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Stabilization of Rigid Body Dynamics by Internal and External Torques*

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The energy-Casimir method from geometric mechanics yields specific internal torque feedback laws that stabilize the dynamics of a rigid body about an otherwise unstable axis and attitude phase drifts can be computed for nearby motions.

Key Words—Stabilizers; feedback control; satellite control; attitude control; dynamic stability.

Abstract—In this paper we discuss the stabilization of the rigid body dynamics by external torques (gas jets) and internal torques (momentum wheels). Our starting point is a generalization of the stabilizing quadratic feedback law for a single external torque recently analyzed in Bloch and Marsden [*Proc.* 27th *IEEE Conf. Dec. and Con.*, pp. 2238–2242 (1989b); *Sys. Con. Letts.*, **14**, 341–346 (1990)] with quadratic feedback torques for internal rotors. We show that with such torques, the equations for the rigid body with momentum wheels are Hamiltonian with respect to a Lie–Poisson bracket structure. Further, these equations are shown to generalize the dual-spin equations analyzed by Krishnaprasad [*Nonlin. Ana. Theory Methods and App.*, **9**, 1011–1035 (1985)] and Sánchez de Alvarez [Ph.D. Diss. (1986)]. We establish stabilization with a single rotor by using the energy-Casimir method. We also show how to realize the external torque feedback equations using internal torques. Finally, extending some work of Montgomery [*Am. J. Phys.*, **59**, 394–398 (1990)], we derive a formula for the attitude drift for the rigid body–rotor system when it is perturbed away from a stable equilibrium and we indicate how to compensate for this.

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1. INTRODUCTION

THE PROBLEM of stabilization of the rigid body and systems of rigid bodies is of importance for numerous practical applications. There has been much work recently on stabilizing the angular momentum equations and attitude equations of the rigid body with $n \geq 2$ torques. Work in this area includes that of Baillieul (1981), Bonnard (1986), Brockett (1976, 1983), Crouch (1986), Aeyels (1985a,b), Krishnaprasad (1985), Sánchez de Alvarez (1986, 1989), Aeyels and Szafranski (1988), Sontag and Sussman (1988), and Byrnes and Isidori (1989).

A related area where there has also been much progress, is the problem of analyzing the *stability* of coupled rigid and flexible bodies. A method of analysis based on ideas of geometric mechanics for these systems (including dissipation) was introduced in Krishnaprasad (1985), Krishnaprasad and Marsden (1987), and Baillieul and Levi (1987). In particular, stability of a rigid body with flexible attachment was analyzed in the body frame by the energy-Casimir method developed by Arnold (1966) and further developed by Holm *et al.* (1985). (The method has a long history, special cases of which can be found already in the last century in the work of Riemann (1860) and Poincaré (1885) on gravitating fluid masses.) In this method, energy and momentum are used together to prove Lyapunov stability. More recently, a variant of this method called the *energy–momentum method,* where the stability analysis is in the material representation, has been developed. See, for example, Simo *et al.* (1990, 1991).

Bloch and Marsden (1989b, 1990) showed that the energy-Casimir method could be used to prove a stabilization result, namely that the angular momentum equations of the rigid body can be stabilized about the intermediate axis of inertia by a single external torque applied about the major or minor axis. Moreover, they showed that there was an interesting Lie–Poisson structure associated with the system. This gave a quite different feedback law and method of analysis for a closely related result originally due to Aeyels (1985).

A striking feature of the analysis is that there are still conserved quantities, *even while torque is being applied.* This leads one to conjecture that the feedback might be realizable as an *internal* torque between the rigid body and an attached rigid body or bodies. We show in this paper that the externally stabilized feedback system may be realized as a rigid body with three internal rotors. Our analysis is based on the original analysis of this system by Krishnaprasad (1985) (see also Sánchez de Alvarez (1986)). In fact, we show that the three internal rotors can realize any external torque feedback for the rigid body. Moreover, we show that a particular choice of internal torques for body–rotor system makes this system behave precisely like the classical *heavy rigid body.*

We go on, however, to do more than this. We analyze the rigid body with internal rotors with certain quadratic feedbacks that make the rigid body–rotor system Hamiltonian with respect to a Lie–Poisson bracket. We show that with a specific choice of feedback, the rotors rotate at constant angular velocity—i.e. we obtain as a special case, the driven dual spin satellite as analyzed by Krishnaprasad (1985) and Sánchez de Alvarez (1986).

Thus, we have produced a further class of systems that, despite having feedback torques, are Hamiltonian. In fact, we show that under certain integrability conditions, even under cubic feedback, we get Hamiltonian systems. Some additional information on when control systems can be expected to be Hamiltonian starting with a Lagrangian point of view and the Lagrange–d'Alembert variational principle is given in Krishnaprasad and Wang (1990).

We then discuss the Hamiltonian structure of the single rotor case and show that we can again prove stabilization of the system about the intermediate axis for sufficiently large torque by using the energy-Casimir method. Related to this Hamiltonian structure is the existence of conserved quantities in other forced systems, such as the kinematic chains of Baillieul (1987) and Baillieul and Levi (1991). The question of stabilization of Hamiltonian control systems has been considered, although in a rather different fashion by van der Schaft (1986) (see also van der Schaft (1982)).

We then show, using the work of Marsden *et al.* (1990), Krishnaprasad (1990) and Montgomery (1991), (see also Levi (1990)) on geometric phases, how an attitude drift can occur if the body is perturbed away from a stable equilibrium. Generalizing a formula of Montgomery (1991), we show how to calculate this drift precisely and indicate how to compensate for it.

The outline of the paper is as follows. First we present the rigid body with a single external torque. We introduce feedback, discuss the Lie–Poisson structure of the system, and recall the stability result proved in Bloch and Marsden (1989b, 1990). We also show how this feedback system is indeed the generalized (possibly indefinite) rigid body. Next, we show how to realize the rigid body with external torque as a system with three internal rotors. We then discuss the body–rotor system with quadratic feedback and its Hamiltonian structure, as well as the cubic feedback case. In the next section we discuss stabilization by one rotor with quadratic feedback using the energy-Casimir method.

Finally, we discuss the question of attitude drift and how to compensate for it. We remark that the work discussed above on phases may be useful here when used in conjunction with the work on chaos in mechanical system using Melnikov's method, as was done in Oh *et al.* (1989).

Another thing that can be done with the ideas here is the following. Suppose that one wanted to control a satellite to rotate stably about its intermediate axis, and that this is done by means of the techniques of this paper, say through internal rotors. Then one uses geometric phases to reorient the body. One way to do this is to relax the control so the body becomes unstable, then let it go unstable so it will swing around its homoclinic orbit, and when it comes back to the opposite saddle point, one imposes the stabilizing control again. One can also help initiate motion along the homoclinic orbit using a linear control near the saddle point, as in Bloch and Marsden (1989a). Some of these ideas are reminiscent of those in Beletskii (1981). This maneuver achieves a reorientation of 180° about the body's long axis, but one can imagine reorienting it using more general phases. We remark that this kind of control could be very useful in certain circumstances as it is both fast and energy-efficient since it uses the natural dynamics of the system to achieve most of the reorientation, rather than control torques. (For further details, see Bloch and Marsden (1989a).) See also Section 6.

2. THE RIGID BODY WITH AN EXTERNAL TORQUE

The rigid body equations with a single torque about the minor axis are given by

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\,\omega_2\omega_3,$$
$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\,\omega_3\omega_1, \tag{2.1}$$
$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\,\omega_1\omega_2 + u,$$

where the I_i are the principal moments of inertia and we assume for the moment that $I_1 > I_2 > I_3$. There are no essential changes in the analysis if the torque is taken about the major axis. Now let us implement the feedback torque

$$u = -\varepsilon\frac{I_1 - I_2}{I_3}\,\omega_1\omega_2. \tag{2.2}$$

Making the transformation to the classical momentum variables, $m_i = I_i\omega_i$, $i = 1, 2, 3$, the equations of motion become

$$\dot{m}_1 = a_1 m_2 m_3,$$
$$\dot{m}_2 = a_2 m_3 m_1, \tag{2.3}$$
$$\dot{m} = a_3(1-\varepsilon) m_1 m_2,$$

where

$$a_1 = \frac{I_2 - I_3}{I_2 I_3}, \quad a_2 = \frac{I_3 - I_1}{I_1 I_3}, \quad \text{and} \quad a_3 = \frac{I_1 - I_2}{I_1 I_2}.$$

Remarkably, this feedback system (2.3) has two constants of motion.

Lemma 2.1. The functions

$$H_F = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_3^2}{I_2} + \frac{m_3^2}{I_3}\frac{1}{1-\varepsilon}\right), \tag{2.4}$$

and

$$M_F^2 = \tfrac{1}{2}(m_1^2(1-\varepsilon) + m_2^2(1-\varepsilon) + m_3^2), \tag{2.5}$$

are constants of the motion (we assume $\varepsilon \neq 1$ for the moment).

The proof is a straightforward computation. Given these constants of the motion—an energy-like quantity and a momentum-like quantity, one might ask whether this feedback system is a Hamiltonian system. This is in fact true—this system, like the free rigid body, is a Lie–Poisson system. To explain this, we recall the theory of Lie–Poisson systems; see also, for example, Holmes and Marsden (1983), Krishnaprasad (1985), and Krishnaprasad and Marsden (1987).

Let G be a Lie group (such as the special orthogonal group $SO(3)$) and let its Lie algebra be denoted by $\mathfrak{g}$ and its dual (as a vector space) by $\mathfrak{g}^*$. For a smooth function $F: \mathfrak{g}^* \to \mathbb{R}$ we define its *functional derivative* $\delta F/\delta\mu: \mathfrak{g}^* \to \mathfrak{g}^*$ by

$$DF(\mu)\cdot\delta\mu = \left\langle \frac{\delta F}{\delta\mu}, \delta\mu \right\rangle,$$

where $DF(\mu)\cdot\delta\mu$ is the directional derivative—the derivative of F at μ in the direction of $\delta\mu \in \mathfrak{g}^*$, where $\delta F/\delta\mu$ is understood to be evaluated at the point $\mu \in \mathfrak{g}^*$, and where $\langle\ ,\ \rangle$ is the pairing between the vector space $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. We then define the $\pm$ *Lie–Poisson brackets* of two functions F and K defined on $\mathfrak{g}^*$ by

$$\{F, K\}_\pm(\mu) = \pm\left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta K}{\delta\mu}\right]\right\rangle,$$

where $[\ ,\ ]$ is the Lie algebra bracket on $\mathfrak{g}$. We denote by $\mathfrak{g}^*_\pm$ the (Poisson) manifolds $\mathfrak{g}^*$ equipped with the Poisson bracket $\{F, K\}_\pm$ or $\{F, K\}_-$. One can identify the space $\mathfrak{g}^*_-$ with the reduced space T^*G/G where T^*G is the cotangent bundle of G. For the rigid body, $G = SO(3)$ is the configuration space and $T^*SO(3)$ is the full phase space of configurations and momenta. The reduced space is $so(3)^* \cong \mathbb{R}^3$ and is identified with the space of body angular momenta, $\mathbf{m}$. The (minus) Lie–Poisson bracket is given on functions of $\mathbf{m}$ by the triple product

$$\{F, G\}(\mathbf{m}) = -\mathbf{m}\cdot(\nabla F \times \nabla G),$$

and the Euler equations of free rigid body motion are simply given by

$$\dot{\mathbf{m}} = \{\mathbf{m}, H\},$$

where H is the rigid body Hamiltonian given by

$$H = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right).$$

In our controlled case, we similarly have.

Proposition 2.2. The controlled system (2.3) is Lie–Poisson with Hamiltonian H_F with respect to the Lie–Poisson structure

$$\{F, G\}_F = -\nabla M_F^2 \cdot (\nabla F \times \nabla G). \tag{2.6}$$

This follows from checking that $\dot{m}_i = \{m_i, H_F\}_F$, $i = 1, 2, 3$. (For a discussion of Lie–Poisson systems see, for example, Krishnaprasad (1985) and references therein.)

There are three other canonical Poisson structures associated with the system (2.3), which are discussed in Bloch and Marsden (1989b). Note also, as pointed out in the latter paper, that for $\varepsilon < 1$, the invariance group of the

Lie–Poisson bracket (2.6) is $SO(3)$, while for $\varepsilon > 1$, it is $SO(2, 1)$. Thus, we have a deformation of the structure with the parameter ε. (Deformations of Poisson structures are discussed in Weinstein (1983).) In fact, we can show that (2.3) are generalized rigid body equations. The generalized rigid body is discussed in Abraham and Marsden (1978) and in Arnold (1978); for $\varepsilon > 1$, we get an indefinite rigid body, which is discussed in the work of Klein (1897).

Theorem 2.3. The equations (2.3) are the generalized rigid body (Euler) equations for the Lie group $G_\Sigma = \{A \mid A \in SL(3), A^T \Sigma A = \Sigma\}$, where Σ is the quadratic form given by $\Sigma = \mathrm{diag}\,(1-\varepsilon)^{-1}, (1-\varepsilon)^{-1}, 1)$.

Proof. We use the fact (see, for example, Marsden *et al.* (1983)) that any Lie–Poisson system is the reduction of the system on T^*G obtained by declaring the Hamiltonian to be left invariant on the corresponding group, in this case, $G = G_\Sigma$. Note that (2.6) is the Lie–Poisson bracket (with a *minus* sign, hence we take a *left* invariant extension) for the group G_Σ with M_F a Casimir. One can check explicitly that the equations may be written in Eulerian form as

$$\dot{\mathbf{m}} = \Lambda_\Sigma \nabla H_F(\mathbf{m}),$$

where Λ_Σ is the Poisson tensor:

$$\Lambda_\Sigma = \begin{bmatrix} 0 & -m_3 & (1-\varepsilon)m_2 \\ m_3 & 0 & -(1-\varepsilon)m_1 \\ -(1-\varepsilon)m_2 & (1-\varepsilon)m_1 & 0 \end{bmatrix}.$$

Bloch and Marsden (1989b) analyzed stabilizability of the free rigid body system about its intermediate axis using the feedback $u = -\varepsilon a_3 m_1 m_2$. this is equivalent to analyzing stability of the closed loop system (2.3). Of course, since the system is Hamiltonian and there is no damping, we are concerned here with *Lyapunov stability, not asymptotic stability.* Recall that an equilibrium point u_0 of a dynamical system $\dot{u} = X(u)$ is said to be Lyapunov stable if for any neighborhood U of u_0, there is a neighborhood V of u_0 such that trajectories $u(t)$ initially in V never leave U. In problems considered here, once Lyapunov stability has been established, it is straightforward to obtain asymptotic stability by the addition of velocity feedback (see Bloch *et al.* (1991)). While this paper considers Lyapunov stability of equilibrium points of the reduced system in momentum space, these points correspond to orbits in the full phase space—such solutions are called *relative equilibria.* Our stability results then imply the Lyapunov orbital stability of rotational motion about a particular axis.

Note first that (2.3), linearized about the (relative) equilibrium $(m_1, m_2, m_3) = (0, M, 0)$, has one zero, one stable and one unstable eigenvalue for $\varepsilon < 1$, but has one zero and two imaginary eigenvalues for $\varepsilon > 1$. Hence we cannot use linearization to conclude stability for $\varepsilon > 1$. To prove stability we will determine a suitable Lyapunov function. For this we will use the energy-Casimir method. We describe this method below and carry out a detailed computation in Section 5. The energy-Casimir method shows in fact that the system is (non-linearly) stable for $\varepsilon > 1$. For $\varepsilon = 1$ the system is gyroscopically stable (all nontrivial orbits are periodic).

Recall that the energy-Casimir method for determining stability provides a systematic procedure for determining a Lyapunov function for Hamiltonian systems defined on Poisson manifolds P, i.e. manifolds equipped with a Poisson bracket operation $\{\,,\}$ on the space of real valued functions on P that makes them into a Lie algebra and which is a derivation in each variable (see for example, Holm *et al.* (1985) and Krishnaprasad and Marsden (1987)). The point is that on such manifolds, there often exist Casimir functions, that is, functions C that commute with every other dynamical variable under the bracket operation. If we have a Hamiltonian system with Hamiltonian H, the equation of motion for any dynamical variable F if given by $\dot{F} = \{F, H\}$. Hence the Casimir functions are all conserved under the flow. The procedure of determining stability is then as follows.

(1) Consider conserved quantities of the form $H + C$, where H is the energy and C is a Casimir function (or, in some examples, a Casimir function plus another conserved quantity).

(2) Choose C such that $H + C$ has a critical point at the (relative) equilibrium of interest.

(3) Definiteness of the second variation of $H + C$ at the critical point is sufficient for Lyapunov stability (at least in finite dimensions—the infinite dimensional case is discussed in the cited references).

In our situation, we use the energy H_F and the Casimir M_F. It is easy to check that M_F is indeed a Casimir—it commutes with every dynamical variable under the Poisson bracket $\{\,,\}_F$. We omit the proof here, but state the theorem.

Theorem 2.4. The rigid body equations with a single torque about the minor axis and with feedback $u = -\varepsilon a_3 m_1 m_2$, i.e. the system (2.3), is stabilized about the relative equilibrium $(m_1, m_2, m_3) = (0, M, 0)$ for $\varepsilon > 1$.

3. RECOVERY OF THE EXTERNALLY TORQUED SYSTEM AND THE HEAVY RIGID BODY

As discussed in the Introduction, the fact that the system of Section 2 with external torque feedback has conserved quantities and a Lie–Poisson structure, leads one to ask if there is a mechanical extension of the system to larger system where the closed loop dynamics is realized by an internal torque feedback. In this section we show that such a mechanical extension does indeed exist—a rigid body carrying three symmetric rotors with associated internal torques. The Lie–Poisson structure and stability of this system was first analyzed by Krishnaprasad (1985) and also by Sánchez de Alvarez (1986).

Consider a rigid body carrying one, two or three symmetric rotors. Denote the system center of mass by $\mathbb{0}$ in the body frame and at $\mathbb{0}$ place a set of (orthonormal) body axes. Assume that the rotor axes are aligned with principal axes of the carrier body.

The configuration space of the system is $SO(3) \times S^1 \times S^1 \times S^1$ with tangent space denoted $T(SO(3) \times S^1 \times S^1 \times S^1)$.

Let $\mathbb{I}_{\text{body}}$ be the inertia tensor of the carrier body, $\mathbb{I}_{\text{rotor}}$ the diagonal matrix of rotor inertias about the principal axes and $\mathbb{I}'_{\text{rotor}}$ the remaining rotor inertias about the other axes. Let $\mathbb{I}_{\text{lock}} = \mathbb{I}_{\text{body}} + \mathbb{I}_{\text{rotor}} + \mathbb{I}'_{\text{rotor}}$ be the locked inertia tensor (i.e. with rotors locked) of the full system; this definition coincides with the usage in Marsden *et al.* (1989).

The Lagrangian (kinetic energy) of the free system is the total kinetic energy of the body plus the total kinetic energy of the rotor, i.e.

$$\begin{aligned} L &= \tfrac{1}{2}(\Omega \cdot \mathbb{I}_{\text{body}}\Omega) + \tfrac{1}{2}\Omega \cdot \mathbb{I}'_{\text{rotor}}\Omega \\ &\quad + \tfrac{1}{2}(\Omega + \Omega_{\text{r}}) \cdot \mathbb{I}_{\text{rotor}}(\Omega + \Omega_{\text{r}}) \qquad (3.1) \\ &= \tfrac{1}{2}(\Omega \cdot \mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})\Omega) \\ &\quad + \tfrac{1}{2}(\Omega + \Omega_{\text{r}}) \cdot \mathbb{I}_{\text{rotor}}(\Omega + \Omega_{\text{r}}), \qquad (3.2) \end{aligned}$$

where Ω is the vector of body angular velocities and Ω_{r} is the vector of rotor angular velocities about the principal axes with respect to a body fixed frame.

By the Legendre transform, the conjugate momenta are:

$$\begin{aligned} m &= \frac{\partial L}{\partial \Omega} = (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})\Omega + \mathbb{I}_{\text{rotor}}(\Omega + \Omega_{\text{r}}) \\ &= \mathbb{I}_{\text{lock}}\Omega + \mathbb{I}_{\text{rotor}}\Omega_{\text{r}}, \qquad (3.3) \\ l &= \frac{\partial L}{\partial \Omega_{\text{r}}} = \mathbb{I}_{\text{rotor}}(\Omega + \Omega_{\text{r}}), \qquad (3.4) \end{aligned}$$

and the equations of motion with internal torques (controls) u in the rotors are

$$\dot{m} = m \times \Omega = m \times (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - l), \qquad (3.5)$$
$$\dot{l} = u. \qquad (3.6)$$

We now show how to recover the feedback equations with a single external torque from Section 2. To do so, it is convenient to use the variables $\pi = m - l$ and l. Let $\mathbb{I} = \mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}}$ (for explicit computation we will let $\mathbb{I}_{\text{body}} = \operatorname{diag}(I_1, I_2, I_3)$ as before and let $\mathbb{I} = \mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}} = \operatorname{diag}(\bar{I}_1, \bar{I}_2, \bar{I}_3)$). Then we get the equations

$$\dot{\pi} = (\pi + l) \times \mathbb{I}^{-1}\pi - u, \qquad (3.7)$$
$$\dot{l} = u. \qquad (3.8)$$

Theorem 3.1. There is a choice of internal torque feedback $u(\pi, l)$ such that the body dynamics in the system (3.7)–(3.8) are precisely those of system (2.3) (with external torque feedback).

Proof. Firstly set

$$u(\pi, l) = l \times \mathbb{I}^{-1}\pi - u'(\pi), \qquad (3.9)$$

so the system (3.7)–(3.8) becomes:

$$\dot{\pi} = \pi \times \mathbb{I}^{-1}\pi + u'(\pi), \qquad (3.10)$$
$$\dot{l} = l \times \mathbb{I}^{-1}\pi - u'(\pi). \qquad (3.11)$$

Now suppose that $\mathbb{I} = \operatorname{diag}(\bar{I}_1, \bar{I}_2, \bar{I}_3)$, $\bar{I}_1 > \bar{I}_2 > \bar{I}_3$ and

$$u'(\pi) = \begin{bmatrix} 0 \\ 0 \\ -a_3\varepsilon\pi_1\pi_2 \end{bmatrix}. \qquad (3.12)$$

Here a_1, a_2, and a_3 are defined as in Section 2, $\pi = (\pi_1, \pi_2, \pi_3)^T$, and $l = (l_1, l_2, l_3)^T$. Then the equations (3.7), (3.8) reduce to

$$\left.\begin{aligned} \dot{\pi}_1 &= a_1\pi_2\pi_3, \\ \dot{\pi}_2 &= a_2\pi_3\pi_1, \\ \dot{\pi}_3 &= a_3(1-\varepsilon)\pi_2\pi_2, \\ \dot{l}_1 &= \frac{l_2\pi_3}{\bar{I}_3} - \frac{l_3\pi_2}{\bar{I}_2}, \\ \dot{l}_2 &= \frac{l_3\pi_1}{\bar{I}_1} - \frac{l_1\pi_3}{\bar{I}_3}, \\ \dot{l}_3 &= \frac{\pi_2}{\bar{I}_2}(l_1 + \varepsilon\pi_1) - \frac{\pi_1}{\bar{I}_1}(l_2 + \varepsilon\pi_2), \end{aligned}\right\} \qquad (3.13)$$

in which the $\dot{\pi}$ equations are precisely (2.3).

In fact, we see from (3.10) and (3.11) that we can realize any external torque feedback for the rigid body with our rigid body plus rotor system.

Consider now the following interesting case. Let $m = \pi + l$ as before and take as controls

$u'(\pi, m)$. Then (3.10) and (3.11) may be written

$$\begin{aligned} \dot{\pi} &= \pi \times \mathbb{I}^{-1}\pi + u'(\pi, m), \\ \dot{m} &= m \times \mathbb{I}^{-1}\pi. \end{aligned} \tag{3.14}$$

In particular we have the following.

Proposition 3.2. For $u'(\pi, m) = -C\chi \times m$ where C is a constant, and χ is a constant (body fixed) vector, the equation (3.14) are the equations for the heavy top. Hence these admit the conserved Hamiltonian $H = \frac{1}{2}\pi \cdot \mathbb{I}^{-1}\pi + C\chi \cdot m$ and kinematic conserved quantities (Casimir functions), $C_1 = \|m\|^2$ and $C_2 = \pi \cdot m$.

This may be verified by referring to the heavy top equations in Marsden *et al.* (1984). The equations are Lie–Poisson on the dual of the Euclidean Lie algebra $se(3)$.

4. THE HAMILTONIAN STRUCTURE OF THE RIGID BODY WITH THREE ROTORS UNDER FEEDBACK

The equations of motion for the three rotor system with internal torque controls are

$$\dot{m} = m \times \Omega = m \times (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - l), \tag{4.1}$$

$$\dot{l} = u. \tag{4.2}$$

We have the following result:

Theorem 4.1. For the feedback

$$u = \mathbb{k}(m \times (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - l)), \tag{4.3}$$

where $\mathbb{k}$ is a constant real matrix such that the matrix $\mathbb{J} = (\mathbb{1} - \mathbb{k})^{-1}(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})$ is symmetric, the system (3.5) reduces to a Hamiltonian system on $so(3)^*$ with respect to the standard Lie–Poisson structure $\{F, G\}(m) = -m \cdot (\nabla F \times \nabla G)$.

Proof. We have

$$\dot{l} = u = \mathbb{k}\dot{m}, \tag{4.4}$$

which is an acceleration feedback. (Note that also $u = \mathbb{k}(\mathbb{I}_{\text{lock}}\Omega + \mathbb{I}_{\text{rotor}}\Omega_r) \times \Omega$.) Therefore,

$$\mathbb{k}m - l = p, \tag{4.5}$$

where p is an arbitrary constant. Hence the closed loop system becomes

$$\begin{aligned} \dot{m} &= m \times (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - l) \\ &= m \times (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - \mathbb{k}m + p) \\ &= m \times (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(\mathbb{1} - \mathbb{k})(m - \xi), \end{aligned} \tag{4.6}$$

where $\xi = -(\mathbb{1} - \mathbb{k})^{-1}p$. Define the $\mathbb{k}$-dependent "inertia tensor"

$$\mathbb{J} = (\mathbb{1} - \mathbb{k})^{-1}(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}}). \tag{4.7}$$

Then the equations become

$$\dot{m} = \nabla C \times \nabla H, \tag{4.8}$$

where

$$C = \tfrac{1}{2}\|m\|^2, \tag{4.9}$$

and

$$H = \tfrac{1}{2}(m - \xi) \cdot \mathbb{J}^{-1}(m - \xi). \tag{4.10}$$

Equations (4.8) are Hamiltonian on $so(3)^*$ with respect to the usual Lie–Poisson structure.

Remark 1. Note that (4.5) is equivalent to

$$\mathbb{k}(\mathbb{I}_{\text{lock}}\Omega + \mathbb{I}_{\text{rotor}}\Omega_r) - \mathbb{I}_{\text{rotor}}(\Omega + \Omega_r) = p,$$

i.e.

$$(\mathbb{I}_{\text{rotor}} - \mathbb{k}\mathbb{I}_{\text{rotor}})\Omega_r = (\mathbb{k}\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})\Omega - p.$$

Thus for

$$\mathbb{k}\mathbb{I}_{\text{lock}} = \mathbb{I}_{\text{rotor}}, \tag{4.11}$$

one specializes to the usual dual spin case (see Krishnaprasad (1985) and Sánchez de Alvarez (1986)) where the acceleration feedback is such that each rotor rotates at *constant* angular velocity relative to the carrier. Also note that the Hamiltonian in (4.10) is indefinite for high gains.

Remark 2. Furthermore, if we set $m_b = m - \xi$, the equations become

$$\dot{m}_b = (m_b + \xi) \times \mathbb{J}^{-1}m_b, \tag{4.12}$$

which is a k-dependent form of the dual-spin equations. Hence one can apply, for example, the Morse-theoretic analysis of Krishnaprasad and Berenstein (1984) to this situation. For a related stability analysis using the energy-momentum method (see Wang (1990)).

It is instructive to consider the case where $\mathbb{k} = \operatorname{diag}(k_1, k_2, k_3)$. Let $\mathbb{I} = (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}}) = \operatorname{diag}(\bar{I}_1, \bar{I}_2, \bar{I}_3)$ and the matrix $\mathbb{J}$ satisfies the symmetry hypothesis of Theorem 4.1. Then $l_i = p_i + k_i m_i$, $i = 1, 2, 3$, and the equations become $\dot{m} = m \times \nabla H$ where

$$H = \frac{1}{2}\left(\frac{((1 - k_1)m_1 + p_1)^2}{(1 - k_1)\bar{I}_1} + \frac{((1 - k_2)m_2 + p_2)^2}{(1 - k_2)\bar{I}_2} + \frac{((1 - k_3)m_3 + p_3)^2}{(1 - k_3)\bar{I}_3}\right). \tag{4.13}$$

It is possible to have more complex feedback mechanisms where the system still reduces to a Hamiltonian system on $so(3)$. If we set $u = \mathbb{k}(m, l)\dot{m}$, the key to reduction is that $\mathbb{k}(m, l)\dot{m} - \dot{l} = 0$ is integrable. The natural case to consider is $\mathbb{k} = \mathbb{k}(m)$.

Theorem 4.2. The system (3.5), (3.6) with

$u = \mathbb{k}(m)\dot{m}$, where

$$\mathbb{k}(m) = \begin{bmatrix} \mathbb{k}_1(m)^T \\ \mathbb{k}_2(m)^T \\ \mathbb{k}_3(m)^T \end{bmatrix},$$

and $\mathbb{k}_i(m) = \nabla\phi_i(m)$ for some smooth ϕ_i, and such that $(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}\mathbb{k}(m)$ is symmetric, is Hamiltonian.

Proof. We have

$$\mathbb{k}_i(m) \cdot \dot{m} - \dot{l}_i = 0 = \nabla\phi_i(m)\dot{m} - \dot{l}_i,$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}[\phi_i(m) - l_i] = 0,$$

or

$$\phi_i(m) - l_i = p_i.$$

Thus, $\mathbb{k}(m) = \partial\Phi/\partial m$, where Φ is the vector potential $[\phi_1, \phi_2, \phi_3]^T$ gives integrability. Then (3.15) and (3.16) reduce to

$$\dot{m} = m \times (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - \Phi(m) + p). \quad (4.14)$$

This is Lie–Poisson on $so(3)^*$ if and only if there exists an $H(m)$ such that

$$\nabla H = (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - \Phi(m) + p).$$

By equality of mixed partial derivatives, this holds if

$$\frac{\partial}{\partial m_j}(e_i(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - \Phi(m) + p))$$
$$= \frac{\partial}{\partial m_i}(e_j(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - \Phi(m) + p)),$$

where e_i is the ith unit vector. This is true if and only if $e_i(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}\mathbb{k}(m)e_j = e_j(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}\mathbb{k}(m)e_i$, i.e. $(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}\mathbb{k}(m)$ is symmetric.

To compute H, we need to solve the set of partial differential equations

$$\nabla H = (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - \Phi(m) + p).$$

Consider the special case where $(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})$ and $\mathbb{k}(m)$ are diagonal, $(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}}) = \operatorname{diag}(\bar{I}_1, \bar{I}_2, \bar{I}_3)$ and $\mathbb{k}(m) = \operatorname{diag}(k_1 m_1, k_2 m_2, k_3 m_3)$. Then $\mathbb{k}(m) = \partial\Phi/\partial m\,(m)$ where

$$\Phi(m) = \frac{1}{2}\begin{bmatrix} k_1 m_1^2 \\ k_2 m_2^2 \\ k_3 m_3^2 \end{bmatrix}.$$

Hence

$$\frac{\partial H}{\partial m_i} = \frac{1}{\bar{I}_i}\left(m_i - \frac{k_i m_i^2}{2} + p_i\right), \quad i = 1, 2, 3.$$

Thus, letting $H = \sum_{i=1}^{3} H_i(m_i)$ we have

$$H_i = \frac{1}{\bar{I}_i}\left(\frac{m_i^2}{2} - \frac{k_i m_i^3}{6} + p_i m_i\right). \quad (4.15)$$

This procedure clearly generalizes to $k_i(m_i)$ an arbitrary polynomial in m_i, yielding a large class of Hamiltonian feedback systems.

5. THE RIGID BODY WITH A SINGLE ROTOR-FEEDBACK STABILIZATION BY THE ENERGY-CASIMIR METHOD

We now consider the equations for a rigid body with a *single* rotor. This reveals the essentials of the dynamics and Hamiltonian structure and, further, we are able to obtain the result that with a single rotor about the third principal axis, quadratic feedback with sufficiently high gain stabilizes the system about its intermediate axis—a result similar to that of the single gas jet (external torque) case.

We adopt the following notation: let the rigid body have moments of inertia $I_1 > I_2 > I_3$ as before and suppose the symmetric rotor is aligned with the third principal axis and has moments of inertia $J_1 = J_2$ and J_3. Let ω_i, $i = 1, 2, 3$, denote the carrier body angular velocities and let $\dot{\alpha}$ denote that of the rotor (relative to a frame fixed on the carrier body). Let

$$\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) = \operatorname{diag}(J_1 + I_1, J_2 + I_2, J_3 + I_3), \quad (5.1)$$

be the locked inertia tensor. Then from (3.3) and (3.4), the natural momenta are

$$\begin{aligned} m_i &= (J_i + I_i) \quad \omega_i = \lambda_i\omega_i, \quad i = 1, 2, \\ m_3 &= I_3\omega_3 + l_3, \\ l_3 &= J_3(\omega_2 + \dot{\alpha}). \end{aligned} \quad (5.2)$$

From (3.5) and (3.6), the equations of motion are:

$$\begin{aligned} \dot{m}_1 m_2 m_3&\left(\frac{1}{I_3} - \frac{1}{\lambda_2}\right) - \frac{l_3 m_2}{I_3}, \\ \dot{m}_2 &= m_1 m_3\left(\frac{1}{\lambda_1} - \frac{1}{I_3}\right) + \frac{l_3 m_1}{I_3}, \\ \dot{m}_3 &= m_1 m_2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right), \\ \dot{l}_3 &= u. \end{aligned} \quad (5.3)$$

Choose $u = k a_3 m_1 m_2$ where

$$a_3 = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1}\right).$$

Theorem 5.1. With $u = k a_1 m_1 m_2$ the equations

(5.3) reduce to the equations

$$\dot{m}_1 = m_2\left(\frac{(1-k)m_3 - p}{I_3}\right) - \frac{m_3 m_2}{\lambda_2},$$
$$\dot{m}_2 = -m_1\left(\frac{(1-k)m_3 - p}{I_3}\right) + \frac{m_3 m_1}{\lambda_1}, \tag{5.4}$$
$$\dot{m}_3 = a_3 m_1 m_2,$$

which are Hamiltonian on $so(2)^*$ with respect to the usual Lie–Poisson structure with Hamiltonian

$$H = \frac{1}{2}\left(\frac{m_1^2}{\lambda_1} + \frac{m_2^2}{\lambda_2} + \frac{((1-k)m_3 - p)^2}{(1-k)I_3}\right) + \frac{1}{2}\frac{p^2}{J_3(1-k)}, \tag{5.5}$$

where p is a constant.

Proof. First note that $p = l_3 - km_3$ is conserved. Eliminating l_3 gives rise to equations (5.3) which are checked to be Hamiltonian with respect to the standard bracket $\{f, g\}(u) = -m \cdot (\nabla f \times \nabla g)$ and the given H.

Remark. When $k = 0$, we get the equations for the free rotor. We get the dual spin case $J_3\dot{\alpha} = 0$, when $\mathbb{k}\mathbb{I}_{\text{lock}} = \mathbb{I}_{\text{rotor}}$ from (4.9), or in this case $k = J_3/\lambda_3$. One can check that for this k, $p = (1-k)\dot{\alpha}$, a multiple of $\dot{\alpha}$.

We can now use the energy-Casimir method to prove:

Theorem 5.2. For $k > 1 - I_3/\lambda_2$ (and $p = 0$) the system (5.4) is stabilized about the middle axis, i.e. about the relative equilibrium $(0, M, 0)$.

Proof. Consider the energy-Casimir function $H + C$ where $C = \varphi(m^2)$, and $m^2 = m_1^2 + m_2^2 + m_3^2$. The first variation is

$$\delta(H + C) = \frac{m_1\,\delta m_1}{\lambda_1} + \frac{m_2\,\delta m_2}{\lambda_2} + \frac{(1-k)m_3 - p}{I_3}\delta m_3 + \varphi'(m^2)(m_1\,\delta m_1 + m_2\,\delta m_2 + m_3\,\delta m_3). \tag{5.6}$$

This is zero if

$$\frac{m_1}{\lambda_1} + \varphi' m_1 = 0,$$
$$\frac{m_2}{\lambda_2} + \varphi' m_2 = 0, \tag{5.7}$$
$$\frac{(1-k)m_3 - p}{I_3} + \varphi' m_3 = 0.$$

Then we compute

$$\delta^2(H + C) = \frac{(\delta m_1)^2}{\lambda_1} + \frac{(\delta m_2)^2}{\lambda_2} + \frac{(1-k)(\delta m_3)^2}{I_3} + \varphi'(m^2)((\delta m_1)^2 + (\delta m_2)^2 + (\delta m_3)^2) + \varphi''(m^2)(m_1\,\delta m_1 + m_2\,\delta m_2 + m_3\,\delta m_3)^2. \tag{5.8}$$

If $p = 0$, i.e. $l_3 = km_3$, $(0, M, 0)$ is a relative equilibrium and (5.7) are satisfied if $\varphi' = -(1/\lambda_2)$. In that case,

$$\delta^2(H + C) = (\delta m_1)^2\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + (\delta m_3)^2\left(\frac{1-k}{I_3} - \frac{1}{\lambda_2}\right) + \varphi''(\delta m_2)^2.$$

Now

$$\frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{I_2 - I_1}{\lambda_1\lambda_2} < 0$$

for $I_1 > I_2 > I_3$. Clearly for $k > 1 - I_3/I_2$, $(1-k)/I_3 - (1/\lambda_2) < 0$, so if we choose $\varphi'' < 0$ we get negative definiteness and hence stability. (Note that for $k = 0$ the second variation is indefinite, as it should be.)

Note that we do not get precisely the gas jet equations as we do in Section 3, but we achieve the same effect—for k greater than a certain value we achieve stability about the middle axis. Also, we are using a much simpler feedback than in Theorem 3.1 of Section 3, as we need not "cancel" the l_3 dynamics. Note also that while the Hamiltonian here is indefinite for k sufficiently large, the conserved momentum here is the usual definite momentum, in contrast to the possibly indefinite "momentum" of the gas jet case in Section 2.

Corresponding to the Hamiltonian (5.5) there is a Lagrangian found using the inverse Legendre transformation

$$\tilde{\omega}_1 = \frac{m_1}{\lambda_1},$$
$$\tilde{\omega}_2 = \frac{m_2}{\lambda_2},$$
$$\tilde{\omega}_3 = \frac{(1-k)m_3 - p}{I_3}, \tag{5.9}$$
$$\dot{\alpha} = -\frac{(1-k)m_3 - p}{(1-k)I_3} + \frac{p}{(1-k)I_3}.$$

Note that $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{\omega}_3$ are the angular velocities ω_1, ω_2, and ω_3 for the free system, but that $\dot{\tilde{\alpha}}$ is not equal to $\dot{\alpha}$. In fact,

$$\dot{\tilde{\alpha}} = \frac{\dot{\alpha}}{(1-k)} - \frac{km_3}{(1-k)J_3}. \tag{5.10}$$

Thus the equation on the tangent bundle $TSO(3)$ determined by (5.9) are the Euler–Lagrange equations for a Lagrangian quadratic in the velocities, so the equations can be regarded as geodesic equations. The torques can be thought of as residing in the velocity shift (5.10). Using the free Lagrangian, the torques appear as generalized forces on the right hand side of the Euler–Lagrange equations. Thus, the d'Alembert principle can be used to describe the Euler–Lagrange equations with the generalized forces. However, this latter approach hides the useful fact that the equations are actually derivable from a Lagrangian (and hence a Hamiltonian) in velocity shifted variables. Also, for problems like the driven rotor, one might think that this is a velocity constraint and should be treated by using constraint theory. Again for the particular problem at hand, this can be circumvented and standard methods are in fact applicable, as we have demonstrated. For a more systematic approach to the Lagrangian side of the story, with emphasis on the variational principles, and for the geometry of gyroscopic systems, see Krishnaprasad and Wang (1992) and Marsden and Scheurle (1991).

6. PHASE SHIFTS

In this final section we discuss an attitude drift that can occur in the system and a method for correcting it.

If the system (4.12) is perturbed from a stable equilibrium, and the perturbation is not too large, the closed loop system executes a periodic motion on a level surface (momentum sphere) of the Casimir function $\|m_b + \xi\|^2$ in the body–rotor feedback system. This leads to an attitude drift which can be thought of as a rotation about the (constant) spatial angular momentum vector.

We will calculate the amount of this rotation. This can be done following a method developed by Montgomery (1991) for calculating the phase shift for the single rigid body. Other pertinent work on phases may be found in Marsden *et al.* (1990) and Krishnaprasad (1990).

Recall that the equations of motion for the rigid body–rotor system with feedback law (4.3) may be written (see 4.12) as

$$\begin{aligned} \dot{m}_b &= (m_b + \xi) \times \mathbb{J}^{-1} m_b \\ &= \nabla C \times \nabla H, \end{aligned} \tag{6.1}$$

where $m_b = m + (1 - \mathbb{k})^{-1}(\mathbb{k}m - l)$ and $\xi = -(1-\mathbb{k})^{-1}(\mathbb{k}m - l)$ and

$$C = \tfrac{1}{2}\|m_b + \xi\|^2, \tag{6.2}$$

$$H = \tfrac{1}{2} m_b \cdot \mathbb{J}^{-1} m_b. \tag{6.3}$$

Here H is the conserved Hamiltonian and C = constant determines a momentum sphere in the reduced phase centered at $\xi \in \mathbb{R}^3 \simeq so(3)^*$.

The attitude equation for the rigid body–rotor system is

$$\dot{A} = A\hat{\Omega},$$

where, in the presence of the feedback law (4.3),

$$\begin{aligned} \Omega &= (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - l) \\ &= (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - \mathbb{k}m + p) \\ &= \mathbb{J}^{-1}(m - \xi) = \mathbb{J}^{-1} m_b. \end{aligned}$$

Therefore the attitude equation may be written

$$\dot{A} = A(\mathbb{J}^{-1} m_b)^\wedge, \tag{6.4}$$

where $^\wedge$ denotes the canonical map from vectors in $\mathbb{R}^3$ to elements of $so(3)$,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix},$$

and A is the attitude matrix, $A \in SO(3)$.

The net spatial (constant) angular momentum vector is now given by

$$\mu = A(m_b + \xi). \tag{6.5}$$

Then we have the following.

Theorem 6.1. Suppose that the solution of

$$m_b = (m_b + \xi) \times \mathbb{J}^{-1} m_b, \tag{6.6}$$

sweeps out a periodic orbit of period T on the momentum sphere, $\|m_b + \xi\|^2 = \|\mu\|^2$, enclosing a solid angle Φ_{solid}. Let Ω_{av} denote the average value of the body angular velocity over this period. Let E denote the constant value of the Hamiltonian and let $\|\mu\|$ denote the magnitude of the angular momentum vector. Then the body undergoes a net rotation $\Delta\theta$ about the spatial angular momentum vector μ given by the formula

$$\Delta\theta = \frac{2ET}{\|\mu\|} + \frac{T}{\|\mu\|}(\xi \cdot \Omega_{\text{av}}) \mp \Phi_{\text{solid}}. \tag{6.7}$$

Proof. Consider the reduced phase space (the momentum sphere)

$$P_\mu = \{m_b \mid \|m_b + \xi\|^2 = \|\mu\|^2\}, \tag{6.8}$$

for μ fixed. Suppose we have a periodic orbit in P_μ of period T. We then compute how $A(t)$ changes. Now

$$m_b(t_0 + T) = m_b(t_0), \tag{6.9}$$

and from momentum conservation

$$\begin{aligned}\mu &= A(T+t_0)(m_b(T+t_0)+\xi)\\ &= A(t_0)(m_b(t_0)+\xi).\end{aligned} \tag{6.10}$$

Hence

$$A(T+t_0)A(t_0)^{-1}\mu = \mu.$$

Thus $A(T+t_0)A(t_0)^{-1}$ is an element of G_μ, the subgroup of the group of rotations that fixes μ, so

$$A(T+t_0)A(t_0)^{-1} = \exp\left(\Delta\theta\frac{\mu}{\|\mu\|}\right), \tag{6.11}$$

for some $\Delta\theta$, which we wish to determine.

Consider a phase trajectory of our system

$$z(t) = (A(t), m_b(t)), \quad z(t_0) = z_0, \tag{6.12}$$

with $A(t_0) = I$ and $m_b(t_0) = \mu - \xi$, the body thus being in the reference configuration at $t = t_0$.

The two curves in phase space

$$C_1 = \{z(t): t_0 \le t \le t_0 + T\}$$

(the dynamical evolution from z_0),

and

$$C_2 = \left\{\exp\left(\theta\frac{\mu}{\|\mu\|}\right)z_0 : 0 \le \theta \le \Delta\theta\right\}$$

intersect at $t = T$.

Thus $C = C_1 - C_2$ is a closed curve in phase space and from Stokes' theorem we have,

$$\int_{C_1} p\,\mathrm{d}q - \int_{C_2} p\,\mathrm{d}q = \iint_\Sigma \mathrm{d}(p\,\mathrm{d}q). \tag{6.13}$$

where $p\,\mathrm{d}q = \sum_{i=1}^{3} p_i\,\mathrm{d}q_i$ and where q_i and p_i are configuration space variables and conjugate momenta in the phase space and Σ is a surface enclosed by the curve C.

Evaluating each of these integrals will give the formula for $\Delta\theta$. To see this, first consider the dynamical evolution along C_1. Letting ω be the spatial angular velocity, we get

$$\begin{aligned}p\frac{\mathrm{d}q}{\mathrm{d}t} &= \mu\cdot\omega = A(\mathbb{J}\Omega+\xi)\cdot A\Omega\\ &= (\mathbb{J}\Omega+\xi)\cdot\Omega\\ &= \mathbb{J}\Omega\cdot\Omega + \xi\cdot\Omega.\end{aligned} \tag{6.14}$$

Hence

$$\begin{aligned}\int_{C_1} p\,\mathrm{d}q &= \int_{C_1} p\cdot\frac{\mathrm{d}q}{\mathrm{d}t}\,\mathrm{d}t\\ &= \int_0^T \mathbb{J}\Omega\cdot\Omega\,\mathrm{d}t + \int_0^T \xi\cdot\Omega\,\mathrm{d}t\\ &= 2ET + (\xi\cdot\Omega_{av})T,\end{aligned} \tag{6.15}$$

since the Hamiltonian is conserved along orbits. Similarly, along C_2 we have,

$$\begin{aligned}\int_{C_2} p\cdot\frac{\mathrm{d}q}{\mathrm{d}t}\,\mathrm{d}t &= \int_{C_2} \mu\cdot\omega\,\mathrm{d}t\\ &= \int_{C_2} \mu\cdot\left\{\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mu}{\|\mu\|}\right\}\mathrm{d}t\\ &= \|\mu\|\int_{C_2}\mathrm{d}\theta = \|\mu\|\,\Delta\theta,\end{aligned} \tag{6.16}$$

where we substituted the spatial angular velocity along C_2.

Finally we note that the map π_μ from the set of points in phase space with angular momentum μ to P_μ satisfies

$$\iint_\Sigma \mathrm{d}(p\,\mathrm{d}q) = \iint_{\pi_\mu(\Sigma)} \mathrm{d}A = \pm\|\mu\|\,\Phi_{\text{solid}}, \tag{6.17}$$

where $\mathrm{d}A$ is the area form on the two-sphere and $\pi_\mu(\Sigma)$ is the spatial cap bounded by the periodic orbit

$$\{m_b(t)\mid t_0 \le t \le t_0 + T\} \subset P_\mu.$$

From (6.15), (6.16) and (6.17) we have the result.

Remark 6.2. (a) When $\xi = 0$, (6.7) reduces to Montgomery's formula (1991). (b) This theorem may be viewed as a special case of a scenario that is useful for other systems, such as rigid bodies with flexible appendages. Phases are usually viewed as occurring in the reconstruction process, which lifts the dynamics from P_μ to $\mathbb{J}^{-1}(\mu)$, where $\mathbb{J}: P \to \mathfrak{g}^*$ is the momentum map for a mechanical system with symmetry and $P_\mu = \mathbb{J}^{-1}(\mu)/G_\mu$ is the reduced space; see Marsden *et al.* (1990). The cotangent bundle reduction theorem states that P_μ itself is a bundle over T^*S, where $S = Q/G$ is shape space. The fiber of this bundle is O_μ, the coadjoint orbit through μ. For a rigid body with three internal rotors, S is the three torus $\mathbb{T}^3$ parametrized by the rotor angles. Controlling them by a feedback or other control and using other conserved quantities associated with the rotors as we have done, leaves one with dynamics on the "rigid variables" O_μ, the momentum sphere in our case. Then the problem reduces to that of lifting the dynamics on O_μ to $\mathbb{J}^{-1}(\mu)$ with the T^*S dynamics given. For $G = SO(3)$ this "reduces" the problem to that for geometric phases for the rigid body given by Montgomery (1991).

Finally, following Krishnaprasad (1990) we show that in the zero total angular momentum case one can compensate for this drift using two rotors.

The total *spatial* angular momentum if one has only two rotors is of the form

$$\mu = A(\mathbb{I}_{\mathrm{lock}}\Omega + b_1\dot{\alpha}_1 + b_2\dot{\alpha}_2), \qquad (6.18)$$

where the scalars $\dot{\alpha}_1$ and $\dot{\alpha}_2$ represent the rotor velocities relative to the body frame. The attitude matrix A satisfies

$$\dot{A} = A\hat{\Omega}, \qquad (6.19)$$

as above. If $\mu = 0$, then from (6.18) and (6.19) we get,

$$\dot{A} = -A((\mathbb{I}_{\mathrm{lock}}^{-1}b_1)^{\wedge}\dot{\alpha}_1 + (\mathbb{I}_{\mathrm{lock}}^{-1}b_2)^{\wedge}\dot{\alpha}_2). \quad (6.20)$$

It is well known (see for instance Brockett (1973) or Crouch (1986)) that if we treat the $\dot{\alpha}_i$, $i = 1, 2$ as *controls,* then attitude controllability holds iff

$$(\mathbb{I}_{\mathrm{lock}}^{-1}b_1)^{\wedge} \quad \text{and} \quad (\mathbb{I}_{\mathrm{lock}}^{-1}b_2)^{\wedge} \quad \text{generate } so(3), \qquad (6.21\mathrm{a})$$

or equivalently, iff the vectors

$$\mathbb{I}_{\mathrm{lock}}^{-1}b_1 \quad \text{and} \quad \mathbb{I}_{\mathrm{lock}}^{-1}b_2 \quad \text{are linearly independent.} \qquad (6.21\mathrm{b})$$

Moreover, one can write the attitude matrix as a reverse path-ordered exponential

$$A(t) = A(0)\cdot\bar{\mathbb{P}}\exp\left[-\int_0^t\left\{(\mathbb{I}_{\mathrm{lock}}^{-1}b_1)^{\wedge}\dot{\alpha}_1(\sigma) + (\mathbb{I}_{\mathrm{lock}}^{-1}b_2)^{\wedge}\dot{\alpha}_2(\sigma)\right\}\mathrm{d}\sigma\right]. \qquad (6.22)$$

A key observation is that the right-hand side of (6.22) depends only on the *path traversed* in the space $\mathbb{T}^2$ of rotor angles (α_1, α_2) and *not* on the history of velocities $\dot{\alpha}_i$. This can be easily checked by carrying out a time rescaling $t \mapsto \beta(t)$. Hence the formula (6.22) should be interpreted as a "geometric phase". Furthermore, the controllability condition can be interpreted as a curvature condition on the principal connection on the bundle $\mathbb{T}^2 \times SO(3) \to \mathbb{T}^2$ defined by the $so(3)$-valued differential 1-form,

$$\theta(\alpha_1, \alpha_2) = -((\mathbb{I}_{\mathrm{lock}}^{-1}b_1)^{\wedge}\,\mathrm{d}\alpha_1 + (\mathbb{I}_{\mathrm{lock}}^{-1}b_2)^{\wedge}\,\mathrm{d}\alpha_2). \quad (6.23)$$

For further details on this geometric picture of multibody interaction see Krishnaprasad (1990) and Krishnaprasad and Wang (1992).

7. CONCLUSIONS

Natural mechanical systems subject to exogenous forces determined by suitable classes of feedback laws can sometimes be modeled as Hamiltonian systems. Often the Hamiltonian structures so derived may be viewed as deformations (by feedback gains) of the Hamiltonian structure governing the open loop unforced system. The methods of geometric mechanics such as reduction phases, energy-Casimir and energy–momentum algorithms for stability analysis, prove to be naturally applicable to such feedback systems. The present paper illustrates this via a careful study of the problem of rigid body control using external torques (as implemented by gas jets) and internal torques (via reaction wheels/rotors), and the relationships between these two methods of control. We reveal a Hamiltonian structure for the controlled system that involves a velocity shift (or a gyroscopic term), a technique that should be of general utility. We especially note that we find a non-obvious but rich Hamiltonian structure for the controlled system, despite the presence of controlling torques.

One of the main accomplishments of the present work is the stabilization analysis of these examples as the feedback parameter is increased. We also calculate the geometric phase (the attitude drift) in the presence of this control when one is executing a periodic motion in the carrier body angular momentum phase space.

A general theory for feedback systems of gyroscopic type appears in the recent thesis of Wang. We note however that, in the examples of the present paper, we go on to consider feedback laws that also effectively alter the metric structure underlying a simple mechanical system. The introduction of geometric phases (a subject of great current interest to physicists) in the analysis of attitude drift due to perturbations from relative equilibria is one of the novel features of the paper. The methods we use for both the phase and stability calculations should be applicable to more complex many-body systems such as those incorporating flexible elements.

In work under way, we are also considering problems of instability in feedback systems. We establish instability theorems based on second variation criteria analogous to the classic work of Kelvin and Chetaev. Details will appear in a forthcoming paper.

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