Generic Bifurcation of Hamiltonian Vector Fields with Symmetry

Michael Dellnitz* and Ian Melbourne†
Department of Mathematics, University of Houston
Houston, Texas 77204-3476, USA

Jerrold E. Marsden‡
Institut für Angewandte Mathematik, Universität Hamburg
D-2000 Hamburg 13, Germany

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Abstract

One of the goals of this paper is to describe explicitly the generic movement of eigenvalues through a one-to-one resonance in a (linearized) Hamiltonian system. We classify this movement, and hence answer the question of when the collisions are "dangerous" in the sense of Krein by using a combination of group theory and definiteness properties of the associated quadratic Hamiltonian. For

*Research is supported by the Deutsche Forschungsgemeinschaft and by NSF/DARPA DMS-8700897. Permanent address: Institut für Angewandte Mathematik, Universität Hamburg, D-2000 Hamburg 13, Germany

†Supported in part by NSF/DARPA DMS-8700897

‡Research partially supported by a Humboldt award. Permanent address: Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, USA

example, for systems with S^1 symmetry, if the representation on an associated four dmensional symplectic space consists of two complex dual subspaces, then generically the eigenvalues split if the Hamiltonian is indefinite, and they pass if the Hamiltonian is definite. The result is to be contrasted with the bifurcation of steady states (zero eigenvalue) where one can use either group theory alone (Golubitsky and Stewart) or definiteness properties of the Hamiltonian (Cartan-Oh) to determine if the eigenvalues split or pass on the imaginary axis. The results are illustrated for the rotating orthogonal double planar pendulum.

1 Introduction

Hamiltonian vector fields can undergo a variety of generic bifurcations as a single bifurcation parameter is varied. Consider the following two types of local bifurcation from an equilibrium.

1. *Steady-state bifurcation* when the linearized vector field at the equilibrium has a zero eigenvalue of multiplicity two.

2. 1–1 *resonance* when the linearization has a pair of purely imaginary eigenvalues of multiplicity 2.

Without loss of generality, we may assume in 2. that these eigenvalues are $\pm i$.

Let ω denote the symplectic form. In case 1, the kernel of the linearization is a two-dimensional symplectic subspace. As the bifurcation parameter is varied, generically the eigenvalues go from purely imaginary to real (or vice versa). In case 2, the sum of the eigenspaces of the eigenvalues $\pm i$ can be written as the sum of two ω-orthogonal two-dimensional symplectic subspaces. This time, generically the eigenvalues go from purely imaginary into the right and left-hand complex plane (or vice versa). We describe the behaviour of the eigenvalues in each of these cases by saying that the eigenvalues *split*, see Figure 1. The $1-1$ resonance with splitting is often called the Hamiltonian Hopf bifurcation, see [9].

It transpires that in many applications the eigenvalues do not behave in the manner described by the generic theory above. Rather than split at 0 or $\pm i$, the eigenvalues remain on the imaginary axis and *pass*, see Figure 2. However it follows from work of Galin [4], that at least three parameters are required for passing to be expected.

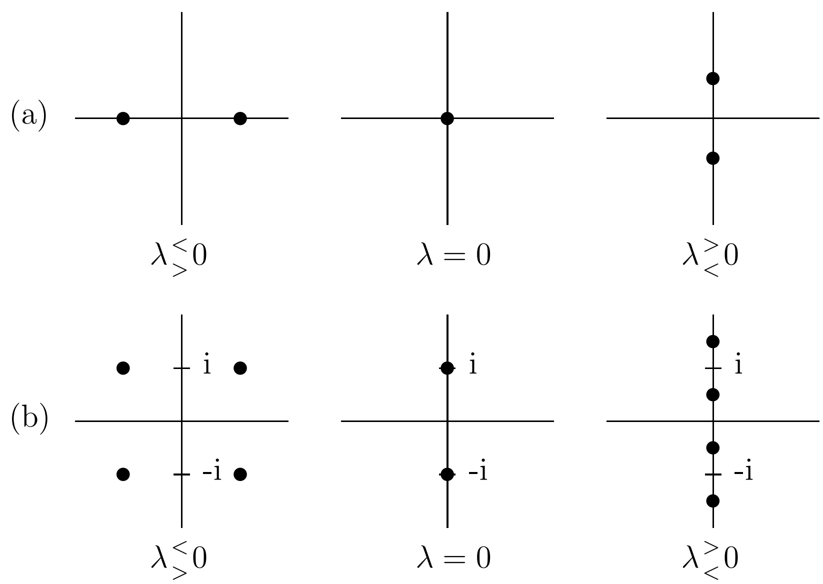

Figure 1: The *splitting* case; (a) for the steady state bifurcation, (b) for the $1-1$ resonance

The reason that passing is seen so often in bifurcations of Hamiltonian vector fields is that in many applications there is symmetry present. As is well known in bifurcation theory (see for example [6]) the presence of symmetry can greatly influence the generic behavior. Indeed, for certain symmetry groups (the most notable example being the circle group S^1), passing of eigenvalues may be generic in a one parameter family.

In the steady state case, the dichotomy in eigenvalue movements can be understood using definiteness properties of the Hamiltonian, a method we shall call

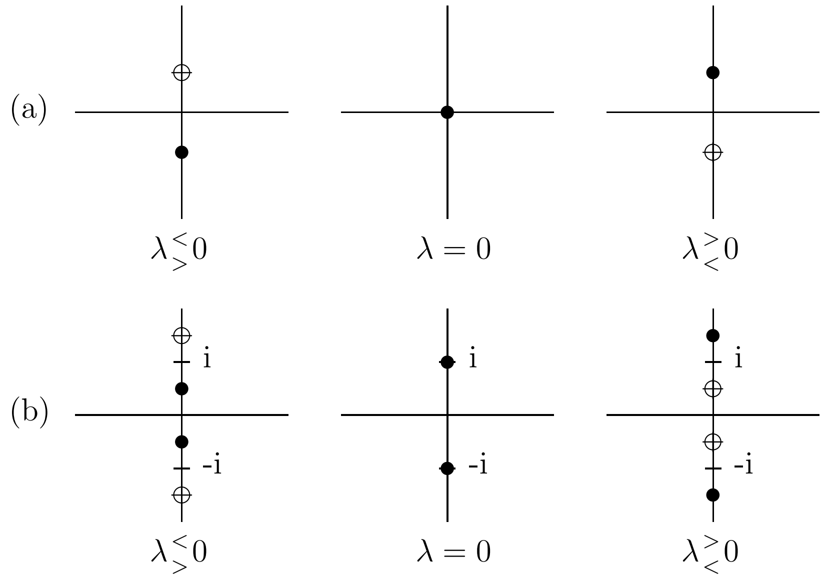

Figure 2: The *passing* case; (a) for the steady state bifurcation, (b) for the $1-1$ resonance

energetics, or group-theoretically (see GOLUBITSKY-STEWART [5]). For the energetics method, see OH [13]. We note that Krein theory uses primarily the energetics approach, but in a way different from that used in this paper. It turns out that energetics or group theory alone is not sufficient to characterize the movement of eigenvalues in the $1-1$ resonance.. One of the main purposes of this paper is to show that a combination of group theory and energetics yields a particularly clean characterization of the splitting and passing cases.

A more basic effect of the symmetry is to force multiplicity of certain eigenvalues, so that the dimensions given above for the various eigenspaces are often invalid even generically. We prove results on the generic structure of the eigenspaces corresponding to the steady-state bifurcation and the $1-1$ resonance (c.f. GOLUBITSKY-STEWART [5, Theorem 3.3] and VAN DER MEER [10, p. 1046]). Assume that the Hamiltonian is invariant under a compact Lie group Γ that preserves the symplectic structure. Theorem 3.2 states that in the case of a steady-state bifurcation, generically the generalized zero eigenspace E_0 is either nonabsolutely Γ-irreducible or the direct sum of two isomorphic absolutely Γ-irreducible subspaces. (A Γ-invariant subspace V is *absolutely* Γ*-irreducible* if the only linear mappings $V \to V$ that commute with the action of Γ are real multiples of the identity. An Γ-irreducible subspace that is not absolutely Γ-irreducible is called *nonabsolutely* Γ*-irreducible*.)

In the case of $1-1$ resonance, Theorem 3.3 states that generically the sum of the generalized eigenspaces of $\pm i$, $E_{\pm i}$, can be written as the sum of two symplectic ω-orthogonal subspaces U_1 and U_2, where each of the U_j is either nonabsolutely Γ-irreducible or the direct sum of two isomorphic absolutely Γ-irreducible subspaces.

Although neither of these results are new, this is the first time that a complete proof has been given. (The proof of the first result in [5] contains nontrivial gaps, and the second result is stated but not proved in [10].)

Our main result, Theorem 4.4, is concerned with the generic movement of eigenvalues in $1-1$ resonance with symmetry. The steady-state bifurcation is well understood both group-theoretically ([5]) and in terms of energetics ([13]). We combine these results in Theorem 4.1. Recall that E_0 is generically either the direct sum of two absolutely irreducible subspaces or is nonabsolutely irreducible. These possibilities correspond precisely to the splitting or passing of eigenvalues. On the other hand, the linearization of the vector field induces a quadratic form on E_0. This quadratic form changes from definite to indefinite in the splitting case, but remains definite in the passing case.

The movement of eigenvalues in the $1-1$ resonance is rather more delicate. The results are summarized in Theorem 4.4. We show that it is necessary to combine the group-theoretic and energetic approaches in order to characterize the dichotomy, splitting or passing, in the eigenvalue movements. The interesting cases are when U_1 and U_2 are isomorphic. If U_1 and U_2 carry distinct representations of Γ then the resonance decouples and the eigenvalues move independently along the imaginary axis.

In order to understand the cases where U_1 and U_2 are isomorphic, we make use of results of MONTALDI-ROBERTS-STEWART [11] on the interrelationship between the symmetric and the symplectic structure. At this stage it becomes necessary to distinguish between the two types of nonabsolutely Γ-irreducible representations: *complex* and *quaternionic*. Provided U_1 and U_2 are not complex irreducibles, generically the eigenvalues split. If the U_j are isomorphic complex irreducibles, then in the terminology of [11] they are either of the *same type* or *dual*. If U_1 and U_2 are complex of the same type, then generically the eigenvalues pass. Finally, in the case of complex duals the eigenvalues can generically pass or split and these possibilities correspond precisely to definiteness and indefiniteness of the quadratic form induced on $U_1 \oplus U_2$ by the linearization.

The paper is organized as follows. First, in Section 2 we review the nonsymmetric case. Using the Galin normal forms listed in [4], it is easy to verify that splitting is generic in the steady-state bifurcation and in the $1-1$ resonance. We also describe the energetic viewpoint in this context.

In Section 3 we formulate results on the generic (group-theoretic) structure of E_0 and $E_{\pm i}$ for a one-parameter family of linear Hamiltonian vector fields. Then, in Section 4 we give group-theoretic and energetic characterizations of the movement of eigenvalues in the steady-state bifurcation. We also state and prove our main theorem, where we combine the group-theoretic and energetic methods to give a complete characterization of eigenvalue movements in the $1-1$ resonance. Finally, in Section 5, we consider an example, namely a rotating orthogonal planar double pendulum.

2 The nonsymmetric case

In this section, we review the situation when there is no symmetry present. The results follow easily from work of GALIN [4]. First, the codimension formula of GALIN (see also ARNOLD [2, Appendix 6]) implies that in a one-parameter family, associated to each eigenvalue is precisely one Jordan block of dimension at most two. Since zero eigenvalues of symplectic matrices have even multiplicity, it follows that in the steady-state bifurcation, generically $\dim E_0 = 2$ and the restriction of the linearization is nilpotent. In the $1-1$ resonance, by definition $\dim E_{\pm i} \geq 4$, so generically this dimension is precisely four. Again, the restriction of the linearization is nilpotent.

Let $A(\lambda)$ denote a one-parameter family of linear Hamiltonian vector fields undergoing one of the above bifurcations at $\lambda = 0$. In each case we can write $A(\lambda)$ in Galin normal form and explicitly compute the eigenvalues. The relevant normal forms in [4] are (36) and (35) respectively.

In the steady-state bifurcation, the Galin normal form of the linearized vector field is

$$A(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} .$$

The eigenvalues are given by $\pm\sqrt{\lambda}$, so as λ increases through zero the eigenvalues move together along the imaginary axis and split onto the real axis.

In the $1-1$ resonance, the Galin normal form is

$$A(\lambda) = \begin{pmatrix} 0 & -1 & \rho & 0 \\ 1 & 0 & 0 & \rho \\ \lambda & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ,$$

where $\rho = \pm 1$. This time a computation yields the eigenvalues

$$\pm\sqrt{\frac{1}{2}\left\{-(\rho\lambda + 2) \pm \sqrt{\lambda(\lambda + 8\rho)}\right\}}.$$

In particular, for λ close to 0, the eigenvalues are purely imaginary precisely when the expression $\lambda(\lambda + 8\rho)$ is positive. Thus the eigenvalues split as required.

We now give a description in terms of energetics. A symplectic linear map A on a symplectic space Z induces a quadratic form Q on Z via the formula

$$Q(z) = \omega(z, Az) \ . \tag{2.1}$$

Moreover $Q(z) = \langle z, JAz\rangle$ so that the quadratic form is represented by the symmetric matrix $B = JA$.

Generalized eigenspaces are symplectic ([14]) and so we may speak of the quadratic form Q induced on E_0 or $E_{\pm i}$ by $A(0)$. In the steady state case we will denote by $Q(\lambda)$ the quadratic form which is induced on the generalized eigenspaces of the eigenvalues going through 0 for $\lambda = 0$. Note that $Q(\lambda)$ is degenerate if and only if $A(\lambda)$ has a zero eigenvalue. In particular, in the case of $1-1$ resonance, Q is nondegenerate. In the steady-state bifurcation, $Q(0)$ is degenerate, but $Q(\lambda)$ is nondegenerate for λ close but not equal to zero.

The following 'stability' theorem is a basic part of Krein theory, see KREIN [8] and MOSER [12].

Theorem 2.1 (Krein) *Suppose that A is a symplectic matrix defined on a symplectic vector space Z. Let Q be the quadratic form induced on Z by A. If Q is definite, then A is semisimple and the eigenvalues of A lie on the imaginary axis.*

Suppose that $A(\lambda)$ undergoes a steady-state bifurcation with $\dim E_0 = 2$. Then definiteness or indefiniteness of the quadratic form $Q(\lambda)$ is governed by the sign (positive or negative) of $\det B(\lambda)$. But in canonical coordinates, $\det J = 1$ so that $\det B(\lambda) = \det A(\lambda)$. It follows that definiteness corresponds to purely imaginary eigenvalues and indefiniteness to real ei genvalues. Thus we have proved the following result.

Theorem 2.2 *Suppose that a Hamiltonian system undergoes a steady-state bifurcation. Let $Q(\lambda)$ denote the quadratic form induced on the corresponding generalized eigenspaces via equation (2.1). Then generically $\dim E_0 = 2$, and the eigenvalues move together along the imaginary axis and then split along the real axis (or vice versa). Simultaneously, the quadratic form Q changes from definite to indefinite (or vice versa).*

Theorem 2.3 *Suppose that a Hamiltonian system undergoes a $1-1$ resonance. Let Q denote the quadratic form induced on $E_{\pm i}$ via equation (2.1). Generically* $\dim E_{\pm i} = 4$*, Q is indefinite, and we have the splitting case.*

Proof It only remains to show that Q is indefinite. But if Q were definite, then by the stability theorem the eigenvalues would be constrained to lie on the imaginary axis and could not split. ∎

3 The generic structure of eigenspaces

In this section, we describe the group-theoretic structure of the generalized eigenspaces E_0 and $E_{\pm i}$ in a generic one-parameter family of linear Hamiltonian vector fields with symmetry. In subsection 3.1 we state the main results of this section. These results are proved in subsection 3.2.

3.1 Statement of results

Let Z be a symplectic vector space with symplectic form ω. Assume that a compact Lie group Γ is acting symplectically on Z, that is,

$$\omega(\gamma v, \gamma w) = \omega(v, w) \quad \forall \gamma \in \Gamma\,; \quad v, w \in Z\,. \tag{3.1}$$

Let $\mathbf{sp}_\Gamma(Z)$ denote the Lie algebra of linear infinitessimally symplectic maps commuting with Γ:

$$B \in \mathbf{sp}_\Gamma(Z) \Longleftrightarrow \begin{cases} (i) & B : Z \to Z \text{ is linear}\,, \\ (ii) & \omega(Bv, w) + \omega(v, Bw) = 0 \quad \forall v, w \in Z\,, \\ (iii) & \gamma B = B\gamma \quad \forall \gamma \in \Gamma\,. \end{cases}$$

Suppose that A is an element of $\mathbf{sp}_\Gamma(Z)$. Let E_0 and $E_{\pm i}$ denote the generalized eigenspaces of A corresponding to the eigenvalues 0 and $\pm i$ respectively.

In this paper we are primarily interested in the behavior associated with A that is generic, or to be expected, in a one-parameter family. However, the generic behavior is nontrivial even when there are no parameters. Of course a zero eigenvalue may be perturbed away so generically $E_0 = 0$. The generic situation for $E_{\pm i}$ is more complicated because purely imaginary eigenvalues occur generically in the context of Hamiltonian systems. Moreover, these eigenvalues may generically have multiplicities forced by Γ-equivariance. Now generically we still have that $E_{\pm i} = 0$ since we can simply scale the eigenvalues along the imaginary axis. However it is convenient to disregard such scalings, since we can always normalize and bring the eigenvalues back to $\pm i$. In this framework, it is generically possible that $E_{\pm i}$ is nontrivial.

Theorem 3.1 *Suppose that A has an eigenvalue i. Then, disregarding the possibility of scaling the eigenvalue, generically either*

(a) *$E_{\pm i}$ is nonabsolutely Γ-irreducible, or*

(b) *$E_{\pm i} = V \oplus V$, V absolutely Γ-irreducible.*

Now we can state our results for one-parameter families. In this case it is possible to have zero eigenvalues or *resonant* purely imaginary eigenvalues. Purely imaginary eigenvalues are in resonance when $E_{\pm i}$ does not have one of the forms listed in Theorem 3.1.

Theorem 3.2 *Suppose that A has a zero eigenvalue. Generically in a one-parameter family, either*

(a) *E_0 is nonabsolutely Γ-irreducible, or*

(b) *$E_0 = V \oplus V$, V absolutely Γ-irreducible.*

Theorem 3.3 *Suppose that A has a resonant eigenvalue i. Generically in a one-parameter family, $E_{\pm i} = U_1 \oplus U_2$ where for $j = 1, 2$ either*

(a) *U_j is nonabsolutely Γ-irreducible, or*

(b) $U_j = V \oplus V$, V *absolutely* Γ*-irreducible.*

Remark 3.4 **(a)** *In Theorem 3.1 the generalized eigenspace* $E_{\pm i}$ *is symplectic, that is* $\omega|_{E_{\pm i}}$ *is nondegenerate (see Proposition 3.7 below). Similarly* E_0 *is symplectic in Theorem 3.2. In Theorem 3.3 the subspaces* U_1 *and* U_2 *may be chosen to be symplectic and also to be* ω*-orthogonal. Recall that two subspaces* U_1 *and* U_2 *are* ω-orthogonal *if* $\omega(u_1, u_2) = 0$ *for all* $u_1 \in U_1$ *and* $u_2 \in U_2$*.*
(b) *When there is no symmetry present, we may take* Γ *to be the trivial group. The irreducible representations are absolutely irreducible and one-dimensional. Hence we recover the expected dimensions of the relevant generalized eigenspaces, as described in Section 2.*

3.2 Proofs

3.2.1 Preliminaries

Let $<,>$ be a Γ-invariant inner product on Z. We may define a linear map $J : Z \to Z$ uniquely by

$$\omega(v, w) = < v, Jw > \quad \text{for all } v, w \in Z. \tag{3.2}$$

Then J is an isomorphism that commutes with Γ and is skew-symmetric, that is, $J^T = -J$. Conversely, given such a J, we may use equation (3.2) to define a symplectic form ω that satisfies (3.1).

A Γ-invariant subspace $W \subset Z$ that is symplectic is called Γ-*symplectic*. The restricted symplectic form $\omega|_W$ induces an isomorphism $J_W : W \to W$. Note that J_W is not the same as $J|_W$. Indeed, J will not in general leave the subspace W invariant. Finally, we recall that a subspace $W \subset Z$ is *isotropic* if $\omega(w_1, w_2) = 0$ for all $w_1, w_2 \in W$.

We now state several basic results from Golubitsky-Stewart [5]. The first four results are Theorem 2.1, Proposition 2.3, Proposition 3.1 and Lemma 2.4(b) of that paper. The fifth result is implicit in the proof of [5, Theorem 2.1], though our choice of JV is more constructive.

Proposition 3.5 *Z has the decomposition into Γ-symplectic ω-orthogonal subspaces*

$$Z = \bigoplus U_i \tag{3.3}$$

Each U_i is either nonabsolutely irreducible or has the form $V \oplus V$ where V is absolutely irreducible.

Proposition 3.6 *If $E \subset Z$ is Γ-symplectic, then there is an ω-orthogonal, Γ-symplectic complement F so that $Z = E \oplus F$,*

Proposition 3.7 *The generalized eigenspaces of a linear map $A \in \mathbf{sp}_\Gamma(Z)$ are Γ-symplectic.*

Proposition 3.8 *If V is Γ-irreducible, then V is either isotropic or symplectic.*

Proposition 3.9 *(a) If V is isotropic, then $V \cap JV = \{0\}$ and $V \oplus JV$ is Γ-symplectic.*

(b) If V is absolutely Γ-irreducible, then V is isotropic.

3.2.2 Proof of the Theorems

We shall begin with the proof of Theorem 3.2. Then the proofs of Theorem 3.1 and Theorem 3.3 are similar. Almost all of this can be done in a coordinate-free setting. However, a computation is required to exclude certain possibilities in Theorems 3.2 and 3.1.

Lemma 3.10 *Let $A(\lambda)$ be a one-parameter family in $\mathbf{sp}_\Gamma(Z)$ such that $A = A(0)$ has a zero eigenvalue. Generically either*

(a) *$E_0 = V$, or*

(b) $E_0 = V \oplus J_U V$

where V is a Γ-irreducible subspace of $\ker A$.

Proof Since the kernel of A is Γ-invariant it possesses a Γ-irreducible subspace V. If V is symplectic, set $U = V$. Otherwise, V is isotropic and we take $U = V \oplus JV$, (where $J = J_{E_0}$). In each case U is Γ-symplectic and has an ω-orthogonal, Γ-symplectic complement Y in E_0 (cf. Propositions 3.7 and 3.6). Also, E_0 has an orthogonal symplectic complement Z_0 in Z. In symbols we have

$$Z = E_0 \oplus Z_0, \quad E_0 = U \oplus Y.$$

If $Y = 0$, there is nothing to do. Otherwise, define $B \in \mathbf{sp}_\Gamma(Z)$ in block-diagonal form as follows:

$$B|_U = B|_{Z_0} = 0, \quad B|_Y = J_Y.$$

Set $A_\epsilon = A + \epsilon B$. Clearly, we have

$$E_0(A_\epsilon) \subset E_0, \quad V \subset \ker A_\epsilon.$$

We claim that $E_0(A_\epsilon)$ is a proper subspace of E_0 for ϵ in a full deleted neighborhood of zero. If the claim is true then we may proceed inductively until $Y = 0$ thus proving the Lemma. Note that V is fixed throughout the induction, but in general U may vary since $J = J_{E_0}$ depends on E_0,

It remains to verify the claim. Choose a nonzero vector $y \in Y$. It is sufficient to show that y is not a generalized eigenvector corresponding to the eigenvalue 0 for all ϵ in a deleted neighborhood of the origin. Suppose for contradiction that $A_\epsilon^k y = 0$ for infinitely many ϵ, where $k = \dim E_0$ say. Expanding A_ϵ^k, we have

$$(P_{k-1}(\epsilon) + \epsilon^k B^k) y = 0,$$

for infinitely many ϵ, where $P_{k-1}(\epsilon)$ is a matrix valued polynomial of degree $k-1$ in ϵ. Equating components in the vector equation, and using properties of polynomials, we see that equality holds for all ϵ. Moreover, comparing coefficients of ϵ^k, we have

$$B^k y = 0.$$

But $\ker B^k \cap Y = \{0\}$ so we have the required contradiction. ∎

Proof of Theorem 3.2 We must show that V is generically nonabsolutely Γ-irreducible in case (a) of the Lemma and absolutely Γ-irreducible in case (b).

Nonabsolute irreducibility in case (a) is automatic by Proposition 3.9 since $E_0 = V$ is symplectic and hence cannot be absolutely Γ-irreducible. Case (b) follows from a relatively tedious computation, see Remark 4.7. ∎

Proof of Theorem 3.1 The proof is completely analogous to that of the previous Lemma and Theorem. This time we choose a Γ-irreducible subspace V in the eigenspace of $\pm i$ and construct U as before. Now write

$$Z = E_{\pm i} \oplus Z_0, \quad E_{\pm i} = U \oplus Y$$

and replace A_ϵ^k by $[A_\epsilon^2 + I]^k$ in the proof of the previous Lemma.

Again U cannot be absolutely Γ-irreducible. Also it cannot be the direct sum of two nonabsolutely Γ-irreducible subspaces, see Remark 4.7. ∎

Lemma 3.11 *Let $A(\lambda)$ be a one-parameter family in $\mathbf{sp}_\Gamma(Z)$ such that $A = A(0)$ has a resonant eigenvalue i. Generically $E_{\pm i} = U_1 \oplus U_2$ where U_1 and U_2 are symplectic ω-orthogonal subspaces and for $j = 1, 2$ either*

(a) $U_j = V$, *or*

(b) $U_j = V \oplus J_{U_j} V$

where V is a Γ-irreducible subspace of the eigenspace of $\pm i$.

Proof The proof is similar to that of the previous Lemma. This time we choose a Γ-irreducible subspace V in the eigenspace of $\pm i$ and construct U_1 as before. By hypothesis, this is not the full generalized eigenspace, so we may construct a second symplectic subspace U_2. Write

$$Z = E_{\pm i} \oplus Z_0, \quad E_{\pm i} = U_1 \oplus U_2 \oplus Y$$

and as before replace A_ϵ^k by $[A_\epsilon^2 + I]^k$ in the proof of the previous Lemma. ∎

Proof of Theorem 3.3 Again we must show that V must be nonabsolutely Γ-irreducible in case (a) and absolutely Γ-irreducible in case (b).

By construction, the U_j are symplectic, so V must be nonabsolutely Γ-irreducible in case (a). It remains to consider the case $E_{\pm i} = U_1 \oplus U_2$ where U_1, say, is of type $V \oplus V$ and V is nonabsolutely Γ-irreducible. In fact we show that this case reduces to $U \oplus U$ where U is of type (a). Begin by perturbing U_2 away as in the proof of the previous Lemma. Then $E_{\pm i} = U_1$ is the sum of two isomorphic Γ-irreducible subspaces, so every Γ-irreducible subspace of $E_{\pm i}$ is isomorphic to V. By Proposition 3.5 we may write $E_{\pm i} = U \oplus U$ where each copy of U is symplectic and isomorphic to V. Hence this one copy of U of type (b) in the Lemma splits into two isomorphic copies of U of type (a) in the theorem. ∎

4 Movement of eigenvalues

Suppose that $A(\lambda)$ is a one-parameter family of linear Hamiltonian vector fields commuting with the action of a compact Lie group Γ. Suppose further that $A(\lambda)$ undergoes a steady-state bifurcation or $1-1$ resonance at $\lambda = 0$. Theorems 3.2 and 3.3 give the generic structure of the generalized eigenspaces E_0 and $E_{\pm i}$.

When there is no symmetry present, these structures reduce to those described in Section 2. Moreover we were able to determine the generic movement of eigenvalues and to give an energetic description. In particular, the eigenvalues generically split off the imagainary axis in each bifurcation.

When there is symmetry present, it is no longer true that the eigenvalues generically split. In this section, we show that the eigenvalues split off the imaginary axis or pass along the axis. Moreover this movement can be completely characterized in terms of group theory and energetics. In fact, in the steady-state bifurcation it is already known that the movement can be characterized using group theory alone ([5]) or by energetics alone ([13]). We combine these two results in Theorem 4.1 below.

The movement of eigenvalues in the $1-1$ resonance is rather delicate and cannot be characterized by group theory alone or energetics alone. Even the statement of the result (Theorem 4.4 below) requires familiarity with the terminology of Montaldi-Roberts-Stewart [11]. In subsection 4.1 we introduce this terminology and state Theorems 4.1 and Theorem 4.4. We also state some of the results in [11] that we shall require, including an equivariant version of Darboux's Theorem. In subsection 4.2 we prove Theorems 4.1 and 4.4.

4.1 Statement of Results

We begin by stating the combined results of [5] and [13] for the steady-state bifurcation.

Theorem 4.1 *Suppose that the hypotheses of Theorem 3.2 hold. Let $Q(\lambda)$ denote the quadratic form induced on the corresponding generalized eigenspaces via equation (2.1). Generically, precisely one of the following occurs:*

(a) *E_0 is nonabsolutely irreducible, $Q(\lambda)$ is definite for $\lambda \neq 0$, and the eigenvalues pass with nonzero speed.*

(b) *E_0 is the direct sum of two isomorphic absolutely irreducible subspaces, $Q(\lambda)$ changes from definite to indefinite, and the eigenvalues split.*

In order to state the corresponding result for the $1-1$ resonance, it is necessary to recall some terminology and results from Montaldi-Roberts-Stewart [11].

If U is a symplectic representation then — by ignoring the symplectic structure — we obtain an ordinary representation, which is called the *underlying representation.* A Γ-*irreducible symplectic representation* is a representation that has no proper nonzero Γ-invarian t symplectic subspaces. It follows from Proposition 3.5 and Proposition 3.9, part (b), that Γ-irreducible symplectic representations are either nonabsolutely Γ-irreducible or the sum of a pair of isomorphic absolutely Γ-irreducible subspaces. Moreover, the following Theorem holds, which is part of [11, Theorem 2.1].

Theorem 4.2 *(a) In the real and quaternionic cases the isomorphism type of the Γ-irreducible symplectic representation is uniquely determined by that of its underlying representation.*

(b) In the complex case there are precisely two isomorphism types of Γ-irreducible symplectic representations for a given complex Γ-irreducible underlying representation. They are said to be **dual** *to each other.*

According to the two different possibilities occuring in part (b) we will speak of *complex irreducibles of the same type* and *complex duals*.

Remark 4.3 *The real, complex and quaternionic cases mentioned in Theorem 4.2 refer to the following well known fact (see e.g. [6]): Let U be Γ-irreducible and $\mathcal{D}$ be the space of linear mappings $U \to U$ which commute with Γ. Then $\mathcal{D}$ is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, where $\mathbb{H}$ denotes the quaternionics. Moreover, U is absolutely Γ-irreducible if $\mathcal{D} \cong \mathbb{R}$ and nonabsolutely Γ-irreducible if either $\mathcal{D} \cong \mathbb{C}$ or $\mathcal{D} \cong \mathbb{H}$.*

After these preparations we state our main Theorem of this Section.

Theorem 4.4 *Suppose that the hypotheses of Theorem 3.3 hold. Let Q denote the quadratic form induced on $E_{\pm i}$ via equation (2.1). Generically, precisely one of the following occurs:*

(a) *U_1 and U_2 are not isomorphic and the eigenvalues pass independently along the imaginary axis. (Q may be indefinite or definite.)*

(b) *$U_1 = U_2 = V \oplus V$, V real, or $U_1 = U_2 = W$, W quaternionic, the eigenvalues split, and Q is indefinite.*

(c) *U_1 and U_2 are complex of the same type, the eigenvalues pass and Q is indefinite.*

(d) *U_1 and U_2 are complex duals and the eigenvalues pass or split depending on whether Q is definite or indefinite.*

Eigenspace structure	Induced quadratic form	
	definite	indefinite
$V \oplus V \oplus V \oplus V$	not generic	splitting
$W \oplus W$ (quaternionic)	not generic	splitting
$W \oplus W$ (complex of the same type)	not generic	passing
$W \oplus W$ (complex duals)	passing	splitting
$U_1 \oplus U_2$ (nonisomorphic)	'independent passing'	

Table 1: Generic eigenvalue movement

The statement of the last Theorem is roughly summarized in Table 1.

Finally we state two more results of [11]. The first is an equivariant version of Darboux's Theorem, which is implicit in Theorem 2.4 of that paper.

Proposition 4.5 *Suppose that U is a Γ-irreducible symplectic representation. Then, up to isomorphism, there is precisely one symplectic form on U in the real and quaternionic cases and precisely two in the complex case.*

In the terminology of [11] a symplectic representation U is said to be **cyclospectral** if every element of $\mathbf{sp}_\Gamma(U)$ has all its eigenvalues on the imaginary axis. Cyclospectral representations are characterized in the following Theorem.

Theorem 4.6 *A symplectic representation U of Γ is cyclospectral if and only if, in its decomposition (3.3)*

(a) There are no real Γ-irreducible representations.

(b) There are no complex duals.

(c) *There are only pairwise nonisomorphic quaternionic* Γ*-irreducible representations.*

4.2 Proofs

Proof of Theorem 4.1 Suppose that E_0 is nonabsolutely irreducible. Then it follows from Remark 4.3 that $A(\lambda) = a(\lambda)I$ where $a(\lambda) \in \mathbb{C}$ or $\mathbb{H}$ and $a(0) = 0$. It follows (see e.g. [7]) that the eigenvalues of $A(\lambda)$ are the same as the eigenvalues of $a(\lambda)$ repeated with multiplicity equal to $\dim E_0$. By Proposition 4.5 we may choose coordinates so that $J = \pm i$ (since these are candidates for J and are distinct if W is complex). The quadratic form $Q(\lambda)$ is represented by the symmetric real matrix $B(\lambda) = \pm ia(\lambda)I$. It follows that $a(\lambda) = ib(\lambda)$ where $b(\lambda)$ is real. In particular, the eigenvalues of $A(\lambda)$ are purely imaginary. In addition, $a'(0) = ib'(0)$ and $b'(0)$ is generically nonzero, so that the eigenvalues pass through zero with nonzero speed. Finally $B(\lambda)$ is a real scalar multiple of the identity and so is definite for $\lambda \neq 0$ as required.

Now suppose that $E_0 = V \oplus V$ where V is real. By Proposition 4.5 we may choose coordinates so that $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Again by [7] we may work with 2×2 matrices provided we include multiplicities equal to $\dim V$. But then we are back in the case where there is no symmetry and we can apply Theorem 2.2. ∎

Proof of Theorem 4.4 (a) Since U_1 and U_2 are nonisomorphic, there is a corresponding blockdiagonal structure of $A(\lambda)$ on $E_{\pm i}$ corresponding to the decomposition $E_{\pm i} = U_1 \oplus U_2$. Since the eigenvalues on each U_j are simple up to multiplicities forced by symmetry, it follows that the eigenvalues belonging to each block remain on the imaginary axis and behave independently as λ is varied. Similarly, the quadratic forms induced on the U_j separately are definite. Depending on whether they are definite of the same sign or of opposite signs, the quadratic form on $E_{\pm i}$ is definite or indefinite.

(b) The case $U_1 = U_2 = V \oplus V$ reduces to the four-dimensional situation of Theorem

2.3. We turn to the quaternionic case which is more difficult since we do not have a list of normal forms. Once we have verified that the eigenvalues split, it follows that Q is indefinite by Theorem 2.1. For simplicity from now on we suppress multiplicities forced by the dimension of the underlying Γ-irreducible representation. Again, we may choose coordinates so that $J : V \oplus V \to V \oplus V$ has the form

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Consequently, on $E_{\pm i}$ the mapping $A(\lambda)$ has the form

$$A(\lambda) = \begin{pmatrix} 0 & a(\lambda) & u_1(\lambda) + iu_2(\lambda) & v_1(\lambda) + iv_2(\lambda) \\ -a(\lambda) & 0 & -v_1(\lambda) + iv_2(\lambda) & u_1(\lambda) - iu_2(\lambda) \\ -u_1(\lambda) - iu_2(\lambda) & v_1(\lambda) - iv_2(\lambda) & 0 & b(\lambda) \\ -v_1(\lambda) - iv_2(\lambda) & -u_1(\lambda) + iu_2(\lambda) & -b(\lambda) & 0 \end{pmatrix},$$

where a, b, u_1, u_2, v_1, v_2 are real-valued functions. The computation of the eigenvalues of $A(\lambda)$ (using MATHEMATICA) leads to

$$\sigma(\lambda) = \pm \frac{1}{\sqrt{2}} \sqrt{p(\lambda) \pm \sqrt{q(\lambda)}},$$

with

$$\begin{aligned} p &= -\left(a^2 + b^2 + 2(u_1^2 - u_2^2 + v_1^2 - v_2^2)\right), \\ q &= p^2 - 4\left(ab - (u_1^2 + u_2^2 + v_1^2 + v_2^2)\right)^2 \\ &= \left((a-b)^2 + 4(u_1^2 + v_1^2)\right)\left((a+b)^2 - 4(u_2^2 + v_2^2)\right). \end{aligned}$$

By assumption $p(0) = -2$, $q(0) = 0$. Since the first factor of q is the sum of three squares, it is generically the case that the second factor vanishes. Using this we compute at 0 that

$$q' = 4((a+b)^2)' - 32(u_2 u_2' + v_2 v_2')$$

We claim that generically $q'(0) \neq 0$ and we have the splitting case. It is clear that generically $q'(0) \neq 0$ whenever

$$a(0) \neq -b(0) \text{ or } u_2(0) \neq 0 \text{ or } v_2(0) \neq 0.$$

Therefore we consider the matrix

$$A(0) = \begin{pmatrix} 0 & a & u & v \\ -a & 0 & -v & u \\ -u & v & 0 & -a \\ -v & -u & a & 0 \end{pmatrix},$$

and show that this situation is not generic. This matrix has semisimple eigenvalues $\pm i(a^2 + u^2 + v^2)$.

We perturb $A(0)$ in the following way:

$$A_\epsilon(0) = \begin{pmatrix} 0 & a + 2\epsilon & u + i\epsilon & v \\ -a - 2\epsilon & 0 & -v & u - i\epsilon \\ -u - i\epsilon & v & 0 & -a \\ -v & -u + i\epsilon & a & 0 \end{pmatrix},$$

where the eigenvalues σ_ϵ of this perturbed matrix are still purely imaginary, namely

$$\sigma_\epsilon = \pm\sqrt{-((a+\epsilon)^2 + u^2 + v^2)}.$$

This completes the proof of the quaternionic case.

(c) In this case Theorem 4.6 guarantees that the eigenvalues remain on the imaginary axis. By Proposition 4.5 we may choose coordinates so that $J = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. A computation similar to that for the case of complex duals below shows that the eigenvalues generically pass with nonzero speed and that Q is indefinite.

(d) By Theorem 4.2 and Proposition 4.5 we may assume that

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Therefore we have to investigate the eigenvalues of the matrix

$$A(\lambda) = \begin{pmatrix} i\alpha_1(\lambda) & a_2(\lambda) \\ \overline{a_2(\lambda)} & i\alpha_4(\lambda) \end{pmatrix}, \tag{4.1}$$

where a_2 is a complex-valued function and α_1, α_4 are real. These eigenvalues are

$$\sigma(\lambda) = p(\lambda) \pm \sqrt{q(\lambda)},$$

where

$$\begin{aligned} p &= \frac{i}{2}(\alpha_1 + \alpha_4), \\ q &= -\frac{1}{4}(\alpha_1 - \alpha_4)^2 + |a_2|^2. \end{aligned}$$

The eigenvalues of the matrix $B = JA(0)$ are

$$\frac{1}{2}\left((\alpha_4 - \alpha_1) \pm \sqrt{(\alpha_1 + \alpha_4)^2 + 4|a_2|^2}\right) .$$

By assumption there are exactly two possibilities:

(i) $p(0) = 0$, $q(0) = -1$:
In this case the eigenvalues of B are given by $-\alpha_1 \pm \sqrt{\alpha_1^2 - 1}$ and B is definite. Hence the eigenvalues of $A(\lambda)$ remain on the imaginary axis and pass with nonzero speed provided $p'(0) = \frac{i}{2}(\alpha_1'(0) + \alpha_2'(0)) \neq 0$.

(ii) $|p(0)| = 1$, $q(0) = 0$.
We claim that in this case generically $q'(0) \neq 0$ and therefore we have the splitting case. We compute

$$q'(0) = -\frac{1}{2}(\alpha_1(0) - \alpha_4(0))(\alpha_1'(0) - \alpha_4'(0)) + 2|a_2(0)||a_2|'(0)$$

and the eigenvalues are generically splitting at $\pm i$ as long as

$$\alpha_1(0) \neq \alpha_4(0) \text{ or } a_2(0) \neq 0.$$

But the situation $\alpha_1 = \alpha_4 = 1$, $a_2(0) = 0$ can be perturbed to

$$A_\epsilon(0) = \begin{pmatrix} i(1+\epsilon) & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & i \end{pmatrix}$$

since $A_\epsilon(0)$ still has the eigenvalue $i(1+\frac{\epsilon}{2})$. Finally B is indefinite since the eigenvalues split.

Remark 4.7 *It is easily seen from the proof of Theorem 4.4 that it is not generic in a one-parameter family for $A(0)$ to have only zero eigenvalues on the space $W \oplus W$ if W is complex or quaternionic Γ-irreducible. To see this one only has to consider the eigenvalues of $A(0)$ for these cases while setting its determinant equal to zero. This is the computation that was required to complete the proof of Theorem 3.2. Similarly it is not generic for a matrix $A \in \mathbf{sp}_\Gamma(Z)$ to have only eigenvalues $\pm i$ on the space $W \oplus W$, as required in Theorem 3.1.*

Example 4.8 *a) We consider a symplectic $\Gamma \cong S^1 \times S^1$-action on $\mathbb{C}^2$, namely*

$$(\theta, \phi)(z_1, z_2) = (e^{i\theta} z_1, e^{i\phi} z_2),$$

where the symplectic form ω is defined by $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Both copies of $\mathbb{C}$ are nonisomorphic Γ-irreducible and therefore we know that always the passing case has to occur.

For example, the Hamiltonian

$$H(z_1, z_2, \lambda) = \frac{1}{2}\left(\lambda |z_1|^2 - |z_2|^2\right),$$

which in real coordinates takes the form

$$H(p_1, p_2, q_1, q_2, \lambda) = \frac{1}{2}\left(\lambda(p_1^2 + q_1^2) - (p_2^2 + q_2^2)\right),$$

is invariant under this action and $A(\lambda)$ *becomes*

$$A(\lambda) = \begin{pmatrix} i\lambda & 0 \\ 0 & i \end{pmatrix}.$$

Independent passing in a $1-1$*-resonance occurs for* $\lambda = \pm 1$ *whereas we have passing in the steady state bifurcation case for* $\lambda = 0$.

Observe that H *is also invariant under the action of the transformations*

$$z_1 \longrightarrow \bar{z}_1\,, \quad z_2 \longrightarrow \bar{z}_2,$$

but these transformations do not commute with J *and therefore lead to non-symplectic actions.*

b) Again we consider the space $\mathbb{C}^2$*, where now the symplectic form* ω *is induced by* $J = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$*. A symplectic* $\Gamma \cong S^1$*-action on* $\mathbb{C}^2$ *is given by*

$$\theta(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2).$$

Since both copies of $\mathbb{C}$ *are complex irreducible of the same type, we have passing, generically with nonzero speed.*

As an example we consider the Γ*-invariant Hamiltonian*

$$H(z_1, z_2, \lambda) = \lambda \left[\frac{1}{2} |z_1|^2 - \mathrm{Im}(z_1 \bar{z}_2) \right] + \mathrm{Re}(z_1 \bar{z}_2),$$

which in real coordinates becomes

$$H(p_1, p_2, q_1, q_2, \lambda) = \lambda \left[\frac{1}{2} \left(p_1^2 + q_1^2 \right) + (p_1 q_2 - p_2 q_1) \right] + (p_1 p_2 + q_1 q_2).$$

Here $A(\lambda)$ *has the form*

$$A(\lambda) = \begin{pmatrix} i\lambda & \lambda + i \\ -\lambda + i & 0 \end{pmatrix}$$

and the eigenvalues of $A(\lambda)$ *are* $i\frac{\lambda}{2} \pm \sqrt{-(\lambda^2 + \frac{5}{4})}$*. A* $1-1$ *resonance occurs for* $\lambda = 0$ *and the eigenvalues pass with nonzero speed.*

c) We take the canonical symplectic structure on $\mathbb{R}^8$ *and consider a symplectic action of* $\mathbf{O}(2)$ *on* $\mathbb{R}^8$*, where* $\mathbf{O}(2)$ *is generated by*

$$R_\varphi = \begin{pmatrix} \cos(\varphi)I & -\sin(\varphi)I & 0 & 0 \\ \sin(\varphi)I & \cos(\varphi)I & 0 & 0 \\ 0 & 0 & \cos(\varphi)I & -\sin(\varphi)I \\ 0 & 0 & \sin(\varphi)I & \cos(\varphi)I \end{pmatrix}, \ \kappa = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}$$

with $I \in \mathbb{R}^{2,2}$.

Using the canonical basis in $\mathbb{R}^8$*, we define the isomorphic* Γ*-irreducible subspaces*

$$V_{1,1} = \mathbb{R}\{e_1, e_3\}, \quad V_{1,2} = \mathbb{R}\{e_5, e_7\},$$
$$V_{2,1} = \mathbb{R}\{e_2, e_4\}, \quad V_{2,2} = \mathbb{R}\{e_6, e_8\},$$

and obtain the decomposition

$$\mathbb{R}^8 = U_1 \oplus U_2, \tag{4.2}$$

where

$$U_1 = V_{1,1} \oplus V_{1,2}\,, \quad U_2 = V_{2,1} \oplus V_{2,2}$$

are Γ*-irreducible symplectic representations. Therefore we know by Theorem 4.4 that generically the splitting cases occur – with an* 8*-dimensional corresponding generalized eigenspace in the* $1-1$*-resonance and a* 4*-dimensional generalized eigenspace in the steady state bifurcation.*

For example, the Hamiltonian

$$H(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4, \lambda) = \lambda \left[\frac{1}{2}(p_1^2 + p_3^2) - (p_1 q_1 + p_3 q_3)\right] -$$
$$- \left[\frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2) + (p_1 q_2 - p_2 q_1) + (p_3 q_4 - p_4 q_3)\right]$$

is invariant under this $\mathbf{O}(2)$*-action and one computes using the coordinates given by the decomposition (4.2)*

$$A(\lambda) = \begin{pmatrix} \lambda I & -I & I & 0 \\ I & 0 & 0 & I \\ \lambda I & 0 & -\lambda I & -I \\ 0 & 0 & I & 0 \end{pmatrix}.$$

Therefore the eigenvalues of $A(\lambda)$ *are*

$$\frac{1}{\sqrt{2}}\left(\pm\sqrt{-2+\lambda+\lambda^2 \pm \sqrt{-8\lambda-3\lambda^2+2\lambda^3+\lambda^4}}\right)$$

and for $\lambda = 0$ *the splitting case in the* $1-1$*-resonance occurs.*

5 An example: The rotating orthogonal planar double pendulum

This example is due to Bridges [3]: We consider a rotating orthogonal planar double pendulum as illustrated in Figure 3.

The angular velocity of the rotation is assumed to be Ω_*. The two masses m_1, m_2 are forced to move in two planes, which are orthogonal to each other. In contrast to the treatment in Bridges [3] we immediately restrict to the case where the two pendulums have equal length, because this will lead to an S^1-symmetry in the problem as we will see down below.

We set

$$\Omega^2 = \frac{\Omega_*^2}{g}\,, \quad m = \frac{m_1}{m_2}\,.$$

Ω will be our bifurcation parameter. Scaling time by $\sqrt{1/g}$ we obtain the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}(m+1)[\dot\theta_1^2 + \Omega^2 \sin^2\theta_1] + \frac{1}{2}[\dot\theta_2^2 + \Omega^2\sin^2\theta_2] \\ &\quad +\dot\theta_1\dot\theta_2 \sin\theta_1\sin\theta_2 + \Omega[\sin\theta_1\cos\theta_2\dot\theta_2 - \cos\theta_1\sin\theta_2\dot\theta_1] \\ &\quad +(m+1)(\cos\theta_1 - 1) + (\cos\theta_2 - 1)\,. \end{aligned}$$

Figure 3: The rotating orthogonal double pendulum

Let

$$\begin{aligned}\phi_1 &= \frac{\partial L}{\partial \dot{\theta}_1} = (m+1)\dot{\theta}_1 + \dot{\theta}_2 \sin\theta_1 \sin\theta_2 - \Omega\cos\theta_1\sin\theta_2 \,, \\ \phi_2 &= \frac{\partial L}{\partial \dot{\theta}_2} = \dot{\theta}_2 + \dot{\theta}_1 \sin\theta_1 \sin\theta_2 + \Omega\sin\theta_1\cos\theta_2\end{aligned}$$

be the conjugate momenta.

The corresponding Hamiltonian $H(\theta_1, \theta_2, \phi_1, \phi_2, \Omega)$ has a Taylor expansion of the following form:

$$H(\theta_1, \theta_2, \phi_1, \phi_2, \Omega) = H_2(\theta_1, \theta_2, \phi_1, \phi_2, \Omega) + H_4(\theta_1, \theta_2, \phi_1, \phi_2, \Omega) + \dots \,,$$

since all terms of odd order have to vanish as a consequence of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry of the spinning orthogonal double pendulum. The representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in $\mathbb{R}^4$

is generated by the transformations (with respect to $(\theta_1, \theta_2, \phi_1, \phi_2)$)

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.1}$$

Observe that this action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not symplectic (cf. (3.1)).

The expression for H_2 is (cf. Bridges [3])

$$\begin{aligned} H_2(\theta_1, \theta_2, \phi_1, \phi_2, \Omega) &= \frac{1}{2(m+1)}\phi_1^2 + \frac{1}{2}\phi_2^2 + \Omega(\frac{1}{m+1}\phi_1\theta_2 - \theta_1\phi_2) \\ &\quad + \frac{1}{2}(m+1)(1 - \frac{m}{m+1}\Omega^2)\theta_1^2 + \frac{1}{2}(1 - \frac{m}{m+1}\Omega^2)\theta_2^2 \,. \end{aligned}$$

Changing coordinates by

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\sqrt{m+1} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{1}{\sqrt{m+1}} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \phi_1 \\ \phi_2 \end{pmatrix}$$

we obtain

$$\begin{aligned} H_2(p_1, p_2, q_1, q_2, \Omega) &= \frac{1}{2(m+1)}(q_1^2 + q_2^2) - \frac{\Omega}{\sqrt{m+1}}(p_1q_2 - p_2q_1) \\ &\quad + \frac{1}{2}(m+1)(1 - \frac{m}{m+1}\Omega^2)(p_1^2 + p_2^2) \,. \end{aligned}$$

Now we can easily see that H_2 is also S^1-invariant, where S^1 is acting symplectically on (p_1, p_2, q_1, q_2) by

$$\begin{pmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix}, \quad \beta \in [0, 2\pi) \,. \tag{5.2}$$

A computation of $A_\Omega = JD^2H_2(0,0,0,0,\Omega)$ yields

$$A_\Omega = \begin{pmatrix} 0 & -\frac{\Omega}{\sqrt{m+1}} & -\frac{1}{m+1} & 0 \\ \frac{\Omega}{\sqrt{m+1}} & 0 & 0 & -\frac{1}{m+1} \\ m+1-m\Omega^2 & 0 & 0 & -\frac{\Omega}{\sqrt{m+1}} \\ 0 & m+1-m\Omega^2 & \frac{\Omega}{\sqrt{m+1}} & 0 \end{pmatrix}. \tag{5.3}$$

Hence $A_\Omega \in \mathbf{sp}_\Gamma(\mathbb{R}^4)$, where the elements of $\Gamma \cong S^1$ are given by (5.2). Since the action of $\mathbb{Z}_2\times\mathbb{Z}_2$ as defined in (5.1) is not symplectic, A_Ω is not $\mathbb{Z}_2\times\mathbb{Z}_2$-equivariant.

Computing the eigenvalues $\lambda(\Omega)$ of A_Ω we obtain

$$\lambda^2(\Omega) = -(1+\frac{1-m}{1+m}\Omega^2) \pm 2\frac{\Omega}{\sqrt{m+1}}\sqrt{1-\frac{m}{m+1}\Omega^2}\,.$$

Furthermore

$$\frac{d\lambda^2}{d\Omega}(0) = \pm 2\sqrt{\frac{1}{m+1}} \neq 0\,.$$

Therefore the eigenvalues pass on the imaginary axis for $\Omega = 0$ with nonzero speed. As pointed out in Bridges [3], also a Hamiltonian Hopf bifurcation occurs for $\Omega = \pm\sqrt{\frac{m+1}{m}}$. Hence we can conclude that case (c) of Theorem 4.4 is underlying in this example. The corresponding eigenspace $E_{\pm i}$ can be decomposed into two complex duals.

Remark 5.1 *In (5.3) A_Ω has not the same structure as $A(\lambda)$ in the proof of part (c) of Theorem 4.4 (see (4.1)). The reason for this is that the spaces $\mathbb{R}\{e_1,e_2\}$ and $\mathbb{R}\{e_3,e_4\}$ are not ω-orthogonal ($(e_1,\dots,e_4)$ denotes the canonical basis of $\mathbb{R}^4$).*

In order to obtain that structure we set

$$W_1 = \mathbb{R}\{e_1-e_4, e_2+e_3\}, \quad W_2 = \mathbb{R}\{e_1+e_4, e_2-e_3\}\,.$$

Then it follows that

$$\mathbb{R}^4 = W_1 \oplus W_2\,,$$

W_1 and W_2 are complex duals, ω-orthogonal and Γ acts by the diagonal action on $W_1 \oplus W_2$.

Correspondingly in these new coordinates

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

as in the proof of Theorem 4.4.

Finally, with

$$\begin{aligned} u(\Omega) &= \frac{1}{2}\left(\frac{2\Omega}{\sqrt{m+1}} - \frac{1}{m+1} - m - 1 + m\Omega^2\right), \\ v(\Omega) &= \frac{1}{2}\left(-\frac{1}{m+1} + m + 1 - m\Omega^2\right), \\ w(\Omega) &= \frac{1}{2}\left(\frac{2\Omega}{\sqrt{m+1}} + \frac{1}{m+1} + m + 1 - m\Omega^2\right) \end{aligned}$$

we obtain after a short calculation that A_Ω now takes the form

$$A_\Omega = \begin{pmatrix} 0 & u(\Omega) & 0 & -v(\Omega) \\ -u(\Omega) & 0 & v(\Omega) & 0 \\ 0 & v(\Omega) & 0 & w(\Omega) \\ -v(\Omega) & 0 & -w(\Omega) & 0 \end{pmatrix}$$

as desired.

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