The Reduced Euler-Lagrange Equations

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August, 1992—this version: May 8, 1994

Fields Institute Communications **1**(1993), 139–164.

Abstract

Marsden and Scheurle [1993] studied Lagrangian reduction in the context of momentum map constraints—here meaning the reduction of the standard Euler-Lagrange system restricted to a level set of a momentum map. This provides a Lagrangian parallel to the reduction of *symplectic* manifolds. The present paper studies the Lagrangian parallel of *Poisson* reduction for Hamiltonian systems. For the reduction of a Lagrangian system on a level set of a conserved quantity, a key object is the Routhian, which is the Lagrangian minus the mechanical connection paired with the fixed value of the momentum map. For unconstrained systems, we use a velocity shifted Lagrangian, which plays the role of the Routhian in the constrained theory. Hamilton's variational principle for the Euler-Lagrange equations breaks up into two sets of equations that represent a set of Euler-Lagrange equations with gyroscopic forcing that can be written in terms of the curvature of the connection for horizontal variations, and into the Euler-Poincaré equations for the vertical variations. This new set of equations is what we call the *reduced Euler-Lagrange equations*, and it includes the Euler-Poincaré and the Hamel equations as special cases. We illustrate this methodology for a rigid body with internal rotors and for a particle moving in a magnetic field.

1 Introduction

The goal of this paper is to study the reduction of Lagrangian systems with symmetries. Marsden and Scheurle [1993] showed how to reduce Lagrangian systems with a fixed value of the momentum map imposed (hereafter called "constraints"), and showed how this procedure is related to the classical procedure of Routh and is the Lagrangian version of symplectic reduction. In this paper, we study the problem

*Research partially supported by the Fields Institute, NSF grant DMS-89-22704 and DOE Contract DE-FGO3-92ER25129.

without the momentum map constraint explicitly imposed, and so it corresponds to the Lagrangian counterpart of Poisson, rather than symplectic reduction.

We begin with a few brief historical comments. In Poincaré [1901, 1910] (see also Hamel [1904]), equations on a general Lie algebra were found that are a Lagrangian counterpart of the Lie-Poisson equations on the dual of a Lie algebra that were implicit in the work of Lie around 1890 (see Marsden and Weinstein [1983], Weinstein [1983], Marsden [1992] and references therein). One can view the resulting ***Euler-Poincaré equations*** as the reduction of the Euler-Lagrange equations from the tangent bundle of a Lie group to its Lie algebra, as we shall see in §4. The Euler-Poincaré equations were combined with the Euler-Lagrange equations by Hamel [1904] and may be regarded as a more general case of Lagrangian reduction. As Hamel [1949] notes, there are many others who also contributed to this theory and so the history is not, in reality, quite as clean as we have suggested; for example, Lagrange himself devoted a good deal of attention to this problem for the rotation group in volume two of *Mechanique Analytique*, with of course, Euler's equations for a rigid body in the background as a key example. Modern references that impact on the present work are too numerous to detail here, but we especially point out that Cendra, Ibort, and Marsden [1987] studied this problem from the variational point of view, Koiller [1992] from the point of view of nonholonomic mechanics, and Weinstein [1993] from the point of view of groupoids.

We start with a configuration manifold Q and a Lagrangian $L : TQ \to \mathbb{R}$. Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Assume that G acts on Q and lift this action to TQ by the tangent operation. Assuming that L is G invariant, there is induced a ***reduced Lagrangian*** $l : TQ/G \to \mathbb{R}$. We can regard TQ/G as a $\mathfrak{g}$ bundle over TS, where $S = Q/G$. We assume that G acts freely and properly on Q, so we can regard $Q \to Q/G$ as a principal G-bundle. Future work is planned to relax this assumption, as the singular case is very important in examples. In fact, substantial work on singular reduction in the Hamiltonian context has been done in, for example, Arms, Marsden, and Moncrief [1981], Sjamaar and Lerman [1992] and references therein and it would be desirable to develop the counterpart of this theory on the Lagrangian side. Progress in this direction has been made by Lewis ([1992] and related works).

An important ingredient in the work is to introduce a connection A on the principal bundle $Q \to S = Q/G$. We discuss below the example of the mechanical connection, which may be chosen for A. This connection allows one to split the variables into a horizontal and vertical part, and as we shall see, this is natural from the point of view of mechanics as well as of mathematics.

Next, we introduce some notation so that we can write the reduced Euler-Lagrange equations in coordinates. We will first discuss the case of the Hamel equations, which does not involve the connection. Let

- x^α, also called "internal variables", be coordinates for shape space Q/G,
- η^a be coordinates for the Lie algebra $\mathfrak{g}$ relative to a chosen basis
- l be the reduced Lagrangian regarded as a function of the variables $x^\alpha, \dot{x}^\alpha, \eta^a$, and

- c^a_{db} be the structure constants of the Lie algebra $\mathfrak{g}$ of G.

If one writes the Euler-Lagrange equations on TQ in a local principal bundle trivialization, with coordinates x^α on the base and η^a in the fiber, then one gets the following system of ***Hamel equations***

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} = 0 \tag{1.1}$$

$$\frac{d}{dt}\frac{\partial l}{\partial \eta^b} - \frac{\partial l}{\partial \eta^a} c^a_{db}\eta^d = 0. \tag{1.2}$$

However, this representation of the equations does not make global intrinsic sense (unless $Q \to S$ admits a global flat connection). The introduction of a connection overcomes this and one can intrinsically and globally split the original variational principle relative to horizontal and vertical variations. One gets from one form to the other by means of the ***velocity shift*** given by replacing η by the vertical part relative to the connection:

$$\xi^a = A^a_\alpha \dot{x}^\alpha + \eta^a$$

Here, A^d_α are the local coordinates of the connection A. As we shall see in the examples, this change of coordinates is motivated from the mechanical point of view since the variables ξ have the interpretation of the ***locked angular velocity***. The resulting ***reduced Euler-Lagrange equations*** have the following form:

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} = \frac{\partial l}{\partial \xi^a}\left(B^a_{\alpha\beta}\dot{x}^\beta + B^a_{\alpha d}\xi^d\right) \tag{1.3}$$

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^b} = \frac{\partial l}{\partial \xi^a}(B^a_{b\alpha}\dot{x}^\alpha + c^a_{db}\xi^d) \tag{1.4}$$

In these equations, $B^a_{\alpha\beta}$ are the coordinates of the curvature B of A, $B^a_{d\alpha} = c^a_{bd}A^b_\alpha$ and $B^a_{d\alpha} = -B^a_{\alpha d}$.

The matrix

$$\begin{bmatrix} B^a_{\alpha\beta} & B^a_{\alpha d} \\ B^a_{d\alpha} & c^a_{bd} \end{bmatrix}$$

is itself the curvature of the connection regarded as residing on the bundle $TQ \to TQ/G$, but we will not pursue this point in the present paper.

The variables ξ^a may be regarded as the rigid part of the variables on the original configuration space, while x^α are the internal variables. As in Simo, Lewis, and Marsden [1991], the division of variables into internal and rigid parts has deep implications for both stability theory and for bifurcation theory, again, continuing along lines developed originally by Riemann, Poincaré and others. The main way this new insight is achieved is through a careful split of the variables, using the (mechanical) connection as one of the main ingredients. This split puts the second variation of the augmented Hamiltonian at a relative equilibrium as well as the symplectic form into "normal form". It is somewhat remarkable that they are *simultaneously* put into a simple form. In another publication, we plan to link the reduced Euler-Lagrange equations with this theory in hopes of obtaining a division of the variables into rigid and internal for the nonlinear theory as well as for the linearized theory.

One of the key results in Hamiltonian reduction theory says that the reduction of a cotangent bundle T^*Q by a symmetry group G is a bundle over T^*S, where $S = Q/G$ is shape space, and where the fiber is either $\mathfrak{g}^*$, the dual of the Lie algebra of G, or is a coadjoint orbit, depending on whether one is doing Poisson or symplectic reduction. We refer to Montgomery, Marsden, and Ratiu [1984] and Marsden [1992] for details and references. Our work gives the analogue of this structure on the tangent bundle with the key reduced equations being given by (1.3) and (1.4). These two sets of equations are coupled through the curvature of a connection on the bundle and the fact that the Lagrangian is, in general, a function of all the variables. Normally one chooses the connection to be the mechanical connection, although any other choice is allowed.

Remarkably, equations (1.3) are formally identical to the equations for a mechanical system with classical nonholonomic velocity constraints (see Neimark and Fufaev [1972] and Koiller [1992].) The connection in that case is the one-form that determines the constraints. This link is made precise in Bloch, Krishnaprasad, Marsden and Murray [1993]. In addition, this structure appears in several control problems, especially the problem of stabilizing controls considered by Bloch, Krishnaprasad, Marsden, and Sanchez [1992].

For systems with a momentum map $\mathbf{J}$ constrained to a specific value μ, the key to the construction of a reduced Lagrangian system is the modification of the Lagrangian L to the Routhian R^μ, which is obtained from the Lagrangian by subtracting off the mechanical connection paired with the constraining value μ of the momentum map. A basic ingredient needed for the reduced Euler-Lagrange equations is a velocity shift in the Lagrangian (the shift is given by a choice of connection, often the mechanical connection), so this velocity shifted Lagrangian plays the role that the Routhian does in the constrained theory.

Acknowledgements We thank Tony Bloch, Mike Enos, P.S. Krishnaprasad, Debra Lewis, Richard Montgomery, Richard Murray, Tudor Ratiu, Juan Simo, Alan Weinstein and the referees for useful comments and inspiration.

2 The Mechanical Connection

We will now recall the definition of the mechanical connection, as it is the one that would often be chosen. We assume, to be specific, that one has a metric $\langle\langle\, , \rangle\rangle$ on Q that is invariant under the group action. (Otherwise, use the second fiber derivative of the Lagrangian as the "metric").

For each $q \in Q$, the ***locked inertia tensor*** is the map $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ defined by

$$\langle \mathbb{I}(q)\eta, \zeta\rangle = \langle\langle \eta_Q(q), \zeta_Q(q)\rangle\rangle. \tag{2.1}$$

where η_Q denotes the infinitesimal generator of the action of G on Q, so that η_Q is a vector field on Q. Since the action is free, $\mathbb{I}(q)$ is an inner product. The terminology comes from the fact that for coupled rigid or elastic systems, $\mathbb{I}(q)$ is the classical moment of inertia tensor of the corresponding instantaneously rigidified system. While most of the results hold in the infinite as well as the finite dimensional case, to expedite the exposition, we give many of the formulas in coordinates. For instance,

$$\mathbb{I}_{ab} = g_{ij} K^i{}_a K^j{}_b, \tag{2.2}$$

where g_{ij} are the components of the metric tensor relative to coordinates $q^i, i = 1, 2, \ldots n$ on Q and where we write

$$[\xi_Q(q)]^i = K^i{}_a(q)\xi^a \tag{2.3}$$

relative to the coordinates on Q and a basis $e_a, a = 1, 2, \ldots, m$ of $\mathfrak{g}$. This equation defines $K^i{}_a$, which we call the ***action coefficients***. In such a basis, the coordinates of $\xi \in \mathfrak{g}$ are defined by writing $\xi = \xi^a e_a$.

Define the map $A : TQ \to \mathfrak{g}$ that assigns to each (q, v) the corresponding angular velocity of the locked system:

$$A(q, v) = \mathbb{I}(q)^{-1}(\mathbf{J}(q, v)). \tag{2.4}$$

where $\mathbf{J} : TQ \to \mathfrak{g}^*$ is the standard momentum map for the lifted action of G on Q given by $\langle \mathbf{J}(q, v), \xi\rangle = \langle\langle v, \xi_Q(q)\rangle\rangle$. (Sometimes we shall regard $\mathbf{J}$ as defined on the cotangent bundle by identification with the tangent bundle using the metric without explicit mention.) In coordinates,

$$A^a = \mathbb{I}^{ab} g_{ij} K^i{}_b v^j \tag{2.5}$$

The ***components*** of A are defined by

$$A^a_j = \mathbb{I}^{ab} g_{ij} K^i{}_b \tag{2.6}$$

so that $A^a = A^a_j v^j$. The map A is, in fact, a connection on the principal G-bundle $Q \to Q/G$ and is called the ***mechanical connection***. In other words, A is G-equivariant and satisfies $A(\xi_Q(q)) = \xi$. While A appears explicitly in Smale [1970] and Abraham and Marsden [1978], the point of view of connections is due to Kummer [1981]. The ***horizontal space*** of the connection A is given by

$$\mathrm{hor}_q = \{(q, v) \mid \mathbf{J}(q, v) = 0\} \subset T_qQ; \tag{2.7}$$

i.e., the space orthogonal to the G-orbits, as in the Yang-Mills construction. The ***vertical space*** consists of vectors that are mapped to zero under the projection $Q \to S = Q/G$; *i.e.,*

$$\mathrm{ver}_q = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}. \tag{2.8}$$

For each $\mu \in \mathfrak{g}^*$, define the 1-form A_μ on Q by

$$\langle A_\mu(q), v\rangle = \langle \mu, A(q, v)\rangle \tag{2.9}$$

i.e.,

$$(A_\mu)_i = g_{ij} K^j{}_b \mu_a \mathbb{I}^{ab}. \tag{2.10}$$

It follows from the identity $A(\xi_Q(q)) = \xi$ that A_μ takes values in $\mathbf{J}^{-1}(\mu)$.

The horizontal-vertical decomposition of a vector $(q, v) \in T_qQ$ is given by

$$v = \mathrm{hor}_q v + \mathrm{ver}_q v \tag{2.11}$$

where

$$\mathrm{ver}_q v = [A(q, v)]_Q(q) \quad \text{and} \quad \mathrm{hor}_q v = v - \mathrm{ver}_q v.$$

Notice that $\mathrm{hor} : TQ \to \mathbf{J}^{-1}(0)$ and that it may be regarded as a velocity shift.

3 The Routh Method

The abelian case of Lagrangian reduction was known to Routh by around 1860; a modern account is given in Arnold [1988]. Marsden and Scheurle [1993] give a geometrization and a generalization of the Routh procedure to the nonabelian case. To achieve this, we incorporated the conservative gyroscopic forces into the variational principle in the sense of Lagrange and d'Alembert. We also employed a Dirac constraint construction to include the cases in which the reduced space is not a tangent bundle (but it is a Dirac constraint set inside one). In this section, we recall some of the key features of the general Routh method.

Given $\mu \in \mathfrak{g}^*$, define the ***Routhian*** $R^\mu : TQ \to \mathbb{R}$ as follows:

$$R^\mu(q, v) = L(q, v) - \langle A(q, v), \mu \rangle \tag{3.1}$$

where A is the mechanical connection. Thus,

$$R^\mu(q^k, \dot{q}^l) = L(q^k, \dot{q}^l) - A^a_j(q^k)\mu_a \dot{q}^j. \tag{3.2}$$

This function is not quite the classical Routhian (which does not make coordinate invariant sense), but is closely related to it as we shall see later. Notice that the Routhian has the form of a *Lagrangian with a gyroscopic term*; see Bloch, Krishnaprasad, Marsden, and Sanchez [1992] and Wang and Krishnaprasad [1992] for information on the use of gyroscopic systems in control theory.

A basic observation about the Routhian is that solutions of the Euler-Lagrange equations for L can be regarded as solutions of the Euler-Lagrange equations for the Routhian, with the addition of "magnetic forces". The chain rule proves the identity

$$\frac{d}{dt}\frac{\partial R^\mu}{\partial \dot{q}^i} - \frac{\partial R^\mu}{\partial q^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} - \mu_a \left(\frac{\partial A^a_i}{\partial q^j} - \frac{\partial A^a_j}{\partial q^i} \right) \dot{q}^j.$$

The last term

$$\beta^a_{ij} = \left(\frac{\partial A^a_j}{\partial q^i} - \frac{\partial A^a_i}{\partial q^j} \right)$$

represents "magnetic" or "gyroscopic" forces. As we shall see below, it is closely related to, but is not quite the curvature B of the connection A, except in the abelian case. This identity proves the following:

$$\left\{ \begin{array}{c} \text{Euler-Lagrange} \\ \text{for } L \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{Lagrange d'Alembert} \\ \text{for } R^\mu \end{array} \right\}$$

i.e.,

$$\delta \int R^\mu(q, \dot{q}) dt = \int \mu \cdot \beta(\dot{q}, \delta q) dt \tag{3.3}$$

This form of the variational principle drops (or reduces) to a variational principle on the orbit space Q/G_μ. For example, this is how one can directly get a reduced variational principle for the Euler rigid body equations. In this principle, the variation of the integral of R^μ is taken over curves satisfying the fixed endpoint condition. The principle holds, in particular, if the curves are constrained to satisfy the condition $\mathbf{J}(q, v) = \mu$. The restriction of R^μ to this level set equals

$$R^\mu = \frac{1}{2} \|\mathrm{hor}_q v\|^2 - V_\mu \tag{3.4}$$

where V_μ is the amended potential defined by $V_\mu = H \circ A_\mu$, and where $H : T^*Q \to \mathbb{R}$ is the Hamiltonian. In fact, one has, in the specific case of Lagrangians of the form kinetic minus potential,

$$V_\mu(q) = V(q) + \frac{1}{2} \left\langle \mu, \mathbb{I}(q)^{-1} \mu \right\rangle .$$

In the variational principle (3.3) dropped to Q/G_μ, the endpoint conditions can be relaxed to the condition that the ends lie on orbits rather than being fixed. This is because the kinetic part now just involves the horizontal part of the velocity, and so the endpoint conditions in the variational principle, which involve the contraction of the momentum $\mathfrak{p}$ with the variation of the configuration variable δq vanish if $\delta q = \zeta_Q(q)$ for some $\zeta \in \mathfrak{g}$, *i.e.,* if the variation is tangent to the orbit. The condition that (q, v) be in the μ level set of $\mathbf{J}$ means that the momentum $\mathfrak{p}$ vanishes when contracted with an infinitesimal generator on Q.

The function R^μ restricted to the μ-level set of $\mathbf{J}$ defines a function on the quotient space $T(Q/G_\mu)$ – that is, it factors through the tangent of the projection map $\tau_\mu : Q \to Q/G_\mu$. The variational principle also drops, therefore, since the curves that join orbits correspond to those that have fixed endpoints on the base. The magnetic term defines a well-defined two form on the quotient as well, as is known from the Hamiltonian case, even though A_μ need not drop to the quotient

To summarize, suppose that $q(t)$ satisfies the (unconstrained) Euler-Lagrange equations for L and satisfies $\mathbf{J}(q, \dot{q}) = \mu$. Then the induced curve on Q/G_μ satisfies the reduced Lagrangian variational principle, i.e., the variational principle of Lagrange-d'Alembert on Q/G_μ with magnetic term β and the Routhian dropped to $T(Q/G_\mu)$.

In the special case of a torus action, i.e., with cyclic variables, this reduced variational principle is equivalent to the Euler-Lagrange equations for the classical Routhian which agrees with the classical procedure of Routh. We shall give more details of the abelian case in §**6** below.

There is a well defined *reconstruction procedure* for these systems. One can horizontally lift a curve in Q/G to a curve $\mathfrak{q}(t)$ in Q (which therefore has zero angular momentum) and then one rotates it by the group action by a time dependent group element solving the equation

$$\dot{g}(t) = g(t)\xi(t)$$

where $\xi(t) = A(\mathfrak{q}(t), \dot{\mathfrak{q}(t)})$, as is used in the development of geometric phases—see Marsden, Montgomery, and Ratiu [1990].

Using our reduced Euler-Lagrange equations, we will get a reduced Lagrangian description in terms of the angular velocity rather than the angular momentum variables. Notice that the above Routh method is partially Hamiltonian in that it directly utilizes the momentum variables for the rigid part and the velocity variables for the internal part. In the context of systems with a momentum map constraint, this is natural. However, the reduced Euler-Lagrange equations provide a reduced Lagrangian description entirely in terms of velocity variables.

4 A variational principle for the Euler-Poincaré equations

We shall study the reduced Euler-Lagrange equations in two special cases before tackling the general case. These are the cases in which $Q = G$, treated in this section and the case in which G is abelian, treated in the next. Later, we will synthesize these two cases.

Let us begin with a discussion of the special case of the rigid body. We regard an element $R \in SO(3)$ giving the configuration of the body as a map of a ***reference configuration*** $\mathcal{B} \subset \mathbb{R}^3$ to the current configuration $R(\mathcal{B})$ taking a reference or label point $X \in \mathcal{B}$ to a current point $x = R(X) \in R(\mathcal{B})$. For a rigid body in motion, the matrix R becomes time dependent and the velocity of a point of the body is $\dot{x} = \dot{R}X = \dot{R}R^{-1}x$. Since R is an orthogonal matrix, $R^{-1}\dot{R}$ and $\dot{R}R^{-1}$ are skew matrices, and so we can write

$$\dot{x} = \dot{R}R^{-1}x = \omega \times x, \tag{4.1}$$

which defines the ***spatial angular velocity*** vector ω. The corresponding ***body angular velocity*** is defined by

$$\Omega = R^{-1}\omega, \quad i.e., \quad R^{-1}\dot{R}v = \Omega \times v \tag{4.2}$$

so that Ω is the angular velocity relative to a body fixed frame. The kinetic energy is

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho(X)\|\dot{R}X\|^2 d^3X, \tag{4.3}$$

where ρ is a given mass density. Since

$$\|\dot{R}X\| = \|\omega \times x\| = \|R^{-1}(\omega \times x)\| = \|\Omega \times X\|,$$

K is a quadratic function of Ω. Writing

$$K = \frac{1}{2}\Omega^T \mathbb{I}\Omega \tag{4.4}$$

defines the ***moment of inertia tensor*** $\mathbb{I}$, which, if the body does not degenerate to a line, is a positive definite 3×3 matrix, or better, a quadratic form. This quadratic form, can be diagonalized, and this defines the ***principal axes and moments of inertia***. In this basis, we write $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$.

From the Lagrangian point of view, the precise relation between the motion in R space and in Ω space is as follows.

Theorem 4.1. *The curve $R(t) \in SO(3)$ satisfies the Euler-Lagrange equations for*

$$L(R, \dot{R}) = \frac{1}{2}\int_{\mathcal{B}} \rho(X)\|\dot{R}X\|^2 d^3X \tag{4.5}$$

if and only if $\Omega(t)$ defined by $R^{-1}\dot{R}v = \Omega \times v$ for all $v \in \mathbb{R}^3$ satisfies Euler's equations:

$$\mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega. \tag{4.6}$$

One instructive way to prove this is to use variational principles. By Hamilton's principle, $R(t)$ satisfies the Euler-Lagrange equations if and only if

$$\delta \int L\, dt = 0.$$

Let $l(\Omega) = \frac{1}{2}(\mathbb{I}\Omega) \cdot \Omega$ so that $l(\Omega) = L(R, \dot{R})$ if R and Ω are related as above. To see how we should transform the variational principle of L, we differentiate the relation $R^{-1}\dot{R}v = \Omega \times v$ with respect to R to get

$$-R^{-1}\delta R R^{-1}\dot{R}v + R^{-1}\delta\dot{R}v = \delta\Omega \times v. \tag{4.7}$$

Let the skew matrix $\hat{\Sigma}$ be defined by

$$\hat{\Sigma} = R^{-1}\delta R \tag{4.8}$$

and define the vector Σ by

$$\hat{\Sigma}v = \Sigma \times v. \tag{4.9}$$

Note that

$$\dot{\hat{\Sigma}} = -R^{-1}\dot{R}R^{-1}\delta R + R^{-1}\delta\dot{R}$$

so

$$R^{-1}\delta\dot{R} = \dot{\hat{\Sigma}} + R^{-1}\dot{R}\hat{\Sigma} \tag{4.10}$$

substituting (4.10) and (4.8) into (4.7) gives

$$-\hat{\Sigma}\hat{\Omega}v + \dot{\hat{\Sigma}}v + \hat{\Omega}\hat{\Sigma}v = \widehat{\delta\Omega}v$$

i.e.,

$$\widehat{\delta\Omega} = \dot{\hat{\Sigma}} + [\hat{\Omega}, \hat{\Sigma}]. \tag{4.11}$$

The identity $[\hat{\Omega}, \hat{\Sigma}] = (\Omega \times \Sigma)\hat{}$ holds by Jacobi's identity for the cross product, and so

$$\delta\Omega = \dot{\Sigma} + \Omega \times \Sigma. \tag{4.12}$$

These calculations prove the following

Theorem 4.2. *Hamilton's variational principle*

$$\delta \int_a^b L \, dt = 0 \tag{4.13}$$

*on $SO(3)$ is equivalent to the **reduced variational principle***

$$\delta \int_a^b l \, dt = 0 \tag{4.14}$$

on $\mathbb{R}^3$ where the variations $\delta\Omega$ are of the form (4.12) with $\Sigma(a) = \Sigma(b) = 0$.

To complete the proof of Theorem **4.1**, it suffices to work out the equations equivalent to the reduced variational principle (4.14). Since $l(\Omega) = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle$, and $\mathbb{I}$ is symmetric, we get

$$\begin{aligned}
\delta \int_a^b l \, dt &= \int_a^b \langle \mathbb{I}\Omega, \delta\Omega\rangle dt \\
&= \int_a^b \langle \mathbb{I}\Omega, \dot{\Sigma} + \Omega \times \Sigma\rangle dt \\
&= \int_a^b \left[\left\langle -\frac{d}{dt}\mathbb{I}\Omega, \Sigma \right\rangle + \langle \mathbb{I}\Omega, \Omega \times \Sigma\rangle \right] \\
&= \int_a^b \left\langle -\frac{d}{dt}\mathbb{I}\Omega + \mathbb{I}\Omega \times \Omega, \Sigma \right\rangle dt
\end{aligned}$$

where we have integrated by parts and used the boundary conditions $\Sigma(b) = \Sigma(a) = 0$. Since Σ is otherwise arbitrary, (4.14) is equivalent to

$$-\frac{d}{dt}(\mathbb{I}\Omega) + \mathbb{I}\Omega \times \Omega = 0,$$

which are Euler's equations. ■

The ***body angular momentum*** is defined in the usual way, by

$$\Pi = \mathbb{I}\Omega$$

so that in principal axes,

$$\Pi = (\Pi_1, \Pi_2, \Pi_3) = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3).$$

Assuming that no external moments act on the body, the spatial angular momentum vector $\pi = R\Pi$ is conserved in time. This follows by general considerations of symmetry, but it can, of course, be checked directly from Euler's equations:

$$\begin{aligned}\frac{d\pi}{dt} &= \dot{R}\mathbb{I}\Omega + R(\mathbb{I}\Omega \times \Omega) = R(R^{-1}\dot{R}\mathbb{I}\Omega + \mathbb{I}\Omega \times \Omega) \\ &= R(\Omega \times \mathbb{I}\Omega + \mathbb{I}\Omega \times \Omega) = 0.\end{aligned}$$

There is a generalization of Theorem 4.1 on $SO(3)$ and $\mathfrak{so}(3)$ to general Lie groups using the Euler-Lagrange equations and the variational principle as a starting point. We shall also make the direct link with the Lie-Poisson equations.

Theorem 4.3. *Let G be a Lie group and $L : TG \to \mathbb{R}$ a left invariant Lagrangian. Let $l : \mathfrak{g} \to \mathbb{R}$ be its restriction to the identity. For a curve $g(t) \in G$, let $\xi(t) = g(t)^{-1} \cdot \dot{g}(t)$; i.e., $\xi(t) = T_{g(t)}L_{g(t)^{-1}}\dot{g}(t)$. Then the following are equivalent*

i *$g(t)$ satisfies the Euler-Lagrange equations for L on G*

ii *the variational principle*

$$\delta \int_a^b L(g(t), \dot{g}(t))dt = 0 \tag{4.15}$$

holds, for variations with fixed endpoints

iii *the* ***Euler-Poincaré equations*** *hold:*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \tag{4.16}$$

iv *the variational principle*

$$\delta \int l(\xi(t))dt = 0 \tag{4.17}$$

holds on $\mathfrak{g}$, using variations of the form

$$\delta\xi = \dot{\eta} + [\xi, \eta] \tag{4.18}$$

where η vanishes at the endpoints.

In coordinates, the Euler-Poincaré equations read as follows

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^d} = c^b_{ad}\frac{\partial l}{\partial \xi^b}\xi^a. \tag{4.19}$$

Let us discuss the proof of this theorem. First of all, the equivalence of **i** and **ii** holds, of course, on the tangent bundle of any configuration manifold Q, and in particular, on G. Secondly, **ii** and **iv** are equivalent. To see this, one needs to compute the variations $\delta\xi$ induced on $\xi = g^{-1}\dot{g} = TL_{g^{-1}}\dot{g}$ by a variation of g. We did this explicitly for $SO(3)$ above. To calculate it in general, we need to differentiate

$g^{-1}\dot{g}$ in the direction of a variation δg. If $\delta g = dg/d\epsilon$ at $\epsilon = 0$, where g is extended to a curve g_ϵ, then, roughly speaking,

$$\delta\xi = \frac{d}{d\epsilon} g^{-1} \frac{d}{dt} g$$

while if $\eta = g^{-1}\delta g$, then

$$\dot{\eta} = \frac{d}{dt} g^{-1} \frac{d}{d\epsilon} g.$$

The difference $\delta\xi - \dot{\eta}$ is thus the commutator, $[\xi, \eta]$.

To complete the proof, we show the equivalence of **iii** and **iv**. Indeed, using the definitions and integrating by parts,

$$\begin{aligned}
\delta \int l(\xi) dt &= \int \frac{\delta l}{\delta \xi} \delta\xi \, dt \\
&= \int \frac{\delta l}{\delta \xi} (\dot{\eta} + \mathrm{ad}_\xi \eta) dt \\
&= \int \left[-\frac{d}{dt} \left(\frac{\delta l}{\delta \xi} \right) + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \right] \eta \, dt
\end{aligned}$$

so the result follows. ■

Since the Euler-Lagrange and Hamilton equations on TQ and T^*Q are equivalent, it follows that the Lie-Poisson and Euler-Poincaré equations are also equivalent. To see this *directly*, we make the following Legendre transformation from $\mathfrak{g}$ to $\mathfrak{g}^*$:

$$\mu = \frac{\delta l}{\delta \xi}, \quad h(\mu) = \langle \mu, \xi \rangle - l(\xi).$$

Note that

$$\frac{\delta h}{\delta \mu} = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \frac{\delta l}{\delta \xi}, \frac{\delta \xi}{\delta \mu} \right\rangle = \xi$$

and so it is now clear that the Euler-Poincaré equations are equivalent to the Lie-Poisson equations on $\mathfrak{g}^*$, namely

$$\frac{d\mu}{dt} = \mathrm{ad}^*_{\delta h/\delta \mu} \mu$$

which is equivalent to $\dot{F} = \{F, h\}$ relative to the Lie-Poisson bracket (see Marsden [1992] for more information and references).

5 The Abelian case

In this section we consider a simple mechanical system with Lagrangian L of the form kinetic minus potential energy, where the configuration manifold Q is a Riemannian manifold. Furthermore, assume that the symmetry group G is abelian. As before, the goal is to reduce the Euler-Lagrange equations for L on TQ to equations on TQ/G where we assume that TQ/G is a smooth manifold. As we have seen in the last section, in the special case $Q = G$, the reduced equations are the Euler-Poincaré equations on the Lie algebra $\mathfrak{g}$ corresponding to G. In the more general case considered here, the reduced equations are going to be a combination of standard Euler-Lagrange-d'Alembert equations for the base variables of TQ/G and the Euler-Poincaré equations for the fiber, which are conservation laws in the case of abelian groups. To make the exposition as transparent as possible, we work out everything in coordinates here. For an abelian group G we identify the symmetry group using a set of cyclic coordinates. We assume that G acts on Q by $x^\alpha \mapsto x^\alpha (\alpha = 1, \ldots, m)$ and $\theta^a \mapsto \theta^a + \varphi^a (a = 1, \ldots k)$ with $\varphi^a \in [0, 2\pi)$, where $x^1, \ldots, x^m, \theta^1, \ldots, \theta^k$ are suitably chosen (local) coordinates on Q. Then G-invariance implies that the Lagrangian $L = L(x, \dot{x}, \dot{\theta})$ does not explicitly depend on the variables θ^a, *i.e.,* these variables are ***cyclic***. In these coordinates we have

$$L(x, \dot{x}, \dot{\theta}) = \frac{1}{2} g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta + g_{a\alpha}(x) \dot{x}^\alpha \dot{\theta}^a + \frac{1}{2} g_{ab}(x) \dot{\theta}^a \dot{\theta}^b - V(x). \tag{5.1}$$

Moreover, the infinitesimal generator ξ_Q on Q of an element $\xi = (\xi^1, \ldots, \xi^k)$ is given by

$$\xi_Q = (0, \ldots, 0, \xi^1, \ldots, \xi^k),$$

where there are m zeros. Therefore we have the following action coefficients—see (2.2):

$$K^\beta{}_a = 0, \quad K^b{}_a = \delta^b{}_a.$$

The components of the momentum map $\mathbf{J} : TQ \to \mathfrak{g}^*$ for G are given by

$$J_a = \frac{\partial L}{\partial \dot{\theta}^a} = g_{a\alpha} \dot{x}^\alpha + g_{ab} \dot{\theta}^b,$$

i.e., they are the classical momenta conjugate to the variables $\dot{\theta}^a$. Note that the momentum map is G-equivariant here, *i.e.,* its components are conserved quantities for the equations of motion. Thus, the level surfaces of $\mathbf{J}$ in TQ are invariant. In the present coordinates this also follows immediately from the standard Euler-Lagrange equations for L

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right) - \frac{\partial L}{\partial x^\gamma} &= 0 \quad (\gamma = 1, \ldots, m) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}^c}\right) &= 0 \quad (c = 1, \ldots, k). \end{aligned} \tag{5.2}$$

We come back to this fact later when we talk about the relation to the Routhian procedure of reduction.

Note that (5.2) are local equations, since the coordinates $x, \dot{x}, \dot{\theta}$ correspond to a (local) trivialization of the bundle TQ/G. Since these equations are independent of θ, they locally drop to the quotient. In order to give the dropped equations a global (intrinsic) meaning, we now introduce the velocity shift. To this end we replace the variables $\dot{\theta}^a$ by ξ^a given by means of the mechanical connection A as follows:

$$\begin{aligned}
\xi^a = A^a &= \mathbb{I}^{ab} g_{ij} K^i{}_b \dot{q}^j \\
&= \mathbb{I}^{ab} g_{bj} \dot{q}^j \\
&= \mathbb{I}^{ab} g_{b\alpha} \dot{x}^\alpha + \mathbb{I}^{ab} g_{bc} \dot{\theta}^c \\
&= \dot{\theta}^a + A^a_\alpha \dot{x}^\alpha,
\end{aligned}$$

where

$$A^a_\alpha = \mathbb{I}^{ab} g_{b\alpha}$$

are the components of the connection. Thus

$$\dot{\theta}^a = \dot{\tilde{\theta}}^a(x, \dot{x}, \xi) = \xi^a - A^a_\alpha \dot{x}^\alpha,$$

In particular, as a Lie algebra valued 1-form A has components

$$A^a = d\theta^a + A^a_\alpha dx^\alpha.$$

Thus, we replace $\dot{\theta}$ by the locked angular velocity $\xi \in \mathfrak{g}$, the generator ξ_Q of which gives the velocity component tangential to the group orbit at a given configuration of the system. In terms of the transformed Lagrangian

$$l(x, \dot{x}, \xi) = L(x, \dot{x}, \dot{\tilde{\theta}}(x, \dot{x}, \xi))$$

the equation of motion (5.2) reads as follows

$$\begin{aligned}
\frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\gamma} - \frac{\partial l}{\partial x^\gamma} &= \frac{\partial l}{\partial \xi^a} B^a_{\gamma\alpha} \dot{x}^\alpha \quad (\gamma = 1, \ldots, m) \\
\frac{d}{dt}\frac{\partial l}{\partial \xi^c} &= 0 \quad (c = 1, \ldots, k)
\end{aligned} \tag{5.3}$$

as a straightforward computation using the chain rule shows. Here

$$B^a_{\gamma\alpha} = A^a_{\alpha,\gamma} - A^a_{\gamma,\alpha}$$

can be viewed as the components of the curvature B of the connection A which in the abelian case, is given by the exterior derivative of A:

$$B^a = [\mathbf{d}A]^a = \sum_{\gamma<\alpha} B^a_{\gamma\alpha} dx^\gamma \wedge dx^\alpha.$$

So, the first set of equations in (5.3) are forced Euler-Lagrange equations with Lagrangian l as a function of the base variables $(x, \dot{x})$. The second set of equations are Poincaré equations in ξ, which are degenerate because we are dealing with an abelian group G here. Since these equations as well as l do not depend on the θ-variables, they drop to the quotient $(TQ)/G$ just as they are stated. As we shall see in the next section, this is the principal structure of the equations on $(TQ)/G$

even for non-abelian groups G. Here one does not even need the assumption of a simple mechanical system.

In our previous paper (Marsden and Scheurle [1993]) we carried out a further reduction to $J^{-1}(\mu)/G_\mu$ in the case of a simple mechanical system and assuming that the quotient is a smooth manifold again, *i.e.,* μ is a regular value of J and G_μ acts freely on the level set $J = \mu$. Recall that G_μ is the isotropy subgroup of μ. For abelian groups, locally, *i.e.,* in coordinates, this reduction amounts to the classical Routhian procedure, which leads to Euler-Lagrange equations for the classical Routhian R^μ_{class}. Note that in the abelian case, $G_\mu = G$ and $J^{-1}(\mu)/G_\mu \approx T(Q/G)$. We have modified this procedure to give it an intrinsic (global) meaning and to include the case of nonabelian groups G. In terms of the setting of this section, the Routhian R^μ is defined by a (partial) Legendre transformation of l with respect to ξ:

$$R^\mu(x, \dot{x}) = [l(x, \dot{x}, \xi) - \langle \mu, \xi \rangle]_{|\xi = \tilde{\xi}(x, \dot{x}, \mu)}$$

where $\tilde{\xi}(x, \dot{x}, \mu)$ is the unique solution for ξ of the equation

$$\frac{\partial l}{\partial \xi}(x, \dot{x}, \xi) = \mu$$

for a particular value of $\mu \in \mathfrak{g}^*$. So, now we replace the variable ξ by μ in the equations (5.3). Then the second set of equations in (5.3) just become

$$\frac{d}{dt}\mu_c = 0 \quad (c = 1, \ldots, k)$$

which expresses the fact, that μ is conserved for the motion. Thus, fixing a particular value for μ, the restriction of (5.2) to the level set $J = \mu$ is given by the first set of equations in (5.3). Rewriting those in terms of the Routhian R^μ rather than l leads to

$$\frac{d}{dt}\frac{\partial R^\mu}{\partial \dot{x}^\gamma} - \frac{\partial R^\mu}{\partial x^\gamma} = \mu_a B^a_{\gamma\alpha}\dot{x}^\alpha = [\mathbf{i}_{\dot{x}} B_\mu]_\gamma \quad (\gamma = 1, \ldots, m), \tag{5.4}$$

i.e., on $J = \mu$ we have Euler-Lagrange equations for the Routhian R^μ with forcing given by the magnetic 2-form B_μ which in the present coordinates is defined as follows:

$$B_\mu = \sum_{\gamma < \alpha} \mu_a B^a_{\gamma\alpha} dx^\gamma \wedge dx^\alpha.$$

Note that $B_\mu = dA_\mu$ is the exterior derivative of the 1-form $A_\mu : Q \to T^*Q$ given by

$$A_\mu = \mu_a d\theta^a + \mu_a A^a_\alpha dx^\alpha$$

(cf. (2.9). Note that in contrast to A_μ, B_μ does not depend on the θ-variables just as R^μ does not. Therefore all terms in (5.4) drop further to the quotient $J^{-1}(\mu)/G$ and the structure of these equations is preserved there.

Moreover, locally these equations are even standard Euler-Lagrange equations on the quotient. Namely, as a closed 2-form B_μ is locally exact. Therefore, locally the right-hand side of (5.4) can be included into the left-hand side by replacing R^μ, by the classical Routhian

$$\begin{aligned} R^\mu_{\text{class}} &= \left[L(x,\dot{x},\dot{\theta}) - \mu_a \dot{\theta}^a\right]_{\dot{\theta}^a = \dot{\tilde{\theta}}^a(x,\dot{x},\mu)} \\ &= R^\mu + \mu_a g_{\alpha a}\dot{x}^\alpha \mathbb{I}^{ca}, \end{aligned}$$

where

$$\dot{\theta}^a = \dot{\tilde{\theta}}^a(x,\dot{x},\mu) = [\mu_c - g_{\alpha c}\dot{x}^\alpha]\mathbb{I}^{ca}$$

is the unique solution of the equation $J = \mu$ with respect to $\dot{\theta}$. It is to be noted that globally, B_μ is not exact on the quotient $J^{-1}(\mu)/G$ in general. In the present context this can be seen from the fact that the 1-form A_μ explicitly depends on the θ-variables.

Also, there is a variational principle behind the equations (5.4), namely in the sense of Lagrange-d'Alembert:

$$\delta \int_a^b R^\mu dt = \int_a^b i_{\dot{x}} B_\mu \delta x.$$

In this variational principle, the variation of the integral of R^μ is taken over curves on Q/G_μ, that satisfy the fixed endpoint condition. This variational principle allows one to generalize the Routhian reduction procedure even for nonabelian groups G. Here is what was proved in Marsden and Scheurle [1993].

Theorem 5.1. *Let Q be a Riemannian manifold and let a (not necessarily abelian) Lie group G act freely on it. Let J be its momentum map on TQ. Assume that $\mu \in \mathfrak{g}^*$ is a regular value of J. Consider a simple mechanical system given by a Lagrangian $L : TQ \to \mathbb{R}$ which is G-invariant. Define the Routhian $R^\mu : TQ \to \mathbb{R}$ to be*

$$R^\mu(q,v) = L(q,v) - \langle A(q,v), \mu\rangle$$

where A is the mechanical connection as defined in (2.4).

*Suppose that $q(t)$ satisfies the Euler-Lagrange equations for L and lies on the level set $J(q(t),v(t)) = \mu$. Then the induced curve on Q/G_μ satisfies the **reduced Lagrangian variational principle**, i.e., the variational principle of Lagrange-d'Alembert on Q/G_μ with magnetic term $B_\mu = dA_\mu$, where A_μ is defined as in (2.9), and the Routhian R^μ dropped to $T(Q/G_\mu)$.*

Remark In the special case $Q = G$, e.g. in the rigid body case, the reduced variational principle becomes degenerate and leads to first order equations on $Q/G_\mu = G/G_\mu$.

6 Bundles and Local Trivializations

In this section we set up the machinery for the derivation of the reduced Euler-Lagrange equations.

As above, let G be a Lie group that acts on the left freely and properly on a (configuration) manifold Q and let $S = Q/G$ be the shape space. Under these hypotheses, the natural projection $\pi_S : Q \to S$ defines a principal G-bundle. In particular, the bundle π_S admits local trivializations; that is, Q is covered by open sets $U \subset Q$ and there are diffeomorphisms

$$\Psi : U \to V \times G$$

where $V \subset S$ is open such that Ψ has the form $\Psi(q) = (\pi_S(q), \Psi_G(q))$ and

$$\Psi(g \cdot q) = (\pi_S(q), g\Psi_G(q)).$$

In other words, in this local trivialization, the action of G is given by the trivial action on the first fiber and left multiplication on the second fiber. This follows readily from the freeness and properness of the action and the fact that π_S is a submersion.

Note that in our conventions, the bundle $\pi_S : Q \to S$ is a *left* principal bundle. Of course it would be a *right* principal bundle if the original action were a right action.

Examples

A If $Q = G$ with G acting by left multiplication, then S is a point and we can choose $U = G$, and $V = S$.

B Let $Q = TG$, with G acting by the lifted left action. Then we can choose $U = TG, V = S = \mathfrak{g}$, the Lie algebra of G and $\Psi = \lambda$

$$\lambda : TG \to \mathfrak{g} \times G$$

given by

$$\lambda(V_g) = (TL_{g^{-1}} \cdot V_g, g),$$

where $L_{g^{-1}}$ denotes left multiplication by g^{-1}. ♦

Under the above circumstances, not only is Q a principal G-bundle, but so is TQ. A local trivialization for Q clearly induces one for TQ given by $(\mathrm{Id} \times \lambda) \circ T\Psi$:

$$\begin{aligned} T\Psi : &\quad TU \subset TQ \to TV \times TG \\ (\mathrm{Id} \times \lambda) : &\quad TV \times TG \to TV \times \mathfrak{g} \times G. \end{aligned}$$

Thus, the base space $(TQ)/G$ is locally diffeomorphic to $TS \times \mathfrak{g}$.

We let q^i denote coordinates on Q and $(q^i, \dot{q}^i)$ be the induced coordinates on TQ. If we use coordinates corresponding to the above local trivialization, we can write coordinates on TQ/G as

$$(x^\alpha, \dot{x}^\alpha, \eta^a) \in TV \times \mathfrak{g}$$

so that TQ itself is coordinatized by

$$(x^\alpha, \dot{x}^\alpha, \eta^a, g^a)$$

where g^a denote coordinates on G. Notice especially that if (x^α, g^a) denote local coordinates on Q, which is locally $S \times G$, then η^a are the Lie algebra coordinates of $g^{-1}\dot{g}$.

Now let $L : TQ \to \mathbb{R}$ be a given left invariant Lagrangian. Then L induces a *reduced Lagrangian*

$$l : TQ/G \to \mathbb{R}.$$

In local coordinates corresponding to a local trivialization, we thus represent l by a function

$$l(x^\alpha, \dot{x}^\alpha, \eta^a).$$

In a local trivialization, L defines a map $L : TV \times TG \to \mathbb{R}$ and G acts only in the second factor. We can take Hamilton's principle and (locally) divide the variations δq of q into those only in x and those only in g. In x, we will get the usual Euler-Lagrange equations, while in g, we can apply the results of §**2** to get the Euler-Poincaré equations. This argument therefore proves that:

Theorem 6.1. *The Euler-Lagrange equations for L on Q are equivalent to the* ***Hamel equations*** *for l in a local principal bundle trivialization:*

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} = 0 \tag{6.1}$$

$$\frac{d}{dt}\frac{\partial l}{\partial \eta^a} = c^b_{ad}\eta^d \frac{\partial l}{\partial \eta^b} \tag{6.2}$$

Notice that in a local trivialization, one can reconstruct the motion on TQ, or locally $TV \times TG$ by solving the equation

$$g^{-1}\dot{g} = \eta,$$

just as in the case of the Euler-Poincaré equations.

7 Connections, Curvature, and the Velocity Shift

To make the Hamel system (6.1), (6.2) independent of the choice of local trivialization as well as to put the Lagrangian into a simpler form, it is useful to introduce a connection. Here, a "simpler form" means, that one can complete the square in the kinetic energy, which is important for studying the stability of relative equilibria. We shall see this simplification explicitly in the examples in the next sections.

We recall that a ***connection*** on Q is a $\mathfrak{g}$-valued one form:

$$A : TQ \to \mathfrak{g}$$

that is

1. equivariant with respect to the action of G on TQ and the adjoint action Ad_g of G on $\mathfrak{g}$, and
2. satisfies $A(\xi_Q(q)) = \xi$ for each $\xi \in \mathfrak{g}$.

Often one chooses A to be the mechanical connection, but in this section, A can be an arbitrary connection.

The ***vertical space*** at $q \in Q$ is spanned by the set of $\xi_Q(q)$ for $\xi \in \mathfrak{g}$ and the ***horizontal space*** is

$$\mathrm{hor}_q = \{v \in T_qQ \mid A(v) = 0\}.$$

Condition **2** can be rephrased as saying that for each $\mu \in \mathfrak{g}^*, \mathbf{J}(A_\mu) = \mu$, where A_μ is the one form obtained by pairing α with μ.

Clearly, every vector $v \in T_qQ$ can be uniquely written as a horizontal vector plus a vertical one; in fact, we can write

$$v = \mathrm{hor}_q v + \mathrm{ver}_q\, v$$

where $\mathrm{ver}_q v = (A(v_q))_Q(q)$ and $\mathrm{hor}_q v = v - \mathrm{ver}_q v$.

Example

If $Q = G$, there is a canonical connection given by the right invariant one form equaling the identity at $g = e$. That is, for $v \in T_gG$, we let

$$A_G : TG \to \mathfrak{g}; \quad A_G(v) = TR_{g^{-1}} \cdot v.$$

Note that A_G is uniquely determined by condition **2**. ♦

In a local trivialization where we can locally write

$$Q = S \times G,$$

then a connection A as a 1-form has the form

$$A(x^\alpha, g^a) = A^a_\alpha(x^\alpha, g^a)dx^\alpha + A_G(g^a)$$

for functions $A^a_\alpha(x^\beta)$, which are called the ***connection coefficients***. If we identify $TG = G \times \mathfrak{g}$ by the map λ, we can write A as a mapping on TQ,

$$A(x^\alpha, \dot{x}^\alpha, g^a, \dot{g}^a) = (0, A^a_\alpha \dot{x}^\alpha + \eta^a)$$

where $\eta = g^{-1}\dot{g}$. The element η is, relative to this local trivialization, the ***body angular velocity***.

In a local trivialization, a vector is vertical if it's $\dot{x}$ component is zero and is horizontal if A applied to it is zero. Thus, the horizontal-vertical decomposition is, at the identity,

$$(\dot{x}^\alpha, \eta^a) = (\dot{x}^\alpha, -A^a_\alpha \dot{x}^\alpha) + (0, A^a_\alpha \dot{x}^\alpha + \eta^a). \tag{7.1}$$

We will, according to previous considerations, call

$$\xi^a = A^a_\alpha \dot{x}^\alpha + \eta^a \tag{7.2}$$

the "locked" angular velocity; *i.e.,* ξ is the value of A.

Now we consider our earlier Lagrangian $l(x^\alpha, \dot{x}^\alpha, \eta^a)$ and rewrite it in terms of ξ. That is, let

$$l_{\text{lock}}(x^\alpha, \dot{x}^\alpha, \xi^a) = l(x^\alpha, \dot{x}^\alpha, \xi^a - A^a_\alpha \dot{x}^\alpha). \tag{7.3}$$

Now let us rewrite the Hamel equations in terms of l_{lock}. In the introduction, we dropped the subscript "lock", but we keep it here to avoid notational confusion. To do this, we compute the Euler-Lagrange derivative in coordinates using the chain rule:

$$\frac{d}{dt}\frac{\partial l_{\text{lock}}}{\partial \dot{x}^\alpha} - \frac{\partial l_{\text{lock}}}{\partial x^\alpha} = \frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial \eta^a} A^a_\alpha\right) - \frac{\partial l}{\partial x^\alpha} + \frac{\partial l}{\partial \eta^a}\frac{\partial A^a_\beta}{\partial x^\alpha}\dot{x}^\beta. \tag{7.4}$$

Now use the Hamel equations to simplify the right-hand side of (7.4). We get

$$\begin{aligned}
&-\frac{d}{dt}\left(\frac{\partial l}{\partial \eta^a}\right) A^a_\alpha - \frac{\partial l}{\partial \eta^a}\frac{\partial A^a_\alpha}{\partial x^\beta}\dot{x}^\beta + \frac{\partial l}{\partial \eta^a}\frac{\partial A^a_\beta}{\partial x^\alpha}\dot{x}^\beta \\
&= \; -c^d_{ba}\eta^b \frac{\partial l}{\partial \eta^d} A^a_\alpha - \frac{\partial l}{\partial \eta^a}\frac{\partial A^a_\alpha}{\partial x^\beta}\dot{x}^\beta + \frac{\partial l}{\partial \eta^a}\frac{\partial A^a_\beta}{\partial x^\alpha}\dot{x}^\beta \\
&= \; \frac{\partial l_{\text{lock}}}{\partial \xi^d}\left(-c^d_{ba}\left[\xi^b - A^b_\beta \dot{x}^\beta\right] A^a_\alpha - \frac{\partial A^d_\alpha}{\partial x^\beta}\dot{x}^\beta + \frac{\partial A^d_\beta}{\partial x^\alpha}\dot{x}^\beta\right) \\
&= \; \frac{\partial l_{\text{lock}}}{\partial \xi^d}\left(-c^d_{ba}\xi^b A^a_\alpha\right) + \frac{\partial l_{\text{lock}}}{\partial \xi^d} B^d_{\alpha\beta}\dot{x}^\beta
\end{aligned}$$

where

$$B^d_{\alpha\beta} = \frac{\partial A^d_\beta}{\partial x^\alpha} - \frac{\partial A^d_\alpha}{\partial x^\beta} + c^d_{ba} A^b_\beta A^a_\alpha$$

are the components of the curvature of A. To summarize,

$$\frac{d}{dt}\frac{\partial l_{\text{lock}}}{\partial \dot{x}^\alpha} - \frac{\partial l_{\text{lock}}}{\partial x^\alpha} = \frac{\partial l_{\text{lock}}}{\partial \xi^d} B^d_{b\alpha}\xi^b + \frac{\partial l_{\text{lock}}}{\partial \xi^d} B^d_{\alpha\beta}\dot{x}^\beta \tag{7.5}$$

where

$$B^d_{b\alpha} = c^d_{ab} A^a_\alpha$$

is the ***interaction*** term. For the Euler-Poincaré part, we get

$$\frac{d}{dt}\frac{\partial l_{\text{lock}}}{\partial \xi^b} = c^a_{db}\frac{\partial l_{\text{lock}}}{\partial \xi^a}\left(\xi^d - A^d_\alpha \dot{x}^\alpha\right)$$

i.e.,

$$\frac{d}{dt}\frac{\partial l_{\text{lock}}}{\partial \xi^b} = c^a_{db}\frac{\partial l_{\text{lock}}}{\partial \xi^a}\xi^d + \frac{\partial l_{\text{lock}}}{\partial \xi^a} B^a_{b\alpha}\dot{x}^\alpha. \tag{7.6}$$

This completes the derivation of the reduced Euler-Lagrange equations, as stated in the introduction. A detailed description of the split of the variational principle into horizontal and vertical parts, and further examples will be given in a forthcoming paper. For the present paper, we limit ourselves to two of the most basic, but still instructive examples given in the following sections.

8 A Particle in a Magnetic Field

In this section we will give a simple, but concrete illustration for the reduced Euler-Lagrange equations. Our example is readily generalized to the case of a particle moving in a Yang-Mills field; see Montgomery [1984] and references therein.

We first review the standard Hamiltonian formulation for the motion of a particle in a magnetic field. Let B be a closed two-form on $\mathbb{R}^3$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$ the associated divergence free vector field, *i.e.,* $\mathbf{i}_{\mathrm{B}}(dx \wedge dy \wedge dz) = B$, or

$$B = B_x dy \wedge dz - B_y dx \wedge dz + B_z dx \wedge dy.$$

Thinking of $\mathbf{B}$ as a magnetic field, the equations of motion for a particle with charge e and mass m are given by the ***Lorentz force law***:

$$m\frac{d\mathbf{v}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B} \tag{8.1}$$

where $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$. On $\mathbb{R}^3 \times \mathbb{R}^3$ *i.e.,* on $(\mathbf{x}, \mathbf{v})$-space, consider the symplectic form

$$\Omega_B = m(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}) - \frac{e}{c}B. \tag{8.2}$$

For the Hamiltonian, take the kinetic energy:

$$H = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{8.3}$$

writing $X_H(u, v, w) = (u, v, w, (\dot{u}, \dot{v}, \dot{w}))$, the condition defining X_H, namely $\mathbf{i}_{X_H}\Omega_B = \mathbf{d}H$ is

$$\begin{aligned} m(ud\dot{x} - \dot{u}dx + vd\dot{y} - \dot{v}dy + wd\dot{z} - \dot{w}dz) & \\ - \frac{e}{c}[B_x vdz - B_x wdy - B_y udz + B_y wdx + B_z udy - B_z vdx] & \\ = \quad m(\dot{x}d\dot{x} + \dot{y}d\dot{y} + \dot{z}d\dot{z}) & \end{aligned} \tag{8.4}$$

which is equivalent to $u = \dot{x}, v = \dot{y}, w = \dot{z}, m\dot{u} = e(B_z v - B_y w)/c, m\dot{v} = e(B_x w - B_z u)/c$, and $m\dot{w} = e(B_y u - B_x v)/c$, *i.e.,* to

$$\begin{aligned} m\ddot{x} &= \frac{e}{c}(B_z\dot{y} - B_y\dot{z}) \\ m\ddot{y} &= \frac{e}{c}(B_x\dot{z} - B_z\dot{x}) \\ m\ddot{z} &= \frac{e}{c}(B_y\dot{y} - B_x\dot{z}) \end{aligned} \tag{8.5}$$

which is the same as (8.1). Thus the *equations of motion for a particle in a magnetic field are Hamiltonian, with energy equal to the kinetic energy and with the symplectic form* Ω_B.

If $B = \mathbf{d}A$; *i.e.,* $\mathbf{B} = \nabla \times \mathbf{A}$, where A is a one-form and $\mathbf{A}$ is the associated vector field, then the map $(\mathbf{x}, \mathbf{v}) \mapsto (\mathbf{x}, \mathbf{p})$ where $\mathbf{p} = m\mathbf{v} + e\mathbf{A}/c$ pulls back the *canonical* form to Ω_B, as is easily checked. *Thus, equations (8.1) are also Hamiltonian relative to the canonical bracket on* $(\mathbf{x}, \mathbf{p})$*-space with the Hamiltonian*

$$H_{\mathbf{A}} = \frac{1}{2m}\|\mathbf{p} - \frac{e}{c}\mathbf{A}\|^2. \tag{8.6}$$

Even in Euclidean space, not every magnetic field can be written as $\mathbf{B} = \nabla \times \mathbf{A}$. For example, the field of a magnetic monopole of strength $g \neq 0$, namely

$$\mathbf{B}(\mathbf{r}) = g\frac{\mathbf{r}}{\|\mathbf{r}\|^3} \tag{8.7}$$

cannot be written this way since the flux of $\mathbf{B}$ through the unit sphere is $4\pi g$, yet Stokes' theorem applied to the two hemispheres would give zero. Thus, one might think that the Hamiltonian formulation involving only B (*i.e.,* using Ω_B and H) is preferable. However, one can recover the magnetic potential A by regarding A as a connection on a nontrivial bundle over $\mathbb{R}^3\backslash\{0\}$.

We now recall how to write the equations of a charged particle in a magnetic field in terms of geodesics; that is, the Kaluza-Klein description using an S^1-reduction. When generalizing the process described here to the case of a particle in a Yang-Mills field, we replace the magnetic potential A by the Yang-Mills connection.

Above, we saw that if $\mathbf{B} = \nabla \times \mathbf{A}$ is a given magnetic field on $\mathbb{R}^3$, then with respect to canonical variables $(\mathbf{q}, \mathbf{p})$, the Hamiltonian is

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2m}\|\mathbf{p} - \frac{e}{c}\mathbf{A}\|^2. \tag{8.8}$$

We can obtain (8.8) via the Legendre transform if we choose

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{q}} \tag{8.9}$$

for then

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A} \tag{8.10}$$

and

$$\begin{aligned}
\mathbf{p}\cdot\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}) &= (m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A})\cdot\dot{\mathbf{q}} - \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 - \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{q}} \\
&= \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 \\
&= \frac{1}{2m}\|\mathbf{p} - \frac{e}{c}\mathbf{A}\|^2 = H(\mathbf{q}, \mathbf{p}).
\end{aligned} \tag{8.11}$$

Thus, the Euler-Lagrange equations for (8.9) reproduce the equations for a particle in a magnetic field. (If an electric field $\mathbf{E} = -\nabla\varphi$ is present as well, subtract $e\varphi$ from L, treating $e\varphi$ as a potential energy.) Let the ***Kaluza-Klein configuration space*** be

$$Q_K = \mathbb{R}^3 \times S^1 \tag{8.12}$$

with variables $(\mathbf{q}, \theta)$ and consider the one-form

$$\omega = A + \mathbf{d}\theta \tag{8.13}$$

on Q_K regarded as a connection one-form. Define the ***Kaluza-Klein Lagrangian*** by

$$\begin{aligned} L_K(\mathbf{q}, \dot{\mathbf{q}}, \theta, \dot{\theta}) &= \frac{1}{2} m \|\dot{\mathbf{q}}\|^2 + \frac{1}{2} \langle \omega, (\mathbf{q}, \dot{\mathbf{q}}, \theta, \dot{\theta}) \rangle^2 \\ &= \frac{1}{2} m \|\dot{\mathbf{q}}\|^2 + \frac{1}{2} (\mathbf{A} \cdot \dot{\mathbf{q}} + \dot{\theta})^2. \end{aligned} \tag{8.14}$$

The corresponding momenta are

$$\mathfrak{p} = m\dot{\mathbf{q}} + (\mathbf{A} \cdot \dot{\mathbf{q}} + \dot{\theta})\mathbf{A} \quad \text{and} \quad p_\theta = \mathbf{A} \cdot \dot{\mathbf{q}} + \dot{\theta}. \tag{8.15}$$

Since (8.14) is quadratic and positive definite in $\dot{\mathbf{q}}$ and $\dot{\theta}$, *the Euler-Lagrange equations are the geodesic equations on* $\mathbb{R}^3 \times S^1$ *for the metric for which* L_K *is the kinetic energy*. Since p_θ is constant in time as can be seen from the Euler-Lagrange equation for $(\theta, \dot{\theta})$, we can define the ***charge*** e by setting

$$p_\theta = e/c; \tag{8.16}$$

then (8.15) coincides with (8.10). The corresponding Hamiltonian on T^*Q_K endowed with the canonical symplectic form is

$$H_K(\mathbf{q}, \mathfrak{p}, \theta, p_\theta) = \frac{1}{2m} \|\mathfrak{p} - p_\theta A\|^2 + \frac{1}{2} p_\theta^2. \tag{8.17}$$

Since p_θ is constant, H_K differs from H only by the constant $p_\theta^2/2$.

These constructions generalize to the case of a particle in a Yang-Mills field where ω becomes the connection of a Yang-Mills field and its curvature measures the field strength which, for an electromagnetic field, reproduces the relation $\mathbf{B} = \nabla \times \mathbf{A}$. We refer to Montgomery [1985] for details and further references. Finally, we remark that the relativistic context is the most natural to introduce the full electromagnetic field. In that setting the construction we have given for the magnetic field will include both electric and magnetic effects.

From the point of view of the present paper, we regard the equations (8.1) as the Euler-Lagrange equations with a curvature term on the right hand side, and regard it as being obtained from L_K by Lagrangian reduction. The reduced Euler-Lagrange equations then become (8.1), corresponding to (1.3), together with the conservation of p_θ, corresponding to (1.4), which, in the abelian case, becomes a conservation law. Of course, in this abelian case, reduction by the Routh procedure is essentially indistinguishable from the general Euler-Lagrange reduction procedure given in the present paper.

9 The Rigid Body with Rotors

In this section, we show the role of the mechanical connection in this system, and why it is useful to employ it.

For a rigid body with three rotors aligned with, say, the principal axes, we take the configuration space to be

$$Q = SO(3) \times S^1 \times S^1 \times S^1$$

with elements denoted

$$(R, \theta_1, \theta_2, \theta_3)$$

where the angles are relative to the carrier. To illustrate the general theory, we shall choose $G = SO(3)$. As in the rigid body, the body angular velocity is given by $\hat{\Omega} = R^{-1}\dot{R}$, which corresponds to η in the general theory, and $\Omega_r = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$, which corresponds to x^α in the theory (with the given choice of G).

The Lagrangian is given on Q by the total kinetic energy:

$$L = \frac{1}{2}\langle \Omega, I\Omega \rangle + \frac{1}{2}\langle \Omega + \Omega_r, K(\Omega + \Omega_r) \rangle$$

where I, K are inertia tensors. Through the given definitions, this may be regarded as a function on TQ, and so the equations of motion for the system are the Euler-Lagrange equations. Clearly, we can also regard L as a function on TQ/G. However, eventually, one wants to add controls to this situation as in Bloch, Krishnaprasad, Marsden, and Sanchez [1992]. This aspect in the present context will be discussed in another publication.

For this example, one checks that the locked inertia tensor is given by

$$\mathbb{I} = R(I + K)R^{-1}$$

while the momentum map is

$$\mathbf{J} = R[(I + K)\Omega + K\Omega_r] = R\Pi = \mu$$

where μ is the spatial (fixed) angular momentum and where

$$\Pi = [(I + K)\Omega + K\Omega_r] = \frac{\partial L}{\partial \Omega}$$

is the body angular momentum. The mechanical connection is computed to be

$$A = R^{-1}\dot{R} + (I + K)^{-1}K\Omega_r.$$

The equations of motion are the following:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}^i} - \frac{\partial L}{\partial \theta^i} &= 0, \\ \frac{d}{dt}\Pi &= \Pi \times \Omega \end{aligned}$$

so that, from the first equation, $l = K(\Omega + \Omega_r) = \text{constant}$. Note that

$$\mu = R(I\Omega + l)$$

and that the shifted, or locked, velocity

$$\xi = \Omega + (I+K)^{-1} K \Omega_r$$

completes the square in L. That is,

$$l_{\text{lock}} = \frac{1}{2}\langle \xi, (I+K)\xi \rangle + \frac{1}{2}\langle K\Omega_r, (I+K)^{-1} I \Omega_r \rangle$$

which "decouples" Ω and Ω_r. If we rewrite the above equations of motion in terms of l_{lock} and $\xi, \theta, \dot{\theta}$, we get the form of the reduced Euler-Lagrange equations that was given in the introduction.

Finally, we notice that to recover the attitude from

$$\Pi = (I+K)\xi,$$

note that

$$(I+K)R^{-1}\dot{R} + K\Omega_r = (I+K)\xi.$$

Regarding ξ (or Π) and Ω_r as known, then $\Omega = R^{-1}\dot{R} = \xi - (I+K)^{-1}K\Omega_r$ where the last term is the connection. Formulas like this are also important in the determination of phases for the rigid body with internal rotors, as in Bloch, Krishnaprasad, Marsden and Sanchez [1992].

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