On the link between umbilic geodesics and soliton solutions of nonlinear PDEs

By Mark S. Alber[1], Roberto Camassa[2], Darryl D. Holm[2] and Jerrold E. Marsden[3]

[1]*Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720, USA and Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA.*

[2]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*

[3]*Control and Dynamical Systems, 104-44, Caltech, Pasadena CA 91125, USA*

In this paper we describe a new class of soliton solutions, called umbilic solitons, for certain nonlinear integrable PDEs. These umbilic solitons have the property that as the space variable x tends to infinity, the solution tends to a periodic wave, and as x tends to minus infinity, it tends to a phase shifted wave of the same shape. The equations admitting solutions in this new class include the Dym equation and equations in its hierarchy. The methods used to find and analyse these solutions are those of algebraic and complex geometry. We look for classes of solutions by constructing associated finite-dimensional integrable Hamiltonian systems on Riemann surfaces. In particular, in this setting we use geodesics on n-dimensional quadrics to find the spatial, or x-flow, which, together with the commuting t-flow given by the equation itself, defines new classes of solutions. Amongst these geodesics, particularly interesting ones are the umbilic geodesics, which then generate the class of umbilic soliton solutions. This same setting also enables us to introduce another class of solutions of Dym-like equations, which are related to elliptic and umbilic billiards.

1. Introduction

The goal of the present paper is to establish a link between umbilic geodesics on n-dimensional quadrics (described in Alber & Marsden (1994*a*) in terms of complex angle representations) and new soliton-like solutions of nonlinear equations such as the following equation in the Dym hierarchy:

$$U_{xxt} = -2U_xU_{xx} - UU_{xxx} + \kappa U_x, \tag{1}$$

which is discussed in Alber *et al.* (1994*a*). Here κ is a real parameter and, as we shall see in §3, there is a fundamental difference between $\kappa = 0$ and $\kappa \neq 0$ in the phase-space geometry. Our solutions are solitons in the standard sense (see, for example, Ablowitz & Segur 1981) that they undergo nonlinear interactions but retain their identity after an interaction, up to a phase shift.

It is known that there are basic links between geodesic flows on quadrics and quasi-periodic solutions of certain nonlinear PDEs, including the KdV equation (for the KdV case, see, for example, Alber & Alber (1987) and Alber & Marsden (1994*a*)).

Proc. R. Soc. Lond. A (1995) **450**, 677–692
Printed in Great Britain

TEX Paper

The geodesic flows we discuss provide the spatial x-flow, and when combined with the commuting t-flow, yield the class of soliton-like PDE solutions we seek.

In what follows, we also demonstrate a 'spiral' type of behaviour of umbilic geodesics which may be interpreted as the geodesics being homoclinic to periodic orbits. The corresponding umbilic solitons approach a spatially periodic wave as $x \to \pm\infty$. This oscillatory behaviour at spatial infinity is similar to that observed by Hunter & Scheurle (1988) for solutions of nonlinear equations that are of perturbed KdV type.

Elliptic billiards can be obtained from the problem of geodesics on quadrics by collapsing along the shortest semiaxis (for details, see Alber (1986)). This process yields Hamiltonians and first integrals for the resulting billiard problem. The reflection conditions of the billiard can be expressed as a jump from one sheet of an associated Riemann surface to another. Here, we use the corresponding Hamiltonian billiard flows to construct new classes of solutions of equations in the Dym hierarchy. Such billiard-type solutions have discontinuous spatial derivative and, thus, are weak solutions for the class of PDEs we consider. Weak solutions of equation (1) for the case $\kappa = 0$ are studied in connection with weakly nonlinear solutions of hyperbolic equations in Hunter & Zheng (1994). In another publication (Alber *et al.* 1994*a*), we characterized the peakon solutions of a shallow-water equation as solutions of billiard type and further extended the class of billiard solutions for this equation.

This association of weak solutions of an equation in the Dym hierarchy with the dynamics of elliptic billiards raises the question of what types of algebraic perturbations of Hamiltonian billiard flows may be associated with quasi-periodic and umbilic-soliton solutions of Dym-like equations. For example, an interesting class of perturbations of the elliptic billiard flow is considered in Levallois & Tabanov (1993). Elliptic billiards are investigated in Moser & Veselov (1991) in connection with the discrete version of the so-called C. Neumann problem, and in Deift *et al.* (1991) in the more general context of loop groups. Also, Cewen (1990) discusses a link between stationary flows of the Dym equation and the problem of geodesics on an ellipsoid. A relation between stationary flows in the Dym hierarchy and certain mechanical problems is also discussed in Ragnisco & Rauch-Wojciechowski (1994).

The present paper is organized as follows: in § 2 we recall the general set-up for Hamiltonian systems on Riemann surfaces. In the context of travelling-wave solutions and first integrals, we consider special billiard solutions for the Dym hierarchy in § 3. In § 4 we set up the Riemann surfaces and the corresponding Hamiltonian systems for the case of the Dym hierarchy. We also discuss billiard solutions for the elliptic case. In § 5 we introduce the angle representations for umbilic geodesics on quadrics and in § 6 we transfer this information to the context of umbilic solitons and examine some properties of these solutions numerically. Some related remarks are made for umbilic geodesics on hyperboloids.

2. Hamiltonian systems on Riemann surfaces

We first briefly recall that quasi-periodic solutions of many integrable nonlinear equations can be described in terms of finite-dimensional Hamiltonian systems on $\mathbb{C}^{2n}$. A complete set of first integrals for such equations are obtainable, for example, by the method of generating equations, as summarized in Alber *et al.* (1994*a*). The method of generating equations has associated with it a finite-dimensional complex phase space $\mathbb{C}^{2n}$ and two commuting Hamiltonian flows. The first Hamiltonian flow gives the spatial evolution, and the other gives the temporal evolution of special classes of solutions of the original PDE. The level sets of the first integrals are Riemann surfaces having branch points parametrized by the choice of values of the first integrals.

We think of $\mathbb{C}^{2n}$ as being the cotangent bundle of $\mathbb{C}^n$, with configuration variables $\mu_1, \ldots, \mu_n$ and with canonically conjugate momenta $P_1, \ldots, P_n$. The two commuting Hamiltonians on $\mathbb{C}^{2n}$ both have the form

$$H = \tfrac{1}{2} g^{jj} P_j^2 + V(\mu_1, \ldots \mu_n),$$

where g^{jj} is a Riemannian metric on $\mathbb{C}^n$. The two Hamiltonians are distinguished by different choices of the metric. The two associated Hamiltonian flows have the same set of first integrals, the zero level sets of which are of the form

$$P_j^2 = K(\mu_j), \quad j = 1, \ldots, n,$$

where K is a rational function of μ_j. Thus, we get two commuting flows on the symmetric product of n copies of the Riemann surface

$$\mathcal{R} : P^2 = K(\mu)$$

defined by the first integrals. These Riemann surfaces can be regarded as complex Lagrangian submanifolds. *We call this set-up the μ-representation of the problem.*

In the following section we will utilize the method of generating equations for the Dym hierarchy of equations.

3. Special solutions for the Dym hierarchy

The integrable shallow-water equation investigated in Camassa & Holm (1993), namely

$$M_t = -(M\partial_x + \partial_x M)U, \quad \text{where } M = U - U_{xx} + \tfrac{1}{2}k, \tag{2}$$

has an associated hierarchy determined from the recursion operator $R = J_2 J_1^{-1}$, where

$$J_1 = \partial_x - \partial_x^3 \quad \text{and} \quad J_2 = M\partial_x + \partial_x M$$

are the first and second Hamiltonian operators. Applying the recursion operator R three times, starting from the shallow-water equation (2), gives the following integrable equation (Camassa & Holm 1993):

$$M_t = -(\partial_x - \partial_x^3)\frac{1}{\sqrt{M}}. \tag{3}$$

In the standard Dym equation (see Kruskal 1975; Wadati *et al.* 1980), the term ∂_x in (3) is absent. Thus, (3) is Dym-like, but there are deep differences from the standard Dym equation in, for example, its underlying complex geometry.

The two Hamiltonian structures for the standard Dym equation are $\tilde{J}_1 = -\partial_x^3$ and J_2 as above, with

$$M = -U_{xx} + \tfrac{1}{2}\kappa.$$

The equation that is located at the same level as the shallow-water equation (2) in the hierarchy of the standard Dym equation is (1); thus, it may be obtained by applying the recursion operator $\tilde{R} = \tilde{J}_1 J_2^{-1}$ three times to the Dym flow. As we shall see, equation (1) is of special interest because it can be viewed as describing the dynamics for a stationary Hamiltonian system with a finite number of degrees of freedom, associated with geodesic flow on n-dimensional quadrics. This connection leads to new interesting classes of solutions of equation (1). Also, every equation in the Dym hierarchy shares the link to the problem of geodesics on quadrics (cf. Cewen 1990). Therefore, new classes of solutions can be constructed in a similar fashion for every member in the hierarchy. Furthermore, the corresponding time flows are different for each particular equation in the hierarchy.

To provide an example of billiard solutions of (1), we consider the first integral in the travelling wave case. After substituting an ansatz of the form $u = U(x - ct)$ into (1) and integrating, one obtains

$$-cU'' + \tfrac{1}{2}(U')^2 - \kappa U + UU'' = C_1. \tag{4}$$

This yields

$$U' = \pm\sqrt{\frac{\kappa U^2 + C_1 U + C_2}{U - c}}, \tag{5}$$

which can be transformed into the integral form

$$\pm\int_{U^0}^{U}\sqrt{\frac{U - c}{\kappa U^2 + C_1 U + C_2}}\,\mathrm{d}U = \pm\int_{U^0}^{U}\sqrt{\frac{U - c}{\kappa(U - a_1)(U - a_2)}}\,\mathrm{d}U = x - ct. \tag{6}$$

This expression is different from the corresponding first integral for the KdV equation in the following way: the genus of the Riemann surface is $g = 1$ in both cases, but the integrand here is an Abelian differential of the second type because of the extra square root in the numerator; as a result, the limiting procedure (namely the coalescence of two roots) that would lead to single-soliton solutions in the KdV case, here leads to unbounded solutions. Because of this, equation (1) does not have solitons of KdV type. We will show, however, that there exists an umbilic soliton which can be obtained from a quasi-periodic solution on a Riemann surface of genus $g = 2$. In this sense, it is a new two-dimensional solution of soliton type. At the same time, equation (1) also has cusped and peaked periodic solutions.

The cusped solutions correspond to a particular distribution of a_1, a_2, c, κ around zero in such a way that the allowed domain for U real includes c. Thus, from (6) it follows that U can be a continuous function whose derivative can be unbounded, i.e. U can have a cusp. The peaked travelling-wave solutions can be described as follows: we choose, for example, a number $\kappa < 0$ and $0 < a_2 < c < a_1$, and then apply the limiting procedure $a_2 \to c$ to the basic polynomial, i.e. to the polynomial in the numerator of (5). This results in the following problem of inversion:

$$\pm\int_{U^0}^{U}\sqrt{\frac{1}{\kappa(U - a_1)}}\,\mathrm{d}U = x - ct. \tag{7}$$

One has to keep in mind that U is still changing between c and a_1. In other words, U jumps from one sheet of the Riemann surface to the other at $U = c$. This kind of solution corresponds to the one discussed in Alber *et al.* (1994*a*) for the shallow-water equation (2), but here a quadratic rather than an exponential profile is present on either side of the peak, at which $U = c$. These peaked solutions constitute the simplest example of the billiard solutions we mentioned above.

The case $\kappa = 0$ is of special interest, both for physical (see Hunter & Zheng 1994) and for mathematical reasons. This can already be appreciated from expression (6) for the travelling-wave solution, since $\kappa = 0$ reduces the order of the basic polynomial and eliminates one of the roots from the problem of inversion (6). Also, when $\kappa = 0$, the approach of Camassa & Holm (1993) for finding solutions of the shallow-water equation with finite jumps in the first derivative can be used for (1). The 'N-peakon' solution in this case is given by

$$U(x,t) = \sum_{j=1}^{N} p_j(t)|x - q_j(t)|, \tag{8}$$

where the p_j and q_j are canonically conjugate variables whose evolution is determined from the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{i,j=1}^{N} p_i p_j |q_i - q_j|. \tag{9}$$

When the p_j are chosen so that the constant of motion $\mathcal{P} = \sum_j^N p_j = 0$, the solution (8) corresponds to the weak solutions analysed in Hunter & Zheng (1994) (see also Alber *et al.* 1994*b*). This N-peakon solution can be obtained by applying the limiting procedure for soliton solutions (see § 5) to a solution of billiard type.

4. Generating equation for the Dym hierarchy

Methods of algebraic and complex geometry enable one to extend the one-dimensional result (6) to the n-dimensional case. (For details about the general method, see Ercolani (1989).) In particular, using generating equations, one obtains umbilic n-soliton solutions and billiard solutions of equation (1).

The method begins in the standard fashion by considering the spectral problem for an associated Schrödinger operator of the form

$$L = -\frac{\partial^2}{\partial x^2} + V(x,t,E), \tag{10}$$

where E is a parameter. In some cases, such as the KdV equation, E appears as an eigenvalue and one ultimately equates the potential with a solution of the nonlinear equation itself. Our case is similar to the nonlinear Schrödinger equation in that the solution U and the potential V are related in a slightly more complicated way, given below. To carry out this procedure, one begins by looking for a solution A of the Lax system

$$L\psi = 0, \quad \left(\frac{\partial L}{\partial t} + [L,A]\right)\psi = 0 \tag{11}$$

of the form

$$A = B\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial B}{\partial x}. \tag{12}$$

Substituting the given form of A into the Lax system, one gets

$$\partial V/\partial t = -\tfrac{1}{2}B''' + 2B'V + BV'. \tag{13}$$

where the prime denotes $\partial/\partial x$. Equation (13) is called the *generating equation*. For different choices of the forms of B and V, usually taken to be rational functions of E with coefficients depending on U and a finite number of space derivatives and one time derivative of U, this procedure will lead to different hierarchies of integrable systems. For instance, choosing

$$B(E,x) = E - U(x,t) \quad \text{and} \quad V = \frac{M}{2E}, \quad \text{where } M = -\frac{\partial^2 U}{\partial x^2} + \tfrac{1}{2}\kappa, \tag{14}$$

yields equation (1). Then, setting B to be a polynomial of degree n in E while keeping V the same, and equating coefficients for the same powers of E after substituting in the generating equation in a way similar to the method described in Alber *et al.* (1994*a*), leads to a 'chain' of recurrence relations which yields the evolution equations (non-local when $n > 1$) in the hierarchy of the flow (1).

Remark. The most important fact about this approach is that the stationary generating equation,

$$-\tfrac{1}{2}B''' + 2B'V + BV' = 0, \tag{15}$$

with V of the form given in (14), coincides with the generating equation obtained in Alber & Alber (1985) for the problem of geodesics on n-dimensional quadrics. This equation was called there the generating equation of inverse KdV type.

After multiplying (15) by $2B$ and integrating, one obtains

$$-B''B + \tfrac{1}{2}B'^2 + 2B^2V = C_{2n}(E)/E. \tag{16}$$

Here, the right-hand side is an integration constant, but we are interested in the case in which the numerator C_{2n} is a polynomial of degree $2n$ in E with constant coefficients, in order to match like powers of E when B is chosen to be a polynomial of degree n in E. Substituting $E = \mu_j$ and $B = \prod_{j=1}^{n}(E - \mu_j(x,t))$, one obtains the system of equations for the x-derivatives of the μ variables, as we shall see in equations (20) below.

The link between the problem of geodesics and the generating equation can be explained as follows: recall that geodesics on n-dimensional quadrics of the form

$$Q : \sum_{j=1}^{n+1} \frac{x_j^2}{l_j} = \sum_{j=1}^{n+1} q_j^2 = 1$$

(so that $q_j^2 = x_j^2/l_j$ and l_j are the squared semi-axes of the quadric) are described by Euler–Lagrange equations for the Lagrangian

$$L = \tfrac{1}{2}\sum_{j=1}^{n+1} l_j(q_j')^2 + \tfrac{1}{2}u(x)\left(\sum_{j=1}^{n+1} q_j^2 - 1\right). \tag{17}$$

These Euler–Lagrange equations are

$$q_j'' - \frac{u}{l_j}q_j = 0, \quad j = 1, \ldots, n+1. \tag{18}$$

It follows from (18), the constraint in (17) and its second derivative that

$$u = -\sum_{j=1}^{n+1} q_j'^2 \Big/ \sum_{j=1}^{n+1} \frac{q_j^2}{l_j}. \tag{19}$$

Thus u plays a role of a 'potential' for the constraint in (17).

Lemma 4.1. *Let $q_j(x)$, $j = 1, \ldots, n+1$, be a solution of the system (18). Then* *the polynomial*

$$B(E,x)=\sum_{j=1}^{n+1}\left(\prod_{k\neq j}(E-l_k)\right)q_j^2$$

is a solution of the generating equation (15).

This lemma shows, in particular, that the sets of first integrals coincide for the two problems and that the problem of geodesics determines the phase-space geometry for the quasi-periodic solutions of the Dym hierachy of equations.

The commuting Hamiltonian x- and t-flows for each member of the Dym hierachy can be introduced using the dynamical generating equation (13), following the approach of Alber *et al.* (1994*a*); one obtains

$$\left.\begin{aligned}\frac{\partial\mu_j}{\partial x}&=2\mathrm{i}\epsilon_j\frac{\sqrt{K(\mu_j)}}{\prod_{l\neq j}(\mu_j-\mu_l)},\\ \frac{\partial\mu_j}{\partial t}&=2\mathrm{i}\epsilon_j D_j\frac{\sqrt{K(\mu_j)}}{\prod_{l\neq j}(\mu_j-\mu_l)},\quad j=1,\dots,n,\end{aligned}\right\}\tag{20}$$

where $$K(E)=-L_0^2\frac{1}{E}\prod_{k=1}^{2n}(E-m_k),$$

and the function D_j, of $\mu_1,\dots,\mu_n$, which is obtained by the method of generating equations, is different for each member of the Dym hierachy. Here L_0 is a constant, $\epsilon_j=\pm1$, and each variable μ_j lies on a copy of the Riemann surface

$$\mathcal{R}_{\text{quasi}}:P^2=K(E).$$

Following the methods of Alber *et al.* (1994*a*), the solution of (1) can be represented as follows:

$$U=\sum_{j=1}^{n}\mu_j+(\kappa+L_0^2)\tfrac{1}{2}x^2+C_1x+C_2.\tag{21}$$

Requiring the boundary condition that U be bounded at spatial infinity gives the condition $\kappa=-L_0^2$ and $C_1=0$. The constant C_2 can be removed by a Galilean boost of the original equation (1).

Notice that in the case of the shallow-water equation (2), the limiting case $k=0$, while special, is not singular. For the Dym equation (1), however, the limit $\kappa\to0$ leads to $L_0=0$ (due to $\kappa=-L_0^2$), i.e. the coefficient of the leading-order term E^{2n-1} of K vanishes, and there is a change in the genus of the associated Riemann surface, so this limit must be treated separately. This illustrates the nature of the singularity of the limit $\kappa\to0$ in the context of the Riemann suface associated to quasi-periodic solutions of (1), and generalizes the observation following the simple case of the travelling-wave solution (6).

The systems (20) are Hamiltonian systems which have the following set of first integrals:

$$P_j^2=\kappa\frac{1}{\mu_j}\prod_{k=1}^{2n}(\mu_j-m_k),\quad j=1,\dots,n.$$

Notice that the limiting process $m_{k_0}\to0$ applied to these integrals results in the following first integrals:

$$P_j^2 = \kappa \prod_{k=1, k\neq k_0}^{2n} (\mu_j - m_k), \quad j = 1, \dots, n$$

and, thus, one also obtains a system of differential equations which describes elliptic (or hyperbolic) billiard solutions of the equations in the Dym hierarchy. Notice that the x-part of the system (20) after this limiting process, describes standard geodesic billiards. In what follows, we will continue to indicate by L_0 the coefficient of the leading-order power of E in the definition of the Riemann surfaces, with the understanding that whenever a formula applies to the PDE (1), $-L_0^2 = \kappa \neq 0$.

5. Umbilic geodesics on n-dimensional quadrics

Soliton solutions for nonlinear integrable equations are typically obtained by coalescing pairs of roots of the basic polynomial of the Riemann surface (see, for example, Previato 1985; Alber & Marsden 1992). By applying this procedure, we now show that classical two-dimensional umbilic geodesic flows on quadrics can also be obtained in this way. Then we generalize this procedure and introduce umbilic geodesic flows on n-dimensional quadrics, thereby obtaining Hamiltonians and angle representations for the new class of umbilic n-soliton-like solutions alluded to above.

We start with the definition of the umbilic points on a two-dimensional surface.

Definition 5.1. Let $K_1(p)$ and $K_2(p)$ be the largest and smallest principal curvatures at a point p of a surface

$$\mathcal{S} : W \to \mathbb{R}^3.$$

A point p_0 on the surface is called an *umbilic point* if $K_1(p_0) = K_2(p_0)$.

The study of umbilic geodesics on two-dimensional quadrics is one of the classical problems of Riemannian geometry. For example, it can be shown that in case of a two-dimensional ellipsoid there are exactly four umbilic points (see Klingenberg 1982). Also, there is a special family of umbilic geodesics on the ellipsoid going through pairs of umbilic points. In Alber & Marsden (1994*b*) we show that they correspond to a special choice of the value of a particular first integral of the quasi-periodic Hamiltonian flow. The value of this first integral should coincide with the length of the intermediate semiaxis of the ellipsoid. In what follows we obtain umbilic Hamiltonians and construct a complete set of umbilic angle variables. Then we generalize the notion of the umbilic geodesics to the n-dimensional case.

Before we do this, we discuss the parametrization of the family of umbilic geodesics. The geodesics corresponding to the quasi-periodic case are geodesics that are tangent to two special curves on the ellipsoid, called caustics: in this case, it is possible to parametrize the geodesics by their intersection points with the caustics. These caustics are the intersection curves of the ellipsoid with confocal hyperboloids. We will show that the umbilic case results from the limit in which the hyperboloid flattens to the region between two branches of a hyperbola. The four umbilic points are the intersection points of the ellipsoid and the hyperbola. As we shall see, the Hamiltonians for the umbilic case can be obtained through the limiting procedure described. Each member of the family of umbilic geodesics passes through two antipodal umbilic points and these are the only points on the caustics through which they pass.

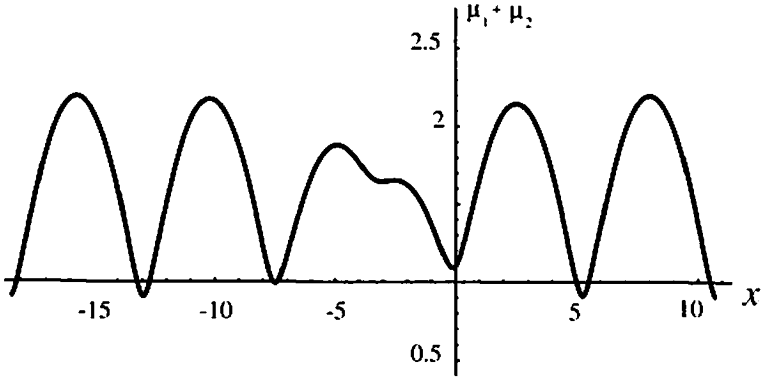

Figure 1. The profile of the umbilic soliton corresponding to figure 2.

These umbilic geodesics can be parametrized by the choice of initial tangent vector at one of the umbilic points. The parameter along the geodesic will correspond to the spatial variable x. As $x \to \pm\infty$, the umbilic geodesic converges to the periodic geodesic through the four umbilic points.

In §4 we showed a link between the problem of geodesics and wave solutions of nonlinear equations. In terms of the spatial shape $u(x,t)$ of the solution, the above geometry translates to the following: as $x \to \infty$, the solution u converges to a periodic function, while as $x \to -\infty$, it tends to the same periodic function, but phase shifted with respect to the first (see figure 1).

Consider a general family of quasi-periodic geodesics on an n-dimensional ellipsoid

$$\sum_{j=1}^{n+1} \frac{x_j^2}{l_j} = 1, \quad 0 < l_{n+1} < \cdots < l_1,$$

which is described (see, for example, Alber & Alber 1985, 1987) in terms of the μ-variables as a solution of the system of equations

$$\mu_j' = \frac{\partial \mu_j}{\partial x} = \frac{L_0}{\prod_{i\neq j}(\mu_j - \mu_i)} \sqrt{\prod_{k=1,k\neq j_0}^{n} (\mu_j - m_k) \prod_{r=1}^{n+1} (\mu_j - l_r)/(-\mu_j)}, \quad j = 1,\ldots,n, \tag{22}$$

where the μ-variables here are elliptic variables on the Riemann surfaces associated with the first integrals of the problem and where we have chosen $\epsilon_j = 1$. These can be obtained by the method of generating equations as described in Alber *et al.* (1994*a*) with the Riemann surface defined by zero-level sets of integrals of motion H_j, given by

$$H_j = P_j^2 + L_0^2 \frac{\prod_{r=1}^{n+1}(\mu_j - l_r)}{\mu_j \prod_{k=1,k\neq j_0}^{n}(\mu_j - m_k)}, \quad j = 1,\ldots,n.$$

Here m_j and L_0 are constants along the solutions of the system (22), and

$$l_{j+1} < m_j < l_j, \quad j \neq j_0, \quad 1 \leqslant j \leqslant n.$$

There are $n - 1$ first integrals m_k for the quasi-periodic flow on quadrics. Each one of them has a value which belongs to one of the intervals with the end points corresponding to the squares of the ellipsoid semiaxis lengths and there are n such intervals. Therefore, in the non-degenerate case, there is no more than one m_k in each interval, and there is always one interval j_0 which does not contain m_k. Different choices of j_0 correspond to different families of quasi-periodic geodesics.

Remark. Instead of using curvature in the definition of umbilics, in the n-dimensional case we consider the angle representation and the μ-representation which will be obtained by a certain limiting process.
Namely, we consider the limiting process

$$m_k \to l_{k+1} = b_k, \quad k = 1, \dots, (j_0 - 1), \tag{23}$$

$$m_k \to l_k = b_k, \quad k = (j_0 + 1), \dots, n, \tag{24}$$

which results in the system of equations

$$\mu_j' = \frac{L_0\sqrt{(\mu_j - l_1)(\mu_j - l_{n+1})}\prod_{r=1, r\neq j_0}^{n}(\mu_j - b_r)}{\sqrt{-\mu_j}\prod_{i\neq j}(\mu_j - \mu_i)}, \quad j = 1, \dots, n, \tag{25}$$

corresponding to different choices of j_0, where $(1 \leqslant j_0 \leqslant n)$. This system yields the following expressions:

$$\left.\begin{aligned}\sum_{j=1}^{n}\frac{\sqrt{-\mu_j}\mu_j'}{(\mu_j - b_k)\sqrt{(\mu_j - l_1)(\mu_j - l_{n+1})}} &= \sum_{j=1}^{n}\frac{L_0\prod_{r=1, r\neq j_0, r\neq k}^{n}(\mu_j - b_r)}{\prod_{i\neq j}(\mu_j - \mu_i)}, \quad k \neq j_0\\ \sum_{j=1}^{n}\frac{\sqrt{-\mu_j}\mu_j'}{L_0\sqrt{(\mu_j - l_1)(\mu_j - l_{n+1})}} &= \sum_{j=1}^{n}\frac{\prod_{r=1, r\neq j_0}^{n}(\mu_j - b_r)}{\prod_{i\neq j}(\mu_j - \mu_i)}, \quad k = j_0.\end{aligned}\right\} \tag{26}$$

Notice that the right-hand sides of these expressions are interpolation formulae of Lagrange type. The left-hand sides are identified with the x-derivatives of the angle variables. Therefore, we set

$$\theta_k' = 0, \quad k = 1, \dots, n, \quad k \neq j_0; \quad \theta_{j_0}' = 1, \quad k = j_0, \tag{27}$$

which, after integration on the Riemann surface, results in the umbilic angle representation

$$\left.\begin{aligned}\theta_k &= \sum_{j=1}^{n}\int_{\mu_j^0}^{\mu_j}\frac{\sqrt{-\mu_j}\,\mathrm{d}\mu_j}{(\mu_j - b_k)\sqrt{(\mu_j - l_1)(\mu_j - l_{n+1})}} = \theta_k^0, \quad k = 1, \dots, n; \quad k \neq j_0\\ \theta_k &= \sum_{j=1}^{n}\int_{\mu_j^0}^{\mu_j}\frac{\sqrt{-\mu_j}\,\mathrm{d}\mu_j}{L_0\sqrt{(\mu_j - l_1)(\mu_j - l_{n+1})}} = x + \theta_k^0, \quad k = j_0.\end{aligned}\right\} \tag{28}$$

Here $\theta_1^0, \dots, \theta_n^0$ are constants, $(\mu_1^0, \dots, \mu_n^0)$ is a base point of integration and $(\mu_1, \dots, \mu_n)$ denotes a point on the invariant variety, i.e. on the symmetric product of n copies of the Riemann surface to be described below. Now we can give the following definition.

Definition 5.2. *Umbilic geodesics* in the n-dimensional case are defined by the following limiting choice of first integrals for the quasi-periodic flow: $m_j = b_j$, resulting in the angle representation (28). *Umbilic points* on n-dimensional quadrics may be defined in terms of the μ-representations by $\mu_j = b_j$, $j = 1, \dots, n$.

In what follows, we will introduce new systems of first integrals for umbilic geodesics and will obtain corresponding exponential Hamiltonians. In the next section we will show that this provides a Hamiltonian x-flow and level sets for a certain class of soliton-like solutions of equations of the Dym hierarchy.

Theorem 5.3. *The system of equations (26) for umbilic geodesics on quadrics define certain solutions for the Hamiltonian system with Hamiltonians given by*

$$\tilde{H} = \sum_{j=1}^{n} \frac{\exp(\mathcal{M}(\mu_j)\tilde{P}_j) - L_0 \prod_{k=1,k\neq j_0}^{n}(\mu_j - b_k)}{\prod_{r\neq j}(\mu_j - \mu_r))}, \tag{29}$$

where

$$\mathcal{M}(\mu) = \sqrt{\frac{(\mu - l_1)(\mu - l_{n+1})}{(-\mu)}}. \tag{30}$$

The resulting Hamiltonian system has a complete set of first integrals $\tilde{H}_j$, of the form

$$\tilde{H}_j = \tilde{P}_j - \frac{1}{\mathcal{M}(\mu_j)} \sum_{k=1,k\neq j_0}^{n} \log[L_0(\mu_j - b_k)], \quad j = 1,\ldots,n, \tag{31}$$

and the angle representation (28) linearizes the corresponding Hamiltonian x-flow. The system (26) is recovered when the Hamiltonian system generated by (29) is considered on the zero-level sets of the integrals $\tilde{H}_j$.

Proof. Upon substituting the expressions (31) for the integrals into the Hamiltonian system generated by (29), we obtain the first part of the proof. Then we consider the action function

$$S = \sum_{j=1}^{n} \int_{\mu_j^0}^{\mu_j} \tilde{P}_j \, d\mu_j,$$

which generates a Lagrangian submanifold of the phase space $\mathbb{C}^{2n}$, and the following system of variables:

$$\left.\begin{aligned} I_k = b_k, \quad k = 1,\ldots,n; \quad k \neq j_0; \quad I_{j_0} = L_0 \\ \theta_k = -\frac{\partial S}{\partial I_k}, \quad k = 1,\ldots,n. \end{aligned}\right\} \tag{32}$$

Even though there are no invariant tori in the phase space, the Hamiltonian flow can be linearized, as we see from (28).

The angle representation (28) has logarithmic singularities. It is similar to the soliton defocusing nonlinear Schrödinger representations and representations for the homoclinic orbits of the C. Neumann problem and can be analysed using multidimensional asymptotic reduction (see, for details, Alber & Marsden 1994*a*, *b*).

Corollary 5.4. *The system of differential equations (25) has a particular solution corresponding to the case when all but one of the root-variables μ_j are constants:*

$$\left.\begin{aligned} \mu_j' = \frac{L_0\sqrt{(\mu_j - l_1)(\mu_j - l_{n+1})}}{\sqrt{-\mu_j}}, \quad l_{n+1} < \mu_j < l_1, \quad j = j_0, \\ \mu_j = b_j, \quad j = 1,\ldots,n; \quad j \neq j_0. \end{aligned}\right\} \tag{33}$$

This result means that the family of umbilic geodesics can asymptotically approach a one-dimensional torus (one of the central ellipses, which can be viewed as a closed geodesic obtained after shrinking the caustics of the family of quasi-periodic geodesics). This gives an example of an orbit approaching a lower-dimensional torus as $x \to \pm\infty$. The system (25) has another particular solution which corresponds to the case when $\mu_j = b_j$, $j = 1, \ldots, k$, $k < n$. This results in umbilic geodesics on $(n-k)$-dimensional quadrics.

6. Umbilic solitons

For example, in the two-dimensional case, a family of umbilic geodesics on an ellipsoid asymptotically approaches a periodic orbit along the middle ellipse. This limiting orbit can be described as follows: $\mu_1 = b$ or $\mu_2 = b$ and

$$\int_{\mu^0}^{\mu} \frac{\sqrt{-\mu}\,\mathrm{d}\mu}{L_0\sqrt{(\mu - l_1)(\mu - l_3)}} = x + \theta^0, \quad l_3 < \mu < l_1. \tag{34}$$

One of the geodesics approaching this limiting orbit is shown in figure 2, which shows the phase plane (μ_1, μ_2) for the system (25) in the two-dimensional case. We set $l_3 = 0.2$, $b = 0.7$ and $l_1 = 1.5$ to provide a specific example, and integrate the system (25) numerically. The umbilic points on the ellipsoid are mapped into $Z = (b, b)$, and the polar points $(\pm\sqrt{l_1}, 0, 0)$ and $(0, 0, \pm\sqrt{l_3})$ are mapped into the points $A_1 = (l_1, b)$ (or $A_2 = (b, l_1)$) and $B_1 = (b, l_3)$ (or $B_2 = (L_3, b)$), respectively. An umbilic geodesic on the ellipsoid is mapped into a trajectory in the plane like the one shown. The trajectory visits the upper and lower rectangles $\mu_1 \in [l_3, b], \mu_2 \in [b, l_1]$ and $\mu_1 \in [b, l_1], \mu_2 \in [l_3, b]$, respectively.

The phase plane consists of four Riemann sheets glued together along the branch cuts at the sides of the square $\mu_1 \in [l_3, l_1], \mu_2 \in [l_3, l_1]$. A typical trajectory touches the vertical and horizontal branch cuts in each rectangle before it goes through the singular point Z and moves to the other rectangle. The trajectory is tangent at the points of contact with the branch cuts. As shown in the figure, as $x \to \infty$ the trajectory quickly approaches the segments of the vertical and horizontal lines $\mu_1 = b$ and $\mu_2 = b$ in a particular order, which in this case is

$$A_2 \to Z \to B_1 \to Z \to A_1 \to Z \to B_2 \to Z \to A_2. \tag{35}$$

The lines $\mu_1 = b$ and $\mu_2 = b$ map to the ellipsoid meridian (which is a closed geodesic) that passes through the umbilic points. This is an unstable periodic orbit for system (25), and the umbilic geodesic is homoclinic to it. For details of the proof of this fact, see Klingenberg (1982). Notice that the umbilic geodesic can be parametrized by a single parameter, e.g., the coordinate μ_1 of a point of tangency along the line $\mu_2 = l_3$, i.e. the point of intersection of the geodesic with the equator.

According to the reconstruction formula (21), the sum $\mu_1 + \mu_2$ provides, up to an additive constant, the profile of the soliton solution $U(x, t)$ of equation (1). Figure 1 shows the umbilic soliton shape, which corresponds to phase portrait shown in figure 2.

In what follows, we construct an angle representation for umbilic 1-soliton solutions. The process of obtaining an umbilic soliton starts with a quasi-periodic Hamiltonian flow on a Riemann surface of genus 2 (rather than genus 1, as for the KdV soliton case) and then one makes a pair of roots coalesce by adjusting the values of the first integrals. The Hamiltonian system that describes the starting quasi-periodic

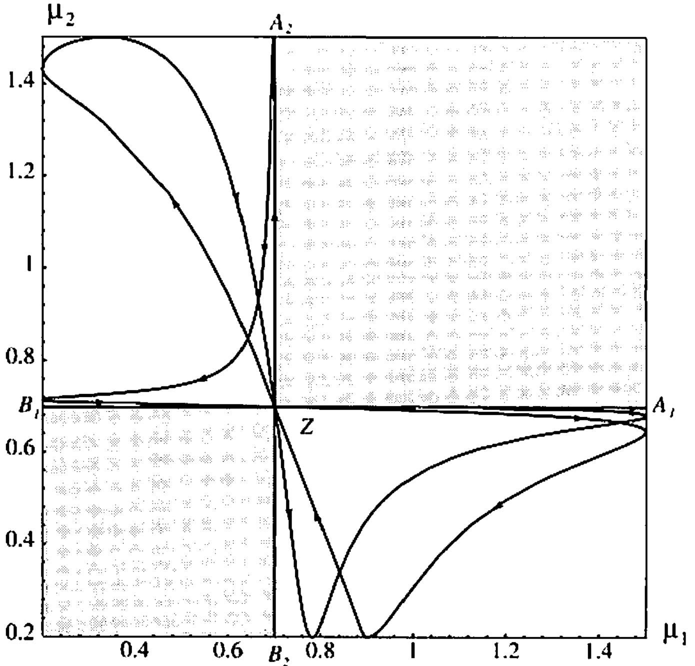

Figure 2. The trajectory in the (μ_1, μ_2) phase plane of an umbilic geodesic on a two-dimensional ellipsoid in $\mathbb{R}^3$. The squares of the ellipsoid semiaxis lengths are chosen to be $l_3 = 0.2$, $b = 0.7$ and $l_1 = 1.5$ and the geodesic flow (25) is integrated numerically starting from the initial condition at the upper edge $\mu_2 = l_1$.

flow for umbilic solitons is two dimensional, in the sense that it is a system on (a two-dimensional Lagrangian submanifold of) phase space $\mathbb{C}^4$. This Lagrangian submanifold consists of two copies of a Riemann surface of genus 2, one for each of μ_1 and μ_2. After coalescing roots, one also gets a new Hamiltonian system on a product of two copies of a Riemann surface determined by first integrals of the same form as those described in § 4 for the problem of umbilic geodesics on quadrics. The new system is still described by two variables, which we again denote μ_1 and μ_2. Using the general method of Alber & Marsden (1992), applied to the resulting system, gives the following angle representation:

$$\left.\begin{aligned} \theta_1 &= \int_{\mu_1^0}^{\mu_1} \frac{\sqrt{-\mu_1}\,\mathrm{d}\mu_1}{(\mu_1-b)\sqrt{(\mu_1-l_1)(\mu_1-l_3)}} + \int_{\mu_2^0}^{\mu_2} \frac{\sqrt{-\mu_2}\,\mathrm{d}\mu_2}{(\mu_2-b)\sqrt{(\mu_2-l_1)(\mu_2-l_3)}} = \theta_1^0 + L_0 t \\ \theta_2 &= \int_{\mu_1^0}^{\mu_1} \frac{\sqrt{-\mu_1}\,\mathrm{d}\mu_1}{L_0\sqrt{(\mu_1-l_1)(\mu_1-l_3)}} + \int_{\mu_2^0}^{\mu_2} \frac{\sqrt{-\mu_2}\,\mathrm{d}\mu_2}{L_0\sqrt{(\mu_2-l_1)(\mu_2-l_3)}} = \theta_2^0 + x - bt. \end{aligned}\right\} \tag{36}$$

Starting with quasi-periodic flow on a Riemann surface of genus 2 in the KdV case, the coalescence of roots produces a two-soliton KdV solution that describes the interaction of two single solitons. In the case of the Dym equation, if one starts with quasi-periodic flow on a Riemann surface of genus 3, coalescence of two pairs of roots produces the following angle representation:

$$\left.\begin{aligned}\theta_1 &= \sum_{j=1}^{3}\int_{\mu_j^0}^{\mu_j}\frac{\sqrt{-\mu_j}\,\mathrm{d}\mu_j}{(\mu_j-b_1)\sqrt{(\mu_j-l_1)(\mu_j-l_4)}} = \theta_1^0 + L_0 t\\ \theta_2 &= \sum_{j=1}^{3}\int_{\mu_j^0}^{\mu_j}\frac{\sqrt{-\mu_j}\,\mathrm{d}\mu_j}{(\mu_j-b_2)\sqrt{(\mu_j-l_1)(\mu_j-l_4)}} = \theta_2^0 + L_0 t\\ \theta_3 &= \sum_{j=1}^{3}\int_{\mu_j^0}^{\mu_j}\frac{\sqrt{-\mu_j}\,\mathrm{d}\mu_j}{L_0\sqrt{(\mu_j-l_1)(\mu_j-l_4)}} = \theta_3^0 + x - (b_1+b_2)t.\end{aligned}\right\} \quad (37)$$

Applying the method of asymptotic reduction of Alber & Marsden (1992) to (36) produces a phase shift of the limiting periodic orbit for $x \to \pm\infty$, given by

$$\Delta\phi = 2\int_{l_3}^{b}\frac{\sqrt{-\mu_1}\,\mathrm{d}\mu_1}{L_0\sqrt{(\mu_1-l_1)(\mu_1-l_3)}}. \quad (38)$$

This technique will be applied to (37), producing other explicit formulas for phase shifts in Alber *et al.* (1994*b*).

To complete the geometric model, we consider umbilic angle representations on various hyperboloids. We demonstrate our approach in the two-dimensional case for simplicity. Different families of quasi-periodic geodesics on a hyperboloid with one sheet can be characterized by the following two distributions of the squares of the semiaxes l_j and first integral m:

$$l_3 < 0 < l_2 < m < l_1, \quad \text{and} \quad m < l_3 < 0 < l_2 < l_1.$$

If we apply the limiting process

$$m \to l_2 = b, \quad b > 0, \quad \text{and} \quad m \to l_3 = b, \quad b < 0,$$

in the first and second case, respectively, to the system of differential equations for quasi-periodic geodesics, we obtain two different families of umbilic geodesics. Their angle representations are similar to those in the case of an ellipsoid. The only difference is that μ_2 is defined in the infinite zone $]-\infty, l_3]$ or $]-\infty, b]$ respectively.

In the first case, umbilic geodesics asymptotically approach the central ellipse on the hyperboloid, which is described by formula (28), where

$$l_2 < \mu_1 < l_3, \quad \mu_2 = b.$$

The second family of geodesics approaches two hyperbolae on the hyperboloid and is described by (28), where

$$\mu_1 = b, \quad \mu_2 < l_3.$$

Umbilic geodesics on hyperboloids correspond to the solutions of the PDEs with one μ-variable defined in the infinite zone and a discrete spectrum which includes both positive and negative elements. The reconstruction formula (21) for the solution of equation (1) then shows that, unlike the elliptic case, umbilic geodesics on hyperboloids do not provide bounded solutions of the PDE.

7. Conclusions

In this paper we have established the existence of, and studied the complex geometry associated with, umbilic solitons for the nonlinear evolution equation (1). These solitons correspond to families of singular geodesics on n-dimensional quadrics, and we have illustrated the procedure concretely for $n = 2$. In a forthcoming paper we will address the question of collisions of two or more such solitons which yields phase shifts and will investigate in detail the special solutions of billiard type for equations in the Dym hierarchy.

We thank Nick Ercolani and Carlos Tomei for useful discussions on elliptic billiards. Mark Alber also thanks the Institute for Advanced Study in Princeton and the Center for Nonlinear Studies at Los Alamos National Laboratory for their hospitality during the Fall of 1993 and during three visits in August 1993 and January and June 1994. Research by M.S.A. was partly supported by NSF grants DMS 9403861 and 9022140, research by R.C. and D.D.H. was partly supported by the DOE, CHAMMP and HPCC programmes and research by J.E.M. was partly supported by the Department of Energy, the Office of Naval Research and the Fields Institute for Research in the Mathematical Sciences.

References

Ablowitz, M. J. & Segur, H. 1981 *Solitons and the inverse scattering transform*. Philadelphia, PA: SIAM.

Alber, M. S. 1986 Hamiltonian formalism for finite-zone solutions of nonlinear integrable equations. In *Proc. VIII Int. Cong. on Mathematical Physics, Marseille*. pp. 447–462. Singapore: World Scientific.

Alber, M. S. & Alber, S. J. 1985 Hamiltonian formalism for finite-zone solutions of integrable equations. *C.r. Acad. Sci., Paris* **301**, 777–781.

Alber, M. S. & Alber, S. J. 1987 Hamiltonian formalism for nonlinear Schrödinger equations and sine-Gordon equations. *J. Lond. math. Soc.* **36**, 176–192.

Alber, M. S., Camassa, R., Holm, D. D. & Marsden, J. E. 1994*a* The geometry of peaked solitons and billiard solutions of a class of integrable PDEs. *Lett. math. Phys.* **32**, 137–151.

Alber, M. S., Camassa, R., Holm, D. D. & Marsden, J. E. 1994*b* (in preparation).

Alber, M. S. & Marsden, J. E. 1992 On geometric phases for soliton equations. *Communs math. Phys.* **149**, 217–240.

Alber, M. S. & Marsden, J. E. 1994*a* Geometric phases and monodromy at singularities. In *Singular limits of dispersive waves*, NATO ASI Series C, (ed. N. Ercolani *et al.*), vol. 320, pp. 273–295. New York: Plenum.

Alber, M. S. & Marsden, J. E. 1994*b* Resonant geometric phases for soliton equations. *Fields Institute Commun.* **3**, 1–26.

Alber, S. J. 1979 Investigation of equations of Korteweg–de Vries type by the method of recurrence relations. *J. Lond. math. Soc.* **19**, 467–480.

Camassa, R. & Holm, D. D. 1993 An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* **71**, 1661–1664.

Camassa, R., Holm, D. D. & Hyman, J. M. 1994 A new integrable shallow water equation. *Adv. appl. Mech.* **31**, 1–33.

Cewen, C. 1990 Stationary Harry-Dyms equation and its relation with geodesics on ellipsoid *Acta math. Sinica* **6**, 35–41.

Deift, P., Li, L. C. & Tomei, C. 1991 Loop groups, discrete versions of some classical integrable systems, and rank 2 extensions. *Mem. Am. math. Soc.* **479**.

Dmitrieva, L. A. 1993 Finite-gap solutions of the Harry Dym equation. *Phys. Lett.* **182A**, 65–70.

Ercolani, N. 1989 Generalized theta functions and homoclinic varieties. In *Proc. Symp. Pure appl. Math.* **49**, 87.

Ercolani, N. & McLaughlin, D. W. 1991 Toward a topological classification of integrable PDEs. In *Proc. of a Workshop on The Geometry of Hamiltonian Systems* (ed. T. Ratiu), vol. 22. MSRI Publications. Berlin: Springer.

Hunter, J. K. & Scheurle, J. 1988 Existence of perturbed solitary wave solutions to a model of water waves. *Physica* **32D**, 235–268.

Hunter, J. K. & Zheng, Y. 1994 On a completely integrable nonlinear hyperbolic variational equation. *Physica* **79D**, 361–386.

Klingenberg, W. 1982 *Riemannian geometry*. New York: de Gruyter.

Kruskal, M. D. 1975 Nonlinear wave equations. In *Dynamical systems, theory and applications*. Springer Lecture Notes in Physics, vol 38. Berlin: Springer.

Levallois, P. & Tabanov, M. B. 1993 Splitting of separatrices of elliptic billiard for a symmetric algebraic perturbation of the ellipse. *C.r. Acad. Sci., Paris* **316**, serie I, 589–592.

Moser, J. & Veselov, A. P. 1991 Discrete versions of some classical integrable systems and factorization of matrix polynomials. *Communs math. Phys.* **139**, 217–243.

Previato, E. 1985 Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation. *Duke math. Jl* **52**, 329–377.

Ragnisco, O. & Rauch-Wojciechowski, S. 1994 Integrable mechanical systems related to the Harry-Dym hierarchy. *J. math. Phys.* **35**, 834–847.

Whitham, G. B. 1974 *Linear and nonlinear waves*, p. 585. New York: Wiley.

Wadati, M., Ichikawa, Y. H. & Shimizu, T. 1980 Cusp soliton of a new integrable nonlinear evolution equation. *Prog. theor. Phys.* **64**, 1959–1967.

Received 30 November 1994; accepted 26 April 1995