsto (q^i, p_i)$, by a Legendre transformation. The Lagrangian is called *regular* when this change of variables is invertible. The Legendre transformation introduces the Hamiltonian

$$H(q^i, p_i, t) = \sum_{j=1}^{n} p_j \dot{q}^j - L(q^i, \dot{q}^i, t). \tag{1.4}$$

One shows that the Euler–Lagrange equations are equivalent to Hamilton's equations:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \tag{1.5}$$

where $i = 1, \dots, n$. There are analogous Hamiltonian partial differential equations for field theories such as Maxwell's equations and the equations of fluid and solid mechanics.

Hamilton's equations can be recast in Poisson bracket form as

$$\dot{F} = \{F, H\}, \tag{1.6}$$

where the canonical Poisson brackets are given by

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} \right). \tag{1.7}$$

Associated to any configuration space Q is a phase space T^*Q called the cotangent bundle of Q, which has coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$. On this space, the canonical Poisson bracket is intrinsically defined in the sense that the value of $\{F, G\}$ is independent of the choice of coordinates. Because the Poisson bracket satisfies $\{F, G\} = -\{G, F\}$ and in particular $\{H, H\} = 0$, we see that $\dot{H} = 0$; that is, energy is conserved along solutions of Hamilton's equations. This is the most elementary of many deep and beautiful conservation properties of mechanical systems.

Poincaré and the Euler equations. Poincaré played an enormous role in the topics treated in the present paper. We mention a few of Poincaré's contributions that are relevant here. First is his work on the gravitating fluid problem, continuing the line of investigation begun by MacLaurin, Jacobi and Riemann. Some solutions of this problem still bear his name today. This work is summarized in Chandrasekhar [1967, 1977] (see Poincaré [1885, 1890, 1892, 1901a] for the original treatments). This background led to his famous paper, Poincaré [1901b], in which he laid out the basic equations of Euler type, including the rigid body, heavy top and fluids as special cases. Abstractly, these equations are determined once one is given a Lagrangian on a Lie algebra. We shall make some additional historical comments on this situation below, after we present a few more mechanical preliminaries. It is because of the paper Poincaré [1901b] that the name ***Euler–Poincaré equations*** is now used for these equations.

To state the Euler–Poincaré equations, let $\mathfrak{g}$ be a given Lie algebra and let $l : \mathfrak{g} \to \mathbb{R}$ be a given function (a Lagrangian), let ξ be a point in $\mathfrak{g}$ and let $f \in \mathfrak{g}^*$ be given forces (whose nature we shall explicate later). Then the evolution of the variable ξ is determined by the Euler–Poincaré equations. Namely,

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + f.$$

The notation is as follows: $\partial l/\partial \xi \in \mathfrak{g}^*$ (the dual vector space) is the derivative of l with respect to ξ; we use partial derivative notation because l is a function of the vector ξ and because shortly l will be a function of other variables as well. The map $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is the linear map $\eta \mapsto [\xi, \eta]$, where $[\xi, \eta]$ denotes the Lie bracket of ξ and η, and where $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is its dual (transpose) as a linear map. In the case that $f = 0$, we will call these equations the ***basic Euler–Poincaré equations***.

These equations are valid for either finite or infinite dimensional Lie algebras. For fluids, Poincaré was aware that one needs to use infinite dimensional Lie algebras, as is clear in his paper Poincaré [1910]. He was aware that one has to be careful with the signs in the equations; for example, for rigid body dynamics one uses the equations as they stand, but for fluids, one needs to be careful about the conventions for the Lie algebra operation ad_ξ; cf. Chetayev [1941].

To state the equations in the finite dimensional case in coordinates, one must choose a basis $e_1, \ldots, e_r$ of $\mathfrak{g}$ (so $\dim \mathfrak{g} = r$). Define, as usual, the structure constants C^d_{ab} of the Lie algebra by

$$[e_a, e_b] = \sum_{d=1}^{r} C^d_{ab} e_d, \tag{1.8}$$

where a, b run from 1 to r. If $\xi \in \mathfrak{g}$, its components relative to this basis are denoted ξ^a. If $e^1, \ldots, e^n$ is the corresponding dual basis, then the components of the differential of the Lagrangian l are the partial derivatives $\partial l/\partial \xi^a$. The Euler–Poincaré equations in this basis are

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^b} = \sum_{a,d=1}^{r} C^d_{ab} \frac{\partial l}{\partial \xi^d} \xi^a + f_b. \tag{1.9}$$

For example, consider the Lie algebra $\mathbb{R}^3$ with the usual vector cross product. (Of course, this is the Lie algebra of the proper rotation group in $\mathbb{R}^3$.) For $l : \mathbb{R}^3 \to \mathbb{R}$, the Euler–Poincaré equations become

$$\frac{d}{dt}\frac{\partial l}{\partial \mathbf{\Omega}} = \frac{\partial l}{\partial \mathbf{\Omega}} \times \mathbf{\Omega} + \mathbf{f},$$

which generalize the Euler equations for rigid body motion.

These equations were written down for a certain class of Lagrangians l by Lagrange [1788, Volume 2, Equation A on p. 212], while it was Poincaré [1901b] who generalized them (without reference to the ungeometric Lagrange!) to an arbitrary Lie algebra. However, it was Lagrange who was grappeling with the derivation and deeper understanding of the nature of these equations. While Poincaré may have understood how to derive them from other principles, he did not reveal this.

Of course, there was a lot of mechanics going on in the decades leading up to Poincaré's work and we shall comment on some of it below. However, it is a curious historical fact that the Euler–Poincaré equations were not pursued extensively until quite recently. While many authors mentioned these equations and even tried to understand them more deeply (see, e.g., Hamel [1904, 1949] and Chetayev [1941]), it was not until the Arnold school that this understanding was at least partly achieved (see Arnold [1966a,c] and Arnold [1988]) and was used for diagnosing hydrodynamical stability (e.g., Arnold [1966b]).

It was already clear in the last century that certain mechanical systems resist the usual canonical formalism, either Hamiltonian or Lagrangian, outlined in the first paragraph. The rigid body provides an elementary example of this. In another example, to obtain a Hamiltonian description for ideal fluids, Clebsch [1857, 1859] found it necessary to introduce certain nonphysical potentials[1].

More about the rigid body. In the absence of external forces, the rigid body equations are usually written as follows:

$$\begin{aligned} I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3, \\ I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1, \\ I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2, \end{aligned} \tag{1.10}$$

where $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the body angular velocity vector and I_1, I_2, I_3 are the moments of inertia of the rigid body. Are these equations as written Lagrangian or Hamiltonian in any sense? Since there are an odd number of equations, they cannot be put in canonical Hamiltonian form.

One answer is to reformulate the equations on $T\mathrm{SO}(3)$ or $T^*\mathrm{SO}(3)$, as is classically done in terms of Euler angles and their velocities or conjugate momenta, relative to which the equations *are* in Euler–Lagrange or canonical Hamiltonian

[1]For modern accounts of Clebsch potentials and further references, see Holm and Kupershmidt [1983], Marsden and Weinstein [1983], Marsden, Ratiu, and Weinstein [1984a,b], Cendra and Marsden [1987], Cendra, Ibort, and Marsden [1987] and Goncharov and Pavlov [1997].

form. However, this reformulation answers a different question for a *six* dimensional system. We are interested in these structures for the equations as given above.

The Lagrangian answer is easy: these equations have Euler–Poincaré form on the Lie algebra $\mathbb{R}^3$ using the Lagrangian

$$l(\mathbf{\Omega}) = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2), \tag{1.11}$$

which is the (rotational) kinetic energy of the rigid body.

One of our main messages is that the Euler–Poincaré equations possess a natural variational principle. In fact, the Euler rigid body equations are equivalent to the ***rigid body action principle***

$$\delta\mathfrak{S}_{\text{red}} = \delta\int_a^b l\,dt = 0, \tag{1.12}$$

where variations of $\mathbf{\Omega}$ are restricted to be of the form

$$\delta\mathbf{\Omega} = \dot{\mathbf{\Sigma}} + \mathbf{\Omega}\times\mathbf{\Sigma}, \tag{1.13}$$

in which $\mathbf{\Sigma}$ is a curve in $\mathbb{R}^3$ that vanishes at the endpoints. As before, we regard the ***reduced action*** $\mathfrak{S}_{\text{red}}$ as a function on the space of curves, but only consider variations of the form described. The equivalence of the rigid body equations and the rigid body action principle may be proved in the same way as one proves that Hamilton's principle is equivalent to the Euler–Lagrange equations: Since $l(\mathbf{\Omega}) = \frac{1}{2}\langle\mathbb{I}\mathbf{\Omega},\mathbf{\Omega}\rangle$, and $\mathbb{I}$ is symmetric, we obtain

$$\begin{aligned}
\delta\int_a^b l\,dt &= \int_a^b \langle\mathbb{I}\mathbf{\Omega},\delta\mathbf{\Omega}\rangle\,dt\\
&= \int_a^b \langle\mathbb{I}\mathbf{\Omega},\dot{\mathbf{\Sigma}}+\mathbf{\Omega}\times\mathbf{\Sigma}\rangle\,dt\\
&= \int_a^b \left[\left\langle -\frac{d}{dt}\mathbb{I}\mathbf{\Omega},\mathbf{\Sigma}\right\rangle + \langle\mathbb{I}\mathbf{\Omega},\mathbf{\Omega}\times\mathbf{\Sigma}\rangle\right]\\
&= \int_a^b \left\langle -\frac{d}{dt}\mathbb{I}\mathbf{\Omega}+\mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega},\mathbf{\Sigma}\right\rangle dt,
\end{aligned}$$

where we used integration by parts and the endpoint conditions $\mathbf{\Sigma}(b) = \mathbf{\Sigma}(a) = 0$. Since $\mathbf{\Sigma}$ is otherwise arbitrary, (1.12) is equivalent to

$$-\frac{d}{dt}(\mathbb{I}\mathbf{\Omega}) + \mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega} = 0,$$

which are Euler's equations.

Let us explain in concrete terms (that will be abstracted later) how to *derive* this variational principle from the *standard* variational principle of Hamilton.

We regard an element $\mathbf{R}\in\mathrm{SO}(3)$ giving the configuration of the body as a map of a reference configuration $\mathcal{B}\subset\mathbb{R}^3$ to the current configuration $\mathbf{R}(\mathcal{B})$; the map $\mathbf{R}$ takes a reference or label point $X\in\mathcal{B}$ to a current point $x = \mathbf{R}(X)\in\mathbf{R}(\mathcal{B})$. When

the rigid body is in motion, the matrix $\mathbf{R}$ is time-dependent and the velocity of a point of the body is $\dot{x} = \dot{\mathbf{R}}X = \dot{\mathbf{R}}\mathbf{R}^{-1}x$. Since $\mathbf{R}$ is an orthogonal matrix, $\mathbf{R}^{-1}\dot{\mathbf{R}}$ and $\dot{\mathbf{R}}\mathbf{R}^{-1}$ are skew matrices, and so we can write

$$\dot{x} = \dot{\mathbf{R}}\mathbf{R}^{-1}x = \boldsymbol{\omega} \times x, \tag{1.14}$$

which defines the ***spatial angular velocity vector*** $\boldsymbol{\omega}$. Thus, $\boldsymbol{\omega}$ is essentially given by *right* translation of $\dot{\mathbf{R}}$ to the identity.

The corresponding body angular velocity is defined by

$$\boldsymbol{\Omega} = \mathbf{R}^{-1}\boldsymbol{\omega}, \tag{1.15}$$

so that $\boldsymbol{\Omega}$ is the angular velocity relative to a body fixed frame. Notice that

$$\begin{aligned} \mathbf{R}^{-1}\dot{\mathbf{R}}X &= \mathbf{R}^{-1}\dot{\mathbf{R}}\mathbf{R}^{-1}x = \mathbf{R}^{-1}(\boldsymbol{\omega} \times x) \\ &= \mathbf{R}^{-1}\boldsymbol{\omega} \times \mathbf{R}^{-1}x = \boldsymbol{\Omega} \times X, \end{aligned} \tag{1.16}$$

so that $\boldsymbol{\Omega}$ is given by *left* translation of $\dot{\mathbf{R}}$ to the identity. The kinetic energy is obtained by summing up $m|\dot{x}|^2/2$ (where $|\bullet|$ denotes the Euclidean norm) over the body:

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho(X)|\dot{\mathbf{R}}X|^2 \, d^3X, \tag{1.17}$$

in which ρ is a given mass density in the reference configuration. Since

$$|\dot{\mathbf{R}}X| = |\boldsymbol{\omega} \times x| = |\mathbf{R}^{-1}(\boldsymbol{\omega} \times x)| = |\boldsymbol{\Omega} \times X|,$$

K is a quadratic function of $\boldsymbol{\Omega}$. Writing

$$K = \frac{1}{2}\boldsymbol{\Omega}^T\mathbb{I}\boldsymbol{\Omega} \tag{1.18}$$

defines the ***moment of inertia tensor*** $\mathbb{I}$, which, provided the body does not degenerate to a line, is a positive-definite (3×3) matrix, or better, a quadratic form. This quadratic form can be diagonalized by a change of basis; thereby defining the principal axes and moments of inertia. In this basis, we write $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$. The function K is taken to be the Lagrangian of the system on $T\mathrm{SO}(3)$ (and by means of the Legendre transformation we obtain the corresponding Hamiltonian description on $T^*\mathrm{SO}(3)$). Notice that K in equation (1.17) is *left* (not right) invariant on $T\mathrm{SO}(3)$. It follows that the corresponding Hamiltonian is also *left* invariant.

In the Lagrangian framework, the relation between motion in $\mathbf{R}$ space and motion in body angular velocity (or $\boldsymbol{\Omega}$) space is as follows: The curve $\mathbf{R}(t) \in \mathrm{SO}(3)$ satisfies the Euler-Lagrange equations for

$$L(\mathbf{R}, \dot{\mathbf{R}}) = \frac{1}{2}\int_{\mathcal{B}} \rho(X)|\dot{\mathbf{R}}X|^2 \, d^3X, \tag{1.19}$$

if and only if $\boldsymbol{\Omega}(t)$ defined by $\mathbf{R}^{-1}\dot{\mathbf{R}}\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$ satisfies Euler's equations

$$\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}. \tag{1.20}$$

An instructive proof of this relation involves understanding how to reduce variational principles using their symmetry groups. By Hamilton's principle, $\mathbf{R}(t)$ satisfies the Euler-Lagrange equations, if and only if

$$\delta \int L\, dt = 0.$$

Let $l(\boldsymbol{\Omega}) = \frac{1}{2}(\mathbb{I}\boldsymbol{\Omega}) \cdot \boldsymbol{\Omega}$, so that $l(\boldsymbol{\Omega}) = L(\mathbf{R}, \dot{\mathbf{R}})$ if $\mathbf{R}$ and $\boldsymbol{\Omega}$ are related as above. To see how we should transform Hamilton's principle, define the skew matrix $\hat{\boldsymbol{\Omega}}$ by $\hat{\boldsymbol{\Omega}}\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^3$, and differentiate the relation $\mathbf{R}^{-1}\dot{\mathbf{R}} = \hat{\boldsymbol{\Omega}}$ with respect to $\mathbf{R}$ to get

$$-\mathbf{R}^{-1}(\delta\mathbf{R})\mathbf{R}^{-1}\dot{\mathbf{R}} + \mathbf{R}^{-1}(\delta\dot{\mathbf{R}}) = \widehat{\delta\boldsymbol{\Omega}}. \tag{1.21}$$

Let the skew matrix $\hat{\boldsymbol{\Sigma}}$ be defined by

$$\hat{\boldsymbol{\Sigma}} = \mathbf{R}^{-1}\delta\mathbf{R}, \tag{1.22}$$

and define the vector $\boldsymbol{\Sigma}$ by

$$\hat{\boldsymbol{\Sigma}}\mathbf{v} = \boldsymbol{\Sigma} \times \mathbf{v}. \tag{1.23}$$

Note that

$$\dot{\hat{\boldsymbol{\Sigma}}} = -\mathbf{R}^{-1}\dot{\mathbf{R}}\mathbf{R}^{-1}\delta\mathbf{R} + \mathbf{R}^{-1}\delta\dot{\mathbf{R}},$$

so

$$\mathbf{R}^{-1}\delta\dot{\mathbf{R}} = \dot{\hat{\boldsymbol{\Sigma}}} + \mathbf{R}^{-1}\dot{\mathbf{R}}\hat{\boldsymbol{\Sigma}}\,. \tag{1.24}$$

Substituting (1.24) and (1.22) into (1.21) gives

$$-\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Omega}} + \dot{\hat{\boldsymbol{\Sigma}}} + \hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\Sigma}} = \widehat{\delta\boldsymbol{\Omega}},$$

that is,

$$\widehat{\delta\boldsymbol{\Omega}} = \dot{\hat{\boldsymbol{\Sigma}}} + [\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\Sigma}}]. \tag{1.25}$$

The identity $[\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\Sigma}}] = (\boldsymbol{\Omega} \times \boldsymbol{\Sigma})\hat{}$ holds by Jacobi's identity for the cross product and so

$$\delta\boldsymbol{\Omega} = \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Omega} \times \boldsymbol{\Sigma}. \tag{1.26}$$

These calculations prove the following:

Theorem 1.1 *Hamilton's variational principle*

$$\delta \mathfrak{S} = \delta \int_a^b L \, dt = 0 \tag{1.27}$$

on $T\mathrm{SO}(3)$ *is equivalent to the* ***reduced variational principle***

$$\delta \mathfrak{S}_{\mathrm{red}} = \delta \int_a^b l \, dt = 0 \tag{1.28}$$

on $\mathbb{R}^3$ *where the variations* $\delta\boldsymbol{\Omega}$ *are of the form (1.26) with* $\boldsymbol{\Sigma}(a) = \boldsymbol{\Sigma}(b) = 0$.

Hamiltonian Form. If, instead of variational principles, we concentrate on Poisson brackets and drop the requirement that they be in the canonical form, then there is also a simple and beautiful Hamiltonian structure for the rigid body equations that is now well known[2]. To recall this, introduce the angular momenta

$$\Pi_i = I_i \Omega_i = \frac{\partial L}{\partial \Omega_i}, \quad i = 1, 2, 3, \tag{1.29}$$

so that the Euler equations become

$$\begin{aligned} \dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3} \Pi_2 \Pi_3, \\ \dot{\Pi}_2 &= \frac{I_3 - I_1}{I_3 I_1} \Pi_3 \Pi_1, \\ \dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2} \Pi_1 \Pi_2, \end{aligned} \tag{1.30}$$

that is,

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega}. \tag{1.31}$$

Introduce the following rigid body Poisson bracket on functions of the $\boldsymbol{\Pi}$'s:

$$\{F, G\}(\boldsymbol{\Pi}) = -\boldsymbol{\Pi} \cdot (\nabla_\Pi F \times \nabla_\Pi G) \tag{1.32}$$

and the Hamiltonian

$$H = \frac{1}{2} \left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3} \right). \tag{1.33}$$

One checks that Euler's equations are equivalent to $\dot{F} = \{F, H\}$.

The rigid body variational principle and the rigid body Poisson bracket are special cases of general constructions associated to any Lie algebra $\mathfrak{g}$. Since we have already described the general Euler–Poincaré construction on $\mathfrak{g}$, we turn next to the Hamiltonian counterpart on the dual space.

[2]See Marsden and Ratiu [1994] for details, references, and the history of this structure.

The Lie-Poisson Equations. Let F, G be real valued functions on the dual space $\mathfrak{g}^*$. Denoting elements of $\mathfrak{g}^*$ by μ, let the functional derivative of F at μ be the unique element $\delta F/\delta\mu$ of $\mathfrak{g}$ defined by

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[F(\mu+\varepsilon\delta\mu)-F(\mu)]=\left\langle \delta\mu, \frac{\delta F}{\delta\mu}\right\rangle, \tag{1.34}$$

for all $\delta\mu \in \mathfrak{g}^*$, where $\langle\,,\rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Define the $(\pm)$ Lie-Poisson brackets by

$$\{F,G\}_\pm(\mu)=\pm\left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right]\right\rangle. \tag{1.35}$$

Using the coordinate notation introduced above, the $(\pm)$ Lie-Poisson brackets become

$$\{F,G\}_\pm(\mu)=\pm\sum_{a,b,d=1}^{r} C^d_{ab}\mu_d\frac{\partial F}{\partial\mu_a}\frac{\partial G}{\partial\mu_b}, \tag{1.36}$$

where $\mu=\sum_{d=1}^r \mu_d e^d$.

The Lie-Poisson equations, determined by $\dot F=\{F,H\}$ read

$$\dot\mu_a=\pm\sum_{b,d=1}^{r} C^d_{ab}\mu_d\frac{\partial H}{\partial\mu_b},$$

or intrinsically,

$$\dot\mu=\mp\,\mathrm{ad}^*_{\partial H/\partial\mu}\,\mu. \tag{1.37}$$

This setting of mechanics is a special case of the general theory of systems on Poisson manifolds, for which there is now an extensive theoretical development. (See Guillemin and Sternberg [1984] and Marsden and Ratiu [1994] for a start on this literature.) There is an especially important feature of the rigid body bracket that carries over to general Lie algebras, namely, *Lie-Poisson brackets arise from canonical brackets on the cotangent bundle* (phase space) T^*G associated with a Lie group G which has $\mathfrak{g}$ as its associated Lie algebra.

For a rigid body which is free to rotate about its center of mass, G is the (proper) rotation group SO(3). The choice of T^*G as the primitive phase space is made according to the classical procedures of mechanics described earlier. For the description using Lagrangian mechanics, one forms the velocity-phase space $T\mathrm{SO}(3)$. The Hamiltonian description on T^*G is then obtained by the Legendre transformation.

The passage from T^*G to the space of $\mathbf{\Pi}$'s (body angular momentum space) is determined by *left* translation on the group. This mapping is an example of a *momentum map*; that is, a mapping whose components are the "Noether quantities" associated with a symmetry group. The map from T^*G to $\mathfrak{g}^*$ being a Poisson (canonical) map *is a general fact about momentum maps*. The Hamiltonian point of view of all this is again a well developed subject.

Geodesic motion. As emphasized by Arnold [1966a], in many interesting cases, the Euler–Poincaré equations on a Lie algebra $\mathfrak{g}$ correspond to *geodesic motion* on the corresponding group G. We shall explain the relationship between the equations on $\mathfrak{g}$ and on G shortly, in theorem 1.2. Similarly, on the Hamiltonian side, the preceding paragraphs explained the relation between the Hamiltonian equations on T^*G and the Lie–Poisson equations on $\mathfrak{g}^*$. However, the issue of geodesic motion is simple: if the Lagrangian or Hamiltonian on $\mathfrak{g}$ or $\mathfrak{g}^*$ is purely quadratic, then the corresponding motion on the group is geodesic motion.

More History. The Lie-Poisson bracket was discovered by Sophus Lie (Lie [1890], Vol. II, p. 237). However, Lie's bracket and his related work was not given much attention until the work of Kirillov, Kostant, and Souriau (and others) revived it in the mid-1960s. Meanwhile, it was noticed by Pauli and Martin around 1950 that the rigid body equations are in Hamiltonian form using the rigid body bracket, but they were apparently unaware of the underlying Lie theory. It would seem that while Poincaré was aware of Lie theory, in his work on the Euler equations he was unaware of Lie's work on Lie-Poisson structures. He also seems not to have been aware of the variational structure of the Euler equations.

The heavy top. Another system important to Poincaré and also for us in this paper is the heavy top; that is, a rigid body with a fixed point in a gravitational field. For the Lie-Poisson description, the underlying Lie algebra, surprisingly, consists of the algebra of infinitesimal Euclidean motions in $\mathbb{R}^3$. These do *not* arise as actual Euclidean motions of the body since the body has a fixed point! As we shall see, there is a close parallel with the Poisson structure for compressible fluids.

The basic phase space we start with is again $T^*\mathrm{SO}(3)$. In this space, the equations are in canonical Hamiltonian form. Gravity breaks the symmetry and the system is no longer SO(3) invariant, so it cannot be written entirely in terms of the body angular momentum $\mathbf{\Pi}$. One also needs to keep track of $\mathbf{\Gamma}$, the "direction of gravity" as seen from the body ($\mathbf{\Gamma} = \mathbf{R}^{-1}\mathbf{k}$ where the unit vector $\mathbf{k}$ points upward and $\mathbf{R}$ is the element of SO(3) describing the current configuration of the body). The equations of motion are

$$\begin{aligned}
\dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 + Mg\ell\,(\Gamma^2\chi^3 - \Gamma^3\chi^2),\\
\dot{\Pi}_2 &= \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1 + Mg\ell\,(\Gamma^3\chi^1 - \Gamma^1\chi^3),\\
\dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 + Mg\ell\,(\Gamma^1\chi^2 - \Gamma^2\chi^1),
\end{aligned} \tag{1.38}$$

or, in vector notation,

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega} + Mg\ell\,\mathbf{\Gamma} \times \boldsymbol{\chi}\,, \tag{1.39}$$

and

$$\dot{\mathbf{\Gamma}} = \mathbf{\Gamma} \times \mathbf{\Omega}, \tag{1.40}$$

where M is the body's mass, g is the acceleration of gravity, $\boldsymbol{\chi}$ is the unit vector on the line connecting the fixed point with the body's center of mass, and ℓ is the length of this segment.

The Lie algebra of the Euclidean group is $\mathfrak{se}(3) = \mathbb{R}^3 \times \mathbb{R}^3$ with the Lie bracket

$$[(\boldsymbol{\xi}, \mathbf{u}), (\boldsymbol{\eta}, \mathbf{v})] = (\boldsymbol{\xi} \times \boldsymbol{\eta}, \boldsymbol{\xi} \times \mathbf{v} - \boldsymbol{\eta} \times \mathbf{u}). \tag{1.41}$$

We identify the dual space with pairs $(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$; the corresponding $(-)$ Lie-Poisson bracket called the ***heavy top bracket*** is

$$\begin{aligned}\{F, G\}(\boldsymbol{\Pi}, \boldsymbol{\Gamma}) = &-\boldsymbol{\Pi} \cdot (\nabla_\Pi F \times \nabla_\Pi G) \\ &- \boldsymbol{\Gamma} \cdot (\nabla_\Pi F \times \nabla_\Gamma G - \nabla_\Pi G \times \nabla_\Gamma F).\end{aligned} \tag{1.42}$$

The above equations for $\boldsymbol{\Pi}, \boldsymbol{\Gamma}$ can be checked to be equivalent to

$$\dot{F} = \{F, H\}, \tag{1.43}$$

where the ***heavy top Hamiltonian***

$$H(\boldsymbol{\Pi}, \boldsymbol{\Gamma}) = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right) + Mg\ell\, \boldsymbol{\Gamma} \cdot \boldsymbol{\chi} \tag{1.44}$$

is the total energy of the body (see, for example, Sudarshan and Mukunda [1974]).

The Lie algebra of the Euclidean group has a structure which is a special case of what is called a *semidirect product*. Here it is the product of the group of rotations with the translation group. It turns out that semidirect products occur under rather general circumstances when the symmetry in T^*G is broken. In particular, there are similarities in structure between the Poisson bracket for compressible flow and that for the heavy top. The general theory for semidirect products will be reviewed shortly.

A Kaluza-Klein form for the heavy top. We make a remark about the heavy top equations that is relevant for later purposes. Namely, since the equations have a Hamiltonian that is of the form kinetic plus potential, it is clear that the equations are *not of Lie-Poisson form on* $\mathfrak{so}(3)^*$*, the dual of the Lie algebra of* SO(3) and correspondingly, are not geodesic equations on SO(3). While the equations *are Lie–Poisson* on $\mathfrak{se}(3)^*$, the Hamiltonian is not quadratic, so again the equations are *not geodesic equations on* SE(3).

However, they can be viewed a different way so that they become Lie-Poisson equations for a different group and with a *quadratic Hamiltonian*. In particular, they are the reduction of geodesic motion. To effect this, one changes the Lie algebra from $\mathfrak{se}(3)$ to the product $\mathfrak{se}(3) \times \mathfrak{so}(3)$. The dual variables are now denoted $\boldsymbol{\Pi}, \boldsymbol{\Gamma}, \boldsymbol{\chi}$. We regard the variable $\boldsymbol{\chi}$ as a momentum conjugate to a new variable, namely a *ghost* element of the rotation group in such a way that $\boldsymbol{\chi}$ is a constant of the motion; in Kaluza-Klein theory for charged particles on thinks of the charge this way, as being the momentum conjugate to a (ghost) cyclic variable.

We modify the Hamiltonian by replacing $\boldsymbol{\Gamma}\cdot\boldsymbol{\chi}$ by, for example, $\boldsymbol{\Gamma}\cdot\boldsymbol{\chi}+\|\boldsymbol{\Gamma}\|^2+\|\boldsymbol{\chi}\|^2$, or any other terms of this sort that convert the potential energy into a positive definite quadratic form in $\boldsymbol{\Gamma}$ and $\boldsymbol{\chi}$. The added terms, being Casimir functions, do not affect the equations of motion. However, now the Hamiltonian is purely quadratic and hence comes from geodesic motion on the group $SE(3) \times \mathrm{SO}(3)$. Notice that this construction is quite different from that of the well known Jacobi metric method.

Later on in our study of continuum mechanics, we shall repeat this construction to achieve geodesic form for some other interesting continuum models. Of course one can also treat a heavy top that is charged or has a magnetic moment using these ideas.

Incompressible Fluids. Arnold [1966a] showed that the Euler equations for an incompressible fluid could be given a Lagrangian and Hamiltonian description similar to that for the rigid body. His approach[3] has the appealing feature that one sets things up just the way Lagrange and Hamilton would have done: one begins with a configuration space Q, forms a Lagrangian L on the velocity phase space TQ and then Legendre transforms to a Hamiltonian H on the momentum phase space T^*Q. Thus, one automatically has variational principles, etc. For ideal fluids, $Q = G$ is the group $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})$ of volume preserving transformations of the fluid container (a region $\mathcal{D}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, or a Riemannian manifold in general, possibly with boundary). Group multiplication in G is composition.

The reason we select $G = \mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})$ as the configuration space is similar to that for the rigid body; namely, each φ in G is a mapping of $\mathcal{D}$ to $\mathcal{D}$ which takes a reference point $X \in \mathcal{D}$ to a current point $x = \varphi(X) \in \mathcal{D}$; thus, knowing φ tells us where each particle of fluid goes and hence gives us the current ***fluid configuration***. We ask that φ be a diffeomorphism to exclude discontinuities, cavitation, and fluid interpenetration, and we ask that φ be volume preserving to correspond to the assumption of incompressibility.

A ***motion*** of a fluid is a family of time-dependent elements of G, which we write as $x = \varphi(X, t)$. The ***material velocity*** field is defined by $\mathbf{V}(X, t) = \partial\varphi(X, t)/\partial t$, and the ***spatial velocity*** field is defined by $\mathbf{v}(x, t) = \mathbf{V}(X, t)$ where x and X are related by $x = \varphi(X, t)$. If we suppress "t" and write $\dot{\varphi}$ for $\mathbf{V}$, note that

$$\mathbf{v} = \dot{\varphi} \circ \varphi^{-1} \quad \text{i.e.,} \quad \mathbf{v}_t = \mathbf{V}_t \circ \varphi_t^{-1}, \tag{1.45}$$

where $\varphi_t(x) = \varphi(X, t)$. We can regard (1.45) as a map from the space of $(\varphi, \dot{\varphi})$ (material or Lagrangian description) to the space of $\mathbf{v}$'s (spatial or Eulerian description). Like the rigid body, the material to spatial map (1.45) takes the canonical bracket to a Lie-Poisson bracket; one of our goals is to understand this reduction. Notice that if we replace φ by $\varphi \circ \eta$ for a fixed (time-independent) $\eta \in \mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})$, then $\dot{\varphi} \circ \varphi^{-1}$ is independent of η; this reflects the *right* invariance of the Eulerian

[3] Arnold's approach is consistent with what appears in the thesis of Ehrenfest from around 1904; see Klein [1970]. However, Ehrenfest bases his principles on the more sophisticated curvature principles of Gauss and Hertz.

description ($\mathbf{v}$ is invariant under composition of φ by η on the right). This is also called the ***particle relabeling symmetry*** of fluid dynamics. The spaces TG and T^*G represent the Lagrangian (material) description and we pass to the Eulerian (spatial) description by right translations and use the (+) Lie-Poisson bracket. One of the things we shall explain later is the reason for the switch between right and left in going from the rigid body to fluids.

The ***Euler equations*** for an ideal, incompressible, homogeneous fluid moving in the region $\mathcal{D}$ are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p \tag{1.46}$$

with the constraint div $\mathbf{v} = 0$ and boundary conditions: $\mathbf{v}$ is tangent to $\partial\mathcal{D}$.

The pressure p is determined implicitly by the divergence-free (volume preserving) constraint div $\mathbf{v} = 0$. The associated Lie algebra $\mathfrak{g}$ is the space of all divergence-free vector fields tangent to the boundary. This Lie algebra is endowed with the *negative* ***Jacobi-Lie bracket*** of vector fields given by

$$[\mathbf{v}, \mathbf{w}]_L^i = \sum_{j=1}^{n} \left(w^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial w^i}{\partial x^j} \right). \tag{1.47}$$

(The subscript L on $[\cdot\,,\cdot]$ refers to the fact that it is the *left* Lie algebra bracket on $\mathfrak{g}$. The most common convention for the Jacobi-Lie bracket of vector fields, also the one we adopt, has the opposite sign.) We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by using the pairing

$$\langle \mathbf{v}, \mathbf{w} \rangle = \int_{\mathcal{D}} \mathbf{v} \cdot \mathbf{w}\, d^3x. \tag{1.48}$$

Hamiltonian structure for fluids. Introduce the (+) Lie-Poisson bracket, called the ***ideal fluid bracket***, on functions of $\mathbf{v}$ by

$$\{F, G\}(\mathbf{v}) = \int_{\mathcal{D}} \mathbf{v} \cdot \left[\frac{\delta F}{\delta \mathbf{v}}, \frac{\delta G}{\delta \mathbf{v}} \right]_L d^3x, \tag{1.49}$$

where $\delta F/\delta \mathbf{v}$ is defined by

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [F(\mathbf{v} + \varepsilon \delta \mathbf{v}) - F(\mathbf{v})] = \int_{\mathcal{D}} \left(\delta \mathbf{v} \cdot \frac{\delta F}{\delta \mathbf{v}} \right) d^3x. \tag{1.50}$$

With the energy function chosen to be the kinetic energy,

$$H(\mathbf{v}) = \frac{1}{2} \int_{\mathcal{D}} |\mathbf{v}|^2\, d^3x, \tag{1.51}$$

one can verify that the Euler equations (1.46) are equivalent to the Poisson bracket equations

$$\dot{F} = \{F, H\} \tag{1.52}$$

for all functions F on $\mathfrak{g}$. For this, one uses the orthogonal decomposition $\mathbf{w} = \mathbb{P}\mathbf{w} + \nabla p$ of a vector field $\mathbf{w}$ into a divergence-free part $\mathbb{P}\mathbf{w}$ in $\mathfrak{g}$ and a gradient. The Euler equations can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbb{P}(\mathbf{v} \cdot \nabla \mathbf{v}) = 0. \tag{1.53}$$

One can also express the Hamiltonian structure in terms of the vorticity as a basic dynamic variable and show that the preservation of coadjoint orbits amounts to Kelvin's circulation theorem. We shall see a Lagrangian version of this property later in the paper. Marsden and Weinstein [1983] show that the Hamiltonian structure in terms of Clebsch potentials fits naturally into this Lie-Poisson scheme, and that Kirchhoff's Hamiltonian description of point vortex dynamics, vortex filaments, and vortex patches can be derived in a natural way from the Hamiltonian structure described above.

Lagrangian structure for fluids. The general framework of the Euler–Poincaré and the Lie-Poisson equations gives other insights as well. For example, this general theory shows that the Euler equations are derivable from the "variational principle"

$$\delta \int_a^b \int_{\mathcal{D}} \frac{1}{2} |\mathbf{v}|^2 \, d^3x = 0$$

which should hold for all variations $\delta \mathbf{v}$ of the form

$$\delta \mathbf{v} = \dot{\mathbf{u}} + [\mathbf{u}, \mathbf{v}]_L$$

where $\mathbf{u}$ is a vector field (representing the infinitesimal particle displacement) vanishing at the temporal endpoints. The constraints on the allowed variations of the fluid velocity field are commonly known as "Lin constraints" and their nature was clarified by Newcomb [1962] and Bretherton [1970]. This itself has an interesting history, going back to Ehrenfest, Boltzmann, and Clebsch, but again, there was little if any contact with the heritage of Lie and Poincaré on the subject.

The Basic Euler–Poincaré Equations. We now recall the abstract derivation of the "basic" Euler–Poincaré equations (i.e., the Euler–Poincaré equations with no forcing or advected parameters) for left–invariant Lagrangians on Lie groups (see Marsden and Scheurle [1993a,b], Marsden and Ratiu [1994] and Bloch et al. [1996]).

Theorem 1.2 *Let G be a Lie group and $L : TG \to \mathbb{R}$ a left (respectively, right) invariant Lagrangian. Let $l : \mathfrak{g} \to \mathbb{R}$ be its restriction to the tangent space at the identity. For a curve $g(t) \in G$, let $\xi(t) = g(t)^{-1}\dot{g}(t)$; i.e., $\xi(t) = T_{g(t)}L_{g(t)^{-1}}\dot{g}(t)$ (respectively, $\xi(t) = \dot{g}(t)g(t)^{-1}$). Then the following are equivalent:*

i *Hamilton's principle*

$$\delta \int_a^b L(g(t), \dot{g}(t)) dt = 0 \tag{1.54}$$

holds, as usual, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii *The curve $g(t)$ satisfies the Euler-Lagrange equations for L on G.*

iii *The "variational" principle*

$$\delta \int_a^b l(\xi(t))dt = 0 \tag{1.55}$$

holds on $\mathfrak{g}$, using variations of the form

$$\delta\xi = \dot{\eta} \pm [\xi, \eta], \tag{1.56}$$

where η vanishes at the endpoints (+ corresponds to left invariance and − to right invariance).[4]

iv *The **basic Euler–Poincaré equations** hold*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \pm \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi}\,. \tag{1.57}$$

Basic Ideas of the Proof. First of all, the equivalence of **i** and **ii** holds on the tangent bundle of any configuration manifold Q, by the general Hamilton principle. To see that **ii** and **iii** are equivalent, one needs to compute the variations $\delta\xi$ induced on $\xi = g^{-1}\dot{g} = TL_{g^{-1}}\dot{g}$ by a variation of g. We will do this for matrix groups; see Bloch, Krishnaprasad, Marsden, and Ratiu [1994] for the general case. To calculate this, we need to differentiate $g^{-1}\dot{g}$ in the direction of a variation δg. If $\delta g = dg/d\epsilon$ at $\epsilon = 0$, where g is extended to a curve g_ϵ, then,

$$\delta\xi = \frac{d}{d\epsilon}g^{-1}\frac{d}{dt}g,$$

while if $\eta = g^{-1}\delta g$, then

$$\dot{\eta} = \frac{d}{dt}g^{-1}\frac{d}{d\epsilon}g.$$

The difference $\delta\xi - \dot{\eta}$ is thus the commutator $[\xi, \eta]$.

To complete the proof, we show the equivalence of **iii** and **iv** in the left-invariant case. Indeed, using the definitions and integrating by parts produces,

$$\begin{aligned}\delta \int l(\xi)dt &= \int \frac{\delta l}{\delta \xi}\delta\xi\, dt = \int \frac{\delta l}{\delta \xi}(\dot{\eta} + \operatorname{ad}_\xi \eta)\, dt \\ &= \int \left[-\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) + \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi}\right]\eta\, dt\,,\end{aligned}$$

so the result follows. ■

[4]Because there are constraints on the variations, this principle is more like a Lagrange d'Alembert principle, which is why we put "variational" in quotes. As we shall explain, such problems are not literally variational.

There is of course a right invariant version of this theorem in which $\xi = \dot{g}g^{-1}$ and the Euler–Poincaré equations acquire appropriate minus signs as in equation (1.57). We shall go into this in detail later.

Since the Euler-Lagrange and Hamilton equations on TQ and T^*Q are equivalent in the regular case, it follows that the Lie-Poisson and Euler–Poincaré equations are then also equivalent. To see this *directly*, we make the following Legendre transformation from $\mathfrak{g}$ to $\mathfrak{g}^*$:

$$\mu = \frac{\delta l}{\delta \xi}, \quad h(\mu) = \langle \mu, \xi \rangle - l(\xi).$$

Note that

$$\frac{\delta h}{\delta \mu} = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \frac{\delta l}{\delta \xi}, \frac{\delta \xi}{\delta \mu} \right\rangle = \xi$$

and so it is now clear that the Lie-Poisson equations (1.37) and the Euler–Poincaré equations (1.57) are equivalent.

We close this paragraph by mentioning the geodesic property of the basic Euler–Poincaré form. When l is a metric on TG, the basic Euler–Poincaré equations are the *geodesic spray equations* for geodesic motion on the group G with respect to that metric. For discussions of this property in applications, see, e.g., Arnold [1966a] for the Euler equations of an incompressible ideal fluid, and Ovsienko and Khesin [1987] for the KdV shallow water equation. (An account of the latter case from the Euler–Poincaré viewpoint may also be found in Marsden and Ratiu [1994].) Zeitlin and Pasmanter [1994] discuss the geodesic property for certain ideal geophysical fluid flows; Zeitlin and Kambe [1993] and Ono [1995a, 1995b] discuss it for ideal MHD; and Kouranbaeva [1997] for the integrable Camassa-Holm equation. From one viewpoint, casting these systems into basic Euler–Poincaré form explains why they share the geodesic property.

Lie-Poisson Systems on Semidirect Products. As we described above, the heavy top is a basic example of a Lie-Poisson Hamiltonian system defined on the dual of a semidirect product Lie algebra. The *general* study of Lie-Poisson equations for systems on the dual of a semidirect product Lie algebra grew out of the work of many authors including Sudarshan and Mukunda [1974], Vinogradov and Kupershmidt [1977], Ratiu [1980], Guillemin and Sternberg [1980], Ratiu [1981, 1982], Marsden [1982], Marsden, Weinstein, Ratiu, Schmidt and Spencer [1983], Holm and Kupershmidt [1983], Kupershmidt and Ratiu [1983], Holmes and Marsden [1983], Marsden, Ratiu and Weinstein [1984a,b], Guillemin and Sternberg [1984], Holm, Marsden, Ratiu and Weinstein [1985], Abarbanel, Holm, Marsden, and Ratiu [1986] and Marsden, Misiolek, Perlmutter and Ratiu [1997]. As these and related references show, the Lie-Poisson equations apply to a wide variety of systems such as the heavy top, compressible flow, stratified incompressible flow, and MHD (magnetohydrodynamics). We review this theory in §2 below.

In each of the above examples as well as in the general theory, one can view the given Hamiltonian in the material representation as one that depends on a param-

eter; this parameter becomes dynamic when reduction is performed; this reduction amounts in many examples to expressing the system in the spatial representation.

Goals of this Paper. The first goal of this paper is to study a Lagrangian analogue of the Hamiltonian semidirect product theory. The idea is to carry out a reduction for a Lagrangian that depends on a parameter and to use the ideas of reduction of variational principles from Marsden and Scheurle [1993a,b] and Bloch, Krishnaprasad, Marsden and Ratiu [1996] to directly reduce the problem to one that parallels Lie-Poisson systems on the duals of semidirect products. We call the resulting equations the Euler–Poincaré equations since, as we have explained, Poincaré [1901b] came rather close to this general picture. These equations generalize the *basic* Euler–Poincaré equations on a Lie algebra in that they depend on a parameter and this parameter in examples has the interpretation of being advected, or Lie dragged, as is the density in compressible flow.

One of the reasons this process is interesting and cannot be derived directly from its Hamiltonian counterpart by means of the Legendre transformation is that in many examples, such as the heavy top, the Hamiltonian describing the Lie-Poisson dynamics is degenerate; that is, the Legendre transformation is not invertible.

A second major goal is to prove a version of the Noether theorem in an action principle formulation that leads immediately to a Kelvin circulation type theorem for continuum mechanics. We call this general formulation the Kelvin-Noether theorem.

Finally, we provide a number of applications of the Euler–Poincaré equations in ideal continuum dynamics which illustrate the power of this approach in unifying various known models, as well as in formulating new models. We also discuss some circumstances when the equations can be cast into the form of geodesics on certain infinite dimensional groups.

Outline of the remainder of this paper. In the next section we review the semidirect product theory for Hamiltonian systems. Then in section 3 we consider the Lagrangian counterpart to this theory. Section 4 discusses the Kelvin-Noether theorem for the Euler–Poincaré equations. Section 5 illustrates the general theory in the example of the heavy top. We introduce the Euler–Poincaré equations for continua in section 6 and consider their applications to compressible flow (including MHD and adiabatic Maxwell-fluid plasmas) in section 7. Various approximate forms of the shallow water equations, such as the Boussinesq equations, the Camassa-Holm equation and its new higher-dimensional variants are developed in section 8. In other publications, the Maxwell-Vlasov equations will be considered as well as a general framework for the theory of reduction by stages.

In the remainder of this paper we assume that the reader is familiar with Lie-Poisson Hamiltonian systems defined on duals of Lie algebras and the Lie-Poisson reduction theorem, reviewed above. We refer to Marsden and Ratiu [1994] for a detailed exposition of these matters.

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2 Hamiltonian Semidirect Product Theory

We first recall how the Hamiltonian theory proceeds for systems defined on semidirect products. We present the abstract theory, but of course historically this grew out of the examples, especially the heavy top and compressible flow.

Generalities on Semidirect Products. We begin by recalling some definitions and properties of semidirect products. Let V be a vector space and assume that the Lie group G acts *on the left* by linear maps on V (and hence G also acts on on the left on its dual space V^*). As sets, the semidirect product $S = G \circledS V$ is the Cartesian product $S = G \times V$ whose group multiplication is given by

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2), \tag{2.1}$$

where the action of $g \in G$ on $v \in V$ is denoted simply as gv. The identity element is $(e, 0)$ where e is the identity in G. We record for convenience the inverse of an element:

$$(g, v)^{-1} = (g^{-1}, -g^{-1}v). \tag{2.2}$$

The Lie algebra of S is the semidirect product Lie algebra, $\mathfrak{s} = \mathfrak{g} \circledS V$, whose bracket has the expression

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2],\, \xi_1 v_2 - \xi_2 v_1)\,, \tag{2.3}$$

where we denote the induced action of $\mathfrak{g}$ on V by concatenation, as in $\xi_1 v_2$.

Below we will need the formulae for the adjoint and the coadjoint actions for semidirect products. We denote these and other actions by simple concatenation; so they are expressed as (see, e.g., Marsden, Ratiu and Weinstein [1984a,b])

$$(g, v)(\xi, u) = (g\xi, gu - (g\xi)v), \tag{2.4}$$

and

$$(g, v)(\mu, a) = (g\mu + \rho_v^*(ga), ga), \tag{2.5}$$

where $(g, v) \in S = G \times V$, $(\xi, u) \in \mathfrak{s} = \mathfrak{g} \times V$, $(\mu, a) \in \mathfrak{s}^* = \mathfrak{g}^* \times V^*$, $g\xi = \mathrm{Ad}_g \xi$, $g\mu = \mathrm{Ad}^*_{g^{-1}}\mu$, ga denotes the induced *left* action of g on a (the *left* action of G on V induces a *left* action of G on V^* — the inverse of the transpose of the action on V), $\rho_v : \mathfrak{g} \to V$ is the linear map given by $\rho_v(\xi) = \xi v$, and $\rho_v^* : V^* \to \mathfrak{g}^*$ is its dual.

Important Notation. For $a \in V^*$, we shall write, for notational convenience,

$$\rho_v^* a = v \diamond a \in \mathfrak{g}^* ,$$

which is a bilinear operation in v and a. Using this notation, the above formula for the coadjoint action reads

$$(g, v)(\mu, a) = (g\mu + v \diamond (ga), ga).$$

We shall also denote actions of groups and Lie algebras by simple concatenation. For example, the $\mathfrak{g}$–action on $\mathfrak{g}^*$ and V^*, which is defined as minus the dual map of the $\mathfrak{g}$–action on $\mathfrak{g}$ and V respectively, is denoted by $\xi\mu$ and ξa for $\xi \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$, and $a \in V^*$.

Using this concatenation notation for Lie algebra actions provides the following alternative expression of the definition of $v \diamond a \in \mathfrak{g}^*$: For all $v \in V$, $a \in V^*$ and $\eta \in \mathfrak{g}$, we define

$$\langle \eta a, v \rangle = - \langle v \diamond a \, , \eta \rangle \, .$$

Left Versus Right. When working with various models of continuum mechanics and plasmas it is convenient to work with *right* representations of G on the vector space V (as in, for example, Holm, Marsden and Ratiu [1986]). We shall denote the semidirect product by the same symbol $S = G \circledS V$, the action of G on V being denoted by vg. The formulae change under these conventions as follows. Group multiplication (the analog of (2.1)) is given by

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + v_1 g_2), \tag{2.6}$$

and the Lie algebra bracket on $\mathfrak{s} = \mathfrak{g} \circledS V$ (the analog of (2.3)) has the expression

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \, v_1\xi_2 - v_2\xi_1), \tag{2.7}$$

where we denote the induced action of $\mathfrak{g}$ on V by concatenation, as in $v_1\xi_2$. The adjoint and coadjoint actions have the formulae (analogs of (2.4) and (2.5))

$$(g, v)(\xi, u) = (g\xi, (u + v\xi)g^{-1}), \tag{2.8}$$

$$(g, v)(\mu, a) = (g\mu + (vg^{-1}) \diamond (ag^{-1}), ag^{-1}), \tag{2.9}$$

where, as usual, $g\xi = \mathrm{Ad}_g \xi$, $g\mu = \mathrm{Ad}^*_{g^{-1}} \mu$, ag denotes the inverse of the dual isomorphism defined by $g \in G$ (so that $g \mapsto ag$ is a *right* action). Note that the adjoint and coadjoint actions are *left* actions. In this case, the $\mathfrak{g}$–actions on $\mathfrak{g}^*$ and V^* are defined as before to be minus the dual map given by the $\mathfrak{g}$–actions on $\mathfrak{g}$ and V and are denoted by $\xi\mu$ (because it is a left action) and $a\xi$ (because it is a right action) respectively.

Lie-Poisson Brackets and Hamiltonian Vector Fields. For a *left* representation of G on V the $\pm$ Lie-Poisson bracket of two functions $f, k : \mathfrak{s}^* \to \mathbb{R}$ is given by

$$\{f,k\}_\pm(\mu,a) \;=\; \pm\left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\right]\right\rangle \pm \left\langle a, \frac{\delta f}{\delta \mu}\frac{\delta k}{\delta a} - \frac{\delta k}{\delta \mu}\frac{\delta f}{\delta a}\right\rangle \tag{2.10}$$

where $\delta f/\delta \mu \in \mathfrak{g}$, and $\delta f/\delta a \in V$ are the functional derivatives of f. The Hamiltonian vector field of $h : \mathfrak{s}^* \to \mathbb{R}$ has the expression

$$X_h(\mu,a) = \mp\left(\mathrm{ad}^*_{\delta h/\delta \mu}\,\mu - \frac{\delta h}{\delta a}\diamond a,\, -\frac{\delta h}{\delta \mu}\,a\right). \tag{2.11}$$

Thus, Hamilton's equations on the dual of a semidirect product are given by

$$\dot\mu \;=\; \mp\,\mathrm{ad}^*_{\delta h/\delta \mu}\,\mu \pm \frac{\delta h}{\delta a}\diamond a\,, \tag{2.12}$$

$$\dot a \;=\; \pm\,\frac{\delta h}{\delta \mu}\,a\,, \tag{2.13}$$

where overdot denotes time derivative. For *right* representations of G on V the above formulae change to:

$$\{f,k\}_\pm(\mu,a) \;=\; \pm\left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\right]\right\rangle \mp \left\langle a, \frac{\delta k}{\delta a}\frac{\delta f}{\delta \mu} - \frac{\delta f}{\delta a}\frac{\delta k}{\delta \mu}\right\rangle, \tag{2.14}$$

$$X_h(\mu,a) = \mp\left(\mathrm{ad}^*_{\delta h/\delta \mu}\,\mu + \frac{\delta h}{\delta a}\diamond a,\, a\,\frac{\delta h}{\delta \mu}\right), \tag{2.15}$$

$$\dot\mu \;=\; \mp\,\mathrm{ad}^*_{\delta h/\delta \mu}\,\mu \mp \frac{\delta h}{\delta a}\diamond a\,, \tag{2.16}$$

$$\dot a \;=\; \mp\, a\,\frac{\delta h}{\delta \mu}\,. \tag{2.17}$$

Symplectic Actions by Semidirect Products. To avoid a proliferation of signs, in *this section* we consider all semidirect products to come from a left representation. Of course if the representation is from the right, there are similar formulae.

We consider a symplectic action of S on a symplectic manifold P and assume that this action has an equivariant momentum map $\mathbf{J}_S : P \to \mathfrak{s}^*$. Since V is a (normal) subgroup of S, it also acts on P and has a momentum map $\mathbf{J}_V : P \to V^*$ given by

$$\mathbf{J}_V = i_V^* \circ \mathbf{J}_S\,,$$

where $i_V : V \to \mathfrak{s}$ is the inclusion $v \mapsto (0,v)$ and $i_V^* : \mathfrak{s}^* \to V^*$ is its dual. We think of this merely as saying that $\mathbf{J}_V$ is the second component of $\mathbf{J}_S$.

We can regard G as a subgroup of S by $g \mapsto (g, 0)$. Thus, G also has a momentum map that is the first component of $\mathbf{J}_S$ but this will play a secondary role in what follows. On the other hand, equivariance of $\mathbf{J}_S$ under G implies the following relation for $\mathbf{J}_V$:

$$\mathbf{J}_V(gz) = g\mathbf{J}_V(z) \tag{2.18}$$

where we denote the appropriate action of $g \in G$ on an element by concatenation, as before. To prove (2.18), one uses the fact that for the coadjoint action of S on $\mathfrak{s}^*$ the second component is just the dual of the given action of G on V.

The Classical Semidirect Product Reduction Theorem. In a number of interesting applications such as compressible fluids, the heavy top, MHD, etc., one has two symmetry groups that do not commute and thus the commuting reduction by stages theorem of Marsden and Weinstein [1974] does not apply. In this more general situation, it matters in what order one performs the reduction, which occurs, in particular for semidirect products. The main result covering the case of semidirect products has a complicated history, with important early contributions by many authors, as we have listed in the introduction. The final version of the theorem as we shall use it, is due to Marsden, Ratiu and Weinstein [1984a,b].

The semidirect product reduction theorem states, roughly speaking, that for the semidirect product $S = G \circledS V$ where G is a group acting on a vector space V and S is the semidirect product, one can first reduce T^*S by V and then by G and thereby obtain the same result as when reducing by S. As above, we let $\mathfrak{s} = \mathfrak{g} \circledS V$ denote the Lie algebra of S. The precise statement is as follows.

Theorem 2.1 (Semidirect Product Reduction Theorem.) *Let $S = G \circledS V$, choose $\sigma = (\mu, a) \in \mathfrak{g}^* \times V^*$, and reduce T^*S by the action of S at σ giving the coadjoint orbit $\mathcal{O}_\sigma$ through $\sigma \in \mathfrak{s}^*$. There is a symplectic diffeomorphism between $\mathcal{O}_\sigma$ and the reduced space obtained by reducing T^*G by the subgroup G_a (the isotropy of G for its action on V^* at the point $a \in V^*$) at the point $\mu|\mathfrak{g}_a$ where $\mathfrak{g}_a$ is the Lie algebra of G_a.*

Reduction by Stages. This result is a special case of a theorem on reduction by stages for semidirect products acting on a symplectic manifold (see Marsden, Misiolek, Perlmutter and Ratiu [1997] for this and more general results and see Leonard and Marsden [1997] for an application to underwater vehicle dynamics).

As above, consider a symplectic action of S on a symplectic manifold P and assume that this action has an equivariant momentum map $\mathbf{J}_S : P \to \mathfrak{s}^*$. As we have explained, the momentum map for the action of V is the map $\mathbf{J}_V : P \to V^*$ given by $\mathbf{J}_V = i_V^* \circ \mathbf{J}_S$

We carry out the reduction of P by S at a regular value $\sigma = (\mu, a)$ of the momentum map $\mathbf{J}_S$ for S in two stages using the following procedure. First, reduce P by V at the value a (assume it to be a regular value) to get the reduced space $P_a = \mathbf{J}_V^{-1}(a)/V$. Second, form the group G_a consisting of elements of G that leave the point a fixed using the action of G on V^*. One shows (and this step is not

trivial) that the group G_a acts on P_a and has an induced equivariant momentum map $\mathbf{J}_a : P_a \to \mathfrak{g}_a^*$, where $\mathfrak{g}_a$ is the Lie algebra of G_a, so one can reduce P_a at the point $\mu_a := \mu|\mathfrak{g}_a$ to get the reduced space $(P_a)_{\mu_a} = \mathbf{J}_a^{-1}(\mu_a)/(G_a)_{\mu_a}$.

Theorem 2.2 (Reduction by Stages for Semidirect Products.) *The reduced space $(P_a)_{\mu_a}$ is symplectically diffeomorphic to the reduced space P_σ obtained by reducing P by S at the point $\sigma = (\mu, a)$.*

Combined with the cotangent bundle reduction theorem (see Abraham and Marsden [1978] and Marsden [1992] for an exposition and references), the semidirect product reduction theorem is a useful tool. For example, using these tools, one sees readily that the generic coadjoint orbits for the Euclidean group are cotangent bundles of spheres with the associated coadjoint orbit symplectic structure given by the canonical structure plus a magnetic term.

Semidirect Product Reduction of Dynamics. There is a technique for reducing dynamics that is associated with the geometry of the semidirect product reduction theorem. One proceeds as follows:

- We start with a Hamiltonian H_{a_0} on T^*G that depends parametrically on a variable $a_0 \in V^*$.
- The Hamiltonian, regarded as a map $H : T^*G \times V^* \to \mathbb{R}$ is assumed to be invariant on T^*G under the action of G on $T^*G \times V^*$.
- One shows that this condition is equivalent to the invariance of the function H defined on $T^*S = T^*G \times V \times V^*$ extended to be constant in the variable V under the action of the semidirect product.
- By the semidirect product reduction theorem, the dynamics of H_{a_0} reduced by G_{a_0}, the isotropy group of a_0, is symplectically equivalent to Lie-Poisson dynamics on $\mathfrak{s}^* = \mathfrak{g}^* \times V^*$.
- This Lie-Poisson dynamics is given by the equations (2.12) and (2.13) for the function $h(\mu, a) = H(\alpha_g, g^{-1}a)$ where $\mu = g^{-1}\alpha_g$.

3 Lagrangian Semidirect Product Theory

Despite all the activity in the Hamiltonian theory of semidirect products, little attention has been paid to the corresponding Lagrangian side. Now that Lagrangian reduction is maturing (see Marsden and Scheurle [1993a,b]), it is appropriate to consider the corresponding Lagrangian question. We shall formulate four versions, depending on the nature of the actions and invariance properties of the Lagrangian. (Two of them are relegated to the appendix.)

It should be noted that *none of the theorems below require that the Lagrangian be nondegenerate.* The subsequent theory is entirely based on variational principles with symmetry and is not dependent on any previous Hamiltonian formulation. We

shall, however, show that this purely Lagrangian formulation is equivalent to the Hamiltonian formulation on duals of semidirect products, provided an appropriately defined Legendre transformation happens to be a diffeomorphism.

The theorems that follow are modelled after the reduction theorem for the basic Euler–Poincaré equations given earlier. However, as we shall explain, they are *not* literally special cases of it. To distinguish the two types of results, we shall use phrases like *basic* Euler–Poincaré equations for the equations (1.57) and simply the Euler–Poincaré equations or the Euler–Poincaré equations *with advection* or the Euler–Poincaré equations *with advected parameters*, for the equations that follow.

The main difference between the left (right) invariant Lagrangians considered in the theorem above and the ones we shall work with below is that L and l depend in addition on another parameter $a \in V^*$, where V is a representation space for the Lie group G and L has an invariance property relative to both arguments. As we shall see below, the resulting ***Euler–Poincaré*** equations are *not* the Euler–Poincaré equations for the semidirect product Lie algebra $\mathfrak{g} \circledS V^*$ or on $\mathfrak{g} \circledS V$, for that matter.

Upcoming Examples. As we shall see in the examples, the parameter $a \in V^*$ acquires dynamical meaning under Lagrangian reduction. For the heavy top, the parameter is the unit vector in the direction of gravity, which becomes a dynamical variable in the body representation. For compressible fluids, the parameter is the density of the fluid in the reference configuration, which becomes a dynamical variable (satisfying the continuity equation) in the spatial representation.

Left Representation and Left Invariant Lagrangian. We begin with the following ingredients:

- There is a *left* representation of Lie group G on the vector space V and G acts in the natural way on the *left* on $TG \times V^*$: $h(v_g, a) = (hv_g, ha)$.
- Assume that the function $L : TG \times V^* \to \mathbb{R}$ is left G–invariant.
- In particular, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) = L(v_g, a_0)$. Then L_{a_0} is left invariant under the lift to TG of the left action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0.
- Left G–invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(g^{-1}v_g, g^{-1}a_0) = L(v_g, a_0).$$

  Conversely, this relation defines for any $l : \mathfrak{g} \times V^* \to \mathbb{R}$ a left G–invariant function $L : TG \times V^* \to \mathbb{R}$.

- For a curve $g(t) \in G$, let

$$\xi(t) := g(t)^{-1}\dot{g}(t)$$

and define the curve $a(t)$ as the unique solution of the following linear differential equation with time dependent coefficients

$$\dot{a}(t) = -\xi(t)a(t),$$

with initial condition $a(0) = a_0$. The solution can be written as $a(t) = g(t)^{-1}a_0$.

Theorem 3.1 *With the preceding notation, the following are equivalent:*

i *With a_0 held fixed, Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0 \tag{3.1}$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii *$g(t)$ satisfies the Euler–Lagrange equations for L_{a_0} on G.*

iii *The constrained variational principle*[5]

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0 \tag{3.2}$$

holds on $\mathfrak{g} \times V^*$, using variations of ξ and a of the form

$$\delta\xi = \dot{\eta} + [\xi, \eta], \quad \delta a = -\eta a, \tag{3.3}$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

iv *The **Euler–Poincaré** equations*[6] *hold on $\mathfrak{g} \times V^*$*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a. \tag{3.4}$$

Proof. The equivalence of **i** and **ii** holds for any configuration manifold and so, in particular, it holds in this case.

Next we show the equivalence of **iii** and **iv**. Indeed, using the definitions, integrating by parts, and taking into account that $\eta(t_1) = \eta(t_2) = 0$, we compute the

[5] As with the basic Euler–Poincaré equations, this is not strictly a variational principle in the same sense as the standard Hamilton's principle. It is more of a Lagrange d'Alembert principle, because we impose the stated constraints on the variations allowed.

[6] Note that these equations are not the basic Euler–Poincaré equations because we are not regarding $\mathfrak{g} \times V^*$ as a Lie algebra. Rather these equations are thought of as a generalization of the classical Euler-Poisson equations for a heavy top, written in body angular velocity variables, as we shall see in the examples. Some authors may prefer the term Euler-Poisson-Poincaré equations for these equations.

variation of the integral to be

$$\begin{aligned}
\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt &= \int_{t_1}^{t_2} \left(\left\langle \frac{\delta l}{\delta \xi}, \delta\xi \right\rangle + \left\langle \delta a, \frac{\delta l}{\delta a} \right\rangle \right) dt \\
&= \int_{t_1}^{t_2} \left(\left\langle \frac{\delta l}{\delta \xi}, \dot{\eta} + \mathrm{ad}_\xi\, \eta \right\rangle - \left\langle \eta a, \frac{\delta l}{\delta a} \right\rangle \right) dt \\
&= \int_{t_1}^{t_2} \left(\left\langle -\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}, \eta \right\rangle + \left\langle \frac{\delta l}{\delta a} \diamond a, \eta \right\rangle \right) dt \\
&= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a, \eta \right\rangle dt
\end{aligned}$$

and so the result follows.

Finally we show that **i** and **iii** are equivalent. First note that the G–invariance of $L : TG \times V^* \to \mathbb{R}$ and the definition of $a(t) = g(t)^{-1}a_0$ imply that the integrands in (3.1) and (3.2) are equal. However, all variations $\delta g(t) \in TG$ of $g(t)$ with fixed endpoints induce and are induced by variations $\delta\xi(t) \in \mathfrak{g}$ of $\xi(t)$ of the form $\delta\xi = \dot{\eta} + [\xi, \eta]$ with $\eta(t) \in \mathfrak{g}$ vanishing at the endpoints; the relation between $\delta g(t)$ and $\eta(t)$ is given by $\eta(t) = g(t)^{-1}\delta g(t)$. This is the content of the following lemma proved in Bloch et al. [1996]. [7]

Lemma 3.2 *Let $g : U \subset \mathbb{R}^2 \to G$ be a smooth map and denote its partial derivatives by*

$$\xi(t, \varepsilon) = TL_{g(t,\varepsilon)^{-1}}(\partial g(t, \varepsilon)/\partial t)$$

and

$$\eta(t, \varepsilon) = TL_{g(t,\varepsilon)^{-1}}(\partial g(t, \varepsilon)/\partial \varepsilon).$$

Then

$$\frac{\partial \xi}{\partial \varepsilon} - \frac{\partial \eta}{\partial t} = [\xi, \eta]\,. \tag{3.5}$$

Conversely, if U is simply connected and $\xi, \eta : U \to \mathfrak{g}$ are smooth functions satisfying (3.5) then there exists a smooth function $g : U \to G$ such that $\xi(t, \varepsilon) = TL_{g(t,\varepsilon)^{-1}}(\partial g(t, \varepsilon)/\partial t)$ and $\eta(t, \varepsilon) = TL_{g(t,\varepsilon)^{-1}}(\partial g(t, \varepsilon)/\partial \varepsilon)$.

Thus, if **i** holds, we define $\eta(t) = g(t)^{-1}\delta g(t)$ for a variation $\delta g(t)$ with fixed endpoints. Then if we let $\delta\xi = g(t)^{-1}\dot{g}(t)$, we have by the above proposition $\delta\xi = \dot{\eta} + [\xi, \eta]$. In addition, the variation of $a(t) = g(t)^{-1}a_0$ is $\delta a(t) = -\eta(t)a(t)$. Conversely, if $\delta\xi = \dot{\eta} + [\xi, \eta]$ with $\eta(t)$ vanishing at the endpoints, we define $\delta g(t) = g(t)\eta(t)$ and the above proposition guarantees then that this $\delta g(t)$ is the general variation of $g(t)$ vanishing at the endpoints. From $\delta a(t) = -\eta(t)a(t)$ it follows that the variation of $g(t)a(t) = a_0$ vanishes, which is consistent with the dependence of L_{a_0} only on $g(t), \dot{g}(t)$. ■

[7] This lemma is simple for matrix groups, as in Marsden and Ratiu [1994], but it is less elementary for general Lie groups.

Cautionary Remarks. Let us explicitly show that *these Euler–Poincaré equations (3.4) are not the Euler–Poincaré equations for the semidirect product Lie algebra* $\mathfrak{g} \circledS V^*$. Indeed, by (1.57) the basic Euler–Poincaré equations

$$\frac{d}{dt}\frac{\delta l}{\delta(\xi,a)} = \mathrm{ad}^*_{(\xi,a)}\,\frac{\delta l}{\delta(\xi,a)}\,, \quad (\xi,a)\in \mathfrak{g}\circledS V^*$$

for $l : \mathfrak{g} \circledS V^* \to \mathbb{R}$ become

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a}\diamond a, \quad \frac{d}{dt}\frac{\delta l}{\delta a} = -\xi\frac{\delta l}{\delta a}\,,$$

which is a *different* system from that given by the Euler–Poincaré equation (3.4) and $\dot a = -\xi a$, even though the first equations of both systems are identical.

The Legendre Transformation. As we explained earlier, one normally thinks of passing from Euler–Poincaré equations on a Lie algebra $\mathfrak{g}$ to Lie–Poisson equations on the dual $\mathfrak{g}^*$ by means of the Legendre transformation. In our case, we start with a Lagrangian on $\mathfrak{g}\times V^*$ and perform a partial Legendre transformation in the variable ξ only, by writing

$$\mu = \frac{\delta l}{\delta \xi}\,, \quad h(\mu,a) = \langle \mu,\xi\rangle - l(\xi,a). \tag{3.6}$$

Since

$$\frac{\delta h}{\delta \mu} = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu}\right\rangle - \left\langle \frac{\delta l}{\delta \xi}\,,\, \frac{\delta \xi}{\delta \mu}\right\rangle = \xi\,,$$

and $\delta h/\delta a = -\delta l/\delta a$, we see that (3.4) and $\dot a(t) = -\xi(t)a(t)$ imply (2.11) for the *minus* Lie–Poisson bracket (that is, the sign + in (2.11)). If this Legendre transformation is invertible, then we can also pass from the the minus Lie–Poisson equations (2.11) to the Euler–Poincaré equations (3.4) together with the equations $\dot a(t) = -\xi(t)a(t)$.

Right Representation and Right Invariant Lagrangian. There are four versions of the preceding theorem, the given left-left version, a left-right, a right-left and a right-right version. For us, the most important ones are the left-left and the right-right versions. We state the remaining two in the appendix.

Here we make the following assumptions:

- There is a *right* representation of Lie group G on the vector space V and G acts in the natural way on the *right* on $TG\times V^*$: $(v_g,a)h = (v_gh, ah)$.
- Assume that the function $L : TG\times V^* \to \mathbb{R}$ is right G–invariant.
- In particular, if $a_0\in V^*$, define the Lagrangian $L_{a_0} : TG\to\mathbb{R}$ by $L_{a_0}(v_g) = L(v_g,a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0.

- Right G–invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(v_g g^{-1}, a_0 g^{-1}) = L(v_g, a_0).$$

  Conversely, this relation defines for any $l : \mathfrak{g} \times V^* \to \mathbb{R}$ a right G–invariant function $L : TG \times V^* \to \mathbb{R}$.

- For a curve $g(t) \in G$, let $\xi(t) := \dot{g}(t)g(t)^{-1}$ and define the curve $a(t)$ as the unique solution of the linear differential equation with time dependent coefficients $\dot{a}(t) = -a(t)\xi(t)$ with initial condition $a(0) = a_0$. The solution can be written as $a(t) = a_0 g(t)^{-1}$.

Theorem 3.3 *The following are equivalent:*

i *Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0 \tag{3.7}$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii *$g(t)$ satisfies the Euler–Lagrange equations for L_{a_0} on G.*

iii *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0 \tag{3.8}$$

holds on $\mathfrak{g} \times V^*$, using variations of the form

$$\delta\xi = \dot{\eta} - [\xi, \eta], \quad \delta a = -a\eta, \tag{3.9}$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

iv *The Euler–Poincaré equations hold on $\mathfrak{g} \times V^*$*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a. \tag{3.10}$$

The same partial Legendre transformation (3.6) as before maps the Euler–Poincaré equations (3.10), together with the equations $\dot{a} = -a\xi$ for a to the plus Lie–Poisson equations (2.16) and (2.17) (that is, one chooses the overall minus sign in these equations).

Generalizations. The Euler–Poincaré equations are a special case of the reduced Euler-Lagrange equations (see Marsden and Scheurle [1993b] and Cendra, Marsden and Ratiu [1998]). This is shown explicitly in Cendra, Holm, Marsden and Ratiu [1998]. There is, however, an easy generalization that is needed in some of the examples we will consider. Namely, if $L : TG \times V^* \times TQ$ and if the group G acts in a trivial way on TQ, then one can carry out the reduction in the same way as

above, carrying along the Euler-Lagrange equations for the factor Q at each step. The resulting reduced equations then are the Euler–Poincaré equations above for the $\mathfrak{g}$ factor, together the Euler-Lagrange equations for the $q \in Q$ factor. The system is coupled through the dependence of L on all variables. (For a full statement, see Cendra, Holm, Hoyle and Marsden [1998], who use this extension to treat the Euler–Poincaré formulation of the Maxwell-Vlasov equations for plasma physics.)

4 The Kelvin-Noether Theorem

In this section, we explain a version of the Noether theorem that holds for solutions of the Euler–Poincaré equations. Our formulation is motivated and designed for ideal continuum theories (and hence the name Kelvin-Noether), but it may also of interest for finite dimensional mechanical systems. Of course it is well known (going back at least to the pioneering work of Arnold [1966a]) that the Kelvin circulation theorem for ideal flow is closely related to the Noether theorem applied to continua using the particle relabelling symmetry group.

There is a version of the theorem that holds for each of the choices of conventions, but we shall pick the left-left conventions to illustrate the result.

The Kelvin-Noether Quantity. We start with a Lagrangian L_{a_0} depending on a parameter $a_0 \in V^*$ as above. We introduce a manifold $\mathcal{C}$ on which G acts (we assume this is also a left action) and suppose we have an equivariant map $\mathcal{K} : \mathcal{C} \times V^* \to \mathfrak{g}^{**}$.

As we shall see, in the case of continuum theories, the space $\mathcal{C}$ will be a loop space and $\langle \mathcal{K}(c,a), \mu \rangle$ for $c \in \mathcal{C}$ and $\mu \in \mathfrak{g}^*$ will be a circulation. This class of examples also shows why we *do not* want to identify the double dual $\mathfrak{g}^{**}$ with $\mathfrak{g}$.

Define the ***Kelvin-Noether quantity*** $I : \mathcal{C} \times \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$I(c, \xi, a) = \left\langle \mathcal{K}(c,a), \frac{\delta l}{\delta \xi}(\xi, a) \right\rangle. \tag{4.1}$$

We are now ready to state the main theorem of this section.

Theorem 4.1 (Kelvin-Noether.) *Fixing $c_0 \in \mathcal{C}$, let $\xi(t), a(t)$ satisfy the Euler–Poincaré equations and define $g(t)$ to be the solution of $\dot{g}(t) = g(t)\xi(t)$ and, say, $g(0) = e$. Let $c(t) = g(t)^{-1}c_0$ and $I(t) = I(c(t), \xi(t), a(t))$. Then*

$$\frac{d}{dt} I(t) = \left\langle \mathcal{K}(c(t), a(t)), \frac{\delta l}{\delta a} \diamond a \right\rangle. \tag{4.2}$$

Proof. First of all, write $a(t) = g(t)^{-1}a_0$ as we did previously and use equivariance to write $I(t)$ as follows:

$$\left\langle \mathcal{K}(c(t), a(t)), \frac{\delta l}{\delta \xi}(\xi(t), a(t)) \right\rangle = \left\langle \mathcal{K}(c_0, a_0), g(t) \left[\frac{\delta l}{\delta \xi}(\xi(t), a(t)) \right] \right\rangle.$$

The g^{-1} pulls over to the right side as g (and not g^{-1}) because of our conventions of always using left representations. We now differentiate the right hand side of this

equation. To do so, we use the following well known formula for differentiating the coadjoint action (see Marsden and Ratiu [1994], page 276):

$$\frac{d}{dt}[g(t)\mu(t)] = g(t)\left[-\operatorname{ad}^*_{\xi(t)}\mu(t) + \frac{d}{dt}\mu(t)\right],$$

where, as usual,

$$\xi(t) = g(t)^{-1}\dot{g}(t).$$

Using this coadjoint action formula and the Euler–Poincaré equations, we obtain

$$\begin{aligned}
\frac{d}{dt}I &= \frac{d}{dt}\left\langle \mathcal{K}(c_0,a_0), g(t)\left[\frac{\delta l}{\delta \xi}(\xi(t),a(t))\right]\right\rangle \\
&= \left\langle \mathcal{K}(c_0,a_0), \frac{d}{dt}\left\{g(t)\left[\frac{\delta l}{\delta \xi}(\xi(t),a(t))\right]\right\}\right\rangle \\
&= \left\langle \mathcal{K}(c_0,a_0), g(t)\left[-\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a}\diamond a\right]\right\rangle \\
&= \left\langle \mathcal{K}(c_0,a_0), g(t)\left[\frac{\delta l}{\delta a}\diamond a\right]\right\rangle \\
&= \left\langle g(t)^{-1}\mathcal{K}(c_0,a_0), \left[\frac{\delta l}{\delta a}\diamond a\right]\right\rangle \\
&= \left\langle \mathcal{K}(c(t),a(t)), \left[\frac{\delta l}{\delta a}\diamond a\right]\right\rangle,
\end{aligned}$$

where, in the last steps, we used the definitions of the coadjoint action, as well as the Euler–Poincaré equation (3.4) and the equivariance of the map $\mathcal{K}$. ■

Corollary 4.2 *For the basic Euler–Poincaré equations, the Kelvin quantity $I(t)$, defined the same way as above but with $I : \mathcal{C}\times\mathfrak{g}\to\mathbb{R}$, is conserved.*

For a review of the standard Noether theorem results for energy and momentum conservation in the context of the general theory, see, e.g., Marsden and Ratiu [1994].

5 The Heavy Top

In this section we shall use Theorem 3.1 to derive the classical Euler–Poisson equations for the heavy top. Our purpose is merely to illustrate the theorem with a concrete example.

The Heavy Top Lagrangian. The heavy top kinetic energy is given by the left invariant metric on $SO(3)$ whose value at the identity is $\langle \mathbf{\Omega}_1, \mathbf{\Omega}_2\rangle = \mathbb{I}\mathbf{\Omega}_1\cdot\mathbf{\Omega}_2$, where $\mathbf{\Omega}_1, \mathbf{\Omega}_2 \in \mathbb{R}^3$ are thought of as elements of $\mathfrak{so}(3)$, the Lie algebra of $SO(3)$, via the isomorphism $\mathbf{\Omega}\in\mathbb{R}^3 \mapsto \hat{\mathbf{\Omega}}\in\mathfrak{so}(3)$, $\hat{\mathbf{\Omega}}\mathbf{v} := \mathbf{\Omega}\times\mathbf{v}$, and where $\mathbb{I}$ is the (time independent) moment of inertia tensor in body coordinates, usually taken as

a diagonal matrix by choosing the body coordinate system to be a principal axes body frame. This kinetic energy is thus left invariant under the full group $SO(3)$. The potential energy is given by the work done in lifting the weight of the body to the height of its center of mass, with the direction of gravity pointing downwards. If M denotes the total mass of the top, g the magnitude of the gravitational acceleration, $\boldsymbol{\chi}$ the unit vector of the oriented line segment pointing from the fixed point about which the top rotates (the origin of a spatial coordinate system) to the center of mass of the body, and ℓ its length, then the potential energy is given by $Mg\ell\,\mathbf{R}^{-1}\mathbf{e}_3\cdot\boldsymbol{\chi}$, where $\mathbf{e}_3$ is the axis of the spatial coordinate system parallel to the direction of gravity but pointing upwards. This potential energy breaks the full $SO(3)$ symmetry and is invariant only under the rotations S^1 about the $\mathbf{e}_3$–axis.

However, for the application of Theorem 3.1 we are supposed to think of the Lagrangian of the heavy top as a function on $TSO(3)\times\mathbb{R}^3\to\mathbb{R}$. That is, we need to think of the potential energy as a function of $(u_{\mathbf{R}},\mathbf{v})\in TSO(3)\times\mathbb{R}^3$. This means that we need to replace the vector giving the direction of gravity $\mathbf{e}_3$ by an arbitrary vector $\mathbf{v}\in\mathbb{R}^3$, so that the potential equals

$$U(u_{\mathbf{R}},\mathbf{v})=Mg\ell\,\mathbf{R}^{-1}\mathbf{v}\cdot\boldsymbol{\chi}.$$

Thought of this way, the potential is $SO(3)$–invariant. Indeed, if $\mathbf{R}'\in SO(3)$ is arbitrary, then

$$\begin{aligned}U(\mathbf{R}'u_{\mathbf{R}},\mathbf{R}'\mathbf{v}) &= Mg\ell\,(\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'\mathbf{v}\cdot\boldsymbol{\chi}\\ &= Mg\ell\,\mathbf{R}^{-1}\mathbf{v}\cdot\boldsymbol{\chi}\\ &= U(u_{\mathbf{R}},\mathbf{v})\end{aligned}$$

and the hypotheses of Theorem 3.1 are satisfied. Thus, the heavy top equations of motion in the body representation are given by the Euler–Poincaré equations (3.4) for the Lagrangian $l:\mathfrak{so}(3)\times\mathbb{R}^3\to\mathbb{R}$.

The Reduced Lagrangian. To compute the explicit expression of l, denote by $\boldsymbol{\Omega}$ the angular velocity and by $\boldsymbol{\Pi}=\mathbb{I}\boldsymbol{\Omega}$ the angular momentum in the body representation. Let $\boldsymbol{\Gamma}=\mathbf{R}^{-1}\mathbf{v}$; if $\mathbf{v}=\mathbf{e}_3$, the unit vector pointing upwards on the vertical spatial axis, then $\boldsymbol{\Gamma}$ is this unit vector viewed by an observer moving with the body. The Lagrangian $l:\mathfrak{so}(3)\times\mathbb{R}^3\to\mathbb{R}$ is thus given by

$$\begin{aligned}l(\boldsymbol{\Omega},\boldsymbol{\Gamma}) &= L(\mathbf{R}^{-1}u_{\mathbf{R}},\mathbf{R}^{-1}\mathbf{v})\\ &= \frac{1}{2}\boldsymbol{\Pi}\cdot\boldsymbol{\Omega}-U(\mathbf{R}^{-1}u_{\mathbf{R}},\mathbf{R}^{-1}\mathbf{v})\\ &= \frac{1}{2}\boldsymbol{\Pi}\cdot\boldsymbol{\Omega}-Mg\ell\,\boldsymbol{\Gamma}\cdot\boldsymbol{\chi}\,.\end{aligned}$$

The Euler–Poincaré Equations. It is now straightforward to compute the Euler–Poincaré equations. First note that

$$\frac{\delta l}{\delta\boldsymbol{\Omega}}=\boldsymbol{\Pi},\quad \frac{\delta l}{\delta\boldsymbol{\Gamma}}=-Mg\ell\,\boldsymbol{\chi}\,.$$

Since

$$\mathrm{ad}^*_{\boldsymbol{\Omega}}\,\boldsymbol{\Pi} = \boldsymbol{\Pi}\times\boldsymbol{\Omega}\,,\quad \mathbf{v}\diamond\boldsymbol{\Gamma} = -\boldsymbol{\Gamma}\times\mathbf{v}\,,$$

and

$$\hat{\boldsymbol{\Omega}}\boldsymbol{\Gamma} = -\boldsymbol{\Gamma}\times\boldsymbol{\Omega}\,,$$

the Euler–Poincaré equations are

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi}\times\boldsymbol{\Omega} + Mg\ell\,\boldsymbol{\Gamma}\times\boldsymbol{\chi}\,,$$

which are coupled to the $\boldsymbol{\Gamma}$ evolution

$$\dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}\times\boldsymbol{\Omega}\,.$$

This system of two vector equations comprises the classical Euler–Poisson equations, which describe the motion of the heavy top in the body representation.

The Kelvin-Noether theorem Let $\mathcal{C} = \mathfrak{g}$ and let $\mathcal{K}:\mathcal{C}\times V^*\to\mathfrak{g}^{**}\cong\mathfrak{g}$ be the map $(\mathbf{W},\boldsymbol{\Gamma})\mapsto\mathbf{W}$. Then the Kelvin-Noether theorem gives the statement

$$\frac{d}{dt}\,\langle\mathbf{W},\boldsymbol{\Pi}\rangle = Mg\ell\,\langle\mathbf{W},\boldsymbol{\Gamma}\times\boldsymbol{\chi}\rangle$$

where $\mathbf{W}(t) = \mathbf{R}(t)^{-1}\mathbf{w}$; in other words, $\mathbf{W}(t)$ is the body representation of a space fixed vector. This statement is easily verified directly. Also, note that $\langle\mathbf{W},\boldsymbol{\Pi}\rangle = \langle\mathbf{w},\boldsymbol{\pi}\rangle$, with $\boldsymbol{\pi} = \mathbf{R}(t)\boldsymbol{\Pi}$, so the Kelvin-Noether theorem may be viewed as a statement about the rate of change of the momentum map of the system (the spatial angular momentum) relative to the full group of rotations, not just those about the vertical axis.

6 The Euler–Poincaré Equations in Continuum Mechanics

In this section we will apply the Euler–Poincaré equations in the case of continuum mechanical systems. We let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{D}$ (or, more generally, a smooth compact manifold with boundary and given volume form or density). We let $\mathrm{Diff}(\mathcal{D})$ denote the diffeomorphism group of $\mathcal{D}$ of some given Sobolev class. If the domain $\mathcal{D}$ is not compact, then various decay hypotheses at infinity need to be imposed. Under such conditions, $\mathrm{Diff}(\mathcal{D})$ is a smooth infinite dimensional manifold and a topological group relative to the induced manifold topology. Right translation is smooth but left translation and inversion are only continuous. Thus, $\mathrm{Diff}(\mathcal{D})$ is not actually a Lie group and the previous theory does not apply, strictly speaking. Nevertheless, if one uses right translations and right representations, the Euler–Poincaré equations of Theorem 3.3 do make sense, as a simple verification shows. We shall illustrate below such computations, by verifying several key facts in the proof.

Let $\mathfrak{X}(\mathcal{D})$ denote the space of vector fields on $\mathcal{D}$ of some fixed differentiability class. Formally, this is the *right* Lie algebra of $\mathrm{Diff}(\mathcal{D})$, that is, its standard *left* Lie algebra bracket is *minus* the usual Lie bracket for vector fields. To distinguish between these brackets, we shall reserve in what follows the notation $[\mathbf{u}, \mathbf{v}]$ for the standard Jacobi-Lie bracket of the vector fields $\mathbf{u}, \mathbf{v} \in \mathfrak{X}(\mathcal{D})$ whereas the notation $\mathrm{ad}_{\mathbf{u}}\,\mathbf{v} := -[\mathbf{u}, \mathbf{v}]$ denotes the adjoint action of the *left* Lie algebra on itself.

We also let $\mathfrak{X}(\mathcal{D})^*$ denote the geometric dual space of $\mathfrak{X}(\mathcal{D})$, that is, $\mathfrak{X}(\mathcal{D})^* := \Omega^1(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$, the space of one–form densities on $\mathcal{D}$. If $\alpha \otimes m \in \Omega^1(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$, the pairing of $\alpha \otimes m$ with $\mathbf{u} \in \mathfrak{X}(\mathcal{D})$ is given by

$$\langle \alpha \otimes m, \mathbf{u} \rangle = \int_{\mathcal{D}} \alpha \cdot \mathbf{u}\, m \tag{6.1}$$

where $\alpha \cdot \mathbf{u}$ is the standard contraction of a one–form with a vector field. For $\mathbf{u} \in \mathfrak{X}(\mathcal{D})$ and $\alpha \otimes m \in \mathfrak{X}(\mathcal{D})^*$, the dual of the adjoint representation is defined by

$$\langle \mathrm{ad}^*_{\mathbf{u}}(\alpha \otimes m), \mathbf{v} \rangle = -\int_{\mathcal{D}} \alpha \cdot [\mathbf{u}, \mathbf{v}]\; m$$

and its expression is

$$\mathrm{ad}^*_{\mathbf{u}}(\alpha \otimes m) = (\pounds_{\mathbf{u}}\alpha + (\mathrm{div}_m\, \mathbf{u})\alpha) \otimes m = \pounds_{\mathbf{u}}(\alpha \otimes m)\,, \tag{6.2}$$

where $\mathrm{div}_m \mathbf{u}$ is the divergence of $\mathbf{u}$ relative to the measure m, that is, $\pounds_{\mathbf{u}} m = (\mathrm{div}_m \mathbf{u})m$. Hence if $\mathbf{u} = u^i \partial/\partial x^i$, $\alpha = \alpha_i dx^i$, the one–form factor in the preceding formula for $\mathrm{ad}^*_{\mathbf{u}}(\alpha \otimes m)$ has the coordinate expression

$$\left(u^j \frac{\partial \alpha_i}{\partial x^j} + \alpha_j \frac{\partial u^j}{\partial x^i} + (\mathrm{div}_m\, \mathbf{u})\alpha_i \right) dx^i \;=\; \left(\frac{\partial}{\partial x^j}(u^j \alpha_i) + \alpha_j \frac{\partial u^j}{\partial x^i} \right) dx^i\,, \tag{6.3}$$

the last equality assuming that the divergence is taken relative to the standard measure $m = d^n\mathbf{x}$ in $\mathbb{R}^n$. (On a Riemannian manifold the metric divergence needs to be used.)

Throughout the rest of the paper we shall use the following conventions and terminology for the standard quantities in continuum mechanics. Elements of $\mathcal{D}$ representing the material particles of the system are denoted by X; their coordinates $X^A, A = 1, ..., n$ may thus be regarded as the particle labels. A ***configuration***, which we typically denote by η, is an element of $\mathrm{Diff}(\mathcal{D})$. A ***motion*** η_t is a path in $\mathrm{Diff}(\mathcal{D})$. The ***Lagrangian*** or ***material velocity*** $\mathbf{V}(X, t)$ of the continuum along the motion η_t is defined by taking the time derivative of the motion keeping the particle labels (the reference particles) X fixed:

$$\mathbf{V}(X, t) := \frac{d\eta_t(X)}{dt} := \left.\frac{\partial}{\partial t}\right|_X \eta_t(X),$$

the second equality being a convenient shorthand notation of the time derivative for fixed X.

Consistent with this definition of velocity, the tangent space to $\mathrm{Diff}(\mathcal{D})$ at $\eta \in \mathrm{Diff}(\mathcal{D})$ is given by

$$T_\eta \,\mathrm{Diff}(\mathcal{D}) = \{\mathbf{V}_\eta : \mathcal{D} \to T\mathcal{D} \mid \mathbf{V}_\eta(X) \in T_{\eta(X)}\mathcal{D}\}.$$

Elements of $T_\eta \,\mathrm{Diff}(\mathcal{D})$ are usually thought of as vector fields on $\mathcal{D}$ covering η. The tangent lift of right and left translations on $T\,\mathrm{Diff}(\mathcal{D})$ by $\varphi \in \mathrm{Diff}(\mathcal{D})$ have the expressions

$$\mathbf{V}_\eta\varphi := T_\eta R_\varphi(\mathbf{V}_\eta) = \mathbf{V}_\eta \circ \varphi \qquad \text{and} \qquad \varphi\mathbf{V}_\eta := T_\eta L_\varphi(\mathbf{V}_\eta) = T\varphi \circ \mathbf{V}_\eta\,.$$

During a motion η_t, the particle labeled by X describes a path in $\mathcal{D}$ whose points $x(X,t) := \eta_t(X)$ are called the ***Eulerian*** or ***spatial points*** of this path. The derivative $\mathbf{v}(x,t)$ of this path, keeping the Eulerian point x fixed, is called the ***Eulerian*** or ***spatial velocity*** of the system:

$$\mathbf{v}(x,t) := \mathbf{V}(X,t) := \left.\frac{\partial}{\partial t}\right|_x \eta_t(X).$$

Thus the Eulerian velocity $\mathbf{v}$ is a time dependent vector field on $\mathcal{D}$: $\mathbf{v}_t \in \mathfrak{X}(\mathcal{D})$, where $\mathbf{v}_t(x) := \mathbf{v}(x,t)$. We also have the fundamental relationship

$$\mathbf{V}_t = \mathbf{v}_t \circ \eta_t\,,$$

where $\mathbf{V}_t(X) := \mathbf{V}(X,t)$.

The representation space V^* of $\mathrm{Diff}(\mathcal{D})$ in continuum mechanics is often some subspace of $\mathfrak{T}(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$, the tensor field densities on $\mathcal{D}$ and the representation is given by pull back. It is thus a *right* representation of $\mathrm{Diff}(\mathcal{D})$ on $\mathfrak{T}(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$. The right action of the Lie algebra $\mathfrak{X}(\mathcal{D})$ on V^* is given by $a\mathbf{v} := \pounds_{\mathbf{v}} a$, the Lie derivative of the tensor field density a along the vector field $\mathbf{v}$.

The Lagrangian of a continuum mechanical system is a function $L : T\,\mathrm{Diff}(\mathcal{D}) \times V^* \to \mathbb{R}$ which is right invariant relative to the tangent lift of right translation of $\mathrm{Diff}(\mathcal{D})$ on itself and pull back on the tensor field densities.

Thus, the Lagrangian L induces a function $l : \mathfrak{X}(\mathcal{D}) \times V^* \to \mathbb{R}$ given by

$$l(\mathbf{v}, a) = L(\mathbf{v} \circ \eta, \eta^* a),$$

where $\mathbf{v} \in \mathfrak{X}(\mathcal{D})$ and $a \in V^* \subset \mathfrak{T}(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$, and where $\eta^* a$ denotes the pull back of a by the diffeomorphism η and $\mathbf{v}$ is the Eulerian velocity. The evolution of a is given by the equation

$$\dot{a} = -\pounds_{\mathbf{v}}\, a.$$

The solution of this equation, given the initial condition a_0, is $a(t) = \varphi_{t*} a_0$, where the lower star denotes the push forward operation and φ_t is the flow of $\mathbf{v}$.

Advected Eulerian quantities are defined in continuum mechanics to be those variables which are Lie transported by the flow of the Eulerian velocity field. Using

this standard terminology, the above equation states that the tensor field density a (which may include mass density and other Eulerian quantities) is advected.

As remarked, $V^* \subset \mathfrak{T}(\mathcal{D}) \otimes \mathrm{Den}(\mathcal{D})$. On a general manifold, tensors of a given type have natural duals. For example, symmetric covariant tensors are dual to symmetric contravariant tensor densities, the pairing being given by the integration of the natural contraction of these tensors. Likewise, k–forms are naturally dual to $(n-k)$–forms, the pairing being given by taking the integral of their wedge product.

The operation $\diamond$ between elements of V and V^* producing an element of $\mathfrak{X}(\mathcal{D})^*$ introduced in section 2 becomes

$$\langle v \diamond a, \mathbf{u} \rangle = -\int_{\mathcal{D}} v \cdot \pounds_{\mathbf{u}}\, a \;, \tag{6.4}$$

where $v \cdot \pounds_{\mathbf{u}}\, a$ denotes the contraction, as described above, of elements of V and elements of V^*. (These operations do *not* depend on a Riemannian structure.)

For a path $\eta_t \in \mathrm{Diff}(\mathcal{D})$ let $\mathbf{v}(x,t)$ be its Eulerian velocity and consider as in the hypotheses of Theorem 3.3 the curve $a(t)$ with initial condition a_0 given by the equation

$$\dot{a} + \pounds_{\mathbf{v}} a = 0. \tag{6.5}$$

Let $L_{a_0}(\mathbf{V}) := L(\mathbf{V}, a_0)$. We can now state Theorem 3.3 in this particular, but very useful, setting.

Theorem 6.1 (Euler–Poincaré Theorem for Continua.) *Consider a path η_t in $\mathrm{Diff}(\mathcal{D})$ with Lagrangian velocity $\mathbf{V}$ and Eulerian velocity $\mathbf{v}$. The following are equivalent:*

i *Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L\left(X, \mathbf{V}_t(X), a_0(X)\right)\, dt = 0 \tag{6.6}$$

holds, for variations $\delta\eta_t$ vanishing at the endpoints.

ii *η_t satisfies the Euler–Lagrange equations for L_{a_0} on $\mathrm{Diff}(\mathcal{D})$.*[8]

iii *The constrained variational principle in Eulerian coordinates*

$$\delta \int_{t_1}^{t_2} l(\mathbf{v}, a)\, dt = 0 \tag{6.7}$$

holds on $\mathfrak{X}(\mathcal{D}) \times V^*$, using variations of the form

$$\delta\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} + [\mathbf{v}, \mathbf{u}], \qquad \delta a = -\pounds_{\mathbf{u}}\, a, \tag{6.8}$$

where $\mathbf{u}_t = \delta\eta_t \circ \eta_t^{-1}$ vanishes at the endpoints.

[8] We do not write these equations explicitly since to do so would require either a coordinatization of the diffeomorphism group, which is not easy to give explictly, or requires more structure, such as an affine connection on this group. Certainly writing the equations formally, imagining that η and $\dot{\eta}$ form valid coordinates in which the Euler–Lagrange equations hold is not correct.

iv *The Euler–Poincaré equations for continua*

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \mathbf{v}} = -\operatorname{ad}^*_{\mathbf{v}}\frac{\delta l}{\delta \mathbf{v}} + \frac{\delta l}{\delta a}\diamond a = -\pounds_{\mathbf{v}}\frac{\delta l}{\delta \mathbf{v}} + \frac{\delta l}{\delta a}\diamond a\,, \tag{6.9}$$

hold, where the $\diamond$ operation given by (6.2) needs to be determined on a case by case basis, depending on the nature of the tensor a. (Remember that $\delta l/\delta \mathbf{v}$ is a one–form density.)

Remarks.

1. Of course, this theorem can be proven directly by imitating the proof of Theorem 3.1 with appropriate modifications for right representations and right actions. For those used to the more concrete language of continuum mechanics as opposed to that of Lie algebras, the following string of equalities shows that **iii** is equivalent to **iv**:

$$\begin{aligned} 0 &= \delta\int_{t_1}^{t_2} l(\mathbf{v},a)dt = \int_{t_1}^{t_2}\left(\frac{\delta l}{\delta \mathbf{v}}\cdot\delta\mathbf{v} + \frac{\delta l}{\delta a}\cdot\delta a\right)dt \\ &= \int_{t_1}^{t_2}\left[\frac{\delta l}{\delta \mathbf{v}}\cdot\left(\frac{\partial \mathbf{u}}{\partial t} - \operatorname{ad}_{\mathbf{v}}\,\mathbf{u}\right) - \frac{\delta l}{\delta a}\cdot\pounds_{\mathbf{u}}\,a\right]dt \\ &= \int_{t_1}^{t_2}\mathbf{u}\cdot\left[-\frac{\partial}{\partial t}\frac{\delta l}{\delta \mathbf{v}} - \operatorname{ad}^*_{\mathbf{v}}\frac{\delta l}{\delta \mathbf{v}} + \frac{\delta l}{\delta a}\diamond a\right]dt\,. \end{aligned} \tag{6.10}$$

2. Similarly, one can deduce by hand the form (6.8) of the variations in the constrained variational principle (6.7) by a direct calculation. This proceeds as follows. One writes the basic relation between the spatial and material velocities, namely $\mathbf{v}(x,t) = \dot\eta(\eta_t^{-1}(x),t)$. One then takes the variation of this equation with respect to η and uses the definition $\mathbf{u}(x,t) = \delta\eta(\eta_t^{-1}(x),t)$ together with a calculation of its time derivative. Of course, one can also do this calculation using the inverse map η_t^{-1} instead of η as the basic variable, see Holm, Marsden, and Ratiu [1986], Holm [1996a,b].

3. As we mentioned in the context of perfect fluids, the preceding sort of calculation for $\delta\mathbf{v}$ in fluid mechanics and the interpretation of this restriction on the form of the variations as the so-called Lin constraints is due to Bretherton [1970].

4. The coordinate expressions for $(\delta l/\delta a)\diamond a$ required to complete the equations of motion are given in the next section for several choices of $a_0(X)$ in three dimensions. Namely, we shall discuss the choices corresponding to scalars, one-forms, two-forms, densities in three dimensions, and symmetric tensors. In the equations of motion, all of these quantities will be advected.

5. As with the general theory, variations of the action in the advected tensor quantities contribute to the equations of motion which follow from Hamilton's principle. At the level of the action l for the Euler–Poincaré equations, the

Legendre transform in the variable $\mathbf{v}$ alone is often nonsingular, and when it is, it produces the Hamiltonian formulation of Eulerian fluid motions with a Lie-Poisson bracket defined on the dual of the semidirect product algebra of vector fields acting amongst themselves by Lie bracket and on tensor fields and differential forms by the Lie derivative. This is a special instance of the more general facts for Lie algebras that were discussed earlier.

6. As mentioned earlier, in the absence of the tensor fields a and when l is the kinetic energy metric, the basic Euler–Poincaré equations are the *geodesic spray equations* for geodesic motion on the diffeomorphism group with respect to that metric. See, e.g., Arnold [1966a], Ovsienko and Khesin [1987], Zeitlin and Kambe [1993], Zeitlin and Pasmanter [1994], Ono [1995a, 1995b] and Kouranbaeva [1997] for details in particular applications of ideal continuum mechanics.

Remarks on the inverse map and the tensor fields a for fluids. In the case of fluids in the Lagrangian picture, the flow of the fluid is a diffeomorphism which takes a fluid parcel along a path from its initial position X, in a "reference configuration" to its current position x in the "container", i.e., in the Eulerian domain of flow. As we have described, this forward map is denoted by $\eta : X \mapsto x$. The inverse map $\eta^{-1} : x \mapsto X$ provides the map assigning the Lagrangian labels to a given spatial point. Interpreted as passive scalars, these Lagrangian labels are simply advected with the fluid flow, $\dot{X} = 0$. In the Lagrangian picture, a tensor density in the reference configuration $a_0(X)$ (satisfying $\dot{a}_0(X) = 0$) consists of *invariant* tensor functions of the initial reference positions and their differentials. These tensor functions are parameters of the initial fluid reference configuration (e.g., the initial density distribution, which is an invariant n-form).

When viewed in the Eulerian picture as

$$a_t(x) := (\eta_{t*}a_0)(x) = (\eta_t^{-1*}a_0)(x),$$

i.e.,

$$a_0(X) := (\eta_t^* a_t)(X) = (\eta_{t*}^{-1}a_0)(X),$$

the time invariant tensor density $a_0(X)$ in the reference configuration acquires advective dynamics in the Eulerian picture, namely

$$\dot{a}_0(X) = \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right) a(x,t) = 0,$$

where $\pounds_{\mathbf{v}}$ denotes Lie derivative with respect to the Eulerian velocity field $\mathbf{v}(x,t)$. This relation results directly from the well known Lie derivative formula for tensor fields. (See, for example, Abraham, Marsden and Ratiu [1988].)

Mapping the time invariant quantity a_0 (a tensor density function of X) to the time varying quantity a_t (a tensor density function of x) as explained above is a special case of the general way we advect quantities in V^* in the general theory.

Specifically, we can view this advection of a_t as being the fluid analogue of the advection of the unit vector along the direction of gravity (a spatially fixed quantity) by means of the body rotation vector in the heavy top example.

Consistent with the fact that the heavy top is a *left invariant* system while continuum theories are *right invariant*, the advected tensor density a_t is a spatial quantity, while the advected direction of gravity is a body quantity. If we were to take the inverse map η^{-1} as the basic group variable, rather than the map η, then continuum theories would also become left invariant.

The continuity equation for the mass density. We will need to impose an additional assumption on our continuum theory. Namely, we assume that amongst the tensor densities being advected, there is a special one, namely the mass density. This of course is a tensor density that occurs in all continuum theories. We denote this density by $\rho d^n x$ and it is advected according to the standard principles discussed above. Thus, ρ satisfies the usual continuity equation:

$$\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{v}) = 0.$$

In the Lagrangian picture we have $\rho d^n x = \rho_0(X) d^n X$, where $\rho_0(X)$ is the mass density in the reference configuration. It will also be convenient in the continuum examples below to define Lagrangian *mass* coordinates $\ell(X)$ satisfying $\rho d^n x = d^n \ell$ with $\dot{\ell} = 0$. (When using Lagrangian mass coordinates, we shall denote the density ρ as D.) We assume that ρ (or D) is strictly positive.

The Kelvin-Noether Circulation Theorem. Let $\mathcal{C}$ be the space of continuous loops $\gamma : S^1 \to \mathcal{D}$ in $\mathcal{D}$ and let the group $\operatorname{Diff}(\mathcal{D})$ act on $\mathcal{C}$ on the left by $(\eta, \gamma) \in \operatorname{Diff}(\mathcal{D}) \times \mathcal{C} \mapsto \eta\gamma \in \mathcal{C}$, where $\eta\gamma = \eta \circ \gamma$.

Next we shall define the ***circulation map*** $\mathcal{K} : \mathcal{C} \times V^* \to \mathfrak{X}(\mathcal{D})^{**}$. Given a one form density $\alpha \in \mathfrak{X}^*$ we can form a one form (no longer a density) by dividing it by the mass density ρ; we denote the result just by α/ρ. We let $\mathcal{K}$ then be defined by

$$\langle \mathcal{K}(\gamma, a), \alpha \rangle = \oint_\gamma \frac{\alpha}{\rho}\,. \tag{6.11}$$

The expression in this definition is called the ***circulation*** of the one–form α/ρ around the loop γ.

This map is equivariant in the sense that

$$\langle \mathcal{K}(\eta \circ \gamma, \eta_* a), \eta_* \alpha \rangle = \langle \mathcal{K}(\gamma, a), \alpha \rangle$$

for any $\eta \in \operatorname{Diff}(\mathcal{D})$. This is proved using the definitions and the change of variables formula.

Given the Lagrangian $l : \mathfrak{X}(\mathcal{D}) \times V^* \to \mathbb{R}$, the Kelvin–Noether quantity is given by (4.1) which in this case becomes

$$I(\gamma, \mathbf{v}, a) = \oint_\gamma \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{v}}\,.$$

With these definitions of $\mathcal{K}$ and I, the statement of Theorem 4.1 becomes the classical Kelvin circulation theorem.

Theorem 6.2 (Kelvin Circulation Theorem.) *Assume that* $\mathbf{v}(x,t)$ *satisfies the Euler–Poincaré equations for continua:*

$$\frac{\partial}{\partial t}\left(\frac{\delta l}{\delta \mathbf{v}}\right) = -\pounds_{\mathbf{v}}\left(\frac{\delta l}{\delta \mathbf{v}}\right) + \frac{\delta l}{\delta a}\diamond a$$

and a *satisfies the advection relation*

$$\frac{\partial a}{\partial t} + \pounds_{\mathbf{v}} a = 0.$$

Let η_t *be the flow of the Eulerian velocity field* $\mathbf{v}$*, that is,* $\mathbf{v}_t = (d\eta_t/dt)\circ\eta_t^{-1}$*. Define* $\gamma_t := \eta_t \circ \gamma_0$ *and* $I(t) := I(\gamma_t, \mathbf{v}_t, a_t)$*. Then*

$$\frac{d}{dt}I(t) = \oint_{\gamma_t} \frac{1}{\rho}\frac{\delta l}{\delta a}\diamond a\ .$$

In this statement, we use a subscript t to emphasise that the operations are done at a particular t and to avoid having to write the other arguments, as in $a_t(x) = a(x,t)$; we omit the arguments from the notation when convenient. Due to the importance of this theorem we shall give here a separate proof tailored for the case of continuum mechanical systems.

Proof. First we change variables in the expression for $I(t)$:

$$I(t) = \oint_{\gamma_t} \frac{1}{\rho_t}\frac{\delta l}{\delta \mathbf{v}} = \oint_{\gamma_0} \eta_t^*\left[\frac{1}{\rho_t}\frac{\delta l}{\delta \mathbf{v}}\right] = \oint_{\gamma_0} \frac{1}{\rho_0}\eta_t^*\left[\frac{\delta l}{\delta \mathbf{v}}\right].$$

Next, we use the Lie derivative formula

$$\frac{d}{dt}\left(\eta_t^*\alpha_t\right) = \eta_t^*\left(\frac{\partial}{\partial t}\alpha_t + \pounds_{\mathbf{v}}\alpha_t\right)\ ,$$

for an arbitrary one–form density α_t. This formula gives

$$\begin{aligned}
\frac{d}{dt}I(t) &= \frac{d}{dt}\oint_{\gamma_0}\frac{1}{\rho_0}\eta_t^*\left[\frac{\delta l}{\delta \mathbf{v}}\right] \\
&= \oint_{\gamma_0}\frac{1}{\rho_0}\frac{d}{dt}\left(\eta_t^*\left[\frac{\delta l}{\delta \mathbf{v}}\right]\right) \\
&= \oint_{\gamma_0}\frac{1}{\rho_0}\eta_t^*\left[\frac{\partial}{\partial t}\left(\frac{\delta l}{\delta \mathbf{v}}\right) + \pounds_{\mathbf{v}}\left(\frac{\delta l}{\delta \mathbf{v}}\right)\right].
\end{aligned}$$

By the Euler–Poincaré equations, this becomes

$$\frac{d}{dt}I(t) = \oint_{\gamma_0}\frac{1}{\rho_0}\eta_t^*\left[\frac{\delta l}{\delta a}\diamond a\right] = \oint_{\gamma_t}\frac{1}{\rho_t}\left[\frac{\delta l}{\delta a}\diamond a\right],$$

again by the change of variables formula. ■

Corollary 6.3 (Kelvin-Noether form.) *Since the last expression holds for every loop γ_t, we may write it as*

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right) \frac{1}{\rho}\frac{\delta l}{\delta \mathbf{v}} = \frac{1}{\rho}\frac{\delta l}{\delta a} \diamond a\,. \tag{6.12}$$

This is the ***Kelvin-Noether form*** of the Euler–Poincaré equations for ideal continuum dynamics.

7 Applications of the Euler–Poincaré Theorem to Continua

Variational formulae in three dimensional Euclidean coordinates. We compute explicit formulae for the variations δa in the cases that the set of tensor fields a consists of elements with the following coordinate functions in a Euclidean basis on $\mathbb{R}^3$,

$$a \in \{b, \mathbf{A}\cdot d\mathbf{x}, \mathbf{B}\cdot d\mathbf{S}, D\, d^3x, S_{ab}\, dx^a \otimes dx^b\}\,. \tag{7.1}$$

These are the tensor fields that typically occur in ideal continuum dynamics. Here, in three dimensional vector notation, we choose $\mathbf{B} = \operatorname{curl}\mathbf{A}$ and $d(\mathbf{A}\cdot d\mathbf{x}) = \mathbf{B}\cdot d\mathbf{S}$. In Euclidean coordinates on $\mathbb{R}^3$, this is $d(A_k dx^k) = A_{k,j} dx^j \wedge dx^k = \frac{1}{2}\epsilon_{ijk}B^i dx^j \wedge dx^k$, where ϵ_{ijk} is the completely antisymmetric tensor density on $\mathbb{R}^3$ with $\epsilon_{123} = +1$. (The two form $\mathbf{B}\cdot d\mathbf{S} = d(\mathbf{A}\cdot d\mathbf{x})$ is the physically interesting special case of $B_{kj}dx^j\wedge dx^k$, in which $B_{kj} = A_{k,j}$, so that $\nabla\cdot\mathbf{B} = 0$.)

We have seen that invariance of the set a in the Lagrangian picture under the dynamics of $\mathbf{v}$ implies in the Eulerian picture that $(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}})\, a = 0$, where $\pounds_{\mathbf{v}}$ denotes Lie derivative with respect to the velocity vector field $\mathbf{v}$. According to Theorem 6.1, equation (6.9), the variations of the tensor functions a at fixed $\mathbf{x}$ and t are also given by Lie derivatives, namely $\delta a = -\pounds_{\mathbf{u}}\, a$, or

$$\begin{aligned}
\delta b &= -\pounds_{\mathbf{u}}\, b = -\mathbf{u}\cdot\nabla b\,,\\
\delta\mathbf{A}\cdot d\mathbf{x} &= -\pounds_{\mathbf{u}}\,(\mathbf{A}\cdot d\mathbf{x}) = -\left((\mathbf{u}\cdot\nabla)\mathbf{A} + A_j\nabla u^j\right)\cdot d\mathbf{x}\\
&= (\mathbf{u}\times\operatorname{curl}\mathbf{A} - \nabla(\mathbf{u}\cdot\mathbf{A}))\cdot d\mathbf{x}\,,\\
\delta\mathbf{B}\cdot d\mathbf{S} &= -\pounds_{\mathbf{u}}\,(\mathbf{B}\cdot d\mathbf{S}) = (\operatorname{curl}(\mathbf{u}\times\mathbf{B}))\cdot d\mathbf{S} \;=\; d(\delta\mathbf{A}\cdot d\mathbf{x})\,,\\
\delta D\; d^3x &= -\pounds_{\mathbf{u}}\,(D\, d^3x) = -\nabla\cdot(D\mathbf{u})\; d^3x\,,\\
\delta S_{ab}\, dx^a\otimes dx^b &= -\pounds_{\mathbf{u}}\,(S_{ab}\, dx^a\otimes dx^b)\\
&= -(u^k S_{ab,k} + S_{kb}u^k_{,a} + S_{ka}u^k_{,b})\, dx^a\otimes dx^b\,.
\end{aligned} \tag{7.2}$$

Hence, Hamilton's principle with this dependence yields

$$
\begin{aligned}
0 &= \delta \int dt\; l(\mathbf{v}; b, \mathbf{A}, \mathbf{B}, D, S_{ab}) \\
&= \int dt\; \left[\frac{\delta l}{\delta \mathbf{v}} \cdot \delta\mathbf{v} + \frac{\delta l}{\delta b}\; \delta b + \frac{\delta l}{\delta D}\; \delta D + \frac{\delta l}{\delta \mathbf{A}} \cdot \delta\mathbf{A} + \frac{\delta l}{\delta \mathbf{B}} \cdot \delta\mathbf{B} + \frac{\delta l}{\delta S_{ab}}\; \delta S_{ab} \right] \\
&= \int dt\; \left[\frac{\delta l}{\delta \mathbf{v}} \cdot \left(\frac{\partial \mathbf{u}}{\partial t} - \mathrm{ad}_{\mathbf{v}}\; \mathbf{u} \right) - \frac{\delta l}{\delta b}\; \mathbf{u} \cdot \nabla b - \frac{\delta l}{\delta D}\; (\nabla \cdot (D\mathbf{u})) \right. \\
&\qquad + \frac{\delta l}{\delta \mathbf{A}} \cdot (\mathbf{u} \times \operatorname{curl} \mathbf{A} - \nabla(\mathbf{u} \cdot \mathbf{A})) + \frac{\delta l}{\delta \mathbf{B}} \cdot (\operatorname{curl} (\mathbf{u} \times \mathbf{B})) \\
&\qquad \left. - \frac{\delta l}{\delta S_{ab}} \left(u^k S_{ab,k} + S_{kb} u^k_{,a} + S_{ka} u^k_{,b} \right) \right] \\
&= \int dt\; \left[\mathbf{u} \cdot \left(-\frac{\partial}{\partial t} \frac{\delta l}{\delta \mathbf{v}} - \mathrm{ad}^*_{\mathbf{v}}\; \frac{\delta l}{\delta \mathbf{v}} - \frac{\delta l}{\delta b}\; \nabla b + D\; \nabla \frac{\delta l}{\delta D} \right. \right. \\
&\qquad\qquad \left. - \frac{\delta l}{\delta \mathbf{A}} \times \operatorname{curl} \mathbf{A} + \mathbf{A}\; \operatorname{div} \frac{\delta l}{\delta \mathbf{A}} + \mathbf{B} \times \operatorname{curl} \frac{\delta l}{\delta \mathbf{B}} \right) \qquad (7.3) \\
&\qquad + u^k \left(-\frac{\delta l}{\delta S_{ab}} S_{ab,k} + (\frac{\delta l}{\delta S_{ab}} S_{kb})_{,a} + (\frac{\delta l}{\delta S_{ab}} S_{ka})_{,b} \right) \Big] \\
&= \int dt\; \mathbf{u} \cdot \left[-\frac{\partial}{\partial t} \frac{\delta l}{\delta \mathbf{v}} - \mathrm{ad}^*_{\mathbf{v}}\; \frac{\delta l}{\delta \mathbf{v}} + \frac{\delta l}{\delta a} \diamond a \right] \\
&= \int dt\; \mathbf{u} \cdot \left[-\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}} \right) \frac{\delta l}{\delta \mathbf{v}} + \frac{\delta l}{\delta a} \diamond a \right],
\end{aligned}
$$

where we have consistently dropped boundary terms arising from integrations by parts, by invoking natural boundary conditions. Thus, for the set of tensor fields a in equation (7.1) we have the following Euclidean components of $\frac{\delta l}{\delta a} \diamond a$,

$$
\begin{aligned}
\left(\frac{\delta l}{\delta a} \diamond a \right)_k &= - \frac{\delta l}{\delta b}\; b_{,k} + D \left(\frac{\delta l}{\delta D} \right)_{,k} \\
&\quad + \left(- \frac{\delta l}{\delta \mathbf{A}} \times \operatorname{curl} \mathbf{A} + \mathbf{A}\; \operatorname{div} \frac{\delta l}{\delta \mathbf{A}} + \mathbf{B} \times \operatorname{curl} \frac{\delta l}{\delta \mathbf{B}} \right)_k \\
&\quad - \frac{\delta l}{\delta S_{ab}} S_{ab,k} + \left(\frac{\delta l}{\delta S_{ab}} S_{kb} \right)_{,a} + \left(\frac{\delta l}{\delta S_{ab}} S_{ka} \right)_{,b}. \qquad (7.4)
\end{aligned}
$$

Stress tensor formulation. For example, if we assume a Lagrangian in the form

$$
l(\mathbf{v}; b, \mathbf{A}, \mathbf{B}, D, S_{ab}) = \int d^3x\; \mathcal{L}(\mathbf{v}, \nabla\mathbf{v}, b, \mathbf{A}, \mathbf{B}, D, S_{ab}), \qquad (7.5)
$$

where $\mathcal{L}$ is a *given* function, then we may use equation (7.4) to express the Euler–Poincaré equations for continua (6.9) in this case in the ***momentum conservation form***,

$$
\frac{\partial}{\partial t} \frac{\delta l}{\delta \mathbf{v}} = - \mathrm{ad}^*_{\mathbf{v}}\; \frac{\delta l}{\delta \mathbf{v}} + \frac{\delta l}{\delta a} \diamond a \quad \Rightarrow \quad \frac{\partial m_i}{\partial t} = - \frac{\partial}{\partial x^j} T^j_i\,, \qquad (7.6)
$$

with ***momentum density components*** m_i, $i = 1, 2, 3$ defined by

$$m_i \equiv \frac{\delta l}{\delta v^i} = \frac{\partial \mathcal{L}}{\partial v^i} - \frac{\partial}{\partial x^k}\left(\frac{\partial \mathcal{L}}{\partial v^i_{,k}}\right), \tag{7.7}$$

and ***stress tensor*** T_i^j given by

$$\begin{aligned} T_i^j &= m_i v^j - \frac{\partial \mathcal{L}}{\partial v^k_{,j}} v^k_{,i} - \frac{\partial \mathcal{L}}{\partial A_j} A_i + \frac{\partial \mathcal{L}}{\partial B^i} B^j - \frac{\partial \mathcal{L}}{\partial S_{jb}} S_{ib} - \frac{\partial \mathcal{L}}{\partial S_{aj}} S_{ia} \\ &\quad + \delta_i^j \left(\mathcal{L} - D\frac{\partial \mathcal{L}}{\partial D} - B^k \frac{\partial \mathcal{L}}{\partial B^k} \right) . \end{aligned} \tag{7.8}$$

Here, in the calculation of T_i^j, we have used the coordinate expression (6.3) for $\mathrm{ad}^*_{\mathbf{u}}(\alpha \otimes m)$.

Kelvin-Noether form. The Euclidean components of the Euler–Poincaré equations for ideal continua may also be summarized in Kelvin-Noether form (6.12) for advected tensor fields a in the set (7.1). We adopt the notational convention of the circulation map $\mathcal{K}$ in equation (6.11) that a one form density can be made into a one form (no longer a density) by dividing it by the mass density D to produce, e.g., the one form in Euclidean components $\frac{1}{D}\frac{\delta l}{\delta v^i} dx^i$ from the one form density $\frac{\delta l}{\delta \mathbf{v}}$. With a slight abuse of notation (but in accord with the usual physics conventions) we write the former coordinate expression as $\frac{1}{D}\frac{\delta l}{\delta \mathbf{v}} \cdot d\mathbf{x}$. We also denote the Lie-derivative relation for the continuity equation as $(\partial/\partial t + \pounds_{\mathbf{v}}) D d^3x = 0$. Then, the Euclidean components of the Euler–Poincaré equations for continua in (7.3) are expressed in Kelvin-Noether form (6.12) as

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right)\left(\frac{1}{D}\frac{\delta l}{\delta \mathbf{v}} \cdot d\mathbf{x}\right) + \frac{1}{D}\frac{\delta l}{\delta b}\nabla b \cdot d\mathbf{x} - \nabla\left(\frac{\delta l}{\delta D}\right) \cdot d\mathbf{x} \\ &\quad + \frac{1}{D}\left(\frac{\delta l}{\delta \mathbf{A}} \times \mathrm{curl}\ \mathbf{A} - \mathbf{A}\ \mathrm{div}\frac{\delta l}{\delta \mathbf{A}} \right) \cdot d\mathbf{x} - \frac{1}{D}\left(\mathbf{B} \times \mathrm{curl}\ \frac{\delta l}{\delta \mathbf{B}}\right) \cdot d\mathbf{x} \\ &\quad + \frac{1}{D}\left(\frac{\delta l}{\delta S_{ab}} S_{ab,k} - (\frac{\delta l}{\delta S_{ab}} S_{kb})_{,a} - (\frac{\delta l}{\delta S_{ab}} S_{ka})_{,b} \right) dx^k = 0 , \end{aligned} \tag{7.9}$$

where the components of the variational derivatives of the Lagrangian l are to be computed according to the usual physics conventions, i.e., as components of Fréchet derivatives as in equation (1.50). In physical applications, the advected Eulerian tensor fields a in (7.1) represent the buoyancy b (or specific entropy, for the compressible case), magnetic vector potential $\mathbf{A}$, magnetic field intensity $\mathbf{B}$, mass density D, and Cauchy-Green strain tensor S_{ab}, respectively. Formula (7.9) is the Kelvin-Noether form of the equation of motion for ideal continua in Euclidean coordinates. This Euclidean component formula is especially convenient for direct calculations in fluid dynamics, to which we turn our attention next.

Eulerian motion equation for an ideal incompressible fluid. In the Eulerian velocity representation we consider fluid motion in an n-dimensional domain and define the reduced action $\mathfrak{S}_{\rm red}$ and reduced Lagrangian $l(\mathbf{v}, D)$ by

$$\mathfrak{S}_{\rm red} = \int dt\, l = \int dt \int d^n x \, \Big[\frac{1}{2} D|\mathbf{v}|^2 - p(D-1)\Big] \,. \tag{7.10}$$

This action produces the following variations at fixed $\mathbf{x}$ and t

$$\frac{1}{D}\frac{\delta l}{\delta \mathbf{v}} = \mathbf{v} \,, \quad \frac{\delta l}{\delta D} = \frac{1}{2}|\mathbf{v}|^2 - p \,, \quad \frac{\delta l}{\delta p} = -\,(D-1) \,. \tag{7.11}$$

Hence, from equation (7.9) for Hamilton principles of this type we find the Eulerian motion equation,

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right)\left(\frac{1}{D}\frac{\delta l}{\delta \mathbf{v}} \cdot d\mathbf{x}\right) \,-\, \nabla\left(\frac{\delta l}{\delta D}\right) \cdot d\mathbf{x} = 0 \,, \quad \text{or} \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 \,, \tag{7.12}$$

for "natural" boundary conditions, $\hat{\mathbf{n}} \cdot \mathbf{v} = 0$ on the boundary, where $\hat{\mathbf{n}}$ is the boundary's outward unit normal vector. This is the Eulerian motion equation for an incompressible fluid in n dimensions. The constraint $D = 1$ (volume or mass preservation) is imposed by varying the Lagrange multiplier p, the pressure. Incompressibility then follows from substituting $D = 1$ into the Lie-derivative relation for D, which closes the ideal incompressible fluid system,

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right) D d^3x = 0, \quad \text{i.e.,} \quad \frac{\partial D}{\partial t} = -\,\nabla\cdot(D\mathbf{v}) \,. \tag{7.13}$$

This relation, together with the constraint $D = 1$ gives incompressibility of the flow, $\nabla \cdot \mathbf{v} = 0$.

Remark on Lagrangian mass coordinates. An alternative way to treat Hamilton's principle for the action (7.10) is to perform variations at fixed $\mathbf{x}$ and t of the *inverse* maps $\mathbf{x} \mapsto \boldsymbol{\ell}$, described by the Lagrangian mass coordinate functions $\ell^A(\mathbf{x}, t)$, $A = 1, 2, \ldots, n$, which determine $\mathbf{v}$ and D by the formulae (in which one sums on repeated indices)

$$\frac{\partial \ell^A}{\partial t} = -v^i D^A_i \,, \quad D^A_i = \frac{\partial \ell^A}{\partial x^i} \,, \quad D = \det(D^A_i) \,. \tag{7.14}$$

As discussed above, the relation of mass coordinates ℓ to the usual Lagrangian coordinates X is given by a change of variables in the fluid reference configuration to make $\rho_0(X) d^n X = d^n \ell$. Variation of an action of the form $\mathfrak{S}_{\rm red}(\mathbf{v}, D)$ with respect to ℓ^A with p imposing volume preservation then yields (Holm, Marsden, and Ratiu [1986], Holm [1996a]),

$$\begin{aligned} \delta\mathfrak{S}_{\rm red} \;=\; & \int dt \int d^n x \, \Big\{ D(D^{-1})^i_A \delta l^A \Big[\frac{d}{dt}\frac{1}{D}\frac{\delta l}{\delta v^i} + \frac{1}{D}\frac{\delta l}{\delta v^j} v^j_{,i} - \left(\frac{\delta l}{\delta D}\right)_{,i} \Big] \\ & \qquad - \delta p (D-1) \Big\} \,, \end{aligned} \tag{7.15}$$

where $d/dt = \partial/\partial t + (\mathbf{v}\cdot\nabla)$ is the material derivative of Eulerian quantities and we again invoke natural boundary conditions when integrating by parts.

Hence, the vanishing of the coefficient of $\delta\ell^A$ in the variational formula (7.15) recovers the Euler–Poincaré equation (7.12) for the Eulerian fluid velocity, $\mathbf{v}$, by stationarity of the action (7.10) with respect to variations of the Lagrangian mass coordinates $\ell^A(\mathbf{x},t)$. Similar arguments based on stationary variations of the action with respect to the Lagrangian mass coordinates ℓ^A at fixed $\mathbf{x}, t$ will also recover the more general Euler–Poincaré equations (7.9) from actions which depend on the velocity $\mathbf{v}$ and the advected quantities in equation (7.1) through their dependence on the $\ell^A(\mathbf{x},t)$.

Adiabatic compressible MHD. In the case of adiabatic compressible magnetohydrodynamics (MHD), the action in Hamilton's principle is given by

$$\mathfrak{S}_{\rm red} = \int dt\, l = \int dt\, d^3x \, \left(\frac{D}{2}\,|\mathbf{v}|^2 - De(D,b) - \frac{1}{2}|\mathbf{B}|^2 \right), \tag{7.16}$$

where $e(D,b)$ is the fluid's specific internal energy, whose dependence on the density D and specific entropy b is given as the "equation of state" and which for an isotropic medium satisfies the thermodynamic first law in the form $de = -pd(1/D) + Tdb$ with pressure $p(D,b)$ and temperature $T(D,b)$. The variation of l in (7.16) is

$$\delta\mathfrak{S}_{\rm red} = \int dt\, d^3x \; D\mathbf{v}\cdot\delta\mathbf{v} - DT\delta b + \left(\frac{1}{2}|\mathbf{v}|^2 - h\right)\delta D - \mathbf{B}\cdot\delta\mathbf{B}. \tag{7.17}$$

The quantity $h = e + p/D$ denotes the specific enthalpy, which thus satisfies $dh = (1/D)dp + Tdb$. The Euler–Poincaré formula in the Kelvin-Noether form (7.9) yields the MHD motion equation as

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right)(\mathbf{v}\cdot d\mathbf{x}) - Tdb + \frac{1}{D}\mathbf{B}\times\text{curl }\mathbf{B}\cdot d\mathbf{x} - d\left(\frac{1}{2}|\mathbf{v}|^2 - h\right) = 0, \tag{7.18}$$

or, in three dimensional vector form,

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{D}\nabla p + \frac{1}{D}\mathbf{B}\times\text{curl }\mathbf{B} = 0. \tag{7.19}$$

By definition, the advected variables $\{b, \mathbf{B}, D\}$ satisfy the the following Lie-derivative relations which close the ideal MHD system,

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right) b = 0, \quad \text{or} \quad \frac{\partial b}{\partial t} &= -\,\mathbf{v}\cdot\nabla\, b\,, \\
\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right)\mathbf{B}\cdot d\mathbf{S} = 0, \quad \text{or} \quad \frac{\partial\mathbf{B}}{\partial t} &= \text{curl}\,(\mathbf{v}\times\mathbf{B}), \\
\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right) D d^3x = 0, \quad \text{or} \quad \frac{\partial D}{\partial t} &= -\,\nabla\cdot(D\mathbf{v}),
\end{aligned} \tag{7.20}$$

and the function $p(D,b) = D^2\partial e/\partial D$ is specified by giving the equation of state of the fluid, $e = e(D,b)$. If the condition $\nabla\cdot\mathbf{B} = 0$ holds initially, then it holds for all time; since this constraint is preserved by the dynamical equation for $\mathbf{B}$.

Adiabatic magneto-elastodynamics. When nonlinear elasticity is also a factor in the MHD evolution, there is an additional Lie-derivative relation,

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right) (S_{ab}\, dx^a \otimes dx^b) = 0\,, \tag{7.21}$$

leading to the dynamical equation for the advected Cauchy-Green strain tensor S_{ab} (which measures nonlinear strain in *spatial* coordinates),

$$\frac{\partial}{\partial t} S_{ab} = -\left(v^k S_{ab,k} + S_{kb} v^k_{,a} + S_{ka} v^k_{,b}\right). \tag{7.22}$$

In this case, additional stress terms appear in the motion equation for $\mathbf{v}$ that arise from the dependence of the specific internal energy $e(D, b, S_{ab})$ on the Cauchy-Green strain tensor S_{ab} in the MHD action (7.16) when the elasticity of the medium is involved. The stress tensor per unit mass σ^{ab} is determined from the equation of state of such an magneto-elastic medium by the Doyle-Erickson formula $\sigma^{ab} \equiv \partial e/\partial S_{ab}$. The Euler–Poincaré equation (7.9) for ideal magneto-elasticity is then seen to be

$$\frac{\partial v_i}{\partial t} + v^j v_{i,j} + \frac{1}{D} p_{,i} + \frac{1}{D} B^j (B_{j,i} - B_{i,j}) - (\sigma^{ab} S_{ib})_{,a} - (\sigma^{ab} S_{ia})_{,b} = 0\,, \tag{7.23}$$

where we have used the specific enthalpy relation for an elastic medium, $dh - T db = D^{-1} dp + \sigma^{ab} dS_{ab}$. Thus, adiabatic magneto-elastodynamics summons all of the advected quantities in equation (7.1) and makes use of the entire Euler–Poincaré equation (7.9).

Adiabatic compressible Maxwell fluid dynamics via the Kaluza-Klein construction. An adiabatic Maxwell fluid (MF) with (nonrelativistic) Eulerian fluid velocity $\mathbf{v}$, density D, specific entropy b and pressure $p(D, b)$ satisfies the following system of equations,

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{D}\nabla p(D,b) = \frac{q}{m}\,(\mathbf{E} + \mathbf{v}\times\mathbf{B}),\\
&\frac{\partial D}{\partial t} = -\,\nabla\cdot(D\mathbf{v}), \qquad \frac{\partial b}{\partial t} = -\,(\mathbf{v}\cdot\nabla)b,\\
&\nabla\cdot\mathbf{E} = \frac{q}{m}D, \qquad \frac{\partial \mathbf{E}}{\partial t} = -\,\frac{q}{m}D\mathbf{v} + \nabla\times\mathbf{B},\\
&\nabla\cdot\mathbf{B} = 0, \qquad \frac{\partial \mathbf{B}}{\partial t} = -\,\nabla\times\mathbf{E}.
\end{aligned} \tag{7.24}$$

This system consists of the motion equation for a charged fluid moving under the combined effects of pressure gradient and Lorentz forces; the continuity equation for the mass density D; advection of the specific entropy, b; and Maxwell's equations for the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ in the moving fluid medium, whose polarizability and magnetization are neglected for simplicity. (For the physically more realistic treatment of moving media with electromagnetic induction in a similar framework, including relativistic effects, see Holm [1987].) The equations for D and b are the

familiar advection laws. The coupling constant q/m is the charge-to-mass ratio of the fluid particles, and the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are defined in terms of the scalar and vector potentials Φ and $\mathbf{A}$ by

$$\mathbf{E} \equiv -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi, \qquad \mathbf{B} \equiv \nabla \times \mathbf{A}\,. \tag{7.25}$$

In the MF equations (7.24), charged fluid motion is the source for the electromagnetic fields which act self-consistently upon the fluid through the Lorentz force. We shall show that equations (7.24) are Euler–Poincaré equations for the gauge invariant action of "Kaluza-Klein" form given by

$$\begin{aligned}\mathfrak{S}_{\rm red} = \int dt\, l \;\; &= \;\; \int dt\, d^3x \;\left(\frac{1}{2}D|\mathbf{v}|^2 + \frac{1}{2}D(\mathbf{A}\cdot\mathbf{v} - \Phi + \tilde{\omega})^2 - De(D,b)\right.\\ &\qquad \left. + \;\frac{1}{2}\Big|\frac{\partial \mathbf{A}}{\partial t} + \nabla\Phi\Big|^2 - \frac{1}{2}|\nabla\times\mathbf{A}|^2\right),\end{aligned} \tag{7.26}$$

where $\tilde{\omega} = \partial\theta/\partial t + \mathbf{v}\cdot\nabla\theta$ for a gauge field θ and $e(D,b)$ is the fluid's specific internal energy, which satisfies the first law of thermodynamics in the form $de = -pd(1/D) + Tdb$ with pressure $p(D,b)$ and temperature $T(D,b)$.

This action principle fits into the general theory with the electromagnetic field variables playing the role of additional configuration variables which are not acted on by the particle relabelling group. They obey the usual Euler-Lagrange equations, coupled to the Euler–Poincaré variables through the Lagrangian. In other words, the primitive unreduced Lagrangian in this case is of the abstract form $L : TG \times V^* \times TQ \times TC \to \mathbb{R}$ in which G, the fluid particle relabelling group, acts trivially on the Maxwell field variables Q and the gauge field $\theta \in C$. Note that the Lagrangian in equation (7.26) is invariant under translations of θ, as well as under the electromagnetic gauge transformations,

$$\mathbf{A} \to \mathbf{A} + \nabla\zeta, \quad \Phi \to \Phi - \partial\zeta/\partial t, \quad \theta \to \theta - \zeta, \tag{7.27}$$

for an arbitary function ζ of $\mathbf{x}$ and t.

We now take variations of the action. The variation of $\mathfrak{S}_{\rm red}$ in equation (7.26) may be written using the definitions of $\mathbf{E}$ and $\mathbf{B}$, and the abbreviated notation $c \equiv \mathbf{A}\cdot\mathbf{v} - \Phi + \tilde{\omega}$, as

$$\begin{aligned}\delta\mathfrak{S}_{\rm red} \;\; &= \;\; \int dt\, d^3x \;\Big[D\left(\mathbf{v} + c\mathbf{A} + c\nabla\theta\right)\cdot\delta\mathbf{v} + \left(\frac{1}{2}|\mathbf{v}|^2 + \frac{1}{2}c^2 - e - \frac{p}{D}\right)\delta D\\ &\qquad - \; DT\;\delta b + \left(cD\mathbf{v} + \frac{\partial\mathbf{E}}{\partial t} - \nabla\times\mathbf{B}\right)\cdot\delta\mathbf{A} + \left(-cD + \nabla\cdot\mathbf{E}\right)\delta\Phi\\ &\qquad - \; \left(\frac{\partial cD}{\partial t} + \nabla\cdot cD\mathbf{v}\right)\;\delta\theta\Big],\end{aligned} \tag{7.28}$$

where terms arising from integration by parts vanish for the natural boundary conditions given by

$$\mathbf{v}\cdot\hat{\mathbf{n}} = 0, \quad \mathbf{E}\cdot\hat{\mathbf{n}} = 0, \text{ and } \hat{\mathbf{n}}\times\mathbf{B} = 0 \text{ on the boundary}, \tag{7.29}$$

and for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints. Stationarity of the action $\mathfrak{S}_{\rm red}$ in (7.26) under variation of the gauge field θ gives the conservation law,

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) c = 0 \quad \text{via the continuity equation,} \quad \frac{\partial D}{\partial t} + \nabla\cdot(D\mathbf{v}) = 0. \tag{7.30}$$

Hence, we may set $c = q/m$ in equation (7.28) and then acquire the two Maxwell equations with sources from stationarity of the action $\mathfrak{S}_{\rm red}$ under variations of $\mathbf{A}$ and Φ. Once the flow velocity $\mathbf{v}$ is known, the relation $c = q/m$ determines the gauge function θ by "quadrature", from the definitions of c and $\tilde{\omega}$ as

$$\tilde{\omega} \equiv \frac{\partial\theta}{\partial t} + \mathbf{v}\cdot\nabla\theta = \frac{q}{m} + \Phi - \mathbf{A}\cdot\mathbf{v}\,. \tag{7.31}$$

The remaining variations of $\mathfrak{S}_{\rm red}$ in $\{\mathbf{v}, D, b\}$ for the Euler–Poincaré dynamics collect into the Kelvin-Noether form of equation (7.9) as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\left(\frac{1}{D}\frac{\delta l}{\delta \mathbf{v}}\right) + \frac{1}{D}\frac{\delta l}{\delta v^j}\nabla v^j \;+\; \frac{1}{D}\frac{\delta l}{\delta b}\nabla b - \nabla\frac{\delta l}{\delta D} = 0\,. \tag{7.32}$$

Specifically, we have

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)(\mathbf{v} + c\mathbf{A} + c\nabla\theta) \;+\; (v_j + cA_j + c\theta_{,j})\nabla v^j$$
$$- \;T\nabla b \;-\; \nabla\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{1}{2}c^2 - e - \frac{p}{D}\right) = 0\,. \tag{7.33}$$

Using the fundamental vector identity of fluid dynamics in three dimensions,

$$(\mathbf{b}\cdot\nabla)\mathbf{a} + a_j\nabla b^j = -\mathbf{b}\times(\nabla\times\mathbf{a}) + \nabla(\mathbf{a}\cdot\mathbf{b})\,, \tag{7.34}$$

with, in this case, $\mathbf{b} = \mathbf{v}$ and $\mathbf{a} = D^{-1}\delta l/\delta\mathbf{v}$, casts the Euler–Poincaré equation (7.32) into its equivalent "curl" form,

$$\frac{\partial}{\partial t}\left(\frac{1}{D}\frac{\delta l}{\delta\mathbf{v}}\right) - \mathbf{v}\times\nabla\times\left(\frac{1}{D}\frac{\delta l}{\delta\mathbf{v}}\right) \;+\; \frac{1}{D}\frac{\delta l}{\delta b}\nabla b + \nabla\Big(\mathbf{v}\cdot\frac{1}{D}\frac{\delta l}{\delta\mathbf{v}} - \frac{\delta l}{\delta D}\Big) = 0\,. \tag{7.35}$$

Similarly, applying the same vector identity with $\mathbf{b} = \mathbf{v}$ and $\mathbf{a} = c(\mathbf{A} + \nabla\theta)$ in the Maxwell fluid motion equation (7.33) yields,

$$\begin{aligned}\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{D}\nabla p \;&=\; c\,\left(-\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi + \mathbf{v}\times(\nabla\times\mathbf{A})\right)\\ &=\; \frac{q}{m}\;(\mathbf{E} + \mathbf{v}\times\mathbf{B})\;,\end{aligned} \tag{7.36}$$

where we have used the thermodynamic first law. Thus we find the Maxwell fluid motion law – the first among the equations in (7.24) – after setting $c = q/m$ and using the definitions of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ in terms of the potentials $\mathbf{A}$ and Φ.

Theorem 7.1 (Kelvin circulation theorem for the Maxwell fluid.) *By the MF motion equation (7.36) and the thermodynamic first law, we have*

$$\frac{dI}{dt} \;=\; \oint_{\gamma_t} T db\,, \tag{7.37}$$

where the circulation integral is given by

$$I \equiv \oint_{\gamma_t} (\mathbf{v} + \frac{q}{m}\mathbf{A}) \cdot d\mathbf{x},$$

for a curve γ_t which moves with the fluid velocity $\mathbf{v}$.

Proof. The proof is the same as for Theorem 6.2; although it is also immediately seen from the motion equation (7.33) after substituting $D^{-1}\delta l/\delta \mathbf{v} = \mathbf{v} + c(\mathbf{A} + \nabla\theta)$ and $D^{-1}\delta l/\delta b = -T$. ■

Corollary 7.2 (Potential vorticity convection for the Maxwell fluid.) *Stokes' theorem, advection of specific entropy b and the continuity equation together imply convection of potential vorticity for the adiabatic Maxwell fluid,*

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q \equiv \frac{dq}{dt} = 0 \quad \text{with} \quad q \equiv \frac{1}{D}\nabla b \cdot \mathrm{curl}(\mathbf{v} + \frac{q}{m}\mathbf{A}). \tag{7.38}$$

Remark. The equation $dq/dt = 0$ for convection of potential vorticity for a general Lagrangian with dependence $l(\mathbf{v}, b, D)$, with

$$q = \frac{1}{D}\nabla b \cdot \mathrm{curl}\Big(\frac{1}{D}\frac{\delta l}{\delta \mathbf{v}}\Big)\,, \tag{7.39}$$

may also be proven directly from the "curl" form of the Kelvin-Noether equation (7.35) in three dimensions, by taking the scalar product of its curl with ∇b and applying the continuity equation for D.

Alternative interpretations of the Maxwell fluid formulation. Note that the first line of the Euler–Poincaré motion equation (7.36) for Maxwell fluids *persists*, when the electromagnetic energy terms are *dropped* from the MF Lagrangian in equation (7.26), to give the action

$$\mathfrak{S}_{\rm red} = \int dt\; l = \int dt\, d^3x\; \left(\frac{1}{2}D|\mathbf{v}|^2 + \frac{1}{2}D(\mathbf{A}\cdot\mathbf{v} - \Phi + \tilde{\omega})^2 - De(D,b)\right)\,. \tag{7.40}$$

The Euler–Poincaré equation which results from this action is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{D}\nabla p = c\;\left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi + \mathbf{v}\times(\nabla\times\mathbf{A})\right)\,. \tag{7.41}$$

The Kelvin Theorem 7.1 and its potential vorticity Corollary 7.2 also persist for the dynamics derived from this truncated action. The "Lorentz force" terms in equation (7.41) in terms of $\mathbf{A}$ and Φ arise *purely* from the Kaluza-Klein coupling term — the

second term in the integrand of the action $\mathfrak{S}_{\rm red}$ in equation (7.40). These "Lorentz forces" may be interpreted physically as noninertial forces resulting from having moved into a frame of reference with a prescribed velocity given by $\mathbf{A}(\mathbf{x},t)$. The velocity $\mathbf{v}$ then represents fluid flow relative to this noninertial frame. This situation reduces to *Faraday driving* of the fluid (Faraday [1831]), when $\mathbf{A}(\mathbf{x},t)$ corresponds to a rigid motion of the fluid container. For a simple example, set $c = f_0$; $\nabla\Phi = 0$; and $\mathbf{A} = \frac{1}{2}\hat{\mathbf{z}} \times \mathbf{x}$. Then $c\mathbf{v} \times (\nabla \times \mathbf{A}) = \mathbf{v} \times f_0\hat{\mathbf{z}}$ gives the Coriolis force and $\mathbf{v}$ corresponds to fluid velocity in a uniformly rotating reference frame with constant angular velocity f_0. This is the typical situation in *geophysical fluid dynamics*.

Alternatively, the right hand side of equation (7.41) may be interpreted as a *vortex force* arising from a given wave field at the surface of an incompressible fluid (for $D = 1$), as in Craik and Leibovich [1976] (see also Holm [1996b]). The Craik-Leibovich equations are formally identical to equation (7.41) when $\mathbf{A}$ is identified as the prescribed mean Stokes drift velocity due to the presence of the wave field.

Fluid motion equations of the same form as (7.41) also appear in the generalized Lagrangian-mean (GLM) formulation of wave, mean-flow interaction theories (see Andrews and McIntyre [1978a,b]), in which $\tilde{\omega}$ is the Doppler-shifted frequency of a wave packet interacting with a Lagrangian-mean flow of velocity $\mathbf{v}$, and $\mathbf{A}$ is the prescribed *pseudomomentum per unit mass* of the wave. For a discussion of *self-consistent* Lie-Poisson Hamiltonian theories of wave, mean-flow interaction in a similar form, see Gjaja and Holm [1996].

Geodesic motion and the Kaluza-Klein construction for incompressible fluids. Hamilton's principle for the action in the "minimal coupling" form

$$\mathfrak{S}' = \int dt\, l = \int dt\, d^3x\, \left(\frac{1}{2}D|\mathbf{v}|^2 + \frac{q}{m}D\mathbf{A}\cdot\mathbf{v} - \frac{q}{m}D\Phi - De(D,b)\right), \qquad (7.42)$$

yields the *same* Euler–Poincaré equation (7.41) as results from Hamilton's principle for the Kaluza-Klein action in equation (7.40). Thus, we see that by introducing the auxiliary gauge field, θ, the Kaluza-Klein construction renders the minimal coupling form of the action for fluid dynamics quadratic in the velocity, while preserving its corresponding Euler–Poincaré equation. The Kaluza-Klein construction for charged particle mechanics is discussed in Marsden and Ratiu [1994]. The historical references are Kaluza [1921], Klein [1926] and Thirry [1948].

In the *incompressible* case, when the Kaluza-Klein action is taken to be

$$\mathfrak{S}_{\rm red} = \int dt\; l = \int dt\, d^3x\, \left(\frac{1}{2}D|\mathbf{v}|^2 + \frac{1}{2}D(\mathbf{A}\cdot\mathbf{v} - \Phi + \tilde{\omega})^2 - p(D-1)\right), \quad (7.43)$$

for arbitrary prescribed functions $\mathbf{A}$ and Φ, the resulting Euler–Poincaré equation (i.e., equation (7.41) with $D = 1$) represents geodesic motion on the group of volume-preserving diffeomorphisms with respect to the conserved kinetic-energy metric given by

$$\|\mathbf{v}\|^2 = \int d^3x\, \left(\frac{1}{2}|\mathbf{v}|^2 + \frac{1}{2}(\mathbf{A}\cdot\mathbf{v} - \Phi + \tilde{\omega})^2\right), \qquad (7.44)$$

where $c \equiv \mathbf{A} \cdot \mathbf{v} - \Phi + \tilde{\omega}$ is an advected quantity, $dc/dt = 0$. This observation extends the geodesic property of incompressible ideal fluid flows established in Arnold [1966a] to the case of incompressible Maxwell fluid flows, as well as to the case of incompressible ideal fluid flows in an arbitrarily moving reference frame. From the Euler–Poincaré point of view, this extension enlarges the particle relabelling group G from the group of diffeomorphisms to the group of automorphisms of the single particle Kaluza-Klein bundle. The total system for the incompressible Maxwell fluid flows is then geodesic motion on the product of this automorphism group with the Maxwell fields themselves. We believe that a similar extension may be involved in the results of Ono [1995a, 1995b].

We should also remark that when one has equations in geodesic form, one can make use of all the attendant geometry to obtain additional interesting results. Examples of this applied to questions of stability and conjugate points are given in the works of Misiolek listed in the references.

8 Approximate Model Fluid Equations which Preserve the Euler–Poincaré Structure

The preceding section demonstrates the applicability of the Euler–Poincaré theorem for ideal continua when the equations of motion are given. Here we discuss approximate fluid models which preserve the Euler–Poincaré structure, and are obtained by making asymptotic expansions and other approximations in Hamilton's principle for a given set of model equations. As examples, in this section we first discuss the derivation of the quasigeostrophic approximation in geophysical fluid dynamics from an approximation of Hamilton's principle for the rotating shallow water equations. Next, we discuss the Boussinesq approximation for dispersive water waves in one dimension. As an example of the type of "bonus" which may appear in making simplifying approximations while preserving mathematical structure, we derive the integrable Camassa-Holm equation (Camassa and Holm [1993], Camassa, Holm and Hyman [1994], Alber et al. [1994, 1995, 1997]), by making asymptotic approximations in the Hamilton's principle for the Boussinesq equations. The Camassa-Holm equation in one dimension is a completely integrable partial differential equation for dispersive water waves that was actually discovered by making structure preserving approximations of this type. This equation turns out to describe geodesic motion on the group of diffeomorphisms of either the real line or the periodic interval, with metric given by the H^1 norm of the velocity. We also derive a multidimensional analogue of the one-dimensional Camassa-Holm equation by invoking the n-dimensional version of this geodesic property. There are also other potential advantages of making structure preserving approximations, e.g., for numerical integrations. However, discussion of these other advantages is deferred to another place. (See Marsden and Wendlandt [1997], Wendlandt and Marsden [1997] and Marsden, Patrick and Shkoller [1997] for recent advances in this direction.)

Rotating shallow water dynamics as Euler–Poincaré equations. We first consider dynamics of rotating shallow water (RSW) in a two dimensional domain with horizontal coordinates $\mathbf{x} = (x_1, x_2)$. RSW motion is governed by the following nondimensional equations for horizontal fluid velocity $\mathbf{v} = (v_1, v_2)$ and depth D,

$$\epsilon \frac{d}{dt}\mathbf{v} + f\hat{\mathbf{z}} \times \mathbf{v} + \nabla\psi = 0\,, \qquad \frac{\partial D}{\partial t} + \nabla \cdot D\mathbf{v} = 0\,, \tag{8.1}$$

with notation

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) \quad \text{and} \quad \psi \equiv \left(\frac{D-B}{\epsilon\mathcal{F}}\right). \tag{8.2}$$

These equations include variable Coriolis parameter $f = f(\mathbf{x})$ and bottom topography $B = B(\mathbf{x})$.

The dimensionless scale factors appearing in the RSW equations (8.1) and (8.2) are the Rossby number ϵ and the rotational Froude number $\mathcal{F}$, given in terms of typical dimensional scales by

$$\epsilon = \frac{\mathcal{V}_0}{f_0 L} \ll 1 \quad \text{and} \quad \mathcal{F} = \frac{f_0^2 L^2}{g B_0} = O(1)\,. \tag{8.3}$$

The dimensional scales $(B_0, L, \mathcal{V}_0, f_0, g)$ denote equilibrium fluid depth, horizontal length scale, horizontal fluid velocity, reference Coriolis parameter, and gravitational acceleration, respectively. Dimensionless quantities in equations (8.1) are unadorned and are related to their dimensional counterparts (primed), according to

$$\begin{aligned} &\mathbf{v}' = \mathcal{V}_0\mathbf{v}, \quad \mathbf{x}' = L\mathbf{x}, \quad t' = \left(\frac{L}{\mathcal{V}_0}\right)t, \quad f' = f_0 f, \\ &B' = B_0 B, \quad D' = B_0 D, \quad \text{and} \quad D' - B' = B_0(D-B). \end{aligned} \tag{8.4}$$

Here, dimensional quantities are: $\mathbf{v}'$, the horizontal fluid velocity; D', the fluid depth; B', the equilibrium depth; and $D' - B'$, the free surface elevation.

For barotropic horizontal motions at length scales L in the ocean, say, for which $\mathcal{F}$ is order $O(1)$ – as we shall assume – the Rossby number ϵ is typically quite small ($\epsilon \ll 1$) as indicated in equation (8.3). Thus, $\epsilon \ll 1$ is a natural parameter for making asymptotic expansions. For example, we shall assume $|\nabla f| = O(\epsilon)$ and $|\nabla B| = O(\epsilon)$, so we may write $f = 1 + \epsilon f_1(\mathbf{x})$ and $B = 1 + \epsilon B_1(\mathbf{x})$. In this scaling, the leading order implications of equation (8.1) are $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\psi$ and $\nabla \cdot \mathbf{v} = 0$. This is geostrophic balance.

A simple calculation using equation (7.12) shows that the RSW equations (8.1) arise as Euler–Poincaré equations from Hamilton's principle with action $\mathfrak{S}_{\rm RSW}$,

$$\mathfrak{S}_{\rm RSW} = \int dt\, l_{\rm RSW} = \int dt \int dx_1 dx_2 \left[D\mathbf{v}\cdot\mathbf{R}(\mathbf{x}) - \frac{(D-B)^2}{2\epsilon\mathcal{F}} + \frac{\epsilon}{2}D|\mathbf{v}|^2\right], \tag{8.5}$$

where $\mathrm{curl}\,\mathbf{R}(\mathbf{x}) \equiv f(\mathbf{x})\hat{\mathbf{z}}$ yields the prescribed Coriolis parameter. The RSW equations (8.1) themselves can be regarded as being derived from asymptotics in Hamilton's principle for three dimensional incompressible fluid motion, see Holm [1996a]. However, this viewpoint is not pursued further here, as we proceed to describe the relation of RSW to the quasigeostrophic approximation of geophysical fluid dynamics.

Quasigeostrophy. The quasigeostrophic (QG) approximation is a useful model in the analysis of geophysical and astrophysical fluid dynamics, see, e.g., Pedlosky [1987]. Physically, QG theory applies when the motion is nearly in geostrophic balance, i.e., when pressure gradients nearly balance the Coriolis force in a rotating frame of reference, as occurs in meso- and large-scale oceanic and atmospheric flows on Earth. Mathematically, the simplest case is for a constant density fluid in a planar domain with Euclidean coordinates $\mathbf{x} = (x_1, x_2)$. QG dynamics for this case is expressed by the following nondimensional evolution equation for the streamfunction ψ of the incompressible geostrophic fluid velocity $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\psi$,

$$\frac{\partial(\Delta\psi - \mathcal{F}\psi)}{\partial t} + [\psi, \Delta\psi] + \beta\frac{\partial\psi}{\partial x_1} = 0\,. \tag{8.6}$$

Here Δ is the Laplacian operator in the plane, $\mathcal{F}$ denotes rotational Froude number, $[a, b] \equiv \partial(a, b)/\partial(x_1, x_2)$ is the Jacobi bracket (Jacobian) for functions a and b defined on $\mathbb{R}^2$ and β is the gradient of the Coriolis parameter, f, taken as $f = 1+\beta x_2$ in the β-plane approximation, with constant β. (Neglecting β gives the f-plane approximation of QG dynamics.) The QG equation (8.6) may be derived from an asymptotic expansion of the RSW equations (8.1) by truncating at first order in the Rossby number, cf. Pedlosky [1987]. Equation (8.6) may be written equivalently in terms of the potential vorticity, q, as in equation (7.38),

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q = 0, \quad \text{where} \quad q \equiv \Delta\psi - \mathcal{F}\psi + f \quad \text{for QG.} \tag{8.7}$$

This form of QG dynamics expresses its basic property of potential vorticity conservation on geostrophic fluid parcels.

The QG approximation to the RSW equations introduces "quasigeostrophic particles" which move with geostrophic velocity $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\psi$ and, thus, trace the geostrophic component of the RSW fluid flow. These QG fluid trajectories are described as functions of Lagrangian mass coordinates $\boldsymbol{\ell} = (\ell_1, \ell_2)$ given by $\mathbf{x}(\boldsymbol{\ell}, t)$ in the domain of flow.

Hamilton's principle derivation of QG as Euler–Poincaré equations. As in Holm and Zeitlin [1997], we consider the following action for QG written in the Eulerian velocity representation with the integral operator $(1 - \mathcal{F}\Delta^{-1})$,

$$\mathfrak{S}_{\rm red} = \int dt\, l = \int dt \int dx_1 dx_2 \, \Big[\frac{\epsilon}{2} D\mathbf{v}\cdot(1 - \mathcal{F}\Delta^{-1})\mathbf{v} + D\mathbf{v}\cdot\mathbf{R}(\mathbf{x}) - \psi(D-1)\Big]\,. \tag{8.8}$$

This choice can be found as an asymptotic approximation of the RSW action $\mathfrak{S}_{\rm RSW}$ in equation (8.5), in the limit of small wave amplitudes of order $O(\epsilon^2)$ and constant mean depth to the same order, when the wave elevation is determined from the fluid velocity by inverting the geostrophic relation, $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\psi$. The variational

derivatives of the reduced Lagrangian $\mathfrak{S}_{\rm red}$ at fixed $\mathbf{x}$ and t are

$$\begin{aligned}
\frac{1}{D}\frac{\delta l}{\delta \mathbf{v}} &= \mathbf{R} + \epsilon\Big[\mathbf{v} - \frac{\mathcal{F}}{2}\Delta^{-1}\mathbf{v} - \frac{\mathcal{F}}{2D}\Delta^{-1}(D\mathbf{v})\Big]\,,\\
\frac{\delta l}{\delta D} &= \frac{\epsilon}{2}\mathbf{v}\cdot(1-\mathcal{F}\Delta^{-1})\mathbf{v} + \mathbf{v}\cdot\mathbf{R} - \psi\,,\\
\frac{\delta l}{\delta \psi} &= -\,(D-1)\,,
\end{aligned} \tag{8.9}$$

where we have used the symmetry of the Laplacian operator and assumed no contribution arises from the boundary when integrating by parts. For example, we may take the domain to be periodic. Hence, the Euler–Poincaré equation (7.12) for action principles of this type and the fundamental vector identity (7.34) combine to give the Eulerian QG "motion equation",

$$\begin{aligned}
\epsilon\frac{\partial}{\partial t}(1-\mathcal{F}\Delta^{-1})\mathbf{v} \;-\; \mathbf{v}\times\mathrm{curl}\left(\epsilon(1-\mathcal{F}\Delta^{-1})\mathbf{v}+\mathbf{R}\right)\\
+\nabla\left(\psi + \frac{\epsilon}{2}\mathbf{v}\cdot(1-\mathcal{F}\Delta^{-1})\mathbf{v}\right) = 0\,,
\end{aligned} \tag{8.10}$$

upon substituting the constraint $D=1$, imposed by varying ψ. The curl of this equation yields

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q + q\nabla\cdot\mathbf{v} = 0\,, \tag{8.11}$$

where the potential vorticity q is given by

$$q = \epsilon\hat{\mathbf{z}}\cdot\mathrm{curl}\,(1-\mathcal{F}\Delta^{-1})\mathbf{v} + f = \epsilon(\Delta\psi - \mathcal{F}\psi) + f\,, \tag{8.12}$$

with

$$f \equiv \hat{\mathbf{z}}\cdot\mathrm{curl}\mathbf{R} = 1 + \beta x_2, \tag{8.13}$$

and β is assumed to be of order $O(\epsilon)$. The constraint $D=1$ implies $\nabla\cdot\mathbf{v}=0$ (from the kinematic relation $\partial D/\partial t + \nabla\cdot D\mathbf{v}=0$) and when $\mathbf{v}=\hat{\mathbf{z}}\times\nabla\psi$ is substituted, the equation for $q=\Delta\psi-\mathcal{F}\psi+f$ yields the QG potential vorticity convection equation (8.7). Thus, the QG approximation follows as the Euler–Poincaré equation for an asymptotic expansion of the action for the RSW equations when the potential energy is modelled by inverting the geostrophic relation. The *same* QG equation follows upon recasting the action (8.8) in the Kaluza-Klein form (7.43) for incompressible fluids,

$$\mathfrak{S}_{\rm red} = \int dt\, l = \int dt\, dx_1 dx_2\, \Big[\frac{\epsilon}{2}D\mathbf{v}\cdot(1-\mathcal{F}\Delta^{-1})\mathbf{v} + \frac{1}{2}D(\mathbf{R}\cdot\mathbf{v}+\tilde{\omega})^2 - \psi(D-1)\Big]\,, \tag{8.14}$$

where $\tilde{\omega}$ is defined as $\tilde{\omega} = d\theta/dt = \partial\theta/\partial t + \mathbf{v}\cdot\nabla\theta$ for the gauge field θ, as in the case of the Maxwell fluid. Thus, the QG motion equation (8.10) with the beta-effect (included in $\mathbf{R}$) describes geodesic motion on the group of area-preserving diffeomorphisms with respect to the conserved kinetic-energy metric given by

$$\|\mathbf{v}\|^2 = \int dx_1 dx_2\, \Big[\frac{\epsilon}{2}\mathbf{v}\cdot(1-\mathcal{F}\Delta^{-1})\mathbf{v} + \frac{1}{2}(\mathbf{R}\cdot\mathbf{v}+\tilde{\omega})^2\Big]\,, \tag{8.15}$$

where $c' \equiv \mathbf{R}\cdot\mathbf{v} + \tilde{\omega}$ is an advected quantity, $dc'/dt = 0$. This observation from the Euler–Poincaré viewpoint confirms the geodesic interpretation of the QG equations for motion in the β-plane established in Zeitlin and Pasmanter [1994].

1D Boussinesq dispersive shallow water equations. For one dimensional shallow water motion with prescribed mean depth $B(x)$ we choose the following action

$$\mathfrak{S}_{\rm red} = \int dt\, l = \int dt\, dx\, \Big[\frac{1}{2}Dv^2 + \frac{\alpha^2}{2}(Dv)_x^2 - \frac{g}{2}(D - B(x))^2\Big], \tag{8.16}$$

in which g and α^2 are constants and subscript x denotes partial derivative. The second term, proportional to α^2, represents the kinetic energy due to vertical motion. The last term is the potential energy. Recall that the surface elevation $h \equiv D - B(x)$ satisfies $\partial h/\partial t = -(Dv)_x$ for shallow water dynamics in one dimension. Thus, the last two terms are analogous to the Lagrangian

$$\frac{\alpha^2}{2}\Big(\frac{\partial h}{\partial t}\Big)^2 - \ \frac{g}{2}\,h^2\,.$$

This is the Lagrangian for a *harmonic oscillator* whose displacement is given by the surface elevation h and whose natural frequency is $\sqrt{g}/\alpha$. For the choice of action in equation (8.16) and for boundary conditions such that $v_x \to 0$ as $|x| \to \infty$, the variation is

$$\delta\mathfrak{S}_{\rm red} = \int dt\, dx\, \Big[D(v - \alpha^2(Dv)_{xx})\delta v + \Big(\frac{v^2}{2} - \alpha^2 v(Dv)_{xx} - gh\Big)\delta D\Big]. \tag{8.17}$$

The Kelvin-Noether form of the Euler–Poincaré equations in (7.12) then gives

$$0 = \Big(\frac{\partial}{\partial t} + \pounds_v\Big)\Big(\frac{1}{D}\frac{\delta l}{\delta v}dx\Big) - d\Big(\frac{\delta l}{\delta D}\Big) = \Big[\frac{\partial}{\partial t}(v - \alpha^2(Dv)_{xx}) + vv_x + gh_x\Big]dx\,. \tag{8.18}$$

Upon inserting the one-dimensional continuity equation into the α^2 term and rearranging slightly, we find the system of equations

$$\frac{\partial v}{\partial t} + vv_x + gh_x + \alpha^2\frac{\partial^2 h_x}{\partial t^2} = 0, \quad \frac{\partial h}{\partial t} + (hv + B(x)v)_x = 0\,, \tag{8.19}$$

the second of which is just a restatement of continuity. According to Whitham [1974] these equations were favored by Boussinesq, who first formulated them by using the method of asymptotic expansions. Here we see that the Boussinesq shallow water equations (8.19) are also Euler–Poincaré equations on $\mathrm{Diff}(\mathbb{R})$ derived from the action (8.16). The term proportional to α^2 in equation (8.19) arises from the kinetic energy due to vertical motion in the action (8.16) and produces the wave dispersion responsible for solitary wave solutions of these equations.

1D Camassa–Holm equation for peakons. In the limit that the potential energy $gh^2/2$ is negligible compared to the kinetic energy (e.g., for weak gravity, or small surface elevation), we may ignore the last term in the action (8.16) for the Boussinesq equations in one dimension, set $D = B(x)$ in the other terms and rescale to $\alpha^2 = 1$, conforming to the notation in Camassa and Holm [1993]. We thereby obtain the following simplified expression for the shallow water action in this regime,

$$\mathfrak{S}_{\rm red} = \int dt\, dx\, B(x) \left(\frac{1}{2}v^2 + \frac{1}{2}v_x^2\right) . \tag{8.20}$$

For this action, when we also assume $B(x) = 1$ (for constant bottom topography) Hamilton's principle implies simply

$$0 = \delta\mathfrak{S}_{\rm red} = \int dt\, l = \int dt\, dx\ (v - v_{xx})\delta v\,, \tag{8.21}$$

for vanishing boundary conditions for v_x on the real line as $|x| \to \infty$. Hence, $\delta l/\delta v = v - v_{xx}$ and the basic Euler–Poincaré equations for this case reduce to

$$\frac{\partial}{\partial t}(v - v_{xx}) = -\,{\rm ad}^*_v(v - v_{xx}) = -v(v - v_{xx})_x - 2(v - v_{xx})v_x\,. \tag{8.22}$$

This is the $\kappa = 0$ case of the completely integrable partial differential equation derived by Camassa and Holm [1993],

$$\frac{\partial}{\partial t}(v - v_{xx}) + 2\kappa v_x = -3vv_x + 2v_xv_{xx} + vv_{xxx}\,. \tag{8.23}$$

For $\kappa = 0$, this equation admits 'peakon' solutions. The peakons are solitons which interact elastically and possess a peak, at which the derivative v_x reverses sign. The simplest case is the single peakon, which is a solitary travelling wave solution given by $v(x,t) = c_0 \exp - |x - c_0t\,|$, with a constant wave speed c_0. The multi-peakon solutions of the Camassa-Holm equation are obtainable from its associated Lax pair and linear isospectral problem, as shown in Camassa and Holm [1993].

Being basic Euler–Poincaré, equation (8.22) describes geodesic motion. Camassa and Holm [1993] note that the integrable dynamics of N peakons interacting nonlinearly via equation (8.22) reduces to finite dimensional geodesic motion on a manifold with N corners. This geodesic property persists to infinite dimensions and although the equation was originally intended to be an approximation of shallow water motion, it turns out equation (8.22) is also the geodesic spray equation for motion on the group of diffeomorphisms of the real line with metric given by the H^1 norm of v, see Kouranbaeva [1997]. The $\kappa \neq 0$ case of the Camassa-Holm equation (8.23) may be obtained formally by shifting $(v - v_{xx})$ by κ in equation (8.22) and retaining homogeneous boundary conditions for $(v - v_{xx})$ as $|x| \to \infty$. The corresponding statement about geodesic motion for $\kappa \neq 0$, however, is rather more technical than for $\kappa = 0$ and involves the Gel'fand-Fuchs co-cycle and the Bott-Virasoro group, see Misiolek [1997] for details. See Alber et al. [1994, 1995, 1997] for discussions of the

periodic solutions of the Camassa-Holm equation and a related integrable shallow water equation in the Dym hierarchy,

$$2\kappa v_x = \frac{\partial}{\partial t} v_{xx} + 2v_x v_{xx} + v v_{xxx} \,. \tag{8.24}$$

This equation is the "high wave number limit" of the Camassa-Holm equation (8.23).

Higher dimensional Camassa–Holm equation. As we have seen, the Camassa-Holm (CH) equation in one dimension describes geodesic motion on the diffeomorphism group with respect to the metric given by the H^1 norm of the Eulerian fluid velocity. Thus, a candidate for its n-dimensional incompressible generalization should be the Euler–Poincaré equation that follows from the Lagrangian given by the H^1 norm of the fluid velocity in n dimensions, subject to volume preservation (for $n \neq 1$),

$$\mathfrak{S}_{\rm red} = \int dt\; l = \int dt \int_{\mathcal{M}} d^n x\, \frac{D}{2}(v_i v^i + \alpha^2 v_i^{\,j} v^i_{\,,j}) - p(D-1)\,, \tag{8.25}$$

where $\mathcal{M}$ is the domain of the fluid motion and where we have restored the length-scale, or aspect-ratio parameter, α. Varying this action at fixed $\mathbf{x}$ and t gives

$$\begin{aligned}\delta\mathfrak{S}_{\rm red} \;&=\; \int dt \int_{\mathcal{M}} d^n x \left[(v_i v^i + \alpha^2 v_i^{\,j} v^i_{\,,j})\delta D - (D-1)\delta p + \Big(D v_i - \alpha^2 \partial_j (D v_i^{\,j})\Big)\,\delta v^i\right] \\ &\qquad + \alpha^2 \int dt \oint_{\partial\mathcal{M}} d^{n-1}x\; \hat{n}_j (D v_i^{\,j} \delta v^i)\,,\end{aligned} \tag{8.26}$$

whose natural boundary conditions on $\partial\mathcal{M}$ are

$$\mathbf{v}\cdot\hat{\mathbf{n}} = 0 \quad \text{and} \quad (\hat{\mathbf{n}}\cdot\nabla)\mathbf{v} \parallel \hat{\mathbf{n}}, \tag{8.27}$$

where $\parallel$ denotes "parallel to" in the second boundary condition, which of course is not imposed when α^2 is absent. (Recall that $\delta\mathbf{v}$ in equation (8.26) is arbitrary except for being tangent on the boundary. This tangency, along with the second condition in equation (8.27) is sufficient for the boundary integral in equation (8.26) to vanish.)

another set of coundary conditions which will guarantee the vanishing of the coundary terms in 8.26 is that $\mathbf{v} = 0$ on $\partial\mathcal{M}$ (and correspondingly, $\delta\mathbf{v} = 0$ on $\partial\mathcal{M}$. An interesting difficulty with the boundary conditions 8.27 is that they do not from a Lie algegra unless the boundary is flat. We shall correct this difficulty shortly with the inclusion of the second fundamental from of the boundary. (See the Section *The Riemannian CH Equations* below.)

By equation (7.9) or (7.32), the Euler–Poincaré equation for the action $\mathfrak{S}_{\rm red}$ in equation (8.25) is

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)(\mathbf{v} - \alpha^2\Delta\mathbf{v}) + (v_j - \alpha^2\Delta v_j)\nabla v^j - \nabla\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{\alpha^2}{2}|\nabla\mathbf{v}|^2 - p\right) = 0\,, \tag{8.28}$$

where $(\nabla \mathbf{v})^i_j = v^i_{,j} \equiv \partial v^i/\partial x^j$, $|\nabla \mathbf{v}|^2 \equiv v^i_{,j} v^j_i = \mathrm{tr}(\nabla \mathbf{v} \cdot \nabla \mathbf{v}^T)$ and superscript $(\bullet)^T$ denotes transpose. We have also used the constraint $D = 1$, which as before implies incompressibility via the continuity equation for D. Requiring the motion equation (8.28) to preserve $\mathrm{div}\, \mathbf{v} = 0$ implies a Poisson equation for the pressure p with a Neumann boundary condition, which is obtained just as in the case of incompressible ideal fluid dynamics by taking the normal component of the motion equation evaluated at the boundary.

Properties of the Camassa–Holm equation. Since the CH action $\mathfrak{S}_{\mathrm{red}}$ in (8.25) is translation invariant, the Noether theorem ensures the CH equation (8.28) conserves a momentum. In fact, by the stress tensor formulae (7.6)–(7.8), equation (8.28) may be rewritten as

$$\frac{\partial m_i}{\partial t} = - \frac{\partial}{\partial x^j} T^j_i \,. \tag{8.29}$$

In this case, the momentum density m_i, $i = 1, 2, 3$ defined in equation (7.7) is given by

$$m_i \equiv \frac{\delta l}{\delta v^i}\Big|_{D=1} = \; v_i - \alpha^2 \Delta v_i \,, \tag{8.30}$$

and the stress tensor T^j_i defined in equation (7.8) is given by

$$T^j_i = (v_i - \alpha^2 \Delta v_i) v^j - \alpha^2 \, v^k_{,i} v^j_k + \delta^j_i p \,. \tag{8.31}$$

Thus, equation (8.29) implies conservation of the total momentum, $\mathbf{M} = \int_{\mathcal{M}} \mathbf{m} \, d^3x$, provided the normal component of the stress tensor T^j_i vanishes on the boundary.

Since the CH equation (8.28) is Euler–Poincaré, it also has a corresponding ***Kelvin-Noether circulation theorem***. Namely, cf. equation (8.32),

$$\frac{d}{dt} \oint_{\gamma_t} (\mathbf{v} - \alpha^2 \Delta \mathbf{v}) \cdot d\mathbf{x} = 0 \,, \tag{8.32}$$

for any closed curve γ_t that moves with the fluid velocity $\mathbf{v}$. This expression for the Kelvin-Noether property of the CH equation in 3D is reminiscent of corresponding expressions in wave, mean-flow interaction theory. This correspondence suggests a physical interpretation of the α^2 term in the Kelvin-Noether circulation integral as a *Lagrangian mean closure relation* for the pseudomomentum of the high frequency (i.e., rapidly fluctuating, turbulent) components of the flow. In this interpretation, α corresponds to the typical length scale at which these high frequency components become important. See Foias, Holm and Titi [1998] for more discussion of using the 3D CH equation (8.28) as the basis for a turbulence closure model.

In three dimensions, we may use the vector identity (7.34) to re-express the CH motion equation (8.28) in its "curl" form, as

$$\begin{aligned} \frac{\partial}{\partial t}(1 - \alpha^2 \Delta)\mathbf{v} \;\; &- \;\; \mathbf{v} \times \big(\nabla \times (1 - \alpha^2 \Delta)\mathbf{v}\big) \\ &+ \;\; \nabla \left(\mathbf{v} \cdot \big(1 - \alpha^2 \Delta\big)\, \mathbf{v} - \frac{1}{2}|\mathbf{v}|^2 - \frac{\alpha^2}{2}|\nabla \mathbf{v}|^2 + p \right) = 0 \,. \end{aligned} \tag{8.33}$$

The inner product of $\mathbf{v}$ with this equation then implies ***conservation of energy***,

$$E = \frac{1}{2}\int_{\mathcal{M}} d^3x \ \left(\mathbf{v}\cdot(1-\alpha^2\Delta)\mathbf{v}\right) = \frac{1}{2}\int d^3x \ \left(|\mathbf{v}|^2 + \alpha^2|\nabla\mathbf{v}|^2\right) , \tag{8.34}$$

upon integrating by parts and using the boundary conditions (8.27). Naturally, this energy is also conserved in n dimensions. In fact, Legendre transforming the action (8.25) gives the following ***Hamiltonian*** (still expressed in terms of the velocity, instead of the momentum density $\mathbf{m} = \delta l/\delta\mathbf{v}$),

$$H = \int_{\mathcal{M}} d^nx \ \Big[\frac{D}{2}\left(|\mathbf{v}|^2 + \alpha^2|\nabla\mathbf{v}|^2\right) + p(D-1)\Big] . \tag{8.35}$$

Thus, when evaluated on the constraint manifold $D = 1$, the Lagrangian and the Hamiltonian for the CH equation coincide in n dimensions. (This, of course, is not unexpected for a stationary principle giving rise to geodesic motion.)

The curl of the 3D Camassa-Holm motion equation (8.33) yields

$$\frac{\partial}{\partial t}\mathbf{q} = \mathbf{q}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{q} \equiv [\mathbf{v},\mathbf{q}\,], \quad \text{where} \quad \mathbf{q} \equiv \mathrm{curl}(\mathbf{v} - \alpha^2\Delta\mathbf{v}) , \tag{8.36}$$

and we have used incompressibility and commutativity of the divergence and Laplacian operators. Thus, $\mathbf{v}$ is the transport velocity for the generalized vorticity $\mathbf{q}$ and the "vortex stretching" term $\mathbf{q}\cdot\nabla\mathbf{v}$ involves $\nabla\mathbf{v}$, whose L^2 norm is *controlled* by the conservation of energy in equation (8.34). Boundedness of this norm will be useful in future analytical studies of the 3D Camassa-Holm equation; for example, in the investigation of the Liapunov stability properties of its equilibrium solutions.

3D periodic CH motion. In a three dimensional periodic domain, the conserved energy E in equation (8.34) may also be expressed as

$$E = \frac{1}{2}\int_{\mathcal{M}} d^3x \ \left(|\mathbf{v}|^2 + \alpha^2|\mathrm{curl}\mathbf{v}|^2\right) , \tag{8.37}$$

upon integrating by parts and using $\mathrm{div}\,\mathbf{v} = 0$. Thus, in the 3D periodic case, the CH energy E may be interpreted as the sum of the kinetic energy and the enstrophy (i.e., the L^2 norm of vorticity) of the Euler fluid.

The inner product of the generalized vorticity $\mathbf{q}$ with the motion equation (8.33) implies conservation of ***helicity***, for three dimensional periodic motion. Namely, the quantity

$$\Lambda \equiv \int_{\mathcal{M}} d^3x \ (1-\alpha^2\Delta)\mathbf{v}\cdot\mathrm{curl}(1-\alpha^2\Delta)\mathbf{v} \quad \text{(helicity)}, \tag{8.38}$$

is also a constant of motion for three dimensional periodic CH motion.

Using the CH vorticity equation (8.36), we see that steady 3D solutions of the CH equation (denoted with subscript e for "equilibrium") are characterized by the vector-field commutation relation $[\mathbf{v}_e, \mathbf{q}_e] = 0$. Thus, the velocity of a steady CH

flow $\mathbf{v}_e$ generates a volume preserving diffeomorphism that leaves invariant its corresponding steady generalized vorticity $\mathbf{q}_e$. For example, the ***CH Beltrami flows*** for equation (8.33) are characterized by $\mathbf{v}_e = \mu\mathbf{q}_e$, for a constant μ. The CH Beltrami flows verify the invariance property; hence, they are steady. These steady solutions are also *critical points* of the sum $E + \Lambda/2\mu$ of the energy E in equation (8.37) and $1/2\mu$ times the conserved helicity in equation (8.38). Hence, they are *relative equilibrium solutions* of the CH equation. The CH Beltrami flows are divergenceless vector eigenfunctions of the product of the curl operator and the Helmholtz operator, $(1 - \alpha^2\Delta)$. They are the CH analogues of "ABC flows" for the ideal Euler fluid.

Constitutive properties of the CH "fluid." Physically, conservation of the energy in equation (8.37) means that the CH fluid can exchange energy between its translational, and its rotational and shear motions. One may ask, what constitutive relation describes such a fluid?

One may verify directly that the 3D Camassa–Holm equation (8.28) in Cartesian coordinates implies the following formula for the geodesic spray form of the CH equations in 3D:

$$(1 - \alpha^2\Delta)\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) v_i = \alpha^2(\Delta v_j)(v^j_{,i} - v^{,j}_i) - \frac{\partial}{\partial x^j}\left[\left(p - \frac{\alpha^2}{2}v^i_{,k}v^{,k}_i\right)\delta^j_i + 2\alpha^2 v^j_{,k}v^{,k}_i\right]. \quad (8.39)$$

In vector notation, this is

$$(1 - \alpha^2\Delta)\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = \alpha^2(\Delta\mathbf{v})\times\operatorname{curl}\mathbf{v} - \operatorname{div}\left[\left(p - \frac{\alpha^2}{2}|\nabla\mathbf{v}|^2\right)\hat{\mathbf{I}} + 2\alpha^2\nabla\mathbf{v}\cdot\nabla\mathbf{v}^T\right], \quad (8.40)$$

where $\hat{\mathbf{I}}$ is the unit tensor. Viewed this way, the CH fluid acceleration $d\mathbf{v}/dt$ is nonlocal, nonlinear, and (as we know from the general Euler–Poincaré theory) geodesic. Foias, Holm and Titi [1998] show, among other things, that the geodesic spray equation (8.40) may be rearranged into a viscoelastic constitutive relation of Jeffreys type.

Discussion. The essential idea of the CH equation is that its specific momentum (i.e., its momentum per unit mass) is transported by a velocity which is smoothed by inverting the elliptic Helmholtz operator $(1 - \alpha^2\Delta)$, where α corresponds to the length scale at which this smoothing becomes important, i.e., when it becomes of order $O(1)$. When the smoothing operator $(1 - \alpha^2\Delta)^{-1}$ is applied to the transport velocity in Euler's equation to produce the CH equation, its effect on length scales smaller than α is that steep gradients of the specific momentum function tend not to steepen much further, and thin vortex tubes tend not to get much thinner as they are transported. And, its effect on length scales that are considerably larger than α

is negligible. Hence, the transport of vorticity in the CH equation is intermediate between that for the Euler equations in 2D and 3D. As for Euler vorticity, the curl of the CH specific momentum is *convected* as an *active* two form, but its transport velocity is the *smoothed, or filtered* CH specific momentum.

The effects of this smoothing or filtering of the transport velocity in the CH equation can be seen quite clearly from its Fourier spectral representation in the periodic case. In this case, we define $\mathbf{m_k}$ as the $\mathbf{k}$-th Fourier mode of the specific momentum $\mathbf{m} \equiv (1 - \alpha^2\Delta)\mathbf{v}$ for the CH equation, so that $\mathbf{m_k} \equiv (1 + \alpha^2|\mathbf{k}|^2)\mathbf{v_k}$. Then the Fourier spectral representation of the CH equation for a periodic three-dimensional domain is expressed as

$$\Pi \left(\frac{d}{dt}\mathbf{m_k} - i \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \frac{\mathbf{m_p}}{1 + \alpha^2|\mathbf{p}|^2} \times (\mathbf{q} \times \mathbf{m_q}) \right) = 0, \tag{8.41}$$

where Π is the Leray projection onto Fourier modes transverse to $\mathbf{k}$. (As usual, the Leray projection ensures incompressibility.) In this Fourier spectral representation of the CH equation, one sees that the coupling to high modes is suppressed by the denominator when $1+\alpha^2|\mathbf{p}|^2 \gg 1$. Consequently, when $|\mathbf{p}| \geq O(1/\alpha)$, the smoothing of the transport velocity suppresses the development of higher modes $|\mathbf{k}| \geq O(2|\mathbf{p}|)$. And, it *also* suppresses the "stochastic backscatter" from higher modes to lower ones, $|\mathbf{k}| = O(1)$. Thus, thinking of "interaction triangles" among the modes, one sees that all $\mathbf{p} + \mathbf{q} = \mathbf{k}$ triangles are suppressed when $|\mathbf{p}| \geq O(1/\alpha)$. Hence, the CH smoothing of the transport velocity suppresses *both* the forward and backward cascades for wave numbers $|\mathbf{p}| \geq O(1/\alpha)$, but leaves the Euler dynamics essentially unchanged for smaller wave numbers. As we have seen, the result is that the vortex stretching term in the dynamics of $\mathbf{q} = \operatorname{curl} \mathbf{m}$ is mollified in the CH model and so the vortices at high wave numbers will tend to be "shorter and fatter" than in the corresponding Euler case.

When the kinetic energy terms are neglected relative to the gradient velocity terms, the CH action $\mathfrak{S}_{\text{red}}$ in (8.25) becomes

$$\mathfrak{S}^{\infty}_{\text{red}} = \int dt\, l = \int dt \int_{\mathcal{M}} d^n x \left[\frac{D}{2} v^i_{,j} v^{,j}_i - p(D-1) \right], \tag{8.42}$$

whose Euler–Poincaré equation in 3D implies the following,

$$\frac{\partial}{\partial t}\Delta\mathbf{v} - \mathbf{v} \times (\nabla \times \Delta\mathbf{v}) + \nabla \left(\mathbf{v} \cdot \Delta\mathbf{v} + \frac{1}{2}|\nabla\mathbf{v}|^2 - p \right) = 0\,. \tag{8.43}$$

This equation is the "high wave number limit" of the 3D Camassa-Holm equation (8.33). Scale invariance is restored in this limiting equation and its corresponding group invariant (e.g., self-similar) solutions may be illuminating.

The Riemannian CH equations. One can formulate the CH equations on a general Riemannian manifold, possibly with boundary. Although this will be the subject of future papers, we will make some comments about some of the features (some of them conjectural) here.

We start with a smooth, oriented, compact Riemannian manifold M, possibly with a smooth boundary. We first define the group $\mathrm{Diff}_{\mathrm{CH}}$ to be the group of diffeomorphisms $\eta : M \to M$ of class H^s, where $s > (n/2) + 2$ with the boundary condition that the tangent map $T\eta : TM \to TM$ takes the outward normal direction to the boundary ∂M at a point $x \in M$ to the outward normal direction at the point $\eta(x)$. In the incompressible case, one imposes the condition that each η be volume preserving. We first **conjecture** that *this group is a smooth manifold and is a Lie group (in the same sense as in Ebin and Marsden [1970]) with Lie algebra the set of vector fields v on M which are tangent to the boundary of M and that satisfy the boundary condition*

$$\langle \nabla_n v, u \rangle = S(u, v)$$

for all vectors u tangent to the boundary. Here $S(u, v)$ is the second fundamental form of the boundary. This condition on the boundary is the CH analogue to the condition of parallel to the boundary in the case of the Euler equations. The condition comes about by differentiation of the condition on η in the definition of $\mathrm{Diff}_{\mathrm{CH}}$ using a routine calculation.

Notice that the boundary conditions are different from those previously (see equation (8.27). This appears, however, to be needed for the group theoretic version of the equations. Now we put a right invariant Lagrangian on $\mathrm{Diff}_{\mathrm{CH}}$ which, at the identity, is given by

$$L(v) = \frac{1}{2} \int_M \left(\|v\|^2 + \|\nabla v\|^2 \right) d\mu - -\frac{1}{2} \oint_{\partial M} S(u, u) dA \tag{8.44}$$

where ∇v is the covariant derivative of v, where $d\mu$ is the Riemannian volume element and where dA is the area element of the boundary. We are using terminology appropriate to the case in which M is three dimensional, but of course there is no restriction on the dimension of M. Also, $\|\nabla v\|^2$ denotes the norm in the sense of the full Riemannian contraction of the tensor ∇v. The associated Laplace operator is usually called the *rough Laplacian.*

At this point there are some choices one can make. One can use a different H^1 metric built out of thinking of v as a one form and using the d and δ operators and the corresponding *Laplace deRham* operator. This leads to a slightly different system in general, but one that has similar analytical properties. One can also use the group Diff^0 of diffeomorphisms that leave the boundary pointwise fixed, corresponding to the boundary conditions $\mathbf{v} = 0$ on $\partial\mathcal{M}$. (This group was studied in Ebin and Marsden [1970].) In this case, one omits the boundary integral in the preceding displayed equation.

The ***Riemannian CH equations*** are, by definition, the Euler–Poincaré equations for this group and this Lagrangian. The boundary term in the Lagrangian is designed to make the boundary conditions in the resulting equations come out agreeing with those for the Lie algebra of the group $\mathrm{Diff}_{\mathrm{CH}}$. Apart from the boundary conditions, the resulting equations agree with the ones we developed in Euclidean space, but in general one replaces the Laplacian with the rough Laplacian. Note

that since the Lagrangian is quadratic in v, the equations on $\mathrm{Diff}_{\mathrm{CH}}$ are geodesic equations (possibly with respect to an indefinite metric).

Conjecture 8.1 *As in the case of the Euler equations (Ebin and Marsden [1970]), the geodesic spray of the Riemannian CH equations is smooth if $s > (n/2) + 2$.*

This conjecture is based on a direct examination of the expression for the spray of the Riemannian CH equations and seeing that there is no derivative loss. If this is true, then other analytic things, including results on the limit of zero viscosity (or viscoelasticity) also hold. We also note that because energy conservation involves a stronger norm than in the Euler equations for ideal flow, one expects other analytic properties of the Riemannian CH equation to be improved. This would include results on stability and long time existence.

Another consequence of this would be that *the spray of the incompressible Riemannian CH equations would also be smooth.* This follows since the projection map is smooth and the fact that the spray of the incompressible equations is given by the composition of the spray of the compressible ones and the tangent of the projection map. (These facts are proved in Ebin and Marsden [1970]).

As we have mentioned, all of these things will be explored in detail in other publications.

2D Camassa–Holm equation. In two dimensions, the curl of the Euler–Poincaré motion equation (8.33) produces a scalar relation for potential vorticity convection, namely,

$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0, \quad \text{where} \quad q \equiv (1 - \alpha^2 \Delta)\Delta\psi \quad \text{for 2dCH.} \tag{8.45}$$

In terms of the stream function ψ, with $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\psi$, the boundary conditions (8.27) in two dimensions with $\hat{\mathbf{s}} = \hat{\mathbf{z}} \times \hat{\mathbf{n}}$ become

$$\psi = \text{const} \quad \text{and} \quad \hat{\mathbf{n}} \cdot \nabla\nabla\psi \cdot \hat{\mathbf{n}} = 0 \Rightarrow \Delta\psi = 0 \quad \text{on the boundary.} \tag{8.46}$$

Potential vorticity convection (8.45) for the 2dCH equation, combined with incompressibility and the first boundary condition in (8.27) imply conservation of the following quantity,

$$C_\Phi \equiv \int d^2x \; \Phi(q)\,, \tag{8.47}$$

for any suitably well-behaved function Φ. (This is a Casimir function, in the Lie-Poisson bracket formulation.) Substituting $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\psi$ and using the divergence theorem yields an expression for the kinetic energy Lagrangian for the 2dCH equation in terms of the stream function, ψ. Namely,

$$\begin{aligned} E &= \frac{1}{2}\int d^2x \;\left(\mathbf{v} \cdot (1 - \alpha^2\Delta)\mathbf{v}\right) = \frac{1}{2}\int d^2x \;\left(\nabla\psi \cdot (1 - \alpha^2\Delta)\nabla\psi\right) \\ &= \frac{1}{2}\sum_i \psi^{(i)} \oint_{\gamma^{(i)}} ds \;\frac{\partial}{\partial n}(1 - \alpha^2\Delta)\psi^{(i)} - \frac{1}{2}\int d^2x \;\left(\psi(1 - \alpha^2\Delta)\Delta\psi\right)\,, \end{aligned} \tag{8.48}$$

where we sum over the connected components of the boundary $\gamma^{(i)}$ and use $\psi^{(i)}$ constant on the ith component. Thus, solutions of the CH equation (8.28) in two dimensions optimize the integrated product of the stream function and potential vorticity, constrained by the circulation of $\mathbf{v} - \alpha^2\Delta\mathbf{v}$ on each connected component of the boundary. The Lagrange multiplier for this circulation is the corresponding boundary component's (constant) stream function.

Steady CH solutions in two dimensions. Steady solutions of the 2dCH equation (8.45) with boundary conditions (8.46) satisfy

$$\hat{\mathbf{z}} \cdot \nabla\psi_e \times \nabla q_e = J(q_e, \psi_e) = 0, \quad \text{for} \quad q_e \equiv (1 - \alpha^2\Delta)\Delta\psi_e \,. \tag{8.49}$$

Thus, steady CH solutions exist when there is a functional relation between the potential vorticity q_e and its associated stream function ψ_e. For example, one could have $q_e = F(\psi_e)$ for function F. In particular, the *linear* steady flows satisfy

$$q_e = \Delta(1 - \alpha^2\Delta)\psi_e = -|\mathbf{k}|^2(1 + \alpha^2|\mathbf{k}|^2)\psi_e. \tag{8.50}$$

These are sines and cosines with wave number $\mathbf{k}$ for periodic boundary conditions. The corresponding fluid velocity for any of these steady 2dCH solutions is found from $\mathbf{v}_e = \hat{\mathbf{z}} \times \nabla\psi_e$.

As another example, the ***point potential-vortex solution*** centered at $z' = x' + iy'$ in the infinite x, y plane has stream function

$$\psi(|z - z'|) = \log(|z - z'|) + K_0(|z - z'|/\alpha), \tag{8.51}$$

where $K_0(|z - z'|/\alpha)$ is the Bessel function of the second kind. This stream function satisfies

$$q = \Delta(1 - \alpha^2\Delta)\psi = 2\pi\delta(|z - z'|),$$

where Δ is the Laplacian operator in the plane and $\delta(|z - z'|)$ is the Dirac delta function. (The proof uses $\Delta\log(|z-z'|) = 2\pi\delta(|z-z'|)$ and $(1-\alpha^2\Delta)K_0(|z-z'|/\alpha) = 2\pi\delta(|z-z'|)$.) The ***circular potential-vortex patch solution***, $q = \Delta(1-\alpha^2\Delta)\psi = C = \text{const}$ for $|z - z'| \le a$ and $q = 0$ for $|z - z'| > a$, has stream function

$$\begin{aligned} \psi &= C\left(\frac{|z - z'|^2}{4} + 1\right) \text{ for } |z - z'| \le a, \\ \psi &= \log(|z - z'|) + \alpha^2 K_0(|z - z'|/\alpha) \text{ for } |z - z'| \ge a \,, \end{aligned} \tag{8.52}$$

where the constant C is chosen so the velocity is continuous at $|z - z'| = a$. The interior of this solution is also a uniformly rotating vortex patch. These special potential-vortex solutions illustrate the "screening" in the vortex interaction dynamics for the 2dCH equation introduced by the Helmholtz operator $(1 - \alpha^2\Delta)$, which modifies the momentum density of the 2dCH flow relative to the standard incompressible Euler equations in the plane. As we have seen, the corresponding screening length α is an additional parameter in the model.

Quasigeostrophic analogue of CH in two dimensions. We extend the nondimensional QG action principle in equation (8.8) for QG dynamics in a periodic two-dimensional domain to include CH α^2 terms, as follows,

$$\mathfrak{S}_{\rm red} = \int dt\, l = \int dt \int dx_1 dx_2 \, \Big[\frac{\epsilon}{2} D\mathbf{v}\cdot(1-\mathcal{F}\Delta^{-1}-\alpha^2\Delta)\mathbf{v} + D\mathbf{v}\cdot\mathbf{R} - \psi(D-1)\Big]\,. \tag{8.53}$$

The corresponding Euler–Poincaré equation is to be compared to the QG motion equation (8.10). We find,

$$\epsilon\frac{\partial}{\partial t}(1-\mathcal{F}\Delta^{-1}-\alpha^2\Delta)\mathbf{v} \;-\; \mathbf{v}\times{\rm curl}\left(\epsilon(1-\mathcal{F}\Delta^{-1}-\alpha^2\Delta)\mathbf{v}+\mathbf{R}\right) \\ +\nabla\left(\psi+\frac{\epsilon}{2}\mathbf{v}\cdot(1-\mathcal{F}\Delta^{-1}-\alpha^2\Delta)\mathbf{v}\right) = 0\,. \tag{8.54}$$

The curl of this equation yields

$$\epsilon\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla(\epsilon q + f) = 0\,, \tag{8.55}$$

where the potential vorticity q is now given by

$$q = \hat{\mathbf{z}}\cdot{\rm curl}\,(1-\mathcal{F}\Delta^{-1}-\alpha^2\Delta)\mathbf{v}\,, \tag{8.56}$$

and we choose

$$f \equiv \hat{\mathbf{z}}\cdot{\rm curl}\mathbf{R} = 1+\epsilon\beta x_2\,. \tag{8.57}$$

The constraint $D=1$ implies $\nabla\cdot\mathbf{v}=0$ as usual and when $\mathbf{v}=\hat{\mathbf{z}}\times\nabla\psi$ is substituted, the equation for $q=\Delta(1-\alpha^2\Delta)\psi-\mathcal{F}\psi$ yields

$$\frac{\partial}{\partial t}(\Delta(1-\alpha^2\Delta)\psi-\mathcal{F}\psi) + [\psi,\Delta(1-\alpha^2\Delta)\psi] + \beta\frac{\partial\psi}{\partial x_1} = 0\,. \tag{8.58}$$

Here $[a,b]\equiv\partial(a,b)/\partial(x_1,x_2)=J(a,b)$ is the Jacobi bracket (Jacobian) for functions a and b defined on the two dimensional domain. Steady solutions of the QG-CH equation (8.55) satisfy

$$J(q_e+\beta x_2,\psi_e)=0, \quad \text{for} \quad q_e=\Delta(1-\alpha^2\Delta)\psi_e-\mathcal{F}\psi_e\,. \tag{8.59}$$

The dispersion relation for plane-wave solutions of equation (8.58) with frequency ω and wavenumber $\mathbf{k}$ is

$$\omega(\mathbf{k}) = \frac{-\beta k_1}{\mathcal{F}+|\mathbf{k}|^2(1+\alpha^2|\mathbf{k}|^2)}\,. \tag{8.60}$$

Such plane-wave solutions are analogous to Rossby waves in QG. As with QG Rossby waves, these plane-wave solutions are *nonlinear* solutions of equation (8.58) in a two dimensional periodic domain. If we choose $\mathcal{F}=O(1)$ and $\alpha^2=o(1)$, then the effect of the α^2 term in this dispersion relation is to significantly reduce the oscillation

frequency and propagation speeds of those waves at wave numbers greater than about α^{-1}. Thus, such short waves are suppressed, and the emerging dynamics of this modified QG theory will tend to possess significant activity only at length scales larger than α. Apparently the dispersion relation for the dynamics at these larger length scales will faithfully approximate the corresponding QG dynamics at these scales. This scenario is, of course, consistent with our earlier discussion of the wave number dynamics of the CH solutions in three dimensions.

Appendix

Left Representation and Right Invariant Lagrangian. There is a version of this theorem for right invariant Lagrangians, but with the representation of G on V still on the left. The proof is, of course, identical so we shall only state this theorem. The set–up is the following:

- There is a *left* representation of Lie group G on the vector space V and G acts in the natural way on the *right* on $TG \times V^*$: $(v_g, a)h = (v_g h, h^{-1}a)$.
- Assume that the function $L : TG \times V^* \to \mathbb{R}$ is right G–invariant.
- In particular, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) = L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G.
- Right G–invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

  $$l(v_g g^{-1}, g a_0) = L(v_g, a_0).$$

  Conversely, this relation defines for any $l : \mathfrak{g} \times V^* \to \mathbb{R}$ a right G–invariant function $L : TG \times V^* \to \mathbb{R}$.
- For a curve $g(t) \in G$, let $\xi(t) := \dot{g}(t)g(t)^{-1}$ and define the curve $a(t)$ as the unique solution of the linear differential equation with time dependent coefficients $\dot{a}(t) = \xi(t)a(t)$ with initial condition $a(0) = a_0$. The solution can be equivalently written as $a(t) = g(t)a_0$.

Theorem 8.2 *The following are equivalent:*

i *Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0 \tag{8.61}$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii *$g(t)$ satisfies the Euler–Lagrange equations for L_{a_0} on G.*

iii *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0 \tag{8.62}$$

holds on $\mathfrak{g} \times V^*$*, using variations of the form*

$$\delta\xi = \dot{\eta} - [\xi, \eta], \quad \delta a = \eta a, \tag{8.63}$$

where $\eta(t) \in \mathfrak{g}$ *vanishes at the endpoints.*

iv *The Euler–Poincaré equations hold on* $\mathfrak{g} \times V^*$

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} - \left(\frac{\delta l}{\delta a}\right) \diamond a. \tag{8.64}$$

Right Representation and Left Invariant Lagrangian. The set up is as follows:

- There is a *right* representation of Lie group G on the vector space V and G acts in the natural way on the *left* on $TG \times V^*$: $h(v_g, a) = (hv_g, ah^{-1})$.
- Assume that the function $L : TG \times V^* \to \mathbb{R}$ is left G–invariant.
- In particular, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) = L(v_g, a_0)$. Then L_{a_0} is left invariant under the lift to TG of the left action of G_{a_0} on G.
- Left G–invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

  $$l(g^{-1}v_g, a_0 g) = L(v_g, a_0).$$

  Conversely, this relation defines for any $l : \mathfrak{g} \times V^* \to \mathbb{R}$ a left G–invariant function $L : TG \times V^* \to \mathbb{R}$.
- For a curve $g(t) \in G$, let $\xi(t) := g(t)^{-1}\dot{g}(t)$ and define the curve $a(t)$ as the unique solution of the linear differential equation with time dependent coefficients $\dot{a}(t) = a(t)\xi(t)$ with initial condition $a(0) = a_0$. The solution can be equivalently written as $a(t) = a_0 g(t)$.

Theorem 8.3 *The following are equivalent:*

i *Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0 \tag{8.65}$$

holds, for variations $\delta g(t)$ *of* $g(t)$ *vanishing at the endpoints.*

ii $g(t)$ *satisfies the Euler–Lagrange equations for* L_{a_0} *on* G*.*

iii *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0 \tag{8.66}$$

holds on $\mathfrak{g} \times V^*$*, using variations of the form*

$$\delta\xi = \dot{\eta} + [\xi, \eta], \quad \delta a = a\eta, \tag{8.67}$$

where $\eta(t) \in \mathfrak{g}$ *vanishes at the endpoints.*

iv *The Euler–Poincaré equations hold on* $\mathfrak{g} \times V^*$

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} - \frac{\delta l}{\delta a} \diamond a. \tag{8.68}$$

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