International Series of Numerical Mathematics, Vol. 104, © 1992 Birkhäuser Verlag Basel

Generic Bifurcations of Pendula

Michael Dellnitz[*]
Department of Mathematics, University of Houston
Houston, TX 77204-3476, USA

Jerrold E. Marsden[†]
Department of Mathematics, University of California at Berkeley
Berkeley, CA 94720, USA

Ian Melbourne[‡]
Department of Mathematics, University of Houston
Houston, TX 77204-3476, USA

Jürgen Scheurle
Institut für Angewandte Mathematik, Universität Hamburg
D-2000 Hamburg 13, Germany

December 9, 1991

1 Introduction

In a parameter dependent Hamiltonian system, an equilibrium might lose its stability via a socalled *Hamiltonian Krein-Hopf bifurcation* ([1], [12]): Two pairs of purely imaginary eigenvalues of the linearised system collide ($1-1$ *resonance*) and split off the imaginary axis into the complex plane. In the following we will refer to this scenario as the *splitting case*, see Figure 1. It is well known that in one parameter problems without external

[*]Research is supported by the Deutsche Forschungsgemeinschaft and by NSF DMS-9101836. Permanent address: Institut für Angewandte Mathematik, Universität Hamburg, D-2000 Hamburg 13, Germany
[†]Research partially supported by a Humboldt award and DOE Contract DE-FG03-88ER25064.
[‡]Supported in part by NSF DMS-9101836

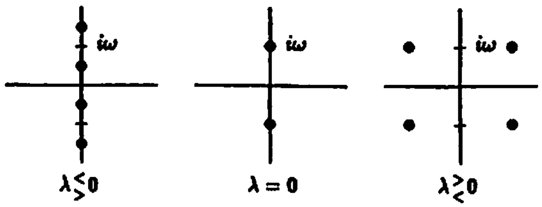

Figure 1: The *splitting* case for the $1-1$ resonance

symmetry this is the only eigenvalue behavior that generically occurs in $1-1$ resonances ([4], [9], [10]).

When there is symmetry present, the situation changes. Under certain circumstances the eigenvalues might also pass while remaining on the imaginary axis (the *passing* case, see Figure 2). In this case the linear stability properties of the corresponding equilibrium

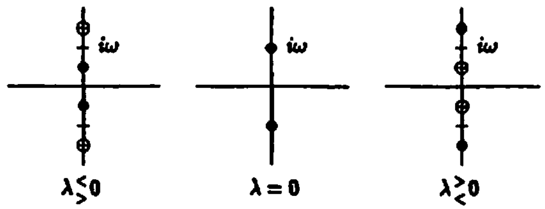

Figure 2: The *passing* case for the $1-1$ resonance

do not change and in this sense the collision is not "dangerous" as in the splitting case. The question arises naturally whether these essentially different eigenvalue movements can be characterised so that the occurence of the one or the other in a given system could in principle be predicted. The answer to this question is given in [3]. There the generic movement of eigenvalues through a $1-1$ resonance is completely classified by use of group theory and energetics.

The main purpose of this paper is to show the usefulness of this type of result for analysing the dynamical behavior of mechanical systems. We describe briefly the main

result of [3] in Section 2 and consider rotating pendula problems in Section 3.

These examples clearly point out the fact that in specific mechanical systems, both passing and splitting can occur generically. In [3] this behavior is explained in the context of systems with *symplectic* symmetries. The examples suggest there is a corresponding result for systems with antisymplectic symmetries as well. We expect that the techniques of [14] will be useful toward this end.

2 Generic movement of eigenvalues

In this section we briefly describe the main result of [3] for the case of $1-1$ resonances in Hamiltonian systems with symmetry.

Let Z be a symplectic vector space with symplectic form ω. Assume there is a compact Lie group Γ acting symplectically on Z, that is,

$$\omega(\gamma v, \gamma w) = \omega(v, w) \quad \text{for all } \gamma \in \Gamma \text{ and } v, w \in Z.$$

We denote by $\mathrm{sp}_\Gamma(Z)$ the Lie algebra of linear infinitesimally symplectic maps commuting with Γ:

$$B \in \mathrm{sp}_\Gamma(Z) \Longleftrightarrow \begin{cases} (i) & \omega(Bv, w) + \omega(v, Bw) = 0 \quad \text{for all } v, w \in Z, \\ (ii) & \gamma B = B\gamma \quad \text{for all } \gamma \in \Gamma. \end{cases}$$

Suppose that $A(\lambda)$ is a one-parameter family in $\mathrm{sp}_\Gamma(Z)$ and that $A(\lambda)$ undergoes a $1-1$ resonance at $\lambda = 0$. After rescaling we may assume that the purely imaginary eigenvalues which are involved are $\pm i$. It is well known (eg [4], [9], [10]) that without symmetry, generically the eigenvalues split off the imaginary axis in a $1-1$ resonance (see Figure 1). When there is symmetry present, this is no longer true: for certain symmetry types the passing case may occur generically (see Figure 2). In [3] it is shown that for symplectic symmetries the generic movement of eigenvalues through a $1-1$ resonance can be completely characterised in terms of group theory and energetics, but by neither of them alone.

To state the corresponding result precisely, it is necessary to recall some terminology from [13]. If U is a symplectic representation of Γ then — by ignoring the symplectic structure of U — we obtain an ordinary representation, which is called the *underlying representation*. A *Γ-irreducible symplectic representation* is a representation that has no proper nonzero Γ-invariant symplectic subspaces. Irreducible symplectic representations are either nonabsolutely irreducible (i.e., are irreducible but some linear map that is not a real multiple of the identity commutes with Γ) or the sum of a pair of isomorphic absolutely irreducible subspaces (see [5]). We now use the fact that the space of linear maps commuting with Γ is isomorphic to $\mathbb{R}$ (the absolutely irreducible case), to $\mathbb{C}$ or to $\mathbb{H}$, the quaternions, see for example [6]. It can be shown (Theorem 2.1 in [13]) that in the real and quaternionic cases the isomorphism type of the irreducible symplectic representation is uniquely determined by that of its underlying representation, whereas in the complex case there are precisely two isomorphism types of irreducible symplectic representations for a given complex irreducible underlying representation. They are said to be *dual* to each other. According to these two different possibilities we will speak of *complex irreducibles of the same type* and *complex duals*.

Theorem 2.1 ([3]) *Let $E_{\pm i}$ be the generalized (real) eigenspace of $A(0)$ belonging to the eigenvalues $\pm i$ and let Q denote the quadratic form induced on $E_{\pm i}$ via*

$$Q(z) = \omega(z, A(0)z)\,. \tag{2.1}$$

Then

$$E_{\pm i} = U_1 \oplus U_2\,,$$

where, generically, precisely one of the following holds:

(a) *U_1 and U_2 are not isomorphic and the eigenvalues pass independently along the imaginary axis. (Q may be indefinite or definite.)*

(b) *$U_1 = U_2 = V \oplus V$, where V is real, or $U_1 = U_2 = W$, where W is quaternionic, the eigenvalues split, and Q is indefinite.*

(c) *U_1 and U_2 are complex of the same type, the eigenvalues pass and Q is indefinite.*

(d) *U_1 and U_2 are complex duals and the eigenvalues pass or split depending on whether Q is definite or indefinite.*

This theorem gives a complete characterisation of the generic eigenvalue movement through 1–1 resonances in Hamiltonian systems with a symplectic symmetry group. The result is summarised in Table 1.

	Eigenspace structure	Induced quadratic form: definite	Induced quadratic form: indefinite
(a)	$U_1 \oplus U_2$ nonisomorphic	"independent passing"	
(b)	$V \oplus V \oplus V \oplus V$ real, or $W \oplus W$ quaternionic	not generic	splitting
(c)	$W \oplus W$ complex of the same type	not generic	passing
(d)	$W \oplus W$ complex duals	passing	splitting

Table 1: Generic eigenvalue movement in 1-1-resonances with symplectic symmetry group

Example 2.2 We consider the space $\mathbb{C}^2$, where the symplectic form ω is induced by $J(z_1, z_2) = (z_2, -z_1)$. Let the group S^1 act symplectically on $\mathbb{C}^2$ by

$$\theta(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2).$$

Then the spaces

$$\mathbb{C}\{z, -iz\}, \quad \mathbb{C}\{z, iz\}$$

are complex duals with respect to this S^1-action: both are irreducible and J is acting as $-i$ on the first and as i on the second subspace. We consider the S^1-invariant quadratic Hamiltonian

$$H(z_1, z_2, \lambda) = \frac{1}{8}|z_1|^2 + \frac{\lambda}{2}\mathrm{Im}(\bar{z}_1 z_2) + 2(1 - \frac{3}{4}\lambda^2)|z_2|^2\,,$$

or in real coordinates,

$$H(q_1, q_2, p_1, p_2, \lambda) = \frac{1}{8}(q_1^2 + q_2^2) + \frac{\lambda}{2}(q_1 p_2 - q_2 p_1) + 2(1 - \frac{3}{4}\lambda^2)(p_1^2 + p_2^2)\,. \tag{2.2}$$

According to part (d) of Theorem 2.1, we expect to see definite passing or indefinite splitting in 1–1 resonances while varying the parameter λ. A computation of the eigenvalues of $A(\lambda) = JD^2H(0,0,\lambda)$ yields

$$\sigma(\lambda) = \frac{i}{2}\left(\lambda \pm \sqrt{1 + 3(1 - \lambda^2)}\right)\,.$$

These together with their complex conjugates are the four eigenvalues for the system induced by (2.2). In fact, definite passing occurs as λ passes through 0 and indefinite splitting occurs as λ passes through $\pm\frac{2}{\sqrt{3}}$ (see also Figure 3). To verify the definiteness properties observe that in this case the induced quadratic form in (2.1) is simply given by H itself.

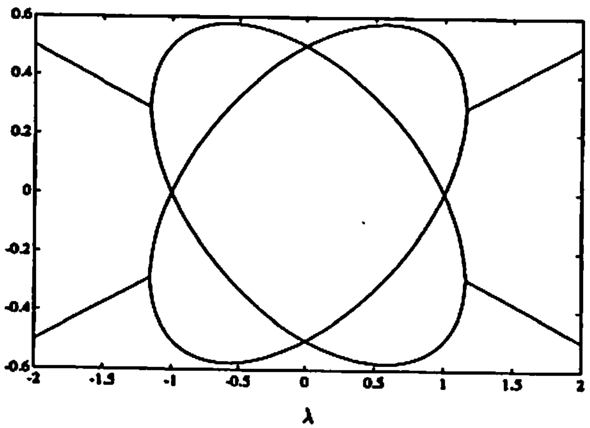

Figure 3: Imaginary parts of all four eigenvalues against λ. The sequence *indefinite splitting* $\rightarrow$ *passing at* 0 $\rightarrow$ *definite passing* $\rightarrow$ *passing at* 0 $\rightarrow$ *indefinite splitting* can be observed as λ is varied.

3 Generic bifurcation of spinning pendula with symmetry

3.1 The forcibly rotated orthogonal planar double pendulum

As in [2] we consider a rotating orthogonal planar double pendulum. The angular velocity of the rotation is assumed to be Ω_*. The two masses m_1 and m_2 are forced to move in two planes, which are orthogonal to each other. The pendula are assumed to have equal

length. We set

$$\Omega^2 = \frac{\Omega_*^2}{g}, \quad m = \frac{m_1}{m_2}.$$

We regard Ω as the bifurcation parameter. After scaling time and making a symplectic change of coordinates (cf [2]) one obtains

$$H_2 = \frac{1}{2(m+1)}(q_1^2 + q_2^2) + \frac{\Omega}{\sqrt{m+1}}(q_1 p_2 - q_2 p_1) + \frac{1}{2}(m+1)\left(1 - \frac{m}{m+1}\Omega^2\right)(p_1^2 + p_2^2)$$

as the quadratic part of the Hamiltonian H describing the behavior of this system. But this Hamiltonian is exactly of the form as the one in (2.2) (set $m = 3$) and, moreover, the underlying symplectic structures are the same. Therefore Example 2.2 shows that both indefinite splitting and definite passing occur in this mechanical system.

Remark 3.1 The S^1 symmetry of H_2 is not a symmetry of the full nonlinear mechanical system which is described in coordinates inside the rotating frame. In fact, this system only possesses a (nonsymplectic) $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Thus, although H_2 clearly has the S^1 symmetry and the results apply, its origin as a mechanical symmetry is not so clear.

First, the S^1 symmetry is only a symmetry at quadratic level which can easily be seen by looking at the higher order terms of H (cf [2]). Although it is not yet completely clear why this symmetry is present, there are the following facts which seem to play an important role:

- the nonsymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is generated by one symplectic and one antisymplectic involution and forces some quadratic terms of the Hamiltonian to vanish.
- the underlying mechanical structure as well as the requirement that there exists a $1-1$ resonance in the problem forces additional restrictions on the quadratic terms of the Hamiltonian.

It will be part of subsequent work to clarify the situation. In particular, this example and the next one suggest it would be useful to extend the above theorem to include antisymplectic symmetries as well.

3.2 The double spherical pendulum

Consider the double spherical pendulum, as shown in Figure 4. The relative equilibria and their stabilities are found in [11]. For angular momentum $\mu \neq 0$ the relative equilibria do not have any obvious symmetry properties (except that they are invariant under reflection in a uniformly rotating plane). Consistent with the generic (nonsymmetric) theory, one only sees eigenvalue splitting.

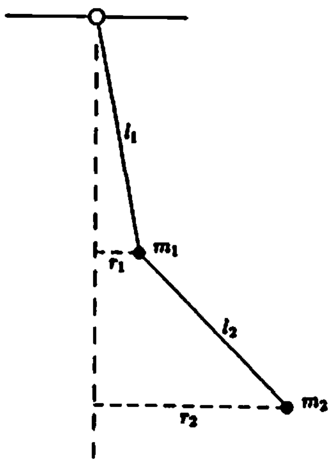

Figure 4: The double spherical pendulum

To get more interesting behavior, we look at the straight down state with $\mu = 0$. This state is however, singular in the sense that the overall S^1 action is not free there and, correspondingly, the set $\mu = 0$ is not a smooth manifold, but has a conic singularity.

To get around this difficulty, we regularise the system near this singular state. This is done as follows. Using ideas of Lagrangian reduction, one finds that the linearised dynamics at a relative equilibrium with angular momentum μ = constant (about the vertical axis) is given by a certain Lagrangian whose quadratic terms we shall denote

$$\mathcal{L}_2(\mu, \delta r_1, \delta r_2, \delta\varphi, \delta\dot{r}_1, \delta\dot{r}_2, \delta\dot{\varphi})$$

where r_1 and r_2 are the distances of the two pendula from the vertical axis and φ is the angle between the two vertical planes through the symmetry axis and the two masses. Variation of these variables are denoted $\delta r_1, \delta r_2$ and $\delta\varphi$.

The regularised Lagrangian at the straight down state is given at quadratic order by

$$\mathcal{L}_2^0(s_1, s_2, \theta, \dot{s}_1, \dot{s}_2, \dot{\theta}) = \lim_{\mu \to 0} \left[\frac{1}{|\mu|} \mathcal{L}_2(\mu, \sqrt{|\mu|}s_1, \sqrt{|\mu|}s_2, \theta, \sqrt{|\mu|}\dot{s}_1, \sqrt{|\mu|}\dot{s}_2, \dot{\theta}) \right].$$

This regularization procedure is akin to those used in celestial mechanics and corresponds to blowing up of singularities in algebraic geometry.

The regular Lagrangian that results from this procedure in this example is given by the following expression

$$\begin{aligned} 2\mathcal{L}_2^0 &= m\dot{s}_1^2 + 2r\dot{s}_1\dot{s}_2 + r^2\dot{s}_2^2 + (m-1)g(g+1)\dot{\theta}^2 \\ &+ \frac{Q}{r}\left(\frac{3m\beta - 4g^2(m-1)}{\beta} + m\right)s_1^2 \\ &+ 2Q\left(\frac{3\beta + 4g(m-1)}{\beta}\right)s_1 s_2 \\ &+ \frac{Q}{r}\left(\frac{3\beta - 4(m-1)}{\beta} + r\right)s_2^2 + \frac{\alpha}{r}\theta^2 \\ &+ \frac{1}{\sqrt{r}}\frac{2g^2(m-1)}{\beta}(\dot{\theta}s_1 - \dot{s}_1\theta) \\ &- \sqrt{r}\frac{2g(m-1)}{\beta}(\dot{\theta}s_2 - \dot{s}_2\theta) \end{aligned}$$

where $r = l_2/l_1$ is the ratio of the lengths,

$$m = (m_1 + m_2)/m_2, \quad g = \frac{1}{2}\left[m(r-1) - \sqrt{m^2(r-1)^2 + 4rm}\right],$$

$$\beta = g^2 + 2g + m, \text{ and } Q = g/(1+g).$$

Notice that $\mathcal{L}_0^2$ still has two free parameters r and m. If one wishes, one can easily get the corresponding Hamiltonian via Legendre transform. The Lagrangian is invariant under the transformation

$$\begin{aligned} s_1 &\mapsto s_1, & s_2 &\mapsto s_2, & \theta &\mapsto -\theta, \\ \dot{s}_1 &\mapsto -\dot{s}_1, & \dot{s}_2 &\mapsto -\dot{s}_2, & \dot{\theta} &\mapsto \dot{\theta}, \end{aligned}$$

which yields an antisymplectic involution on the Hamiltonian side. This symmetry appears to be crucial to what follows and is a reason for suggesting a generalisation of the results of Sec. 2 to include antisymplectic transformations. In addition, this may help put the previous example (Sec. 3.1) into a better context.

The equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}_2^0}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}_2^0}{\partial q^i} = 0$$

for $(q^i) = (s_1, s_2, \theta)$ have the form

$$M\ddot{q} + S\dot{q} + \Lambda q = 0$$

for 3×3 matrices M, S and Λ. The characteristic polynomial of the system is defined by

$$\det[\lambda^2 M + \lambda S + \Lambda] = 0$$

and is a polynomial $p(x)$ in $x = \lambda^2$. It is explicitly given as follows. Defining r, m, g and Q as above, further define the following quantities

$$\begin{array}{rclrcl}
G & = & 2\frac{m-1}{m+g(g+2)} & A & = & \left(1+\frac{g}{m}\right)^2 Q^2 \\
f & = & \frac{A-r^2}{A-g^2} & R & = & r^2 - g^2 f \\
L & = & 1-f & a & = & QL^{\frac{3}{2}}(3m - 2g^2G) + m\sqrt{R} \\
b & = & QL\sqrt{R}(3+2gG) & c & = & QR\sqrt{L}(3-2G) + r^2\sqrt{L} \\
s & = & \sqrt{LR} & d & = & s + g^2 f \\
e & = & mr^2 - d^2 & B & = & mc + ar^2 - 2bd \\
F & = & ac - b^2 & U & = & (m-1)(g+1)R^{\frac{3}{2}}eL \\
V & = & \multicolumn{4}{l}{eRL + (m-1)(g+1)RB\sqrt{L} + RLG^2g(mR + g(2ds + r^2gL))} \\
W & = & \multicolumn{4}{l}{sB + (m-1)(g+1)F\sqrt{R} + sG^2g(aR + g(2bs + cgL)).}
\end{array}$$

Then a straightforward, but lengthy calculation shows that

$$p(x) = Ux^3 + Vx^2 + Wx + F.$$

Using this expression for p we compute numerically $1-1$ resonances for the straight down state in the double spherical pendulum. The results are shown in Figure 5. Note especially that both the splitting and the passing cases occur generically.

The phenomena seen in this example of the straight down state of the double spherical pendulum can be expected to be ubiquitous in mechanical systems with symmetry at symmetric solutions. For example, we hope that techniques like this will be useful for rotating elastic and fluid masses in three dimensions (see [7] and [8]).

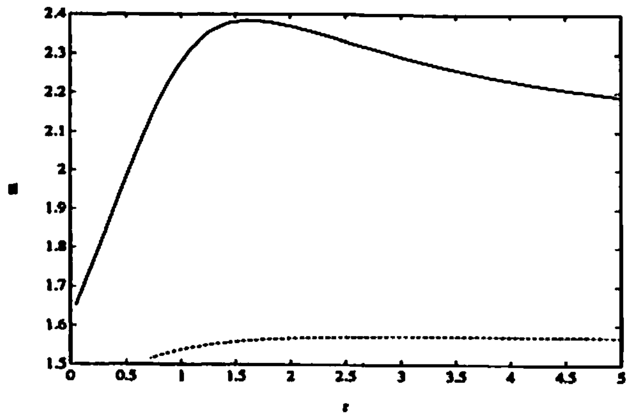

Figure 5: Curves of splitting (solid) and passing cases (dashed) in the spherical double pendulum

References

[1] R. Abraham and J. Marsden [1978] *Foundations of Mechanics*, 2nd ed., Addison-Wesley, New York.

[2] T.J. Bridges [1990] Branching of periodic solutions near a collision of eigenvalues of opposite signature. *Math. Proc. Camb. Phil. Soc.* 108, 575-601.

[3] M. Dellnitz, I. Melbourne and J.E. Marsden [1991] Generic bifurcation of Hamiltonian vector fields with symmetry. University of Houston (preprint).

[4] D.M. Galin [1982] Versal deformations of linear Hamiltonian systems. *AMS Transl.* 2 118, 1-12. (1975 Trudy Sem. Petrovsk. 1 63-74).

[5] M. Golubitsky and I. Stewart [1987] Generic bifurcation of Hamiltonian systems with symmetry. *Physica* 24D, 391-405.

[6] M. Golubitsky, I. Stewart and D. Schaeffer [1988] *Singularities and Groups in Bifurcation Theory*. Vol. 2, Springer.

[7] D.R. Lewis [1989] Nonlinear stability of a rotating planar liquid drop. *Arch. Rat. Mech. Anal.* 106, 287-333.

[8] D.R. Lewis and J.C. Simo [1990] Nonlinear stability of rotating pseudo-rigid bodies. *Proc. Roy. Soc. Lon.* A 427, 281-319.

[9] R.S. MacKay [1986] Stability of equilibria of Hamiltonian systems. In *Nonlinear Phenomena and Chaos*, edited by S. Sarkar, 254-270.

[10] R.S. MacKay and P.G. Saffman [1986] Stability of water waves. *Proc. Roy. Soc. Lond.* A 406, 115-125.

[11] J.E. Marsden and J. Scheurle [1991] Lagrangian reduction and the double spherical pendulum (preprint).

[12] J.C. van der Meer [1985] *The Hamiltonian Hopf Bifurcation*. Lecture Notes in Mathematics 1160.

[13] J. Montaldi, M. Roberts and I. Stewart [1988] Periodic solutions near equilibria of symmetric Hamiltonian systems. *Phil. Trans. R. Soc.* 325, 237-293.

[14] Y.H. Wan [1989] Versal deformations of infinitesimally symplectic transformations with involutions. State University of New York at Buffalo (preprint).

[15] Y.H. Wan [1989] Codimension two bifurcations of symmetric cycles in Hamiltonian systems with an antisymplectic involution. State University of New York at Buffalo (preprint).