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Stability of Relative Equilibria. Part I: The Reduced Energy-Momentum Method

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Contents

Glossary: Simple mechanical systems with symmetry

Q	Configuration space, with elements denoted by $q \in Q$.
TQ	State space. Points $(q, \dot{q}) \in TQ$ are configurations and velocities.
$P = T^*Q$	Phase space. Points $z = (q, \mathbf{p}) \in P$ are configurations and momenta.

$\delta z = (\delta q, \delta \boldsymbol{p})$	*Configuration-momentum variations;* where $\delta q \in T_qQ$, and $\delta \boldsymbol{p} \in T_q^*Q$.
$\langle \cdot, \cdot \rangle$	Non-degenerate duality pairing between T_qQ and T_q^*Q.
G	Lie group acting freely on Q on the left. The action of G on P is symplectic, obtained by cotangent lifts.
$\mathscr{G}$	Lie algebra of G, with bracket denoted by $[\cdot, \cdot]$.
$\mathscr{G}^*$	Dual of $\mathscr{G}$, with duality pairing denoted by a dot. Thus $\boldsymbol{\mu} \cdot \boldsymbol{\eta} \in \mathbb{R}$, $\forall (\boldsymbol{\eta}, \boldsymbol{\mu}) \in \mathscr{G} \times \mathscr{G}^*$.
Ad_g	Adjoint action of G on $\mathscr{G}$; $\mathrm{Ad}_g\, \boldsymbol{\eta} = \left.\frac{d}{d\varepsilon}\right\|_{\varepsilon=0} g(\exp(\varepsilon\boldsymbol{\eta}))\, g^{-1}$.
$\mathrm{Ad}^*_{g^{-1}}$	Coadjoint action of G on $\mathscr{G}^*$; $(\mathrm{Ad}^*_{g^{-1}}\, \boldsymbol{\mu}) \cdot \boldsymbol{\eta} = \boldsymbol{\mu} \cdot \mathrm{Ad}_{g^{-1}}\, \boldsymbol{\eta}$.
ad_ν	Infinitesimal adjoint action of $\mathscr{G}$ on $\mathscr{G}$; $\mathrm{ad}_\nu \boldsymbol{\eta} = [\boldsymbol{\nu}, \boldsymbol{\eta}] = \left.\frac{d}{d\varepsilon}\right\|_{\varepsilon=0} \mathrm{Ad}_{(\exp(\varepsilon\nu))}\, \boldsymbol{\eta}$.
ad^*_ν	Infinitesimal coadjoint action of $\mathscr{G}$ on $\mathscr{G}^*$ $(\mathrm{ad}^*_\nu\, \boldsymbol{\mu}) \cdot \boldsymbol{\eta} = \boldsymbol{\mu} \cdot \mathrm{ad}_\nu\, \boldsymbol{\eta}$.
$\boldsymbol{\eta}_Q(q)$	Infinitesimal generator; $\boldsymbol{\eta}_Q(q) = \left.\frac{d}{d\varepsilon}\right\|_{\varepsilon=0} \exp(\varepsilon\boldsymbol{\eta}) \cdot q$.
$\boldsymbol{J} : P \to \mathscr{G}^*$	Momentum map; $\boldsymbol{J}(q, \boldsymbol{p}) \cdot \boldsymbol{\eta} = \langle \boldsymbol{p}, \boldsymbol{\eta}_Q(q) \rangle$.
$V : Q \to \mathbb{R}$	G-invariant potential energy.
$K : P \to \mathbb{R}$	G-invariant invariant kinetic energy.
$H : P \to \mathbb{R}$	Hamiltonian function: $H(z) = V(q) + K(z)$.
$H_{\mu_e} : P \times \mathscr{G} \to \mathbb{R}$	Energy-momentum functional: $H_{\mu_e}(z, \boldsymbol{\xi}) = V(q) + K(z) - (\boldsymbol{J}(z) - \boldsymbol{\mu}_e) \cdot \boldsymbol{\xi}$.
$\langle \cdot, \cdot \rangle_g$	Positive-definite form on Q associated with the kinetic energy.
$\mathrm{FL} : TQ \to T^*Q$	Legendre transformation; $\langle \mathrm{FL}(v_q), w_q \rangle = \langle v_q, w_q \rangle_g$.
$\mathscr{I}(q) : \mathscr{G} \to \mathscr{G}^*$	*Locked* inertia tensor defined as $\zeta \cdot \mathscr{I}(q)\, \boldsymbol{\eta} := \langle \boldsymbol{\eta}_Q(q), \zeta_Q(q) \rangle_g$, $\forall\, \boldsymbol{\eta}, \zeta \in \mathscr{G}$.
$\Sigma : P \to \boldsymbol{J}^{-1}(0)$	Shifting map: $\Sigma(q, \boldsymbol{p}) := (q, \boldsymbol{p} - \boldsymbol{p}_J(q, \boldsymbol{p}))$, where $\boldsymbol{p}_J(q, \boldsymbol{p})) := \mathrm{FL}([\mathscr{I}^{-1}(q)\, \boldsymbol{J}(q, \boldsymbol{p})]_Q(q))$.
$V_\xi = V - \frac{1}{2}\, \xi \cdot \mathscr{I}\xi$	Augmented potential.
$V_\mu = V + \frac{1}{2}\, \boldsymbol{\mu} \cdot \mathscr{I}^{-1}\, \boldsymbol{\mu}$	Amended potential.
$h_{\mu_e} : \boldsymbol{J}^{-1}(\boldsymbol{0}) \to \mathbb{R}$	Reduced Hamiltonian: $h_{\mu_e}(z) = V_{\mu_e}(q) + K(z)$.
$\mathscr{G}_{\mu_e}$	Isotropy subalgebra of $\boldsymbol{\mu}_e \in \mathscr{G}^*$ under the coadjoint action.
$\mathscr{G}^\perp_{\mu_e} \subset \mathscr{G}_{\mu_e}$	Orthogonal complement to $\mathscr{G}_{\mu_e}$ with respect to $\mathscr{I}(q_e)$ at a given $q_e \subset Q$.
$\mathscr{V} \subset T_{q_e}Q$	Space of admissible configuration variations modulo variations generated by $\mathscr{G}_{\mu_e}$. Thus, $\delta q \in T_{q_e}Q$ is in $\mathscr{V}$ if and only if $\langle \delta q, \boldsymbol{\eta}_Q(q_e) \rangle_g = 0$ for all $\boldsymbol{\eta} \in \mathscr{G}_{\mu_e}$.
$\mathscr{V}_{RIG} \subset \mathscr{V}$	Space of 'rigid' configuration variations $\mathscr{V}_{RIG} := \{\boldsymbol{\eta}_Q(q_e) \mid \boldsymbol{\eta} \in \mathscr{G}^\perp_{\mu_e}\}$.
$\mathrm{ident}_\xi : \mathscr{V} \to \mathscr{G}^*$	(Minus) linearized 'angular' momentum in the direction $\delta q \in \mathscr{V}$ for fixed locked velocity $\xi \in \mathscr{G}$, i.e., $\mathrm{ident}_\xi\, (\delta q) := -\, [D\mathscr{I}(q_e) \cdot \delta q] \cdot \xi$.

$\mathscr{V}_{INT} \subset \mathscr{V}$ Space of 'internal' configuration variations
$\mathscr{V}_{INT} := \{\delta q \in \mathscr{V} \mid \boldsymbol{\eta} \cdot \mathrm{ident}_{\xi_e}(\delta q) = 0 \quad \forall\, \boldsymbol{\eta} \in \mathscr{G}_{\mu_e}^{\perp}\}.$

$\mathscr{S} \subset T_{z_e}P$ Space of admissible configuration-momentum variations modulo variations generated by $\mathscr{G}_{\mu_e}$. The variation $\delta z = (\delta q, \boldsymbol{\delta p}) \in T_{z_e}P$ is an element of $\mathscr{S}$ if and only if $T_{z_e}\boldsymbol{J} \cdot \delta z = \mathbf{0}$ and $\delta q \in \mathscr{V}$.

D Vector tangent map; given a map $\phi : M \to V$ from a manifold M to a vector space V, $D\phi(q) : T_qM \to V$ is given by $D\phi(q) \cdot \delta q = \left.\dfrac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(q_\varepsilon)$ for any curve q_ε tangent to δq at q.

§ 0. Introduction

§ 0.A. *Background and motivation*

A general mechanical system with a Hamiltonian that is the sum of the kinetic and potential energy is said to possess symmetry if the Hamiltonian function is invariant under the action of a group acting on the canonical phase space by canonical transformations. In the terminology of SMALE [1970], one speaks of a *simple mechanical system with symmetry.* A central problem in classical mechanics is the stability analysis of the *relative equilibria* of a simple mechanical system. Relative equilibria are particular solutions for which the dynamic orbit generated by Hamilton's equation coincides with a one-parameter group orbit. A classical example is the stability of rotating self-gravitating fluid masses restricted to affine deformations, a problem in celestial mechanics considered by NEWTON, JACOBI, LIOUVILLE, DIRICHLET [1861], RIEMANN [1861], POINCARÉ and CARTAN among others; see e.g., CHANDRASEKHAR [1987, Chap. 1] for a historical review. RIEMANN's treatment of rotating gravitating fluid masses is particularly relevant to our work due to his geometric approach, which emphasizes symmetries and associated conservation laws.

Another classical example of crucial importance is the stability of rotating, inviscid, incompressible (perfect) fluids considered by RAYLEIGH [1920] and ARNOLD [1966]. A direct precedent of our stability results is found in the pioneering work of ARNOLD [1966], which constitutes the point of departure of modern geometric methods in hydrodynamics. ARNOLD derives an explicit stability condition for mechanical systems in which the configuration space is a group which coincides with the symmetry group of the mechanical system. This situation is relevant to both classical rigid-body mechanics governed by Euler's equations and classical hydrodynamics governed by the incompressible Euler equations. In the former example, the symmetry group and configuration space coincide with the proper orthogonal group, whereas in the latter example the relevant group is the group of volume-preserving diffeomorphisms. ARNOLD's nonlinear stability result reproduces the classical stability conditions for a rigid body and shows that RAYLEIGH's linear stability criterion for planar rotating perfect fluids constitutes, in fact, a nonlinear stability theorem.

Unfortunately, the framework set forth in ARNOLD [1966] is too restrictive to accommodate most simple mechanical systems of interest, for which the configuration space *does not* coincide with the symmetry group. Typical examples include classical three-dimensional elasticity and generalized models of the COSSERAT type, as discussed in ERICKSEN & TRUESDELL [1958], TOUPIN [1964], ANTMAN [1976a, b] and references therein. Attempts to extend this approach have resulted in the Energy-Casimir method, which is described in HOLM, MARSDEN, RATIU & WEINSTEIN [1985], where the technique is applied to fluids and plasmas. The Energy-Casimir method is applied to study the stability of certain rotating structures in KRISHNAPRASAD & MARSDEN [1987], and POSBERGH, KRISHNAPRASAD & MARSDEN [1987], and to axisymmetric three-dimensional fluid flow in SZERI & HOLMES [1988]; see also MCINTYRE & SHEPARD [1987] and FINN & SUN [1987] for further examples and references. A crucial difficulty, however, makes this method unduly restrictive: Conserved quantities in the *reduced* space (the Casimir functions) may be difficult to characterize explicitly or, in fact, may not exist at all. A main objective of this paper is to present a general approach to the stability analysis of relative equilibira which results in *explicit* stability conditions applicable to *any* simple mechanical system.

An alternative approach to the problem of relative equilibria is considered by SMALE [1970a, b] for a class of simple mechanical systems. (SMALE does not address the case in which the group action fails to be free, i.e., in which there are symmetric configurations.) There, it is shown that the relative equilibria of a simple mechanical system coincide with the critical points of an *amended potential* function, denoted by V_{μ_e}. As noted in ARNOLD et al. [1988, p. 103], this result has been used in concrete situations by several authors; the amended potential already appears in the fundamental work of RIEMANN [1861]. A cornerstone of the theory of symmetric mechanical systems is the work of ROUTH [1877], which treats the general case of a mechanical system which is invariant under spatial rotations about a fixed axis (i.e., a left S^1 action). Although a systematic procedure for the actual computation of the relative equilibria is effectively contained in SMALE [1970a, p. 322] and SMALE [1970b, p. 51], explicit and implementable stability conditions comparable to those contained in ARNOLD [1966] appear to be lacking. As pointed out above, one of the main goals of this paper is precisely to develop such techniques.

A crucial result employed in our work is a block-diagonalization procedure which decouples, as far as the stability analysis is concerned, '*rotational*' perturbations generated by the symmetry group from a complementary space of '*internal*' (deformation) perturbations, which are precisely defined below. The identification of the defining conditions on the space of variations leading to this block diagonalization result is by no means obvious, since rotational and vibrational modes are typically dynamically coupled. The stability conditions associated with the rotational variations are explicit, do not depend on the potential energy of the system, and recapture the stability result of ARNOLD [1966] for the case in which the configuration space coincides with the symmetry group. The stability conditions associated with the internal modes, on the other hand, can often be cast in terms of a standard eigenvalue problem for the second variation of SMALE's amended potential. The preceding block-diagonalization procedure was introduced in SIMO, POSBERGH & MARSDEN [1989] and LEWIS & SIMO [1990], where the extension of the method to account for further symmetries of the Hamiltonian is considered. Further geometric aspects of this approach are addressed in the general context of Hamiltonian systems with symmetry in MARSDEN, SIMO, LEWIS & POSBERGH [1989].

The approach presented in Part I and Part II of this paper, referred to as the *reduced energy-momentum method* in what follows, constitutes a significant improvement over these techniques in the sense that it involves only the configuration space, not the full phase space. Aside from the reduction in dimension of the problem (which may be of practical importance for large systems), the method automatically enforces the constraint of constant (angular) momentum without resort to Lagrange multipliers. As a result, many of the more tedious computations involved in the case-by-case application of the original energy-momentum method have been eliminated in the current approach. (Of course, the final stability conditions obtained by the reduced energy-momentum method are identical to those obtained by the earlier method.) Athough the geometric constructs of Hamiltonian mechanics are used extensively in the *derivation* of the method; they are not required for the *application* of the method. Consequently, the present method is more closely related to the techniques of ARNOLD [1966] and SMALE [1970a, b] than our previous work alluded to above. In fact, it can be viewed as a synthesis of the results of ARNOLD [1966] and SMALE [1970].

§ 0.B. Summary of main results and outline of the paper

Our main result is an explicit and readily implementable criterion for rigorous nonlinear stability of relative equilibria which is formulated exclusively in terms configuration variables. For this purpose, the amended potential V_μ introduced by SMALE plays a key role. We remark that momenta (and momenta variations) play no role in the final result. Furthermore, the symmetry of the mechanical system, induced by the action of a group G, is exploited in a crucial manner to split the tangent space of variations into 'rotational' and 'internal' modes.

A main implication of the preceding result is the separation of the rotational and internal modes present in a coupled mechanical system near a relative equilibrium. This has long been recognized as an important problem in mechanics (see, e.g., WILSON, DECIUS & CROSS [1955] and JELLINEK & LI [1989]). Therefore, we believe that the techniques developed in this work will play an important role in a number of related areas such as bifurcation of relative equilibria and geometric phases.

The outline of the paper is as follows. In Part I, we formulate the method in full generality in the context of simple mechanical systems with symmetry in the sense of SMALE [1970], making use of modern geometric methods. In § 1 we summarize some basic results in the geometric theory of Hamiltonian systems with symmetry, and discuss the notion and characterization of relative equilibria.

In § 2 we introduce and motivate the *reduced* energy-momentum method. In § 3 we apply the method to a concrete, nontrivial example considered by LEWIS & SIMO [1990]: homogeneous elasticity. In this example we have attempted to illustrate the abstract geometric concepts in a concrete setting, and explain the actual implementation of the method. For another nontrivial application of the energy-momentum method, we refer to the thesis of PATRICK [1990], which contains an analysis of the dynamics of two non-symmetric coupled rigid bodies.

In Part II, we illustrate in detail the formulation and application of the method in the specific context of classical three-dimensional nonlinear elasticity. An attempt has been made to correlate the concrete results in Part II with the abstract constructions introduced in Part I. However, either part may be read first, depending on the interests of the reader.

§ 1. Relative equilibria in Hamiltonian systems with symmetry

In this section we provide a concise summary of some basic results on the geometry of Hamiltonian systems with symmetry and the notion of relative equilibria. Our presentation is deliberately brief and restricted to those notions needed for the discussion of the basic method and its subsequent application to elasticity. For comprehensive treatments of the subject, we refer to ABRAHAM & MARSDEN [1978, Chap. 4], ARNOLD [1978, App. 5] and ARNOLD et al. [1988, Chap. 3].

§ 1.A. Hamiltonian systems with symmetry

In what follows, we shall be concerned with an abstract mechanical system with *configuration manifold* Q and *canonical phase space* the cotangent bundle $P = T^*Q$. The phase space is endowed with the *symplectic structure* induced by the *canonical symplectic two-form* Ω.

We assume that the mechanical system under consideration is Hamiltonian, with Hamiltonian function denoted by $H: P \to \mathbb{R}$; H represents the *total energy* of the system. Let $X_H: P \to TP$ be the Hamiltonian vector field associated with H; i.e.,

$$DH(z) \cdot \delta z = \Omega(z)\,(X_H(z), \delta z), \qquad \text{for all } z \in P \text{ and } \delta z \in T_z P. \tag{1.1}$$

If $F_t: [0, T] \times P \to P$ denotes the flow of X_H, then *Hamilton's equations* take the following abstract form

$$\frac{d}{dt} F_t(z) = X(F(z)). \tag{1.2}$$

We assume that the Hamiltonian system possesses *symmetry* induced by a Lie group G, with Lie algebra $\mathcal{G}$, which acts on P by canonical transformations. Denoting by $\Psi_g: P \to P$ the action of G on P for each $g \in G$, we thus assume that

$$H(\Psi_g(z)) = H(z) \qquad \text{for all } g \in G. \tag{1.3}$$

We denote by q an element of the configuration manifold Q and let $p \in T_q^*Q$ be the associated momentum. The tangent space T_qQ at q and T_q^*Q are in duality via a non-degenerate pairing denoted by $\langle \cdot, \cdot \rangle$. We identify the pair $(q, p) \in Q \times T_q^*Q$ with the element $z \in T^*Q$. We shall restrict our attention to systems for which the symplectic action $\Psi_g : P \to P$ is the cotangent lift of an action $(g, q) \mapsto g \cdot q$ of G on Q, for $(g, q) \in G \times Q$ (see ABRAHAM & MARSDEN [1978] for a precise definition of this construction). Associated to the action of G on Q are the *infinitesimal generators*, given by the standard expression

$$\zeta_Q(q) := \frac{d}{d\varepsilon} [\exp [\varepsilon\zeta] \cdot q]_{\varepsilon=0} \quad \text{for } (\zeta, q) \in \mathcal{G} \times Q. \tag{1.4}$$

We use the notation

$$\mathcal{G} \cdot q := \{\zeta_Q(q) \mid \zeta \in \mathcal{G}\} \subset T_qQ \tag{1.5}$$

to denote the tangent space to the group orbit $G \cdot q$. We assume that G acts freely on Q; it follows that $G \cdot q \simeq G$. This implies that $\zeta_Q(q) = 0$ if and only if $\zeta = \mathbf{0}$. For systems possessing material frame-indifference, the relevant group $G = \mathrm{SO}(3)$ is the special orthogonal group, $\mathcal{G} = so(3)$ is the Lie algebra of skew-symmetric matrices with Lie bracket the ordinary matrix commutator, and the group action is *left* matrix multiplication. As discussed below, an element $\xi_Q(q)$ of $\mathcal{G} \cdot q$ is interpreted as a superposed infinitesimal group motion on q. In the context of elasticity, the duality pairing $\langle \cdot, \cdot \rangle$ is chosen as the L_2-pairing (up to boundary terms).

Let $\mathcal{G}^*$ denote the dual of the Lie algebra $\mathcal{G}$. We denote by $\boldsymbol{J} : P \to \mathcal{G}^*$ the *momentum map* for the action of G on P. The momentum map, as shown in Part II, reproduces as special cases the usual angular and linear momenta in the context of three-dimensional elasticity. The following facts regarding the momentum map will be used in our subsequent development of the energy-momentum method.

i. For cotangent lifts, the case of interest here, the function $\boldsymbol{J}$ is determined by the formula

$$\boldsymbol{J}(z) \cdot \xi = \langle \boldsymbol{p}, \xi_Q(q) \rangle \tag{1.6}$$

for all $\xi \in \mathcal{G}$, where a dot denotes the duality pairing between $\mathcal{G}$ and $\mathcal{G}^*$.

ii. The momentum map associated to the lifted action on a cotangent bundle is *equivariant* in the following standard sense. The group G acts on $\mathcal{G}^*$ by the coadjoint action $\mathrm{Ad}^* : G \times \mathcal{G}^* \to \mathcal{G}^*$ with infinitesimal generator denoted by $\mathrm{ad}^* : \mathcal{G} \times \mathcal{G}^* \to \mathcal{G}^*$. Equivariance means that the following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{\boldsymbol{J}} & \mathcal{G}^* \\ \Psi_g \downarrow & & \downarrow \mathrm{Ad}^*_{g^{-1}} \\ P & \xrightarrow[\boldsymbol{J}]{} & \mathcal{G}^* \end{array}$$

Equivalently,

$$\boldsymbol{J}(\Psi_g(z)) = \mathrm{Ad}^*_{g^{-1}}(\boldsymbol{J}(z)) \quad \text{for all } g \in G. \tag{1.7}$$

iii. For any Lie algebra element $\zeta \in \mathscr{G}$, the infinitesimal generator ζ_P: $P \to TP$ of the G-action defined by (1.4) is a Hamiltonian vector field with Hamiltonian function $J_\zeta : P \to \mathbb{R}$ defined in terms of the momentum map by the relation

$$J_\zeta(z) := J(z) \cdot \zeta. \tag{1.8}$$

Therefore, as in (1.1), we have $DJ_\xi(z) \cdot \delta z = \Omega(z)\,(\zeta_Q(z), \delta z)$.

Warning: When referring to various group-related constructs, we shall use the terminology appropriate to the rotation group $SO(3)$, e.g., angular momentum and rigid motions, even though our results are applicable to a general Lie group G. By doing so, we hope to emphasize the natural mechanical interpretations of these abstracts constructions.

§ *1.B. Relative equilibria. Energy-momentum map*

Next, we recall the notion of *relative equilibria*, a terminology due to POINCARÉ. A point $z_e \in P$ is a relative equilibrium of a Hamiltonian system with symmetry group G if the trajectory of Hamilton's equations through z_e is given by

$$F_t(z) = \Psi_{\exp[t\xi_e]}(z_e) \quad \text{for some } \xi_e \in \mathscr{G}, \tag{1.9}$$

a condition which states that the dynamic orbit through the point z_e coincides with the orbit through z_e of the one-parameter subgroup $\exp[\varepsilon \xi_e]$. Differentiation of (1.9) with respect to time, combined with Hamilton's equations (1.2) and the definition of the infinitesimal generator, yields the condition

$$X_H(z_e) = (\xi_e)_P(z_e), \tag{1.10}$$

which constitutes the infinitesimal-counterpart of (1.9).

In classical rigid-body dynamics, relative equilibria are stationary rotations about the principal axes of inertia. For elasticity, possible relative equilibria are characterized in Part II. The following result, known as the *relative equilibrium theorem*, provides a convenient variational characterization of the relative equilibria. Early accounts of this well-known result can be found in SMALE [1970a, b] and MARSDEN & WEINSTEIN [1974]. See also ARNOLD [1978, p. 380], ABRAHAM & MARSDEN [1978, Chap. 4], and ARNOLD et al. [1988, Chap. 3].

Theorem 1.1. *$z_e \in P$ is a relative equilibrium of a mechanical system with Hamiltonian H and momentum map J for the symplectic action of a Lie group G on the phase space $P = T^*Q$ if and only if there is exists a $\xi_e \in \mathscr{G}$ such that (z_e, ξ_e) is a critical point of the energy-momentum functional $H_{\mu_e} : P \times \mathscr{G} \to \mathbb{R}$, defined as*

$$H_{\mu_e}(z, \xi) := H(z) - (J(z) - \mu_e) \cdot \xi, \tag{1.11}$$

where $\mu_e = J(z_e)$ is the value of the momentum map at z_e. (μ_e is assumed to be a regular value of J.)

Thus, relative equilibria are the stationary points of the Hamiltonian H restricted to the level set $J^{-1}(\mu_e) \subset P$; i.e., subject to the constraint of having a constant momentum map. The energy-momentum functional H_{μ_e} in Theorem 1.1 is then simply the associated Lagrangian, in the standard sense of optimization theory, namely, the objective function $H(z)$ plus the constraint function $(J(z) - \mu_e) \cdot \xi$, where the Lie algebra element ξ is a Lagrange multiplier. We remark that the determination of the multiplier ξ_e at equilibrium is part of the optimization process.

According to the method of Lagrange multipliers, the second variation of the *constrained* variational problem is definite if the second variation of the Lagrangian for the associated *unconstrained* variational problem is definite on the subspace $\mathscr{S}$ of variations satisfying the linearized constraint. However, in the present context, even the constrained second variation fails to be definite due to the invariance properties of the energy-momentum function. In particular, since H is G-invariant, the neutral directions of $D_1^2 H_{\mu_e}(z_e, \xi_e)$ are precisely the intersection of the tangent space $\mathscr{G} \cdot z_e$ to the group orbit $G \cdot z_e$ with $\ker [T_{z_e} J]$. A result of MARSDEN & WEINSTEIN [1974] shows that this intersection is given by

$$\mathscr{G}_{\mu_e} \cdot z_e = \mathscr{G} \cdot z_e \cap \ker [T_{z_e} J]. \tag{1.12}$$

Here $\mathscr{G}_{\mu_e} \cdot z_e$ denotes the tangent space to the orbit $G_{\mu_e} \cdot z_e$, where G_{μ_e} denotes the isotropy subgroup of μ_e under the co-adjoint action, with Lie algebra

$$\mathscr{G}_{\mu_e} := \{\zeta \in \mathscr{G} \mid \mathrm{ad}_\zeta^* \, \mu_e = 0\}. \tag{1.13}$$

Note that for any $\boldsymbol{\nu} \in \mathscr{G}$ and $\zeta \in \mathscr{G}_{\mu_e}$

$$\mathrm{ad}_\eta^* \, \mu_e \cdot \zeta = \mu_e \cdot [\eta, \zeta] = -\mu_e \cdot [\zeta, \eta] = -\mathrm{ad}_\zeta^* \, \mu_e \cdot \eta = 0. \tag{1.14}$$

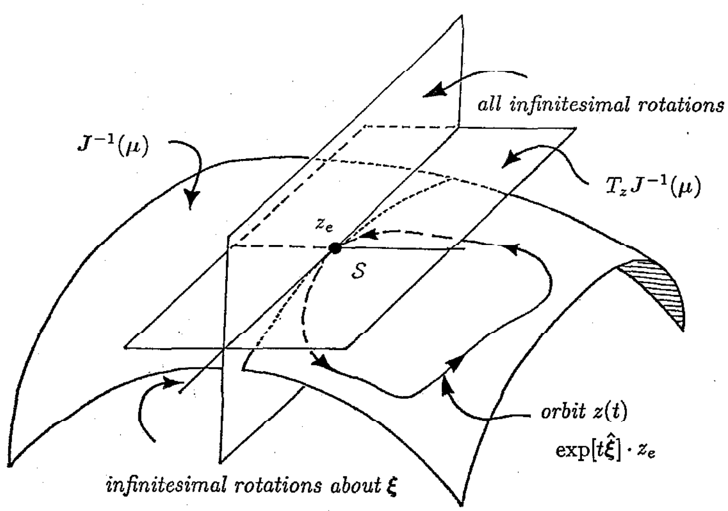

Fig. 1.1. Geometric illustration of the reduction lemma of MARSDEN & WEINSTEIN [1974].

Relation (1.12) follows immediately from the equivariance condition (1.7): Choose $g = \exp[\varepsilon\zeta]$ for an arbitrary $\zeta \in \mathscr{G}$ and differentiate (1.7) to obtain

$$\begin{aligned} T_{z_e}J \cdot \zeta(z_e) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J(\Psi_{\exp[\varepsilon\zeta]}(z_e)) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{Ad}^*_{\exp[-\varepsilon\zeta]}(J(z_e)) = \mathrm{ad}^*_\zeta\, \mu_e. \end{aligned} \tag{1.15}$$

Thus, $\zeta_P(z_e) \in \mathscr{G} \cdot z_e$ is in $\ker[T_{z_e}J]$ if and only if $\mathrm{ad}^*_\zeta\, \mu_e = 0$, i.e., if $\zeta \in \mathscr{G}_{\mu_e}$; hence (1.12) holds. A geometric illustration of (1.12) is given in Figure 1.1. The finite (group-action) version of (1.12) is the fact that the level set $J^{-1}(\mu_e)$ is invariant under the action of G_{μ_e}.

Before stating the precise definition of the subspace $\mathscr{S}$, we summarize the main invariance properties of the energy-momentum functional H_{μ_e} under the action of G.

Proposition 1.2. *The following invariance properties of* $H_{\mu_e}: P \times \mathscr{G} \to \mathbb{R}$ *hold:*
i. *Let G act on $P \times \mathscr{G}$ by $g \cdot (z, \xi) := (\Psi_g(z), \mathrm{Ad}_g\, \xi)$. Then H_{μ_e} is left-invariant under the action of G_{μ_e}.*
ii. *At a relative equilibrium $z_e \in P$ with generator ξ_e and momentum $\mu_e = J(z_e)$, i.e., at a critical point (z_e, ξ_e) of H_{μ_e},*

$$\mathrm{ad}^*_{\xi_e}\, \mu_e = 0, \tag{1.16}$$

so that $\xi_e \in \mathscr{G}_{\mu_e}$.
iii. *Let G act on $P \times \mathscr{G}$ by $g \cdot (z, \xi) := (\Psi_g(z), \xi)$. Then the function $H_{\mu_e}|_{J^{-1}(\mu_e)\times\mathscr{G}}$ obtained by restricting the energy-momentum functional to the level set $J^{-1}(\mu_e)$ is G_{μ_e} invariant.*

Proof. Consider arbitrary elements $g \in G$, $z \in P$, and $\xi \in \mathscr{G}$. By assumption, $H: P \to \mathbb{R}$ is G-invariant, so that $H(\Psi_g(z)) = H(z)$. The equivariance condition (1.7), along with the definition of the adjoint action, thus yields

$$\begin{aligned} H_{\mu_e}(\Psi_g(z), \mathrm{Ad}_g\, \xi) &= H(z) - (\mathrm{Ad}^*_{g^{-1}}\, J(z) - \mu_e) \cdot \mathrm{Ad}_g\, \xi \\ &= H_{\mu_e}(z, \xi) + (\mathrm{Ad}^*_{g^{-1}}\, \mu_e - \mu_e) \cdot \xi. \end{aligned} \tag{1.17}$$

Hence, $H_{\mu_e}(\Psi_g(z), \mathrm{Ad}_g\, \xi) = H_{\mu_e}(z, \xi)$ for all $z \in P$ and $\xi \in \mathscr{G}$ if and only if $\mathrm{Ad}^*_{g^{-1}}\, \mu_e = \mu_e$, which proves i. To prove ii, take $g = \exp[\varepsilon\zeta]$ in (1.17), differentiate (1.17) with respect to ε at $\varepsilon = 0$, and use (1.6) to obtain

$$\begin{aligned} DH_{\mu_e}(z, \xi) \cdot \zeta_P(z) &= J(z) \cdot \mathrm{ad}_\zeta\, \xi \\ &= -J(z) \cdot \mathrm{ad}_\xi\, \zeta \\ &= -\mathrm{ad}^*_\xi\, J(z) \cdot \zeta. \end{aligned} \tag{1.18}$$

At a relative equilibrium, $DH_{\mu_e}(z_e, \xi_e) = 0$. Since $\zeta \in \mathscr{G}$ is arbitrary, ii follows from (1.18). Property (iii) results directly from the equivariance of $\boldsymbol{J}$, since $\boldsymbol{J}(\Psi_g(z)) = \mathrm{Ad}^*_{g^{-1}}(\boldsymbol{J}(z)) = \mathrm{Ad}^*_{g^{-1}} \boldsymbol{\mu}_e = \boldsymbol{\mu}_e$ for $z \in \boldsymbol{J}^{-1}(\boldsymbol{\mu}_e)$ and $g \in G_{\mu_e}$. □

Since $H_{\mu_e}|_{J^{-1}(\mu_e)}$ is G_{μ_e}-invariant, we have $D^2H_{\mu_e}(z_e, \xi_e) \cdot ((\Delta z, \mathbf{0}), (\delta z, \mathbf{0})) = 0$ for any $\Delta z \in \mathscr{G}_{\mu_e} \cdot z_e$ and $\delta z \in T_{z_e}\boldsymbol{J}^{-1}(\boldsymbol{\mu}_e)$. Hence, a relative equilibrium cannot be a strict local extremum of the energy-momentum functional; however, it may be a local extremum modulo the symmetry group G_{μ_e}. To explore this possibility, we introduce the concept of *formal stability*. A relative equilibrium z_e is *formally stable* if the restriction of the second variation $D^2H_{\mu_e}(z_e, \xi_e)$ to a subspace $\mathscr{S} \subset T_{z_e}P$ associated with the constraint $\boldsymbol{J}(z_e) = \boldsymbol{\mu}_e$ is definite. The constraint subspace $\mathscr{S}$ is characterized by the following two conditions:

i. $\mathscr{S}$ is a subspace of the tangent space to the level set $\boldsymbol{J}^{-1}(\boldsymbol{\mu}_e) \subset P$; equivalently, $\mathscr{S} \subset \ker [T_{z_e}\boldsymbol{J}]$.
ii. Neutral directions of $D^2H_{\mu_e}(z_e, \xi_e)$ in $\ker [T_{z_e}\boldsymbol{J}]$ due to group invariance of H_{μ_e} are eliminated from $\mathscr{S}$.

Condition (ii) above and result (1.12) imply that $\mathscr{S}$ is *isomorphic* to the quotient space

$$\mathscr{S} \cong \ker [T_{z_e}\boldsymbol{J}]/(\mathscr{G}_{\mu_e} \cdot z_e). \tag{1.19}$$

The test for *formal stability* of the relative equilibrium, i.e., stability modulo perturbations induced by the group action, then reduces to the following test for definiteness of the second variation $D^2H_{\mu_e}$ on the constrained subspace $\mathscr{S}$:

$z_e \in P$ is formally stable $\Leftrightarrow$

$$D^2H_{\mu_e}(z_e, \xi_e) \cdot ((\delta z, \mathbf{0}), (\delta z, \mathbf{0})) > (<)\, 0 \quad \text{for all } \delta z \in \mathscr{S}. \tag{1.20}$$

If appropriate technical conditions are satisfied, definiteness of the second variation of the energy-momentum functional on the subspace $\mathscr{S}$ implies that the relative equilibrium z_e is nonlinearly stable modulo the action of the isotropy subgroup G_{μ_e}. Finally, observe that definition (1.19) implies that $\mathscr{S}$, regarded as a subspace of $T_{z_e}P$, satisfies

$$\operatorname{codim} [\mathscr{S}] = \operatorname{codim} [\ker [T_{z_e}\boldsymbol{J}]] + \dim [\mathscr{G}_{\mu_e}] \tag{1.21}$$

provided that $\mathscr{G}_{\mu_e}$ is finite-dimensional. For elasticity, we shall see in Part II that $\operatorname{codim} [\mathscr{S}] = 4$.

§ *1.C. The augmented potential and the locked inertia tensor*

In what follows we restrict our attention to *simple mechanical systems*; i.e., Hamiltonian systems with Hamiltonian $H = V + K$ the sum of potential and kinetic energy, which are denoted by $V: Q \to \mathbb{R}$ and $K: P \to \mathbb{R}$, respectively. We further assume that the kinetic energy is a positive-definite quadratic form in the velocity field, and hence is associated with a Riemannian metric on Q denoted by $\langle\cdot,\cdot\rangle_g$. (The subscript g is introduced to distinguish the metric pairing from the duality pairing.)

The metric $\langle\cdot\,\cdot\rangle_g$ induces a inner product on the fibers of T^*Q, denoted by $\langle\cdot,\cdot\rangle_{g^{-1}}$, and constructed as follows. First, we define the Legendre transformation $\mathsf{FL}: TQ \to T^*Q$ by the relation

$$\langle \mathsf{FL}(\delta q), \boldsymbol{v}_q\rangle := \langle \delta q, \boldsymbol{v}_q\rangle_g \quad \text{for all } \boldsymbol{v}_q \in T_qQ. \tag{1.22}$$

Then, we define $\langle\cdot,\cdot\rangle_{g^{-1}}$ by the expression

$$\langle \boldsymbol{p}, \tilde{\boldsymbol{p}}\rangle_{g^{-1}} := \langle \mathsf{FL}^{-1}(\boldsymbol{p}), \mathsf{FL}^{-1}(\tilde{\boldsymbol{p}})\rangle_g \tag{1.23}$$

for all $\boldsymbol{p}, \tilde{\boldsymbol{p}} \in T^*Q$. The kinetic energy is then given by $K = \frac{1}{2}\langle \boldsymbol{p}, \boldsymbol{p}\rangle_{g^{-1}}$. For finite-dimensional systems we have, in coordinates,

$$K := \tfrac{1}{2}\langle \boldsymbol{p}, \boldsymbol{p}\rangle_{g^{-1}} = \tfrac{1}{2} p_i g^{ij}(q)\, p_j,$$
$$\boldsymbol{p} = \mathsf{FL}(\dot{q}) \iff p_i = g_{ij}(q)\, \dot{q}_j, \tag{1.24}$$

where $g(q) = g_{ij}(q)\, dq^i \otimes dq^j$ is a given Riemannian metric on Q, and $g^{ij}(q)$ is the inverse matrix of $g_{ij}(q)$.

We assume that G is the symmetry group of the simple mechanical system under consideration in the following sense:

i. The potential energy $V: Q \to \mathbb{R}$ is *left-invariant* under the action of G on Q i.e.,

$$V(g \cdot q) = V(q), \quad \text{for all } g \in G \text{ and } q \in Q. \tag{1.25}$$

ii. G acts on Q by *isometries* relative to the metric $\langle\cdot,\cdot\rangle_g$.

For simple mechanical systems, the energy-momentum map $H_{\mu_e}: P \times \mathscr{G} \to \mathbb{R}$ can be expressed in an alternative form, which is central to our subsequent developments.

Proposition 1.3. *Given* $\xi \in \mathscr{G}$, *let* $V_\xi: Q \to \mathbb{R}$ *denote the* augmented potential *defined by*

$$V_\xi(q) := V(q) - \tfrac{1}{2}\, \xi \cdot \mathscr{I}(q)\, \xi, \tag{1.26}$$

where the locked inertia tensor $\mathscr{I}(q): \mathscr{G} \to \mathscr{G}^*$ *is defined by*

$$\mathscr{I}(q)\, \boldsymbol{\nu} := J(q, \mathsf{FL}(\boldsymbol{\nu}_Q(q))), \tag{1.27}$$

i.e.,

$$\boldsymbol{\eta} \cdot \mathscr{I}(q)\, \boldsymbol{\nu} := \langle \boldsymbol{\eta}_Q(q), \boldsymbol{\nu}_Q(q)\rangle_g, \quad \forall\ \boldsymbol{\eta}, \boldsymbol{\nu} \in \mathscr{G}. \tag{1.28}$$

Further, define the augmented kinetic energy function $K: P \to \mathbb{R}$ *by the expression*

$$K_\xi(z) := \tfrac{1}{2}\, |\boldsymbol{p} - \mathsf{FL}(\xi_Q(q))|^2_{g^{-1}}. \tag{1.29}$$

Then

$$H_{\mu_e}(z, \xi) = V_\xi(z) + K_\xi(z) + \mu_e \cdot \xi. \tag{1.30}$$

Proof. From definition (1.6) of the momentum map and expression (1.22) for the Legendre transformation, we have

$$\begin{aligned} -J(z) \cdot \xi &= -\langle p, \xi_Q(q)\rangle = -\langle p, \mathbf{FL}(\xi_Q(q))\rangle_{g^{-1}} \\ &= -\langle p, \mathbf{FL}(\xi_Q(q))\rangle_{g^{-1}} + \tfrac{1}{2}\langle \mathbf{FL}(\xi_Q(q)), \mathbf{FL}(\xi_Q(q))\rangle_{g^{-1}} \\ &\quad - \tfrac{1}{2}\langle \xi_Q(q), \xi_Q(q)\rangle_g . \end{aligned} \tag{1.31}$$

Inserting (1.31) into (1.11) and completing a square yields the result. □

Observe that $\mathscr{I}(q) : \mathscr{G} \to \mathscr{G}^*$ is an isomorphism, with inverse denoted by $\mathscr{I}^{-1}$, since the action of G on Q is assumed to be free. The augmented potential V_ξ was introduced by Smale [1970b, Theorem 1.1] in a general context, although in concrete situations it appears to have been used earlier by various authors; see Arnold et al. [1988, pp. 88 and 103]. Expression (1.30) provides a rather practical means of computing the relative equilibria. In fact, we have the following result:

Proposition 1.4. *The critical points* $z_e = (q_e, p_e)$ *of* H_{μ_e} *are characterized by the optimality conditions*

$$\left.\frac{\delta V_\xi}{\delta q}\right|_{q=q_e} = \mathbf{0}, \quad \text{and} \quad p_e = \mathbf{FL}((\xi_e)_Q(q_e)), \tag{1.32}$$

where the multiplier $\xi_e \in \mathscr{G}$ *is given by*

$$\xi_e = \mathscr{I}^{-1}(q_e)\, J(z_e). \tag{1.33}$$

A critical point z_e *of* H_{μ_e} *satisfies*

$$K_{\xi_e}(z_e) = 0 \quad \textit{and} \quad J(z_e) = \mu_e .$$

Proof. That K_{ξ_e}, as defined by (1.29), is a quadratic functional in $p - \mathbf{FL}((\xi_e)_Q(q))$ yields (1.32). Using the definition of momentum map and (1.32), we obtain

$$\begin{aligned} J(z_e) \cdot \zeta &= \langle p_e, \zeta_Q(q_e)\rangle = \langle \mathbf{FL}((\xi_e)_Q(q_e)), \zeta_Q(q_e)\rangle \\ &= \langle (\xi_e)_Q(q_e), \zeta_Q(q_e)\rangle_g = \zeta \cdot \mathscr{I}(q_e)\, \xi_e \end{aligned} \tag{1.34}$$

for any $\zeta \in \mathscr{G}$, which implies (1.33). □

In the result above, $\delta V_{\xi_e}/\delta q$ denotes the functional derivative of V_{ξ_e} defined in terms of the duality pairing by the standard expression

$$DV_{\xi_e}(q) \cdot \delta q = \left\langle \delta q, \frac{\delta V_{\xi_e}}{\delta q} \right\rangle \quad \text{for all } q \in Q. \tag{1.35}$$

In finite dimensions, $\dfrac{\delta V_{\xi_e}}{\delta q}$ coincides with the vector of partial derivatives. In infinite dimensions, however, we may wish to use a variety of spaces in a given duality with TQ in place of the abstract dual space. In this case, the functional derivative $\dfrac{\delta V_{\xi_e}}{\delta q}$ depends on the particular choice of dual space, while the directional derivative DV_{ξ_e} does not. We shall see specific examples of this in Part II.

Propositions 1.3 and 1.4 suggest a compelling mechanical interpretation of the locked inertia tensor illustrated in the following examples.

i. For rigid-body mechanics $(\xi_c, \boldsymbol{\mu}_c) \in \mathscr{G} \times \mathscr{G}^*$ are the spatial angular velocity and the total angular momentum (at equilibrium) of the uniformly rotating state $(q_e, \boldsymbol{p}_e)$, $\mathscr{I}(q_e)$ is the (equilibrium) inertia dyadic in *spatial* coordinates, and (1.33) is the standard relation between angular velocity and angular momentum.
ii. For nonlinear elasticity, the pair $(\xi_e, \boldsymbol{\mu}_e)$ corresponds to the angular velocity and the total angular momentum of an 'equivalent' rigid body with shape defined by $q_e \in Q$; i.e., the elastic body 'locked' at the equilibrium configuration q_e. The locked inertia tensor then becomes the inertia dyadic associated with this 'locked' rigid body. Formula (1.33) is in agreement with the preceding mechanical interpretation. An essentially identical interpretation holds for a general simple mechanical system.

The preceding results can be effectively exploited in specific stability analyses, as suggested by the following observations.

1. In general, it is possible to give a precise estimate of condition (1.20) on the subspace of superposed group motions; this leads to *sharp* stability requirements that generalize the classical stability conditions for a rigid body in stationary (steady) rotation. The precise statement is given in § 2.D.2, and can be viewed as a generalization of a result of ARNOLD [1966].

2. It is also possible to give a precise estimate of the term $D^2 V_{\xi_e}(q_e)$ on the subspace of configuration variations by means of a generalized Poincaré type of inequality which involves the solution of a classical eigenvalue problem. However, a precise estimate for the term $D^2 K_{\xi_e}(z_e)$ requires an explicit determination of the coupling between configuration and momentum variations induced by the momentum constraint. Direct computation of the induced coupling is tractable only in special cases; see for example LEWIS & SIMO [1990]. A non-optimal estimate of condition (1.20) can, nevertheless, be obtained as follows.

From expression (1.29) and the relative equilibrium condition (1.32), it follows that $D^2 K_{\xi_e}(q_e) \geqq 0$ In particular,

$$D^2 K_{\xi_e}(z_e)\,((\mathbf{0}, \boldsymbol{\delta p}))^2 = |\boldsymbol{\delta p}|^2_{g^{-1}} > 0 \tag{1.36}$$

for $\boldsymbol{\delta p} \neq \mathbf{0}$. Consequently, $D^2 H_{\mu_e}(z_e, \xi_e)$ can be bounded from below by the second variation of the augmented potential:

$$D^2 H_{\mu_e}(z_e, \xi_e) \geqq D^2 V_{\xi_e}(q_e). \tag{1.37}$$

Thus, positive-definiteness of $D^2 V_{\xi_e}(q_e)$ ensures positive-definiteness of $D^2 H_{\mu_e}(z_e, \xi_e)$. This criterion, however, is unduly conservative and only provides a *sufficient condition* for formal stability, which *need not be necessary.*

3. The '*reduced*' energy-momentum method developed in the next section bypasses the difficulty alluded to above, and leads to *sharp* and tractable conditions for the stability of relative equilibria.

§ 2. The reduced energy-momentum method

In this section, we present a stability analysis of relative equilibria which offers important advantages in comparison with the procedure outlined in the previous section. In particular:

1. One works directly with the configuration manifold Q, rather than the phase space $P = T^*Q$. Explicit consideration of the momenta is circumvented by exploiting the a priori known positive-definite character of the second variation with respect to momentum perturbations of an appropriate configuration-dependent functional. This feature makes the method particularly attractive for systems in which the dimension of Q is large and explicit numeric or symbolic computations are necessary, e.g., finite-element approximations to infinite-dimensional Hamiltonian systems. The present technique also avoids the inversion of the matrix associated with the metric $\langle \cdot, \cdot \rangle_g$ which defines the kinetic energy of the simple mechanical system. For large finite-dimensional systems the need for this inversion is often viewed as a disadvantage of the Hamiltonian versus the Lagrangian formalism.

2. The space of admissible configuration variations can be decomposed into two subspaces, of dimensions $\dim G - \dim G_{\mu_e}$ and $\dim Q - \dim G$, with respect to which the second variation of the energy-momentum functional block-diagonalizes. This crucial result makes the method particularly easy to apply to concrete examples. A nontrivial application is presented in § 3.

3. The application of the method discussed below does not depend on the explicit characterization of conserved quantities in the reduced space (Casimirs), thus by-passing the difficulties associated with the application of ARNOLD's Energy-Casimir method to infinite-dimensional examples, such as nonlinear elasticity, rods or shells.

4. In contrast with the full energy-momentum method, in the context of the *reduced* energy-momentum method the second variation $D^2 K_{\xi_e}(z_e)$ (with a structure discussed in detail below) can be precisely estimated, leading to sharp stability conditions for relative equilibria.

Before considering the formal development of the reduced energy-momentum method, we first motivate the central construction underlying the method.

§ 2.A. *Motivation: Reparametrization of the energy-momentum map*

The key idea in the development that follows is the reparametrization of the Hamiltonian $H: P \to \mathbb{R}$ in terms of the momentum map $\boldsymbol{J}: P \to \mathscr{G}^*$ by means of the mapping

$$z \in P \mapsto \xi(z) := \mathscr{I}^{-1}(q)\, J(z) \in \mathscr{G}, \tag{2.1}$$

which possesses the following crucial properties:

i. In view of (1.33), the mapping (2.1) gives the value of the Lagrange multiplier at a relative equilibrium, i.e., $\xi(z_e) = \xi_e$. This property is exploited to effectively eliminate the Lagrange multiplier from the energy-momentum functional.
ii. The mapping (2.1) has a compelling mechanical interpretation. If point $z = (q, \boldsymbol{p}) \in P$ has total angular momentum $J(z)$, then $\xi(z)$ gives the corresponding angular velocity of a locked (rigid) body with angular momentum $J(z)$, with shape defined by the configuration q, and with 'equivalent' inertia dyadic determined by the *locked inertia* tensor $\mathscr{I}(q)$.
iii. In view of the preceding interpretation, at a point $z = (q, \boldsymbol{p}) \in P$ the mapping (2.1) determines a *locked velocity* field with associated momenta defined via the Legendre transformation as

$$z = (q, \boldsymbol{p}) \mapsto (q, \boldsymbol{p}_J(z)), \tag{2.2}$$

where

$$p_J(z) := \mathbf{F}L[\xi(z)_Q\,(q)] \in T_q^* Q. \tag{2.3}$$

The reparametrization of the Hamiltonian in terms of the momentum map then takes the following form:

Proposition 2.1. *The Hamiltonian* $H: P \to \mathbb{R}$ *can be expressed as*

$$H(z) = V(q) + \tfrac{1}{2} J(z) \cdot \mathscr{I}^{-1}(q)\, J(z) + \tfrac{1}{2} |\boldsymbol{p} - \boldsymbol{p}_J(z)|^2_{g^{-1}}. \tag{2.4}$$

Proof. Using the definition of the momentum map along with the expression for the Legendre transformation, and expanding the last term in (2.4), one obtains

$$\begin{aligned} \tfrac{1}{2} |\boldsymbol{p} - \boldsymbol{p}_J(z)|^2_{g^{-1}} &= \tfrac{1}{2} |\boldsymbol{p}|^2_{g^{-1}} - \langle \boldsymbol{p}, \xi(z)_Q\,(q)\rangle + \tfrac{1}{2} |\xi(z)_Q\,(q)|^2_g \\ &= \tfrac{1}{2} |\boldsymbol{p}|^2_{g^{-1}} - J(z) \cdot \xi(z) + \tfrac{1}{2} \xi(z) \cdot \mathscr{I}(q)\, \xi(z) \\ &= \tfrac{1}{2} |\boldsymbol{p}|^2_{g^{-1}} - \tfrac{1}{2} J(z) \cdot \mathscr{I}^{-1}(q)\, J(z). \end{aligned} \tag{2.5}$$

Rearranging terms yields

$$V(q) + \tfrac{1}{2} J(z) \cdot \mathscr{I}^{-1}(q)\, J(z) + \tfrac{1}{2} |\boldsymbol{p} - \boldsymbol{p}_J(z)|^2_{g^{-1}} = V(q) + \tfrac{1}{2} |\boldsymbol{p}|^2_{g^{-1}}; \tag{2.6}$$

i.e., the Hamiltonian $H(z)$. □

As pointed out above, the key constraint condition $z \subset J^{-1}(\mu_e)$ can be enforced at the outset by exploiting the preceding reparametrization. In fact, the substitution of (2.1) and (2.4) into expression (1.11) yields

$$\begin{aligned} H_{\mu_e}(z, \xi(z)) &= H(z) - [J(z) - \mu_e] \cdot \xi(z) \\ &= V(q) + [\mu_e - \tfrac{1}{2} J(z)] \cdot \xi(z) + \tfrac{1}{2} |p - p_J(z)|^2_{g^{-1}}. \end{aligned} \tag{2.7}$$

Hence, if $z \in J^{-1}(\mu_e)$, the energy-momentum functional takes the form:

$$\begin{aligned} H_{\mu_e}(z, \zeta) &= V_{\mu_e}(q) + \tfrac{1}{2} |\tilde{p}|^2_{g^{-1}}, \\ V_{\mu_e}(q) &:= V(q) + \tfrac{1}{2} \mu_e \cdot \mathscr{I}^{-1}(q)\, \mu_e, \end{aligned} \tag{2.8}$$

where $\tilde{p} := p - p_J(z)$. Observe that the multiplier $\zeta \in \mathscr{G}$ does not appear on the right-hand side of expression $(2.8)_1$. The functional $V_{\mu_e}(q)$ is referred to as *Smale's amended potential* and plays a central role in our subsequent developments. In particular, we show below that a complete stability analysis of relative equilibria can be carried out by considering only the amended potential. The last term in expression $(2.8)_1$ plays no role in the implementation of this stability analysis. Consequently, the reduced energy-momentum method, which is based on a systematic exploitation of the properties of the amended potential, operates on the configuration space Q and not on the phase space $P = T^*Q$; the momenta play no role in the stability analysis.

More importantly, the stability analysis is remarkably simplified by a block-diagonalization procedure that gives explicit closed-form stability conditions for those configuration variations associated with the action of the symmetry group on the configuration space. These stability conditions contain, as a particular case, those first derived by ARNOLD [1966] for the case in which the configuration space coincides with the symmetry group (for example, the Euler equations for an incompressible inviscid fluid).

§ 2.B. *The reduced energy-momentum map and relative equilibria*

The mapping (2.1), which defines the locked velocity field with associated momentum defined by (2.3), can be interpreted geometrically in terms of a *shifting map* $\Sigma: P \to P$ defined by

$$\Sigma(q, p) := (q, p - p_J(z)). \tag{2.9}$$

We shall refer to $\tilde{p} := p - p_J(z)$ as the shifted momenta. The terminology 'shifting map' assigned to (2.9) is motivated by the following result:

Proposition 2.2. *The map $\Sigma: P \to P$ shifts P onto the level set $J^{-1}(0) \subset P$ of zero total (angular) momentum; i.e.,*

$$\Sigma(P) = J^{-1}(0). \tag{2.10}$$

The values of the Hamiltonian at the shifted and unshifted variables are related as follows

$$H(z) = H(\Sigma(z)) + \tfrac{1}{2}\, J(z) \cdot \mathscr{I}^{-1}(q)\, J(z). \tag{2.11}$$

Proof. Consider an arbitrary $\boldsymbol{\nu} \in \mathscr{G}$, and let $z = (q, \boldsymbol{p}) \in P$. From (2.9), along with expression (1.6) for the momentum map, we have

$$\begin{aligned} J(\Sigma(z)) \cdot \boldsymbol{\nu} &= \langle \tilde{\boldsymbol{p}}, \boldsymbol{\nu}_Q(q) \rangle \\ &= \langle \boldsymbol{p}, \boldsymbol{\nu}_Q(q) \rangle - \langle \boldsymbol{p}_J(z), \boldsymbol{\nu}_Q(q) \rangle \\ &= J(z) \cdot \boldsymbol{\nu} - \langle \mathbb{F}L([\mathscr{I}^{-1}(q)\, J(z)]_Q\,(q)), \boldsymbol{\nu}_Q(q) \rangle. \end{aligned} \tag{2.12}$$

Relations (1.22) and (1.27) then yield

$$\begin{aligned} J(\Sigma(z)) \cdot \boldsymbol{\nu} &= J(z) \cdot \boldsymbol{\nu} - \langle\langle [\mathscr{I}^{-1}(q)\, J(z)]_Q\,(q), \boldsymbol{\nu}_Q(q) \rangle\rangle_g \\ &= J(z) \cdot \boldsymbol{\nu} - \mathscr{I}(q)\, \boldsymbol{\nu} \cdot (\mathscr{I}^{-1}(q)\, J(z)) = 0, \end{aligned} \tag{2.13}$$

for any $\boldsymbol{\nu} \in \mathscr{G}$; hence $J(\Sigma(z)) = \mathbf{0}$. Expression (2.11) follows at once from (2.4) and (2.9). □

With the preceding interpretation in hand, we summarize below the consequences of the reparametrization induced by the mapping (2.1) when the Hamiltonian (and the energy-momentum functional) are restricted to the level set associated with a prescribed value $\boldsymbol{\mu}_e \in \mathscr{G}^*$ of the momentum.

i. The restriction Σ_{μ_e} of the shifting map to the level set $J^{-1}(\boldsymbol{\mu}_e)$ is invertible, with inverse $\Sigma_{\mu_e}^{-1} : J^{-1}(\mathbf{0}) \to J^{-1}(\boldsymbol{\mu}_e)$ given by

$$\Sigma_{\mu_e}^{-1}(q, \tilde{\boldsymbol{p}}) = (q, \tilde{\boldsymbol{p}} + \boldsymbol{p}_{\mu_e}(q)), \tag{2.14}$$

where

$$\boldsymbol{p}_{\mu_e}(q) := \mathbb{F}L[(\mathscr{I}^{-1}(q)\, \boldsymbol{\mu}_e)_Q\,(q)]. \tag{2.15}$$

In mechanical terms, for each $q \in Q$, the map (2.15) gives the momenta of an 'equivalent locked (rigid) body', with shape defined by q and inertia dyadic $\mathscr{I}(q)$, possessing the prescribed (angular) momentum $\boldsymbol{\mu}_e$.

ii. On the level set $J^{-1}(\boldsymbol{\mu}_e)$, the Hamiltonian and the energy-momentum function coincide and are given by expression (2.8). Furthermore, we can regard $\tilde{z} = (q, \tilde{\boldsymbol{p}}) \in J^{-1}(\mathbf{0})$ as a new *independent* variable and view (2.8) as the definition of a reduced Hamiltonian $h_{\mu_e} : J^{-1}(\mathbf{0}) \to \mathbb{R}$ on the level set of zero momentum; i.e.,

$$h_{\mu_e}(\tilde{z}) := V_{\mu_e}(q) + \tfrac{1}{2}\, |\tilde{\boldsymbol{p}}|^2_{g^{-1}}, \tag{2.16}$$

Note that this function is identical to the reduced Hamiltonian arising in the context of reduction of simple mechanical systems; see ABRAHAM & MARSDEN [1978, p. 347].

iii. Given any $z = (q, p) \subset J^{-1}(\mu_e)$ and its image $\tilde{z} := \Sigma(z) = (q, \tilde{p}) \subset J^{-1}(0)$ under the shifting map, the functions $H(z)$, $H_{\mu_e}(z)$, $h_{\mu_e}(z)$ and $V_{\mu_e}(q)$ are connected by the following relations

$$H(z) = H_{\mu_e}(z, \zeta) = V_{\mu_e}(q) + |\tilde{p}|^2_{g^{-1}} = h_{\mu_e}(z), \tag{2.17}$$

which hold for any $\zeta \in \mathcal{G}$. These relations are central to the block-diagonalization result presented in § 2.E.

We are now in a position to discuss in detail the proposed method for the stability analysis of relative equilibria associated with *any* simple mechanical system with symmetry.

§ 2.C. *The reduced energy-momentum method and Smale's amended potential*

It was pointed out in § 2.A that the stability analysis of relative equilibria can be carried out solely in terms of the amended potential V_{μ_e}. The objective of this section is to provide a detailed step-by-step justification of this statement.

2.C.1. Step 1. Relative Equilibria. In view of (2.16), the critical points of $h_{\mu_e}(\tilde{z})$ are defined by the conditions

$$\frac{\delta V_{\mu_e}(q_e)}{\delta q} = \mathbf{0} \quad \text{and} \quad \tilde{p}_e = \mathbf{0}. \tag{2.18}$$

These conditions are related to conditions (1.32) in Proposition 1.4 by noting that

$$\begin{aligned} DV_{\mu_e}(q_e) \cdot \delta q &= DV(q_e) \cdot \delta q - \tfrac{1}{2} (\mathcal{I}^{-1}(q_e)\, \mu_e) \cdot [D\mathcal{I}(q_e) \cdot \delta q]\, (\mathcal{I}^{-1}(q_e)\, \mu_e) \\ &= DV(q_e) \cdot \delta q - \tfrac{1}{2}\, \xi_e \cdot [D\mathcal{I}(q_e) \cdot \delta q]\, \xi_e \\ &= DV_{\xi_e}(q_e) \cdot \delta q, \end{aligned} \tag{2.19}$$

since $\xi_e := \mathcal{I}^{-1}(q_e)\, \mu_e$. Hence the critical points of V_{μ_e} coincide with the critical points of V_{ξ_e} and conditions (2.18) are identical to conditions (1.32).

2.C.2. Step 2. The space of admissible variations. Since H is G invariant, all infinitesimal group motions correspond to neutral variations of H. Here, we are concerned only with the restriction $H_{J^{-1}(\mu_e)}$; therefore, we need only consider group motions that preserve $J^{-1}(\mu_e)$. Equivariance of J, defined by (1.7), implies that given $z \in J^{-1}(\mu_e)$

$$g \cdot z \in J^{-1}(\mu_e) \Leftrightarrow g \in G_{\mu_e}. \tag{2.20}$$

Consequently, we define the space of admissible configuration variations $\mathcal{V} \subset T_{q_e}Q$ as the tangent space to the orbit space Q/G_{μ_e} at q_e. This space can be realized explicitly as the orthogonal complement to $\mathcal{G}_{\mu_e} \cdot q_e$; relative to the metric $\langle \cdot, \cdot \rangle_g$, i.e.,

$$T_{q_e}Q/(\mathcal{G}_{\mu_e} \cdot q_e) \cong \mathcal{V} := \{\delta q \in T_{q_e}Q \mid \langle \delta q, \zeta_Q(q_e) \rangle_g = 0 \quad \forall\ \zeta \in \mathcal{G}_{\mu_e}\}. \tag{2.21}$$

As shown in § 3 and in Part II, for frame-indifferent systems, i.e., for mechanical systems invariant under the left action of SO(3), $\boldsymbol{\mu}_e$ is the total angular momentum at the relative equilibrium z_e, and G_{μ_e} is the subgroup of rotations about $\boldsymbol{\mu}_e$. In this context, $\mathcal{G}_{\mu_e} \cdot z_e$ consists of superposed infinitesimal rotations of z_e *with axis* $\boldsymbol{\mu}_e$, whereas $\mathcal{G} \cdot z_e$ is the space of *all* superposed infinitesimal rotations of z_e.

We now characterize the space of variations for the shifted momenta $\tilde{\boldsymbol{p}}$. Let $z_e = (q_e, \boldsymbol{p}_e) \in \boldsymbol{J}^{-1}(\boldsymbol{\mu}_e)$ be a given relative equilibrium with shifted value $\tilde{z}_e := \Sigma(z_e) = (q_e, \mathbf{0}) \in \boldsymbol{J}^{-1}(\mathbf{0})$ and let $\widetilde{\delta z} = (\delta q, \widetilde{\delta \boldsymbol{p}})$ be an admissible variation of $\tilde{z}_e$. Accordingly, $\widetilde{\delta z} \in T_{\tilde{z}_e} P$ must satisfy the linearized constraint condition $\widetilde{\delta z} \in T_{\tilde{z}_e} \boldsymbol{J}^{-1}(\mathbf{0})$ which, since the momentum map $\boldsymbol{J}$ is linear with respect to the momenta $\boldsymbol{p}$, implies

$$[T_{\tilde{z}_e} \boldsymbol{J} \cdot (\delta q, \widetilde{\delta \boldsymbol{p}})] \cdot \boldsymbol{\eta} = \boldsymbol{J}(q_e, \widetilde{\delta \boldsymbol{p}}) \cdot \boldsymbol{\eta} = \langle \widetilde{\delta \boldsymbol{p}}, \boldsymbol{\eta}_Q(q_e) \rangle = 0 \tag{2.22}$$

for all $\boldsymbol{\eta} \in \mathcal{G}$. Consequently, the constrained subspace of admissible variations, denoted here by the symbol $\mathcal{S}_0$, is given by

$$\mathcal{S}_0 := \{(\delta q, \widetilde{\delta \boldsymbol{p}}) \in \mathcal{V} \times T^*_{q_e} Q \mid \langle \widetilde{\delta \boldsymbol{p}}, \boldsymbol{\eta}_Q(q_e) \rangle = 0 \text{ for } \boldsymbol{\eta} \in \mathcal{G}\}. \tag{2.23}$$

Observe that the condition on $\widetilde{\delta \boldsymbol{p}}$ in (2.23) is equivalent to the requirement that $\widetilde{\delta \boldsymbol{p}} \in T^*_{q_e} Q$ be in the *annihilator* to the tangent space $\mathcal{G} \cdot q_e$ to the orbit $G \cdot q_e$, denoted in what follows by $(\mathcal{G} \cdot q_e)^A$. Therefore, expression (2.23) is equivalent to the characterization of $\mathcal{S}_0$ as

$$\mathcal{S}_0 \approx \mathcal{V} \otimes (\mathcal{G} \cdot q_e)^A. \tag{2.24}$$

Observe finally that $\operatorname{codim}[(\mathcal{G} \cdot q_e)^A] = \dim [\mathcal{G} \cdot q_e]$.

2.C.3. Step 3. The second variation of h_{μ_e}. The expression for $D^2 h_{\mu_e}(\tilde{z}_e)$ is computed from (2.16) and (2.18) as

$$D^2 h_{\mu_e}(\tilde{z}_e) \cdot (\widetilde{\delta z}, \widetilde{\delta z}) = D^2 V_{\mu_e}(q_e)\,(\delta q, \delta q) + |\widetilde{\delta \boldsymbol{p}}|^2_{g^{-1}} \tag{2.25}$$

for all $\widetilde{\delta z} \in \mathcal{S}_0$. In actual implementations, it is more convenient to recast the second variation $D^2 V_{\mu_e}(q_e)$ in terms of the second variation of V_{ξ_e} to avoid inversion of the locked inertia dyadic away from the relative equilibrium. For this purpose, define the map $\operatorname{ident}_{\xi_e} : \mathcal{V} \to \mathcal{G}^*$ by

$$\operatorname{ident}_{\xi_e}(\delta q) := -[D\mathbb{I}(q_e) \cdot \delta q] \cdot \xi_e \tag{2.26}$$

for $\delta q \in T_{q_e} Q$. The chain rule gives

$$[D\mathbb{I}^{-1}(q_e) \cdot \delta q]\, \boldsymbol{\mu}_e = \mathbb{I}^{-1}(q_e) \operatorname{ident}_{\xi_e}(\delta q). \tag{2.27}$$

Differentiation of (2.18), along with the equilibrium condition $\mu_e = \mathscr{I}(q_e)\,\xi_e$, then yields the desired result: i.e.,

$$D^2 V_{\mu_e}(q_e)\,(\delta q, \delta q) = D^2 V_{\xi_e}(q_e)\,(\delta q, \delta q) + \mathrm{ident}_{\xi_e}(\delta q) \cdot \mathscr{I}^{-1}(q_e)\,\mathrm{ident}_{\xi_e}(\delta q). \tag{2.28}$$

This is the most convenient expression of the second variation of V_{μ_e} for applications to specific examples, as illustrated in § 3.

2.C.4. *Step 4. Formal G_{μ_e}-orbital stability.* To assess formal stability of a relative equilibrium $\tilde{z}_e = (q_e, \mathbf{0}) \in J^{-1}(\mathbf{0})$ it suffices to show that $D^2 h_{\mu_e}(\tilde{z}_e)$ is definite (either positive- or negative-definite) on the constrained subspace $\mathscr{S}_0 = \mathscr{V} \otimes (\mathscr{G} \cdot q_e)^A$. Two situations may arise:

i. $\dim\,[(\mathscr{G} \cdot q_e)^A] > 0$ (hence $\dim Q > \dim G$). In this case, there exists $\widetilde{\boldsymbol{\delta p}} \neq \mathbf{0}$ such that $(\mathbf{0}, \widetilde{\boldsymbol{\delta p}}) \in \mathscr{S}_0$. However, since

$$D^2 h_{\mu_e}(\tilde{z}_e)\,((\mathbf{0}, \boldsymbol{\delta p}), (\mathbf{0}, \boldsymbol{\delta p})) = |\boldsymbol{\delta p}|^2_{g^{-1}} > 0, \tag{2.29}$$

it follows that $D^2 h_{\mu_e}(\tilde{z}_e)$ cannot be negative-definite. We therefore conclude that $D^2 h_{\mu_e}(\tilde{z}_e)$ is definite if and only if $D^2 V_{\mu_e}(q_e)$ is positive-definite on $\mathscr{V}$.

ii. $\dim\,[(\mathscr{G} \cdot q_e)^A] = 0$ or, equivalently, $\dim Q = \dim G$. In this case, the constrained subspace $\mathscr{S}_0$ collapses to

$$\mathscr{S}_0 = \{(\delta q, \mathbf{0}) \mid \delta q \in \mathscr{V}\}. \tag{2.30}$$

Since there are no nontrivial momentum variations, positive-definiteness of the kinetic energy does not guarantee the existence of positive second variations.

$D^2 h_{\mu_e}(\tilde{z}_e)$ is positive- (negative-) definite on $\mathscr{S}_0$ if and only if $D^2 V_{\mu_e}(q_e)$ is positive- (negative-) definite on $\mathscr{V}$.

To summarize, we have proved that *positive-definiteness* of $D^2 V_{\mu_e}(q_e)$ implies *formal stability* of the relative equilibrium $z_e = (q_e, \boldsymbol{p}_e) \in P$; i.e.,

$$D^2 V_{\mu_e}(q_e)|_{\mathscr{V} \times \mathscr{V}} > 0 \quad \Rightarrow \quad z_e = (q_e, \boldsymbol{p}_e) \in P \text{ is formally stable.} \tag{2.31}$$

If $\dim Q = \dim G$, then definiteness (either positive or negative) of $D^2 V_{\mu_e}(q_e)|_{\mathscr{V} \times \mathscr{V}}$ implies formal stability. We refer to SIMO, POSBERGH & MARSDEN [1990], and the thesis of PATRICK [1990], for a discussion of the notions formal and orbital stability.

The implementation of this test is remarkably simplified by introducing the following block-diagonalization procedure.

§ 2.D. *Block-diagonalization of the amended potential*

The goal of this subsection is the simplification of the stability test (2.31) for the second variation $D^2 V_{\mu_e}(q_e)$ through the exploitation of the symmetry properties of the amended potential. As alluded to in the introduction, the key idea is the decomposition of the configuration-variation space $\mathscr{V}$ into rigid (group) and internal variations

$$\mathscr{V} = \mathscr{V}_{RIG} \otimes \mathscr{V}_{INT}, \tag{2.32}$$

which we shall specify below. Relative to these subspaces, the second variation $D^2 V_{\mu_e}(q_e)$ takes a block-diagonal structure that results in a substantial simplification of the stability analysis. In fact, the stability conditions associated with $\mathscr{V}_{RIG}$ can be stated in an explicit form that is independent of the potential V. In addition, the actual computation of the stability conditions associated with $\mathscr{V}_{INT}$ is substantially simplified if use is made of alternative expressions for the form $D^2 V_{\mu_e}(q_e)$ given below that only involve inversion of the locked inertia dyadic $\mathscr{I}^{-1}$ at the relative equilibrium configuration q_e.

2.D.1. The decomposition of the space $\mathscr{V}$. Our first objective is the explicit construction of the decomposition (2.32) of the space $\mathscr{V}$ of admissible configuration variations which *decouples*, in the precise sense described below, rotational modes from internal (vibrational) modes of the system. To motivate our developments we remark that for $G = \mathrm{SO}(3)$ with the usual left action, the space $\mathscr{V}_{RIG} \subset \mathscr{V}$ corresponds to the subspace of infinitesimal rigid body variations of the equilibrium configuration $q_e \in Q$. The subspace $\mathscr{V}_{INT} \subset \mathscr{V}$, on the other hand, spans those variations in $\mathscr{V}$ not contained in $\mathscr{V}_{RIG}$, which can be interpreted as the internal deformation modes of the system.

i. *The subspace $\mathscr{V}_{RIG}$ of rotational modes.* In the general case, the space $\mathscr{V}_{RIG}$ is defined as follows. Let $\mathscr{G}_{\mu_e}^{\perp} \subset \mathscr{G}$ be the orthogonal complement of $\mathscr{G}_{\mu_e}$ with respect to the locked inertia metric at the equilibrium configuration q_e, i.e.,

$$\mathscr{G}_{\mu_e}^{\perp} := \{\eta \in \mathscr{G} \mid \eta \cdot \mathscr{I}(q_e)\, \zeta = 0 \quad \forall\, \zeta \in \mathscr{G}_{\mu_e}\}, \tag{2.33}$$

so that $\mathscr{G} = \mathscr{G}_{\mu_e} \otimes \mathscr{G}_{\mu_e}^{\perp}$.

Recall that an infinitesimal G-variation at q_e is of the form $\eta_Q(q_e) \in T_{q_e}Q$ for some $\eta \in \mathscr{G}$. Thus, in view of (2.21) and (2.32), we set

$$\begin{aligned}\mathscr{V}_{RIG} &:= \{\eta_Q(q_e) \in T_{q_e}Q \mid \eta \in \mathscr{G}_{\mu_e}^{\perp}\}\\ &= \{\eta_Q(q_e) \in T_{q_e}Q \mid \langle \eta_Q(q_e), \zeta_Q(q_e)\rangle_g = 0, \quad \forall\, \zeta \in \mathscr{G}_{\mu_e}\}.\end{aligned} \tag{2.34}$$

Note that the requirement that $\eta \in \mathscr{G}_{\mu_e}^{\perp}$ ensures that $\mathscr{V}_{RIG} \subset \mathscr{V}$.

ii. *The space $\mathscr{V}_{INT} \subset \mathscr{V}$ of internal vibration modes.* The subspace $\mathscr{V}_{INT}$ is chosen as a certain 'energy orthogonal' complement to $\mathscr{V}_{RIG}$, in the sense that variations of an appropriately defined energy functional decouple with respect to $\mathscr{V}_{RIG}$ and

$\mathscr{V}_{INT}$. Specifically,

$$\mathscr{V}_{INT} := \{\delta q \in \mathscr{V} \mid \boldsymbol{\eta} \cdot \text{ident}_{\xi_e}(\delta q) = 0 \quad \forall\, \boldsymbol{\eta} \in \mathscr{G}^{\perp}_{\mu e}\}, \tag{2.35}$$

where $\text{ident}_{\xi_e}(\cdot)$ is defined by (2.26). Equivalently, we have

$$\mathscr{V}_{INT} = \{\delta q \in \mathscr{V} \mid \mathscr{I}^{-1}(q_e)\, \text{ident}_{\xi_e}(\delta q) \in \mathscr{G}_{\mu_e}\}. \tag{2.36}$$

For frame-indifferent continuum systems, with $G = SO(3)$, $\mathscr{V}_{INT}$ may be characterized by the property that the restriction of the body force $\delta V_{\xi_e}/\delta q(q_e)$ to $\mathscr{V}_{INT}$ "looks objective". (See Part II and SIMO, POSBERGH & MARSDEN [1990], for a discussion of this interpretation of $\mathscr{V}_{INT}$.)

Next, we determine the form of $\text{ident}_{\xi_e}(\delta q)$ for $\delta q = \boldsymbol{\eta}_Q(q_e) \in \mathscr{V}_{RIG}$ in order to to apply formula (2.28) and compute the second variation of V_{μ_e} restricted to $\mathscr{V}_{RIG}$.

Proposition 2.3. *The following relations hold for all* $\boldsymbol{\eta} \in \mathscr{G}$:

i. *For all* $\boldsymbol{\nu}, \boldsymbol{\omega} \in \mathscr{G}$ *and all* $q \in Q$,

$$\boldsymbol{\omega} \cdot (D\mathscr{I}(q) \cdot \boldsymbol{\eta}_Q(q))\, \boldsymbol{\nu} = [\boldsymbol{\omega}, \boldsymbol{\eta}] \cdot \mathscr{I}(q)\, \boldsymbol{\nu} + [\boldsymbol{\nu}, \boldsymbol{\eta}] \cdot \mathscr{I}(q)\, \boldsymbol{\omega}. \tag{2.37}$$

ii. *At a relative equilibrium* $q_e \in Q$:

$$\text{ident}_{\xi_e}(\boldsymbol{\eta}_Q(q_e)) = \text{ad}^*_{\eta}\, \boldsymbol{\mu}_e + \mathscr{I}(q_e)\, [\boldsymbol{\eta}, \xi_e] \tag{2.38}$$

and

$$0 = \xi_e \cdot \text{ident}_{\xi_e}(\boldsymbol{\eta}_Q(q_e)). \tag{2.39}$$

Proof. Given $\boldsymbol{\eta} \in \mathscr{G}$, define the curve $q_\varepsilon := \exp[\varepsilon \boldsymbol{\eta}] \cdot q_e$. Using the facts that G acts on Q by isometries and that the map taking the Lie-algebra element $\boldsymbol{\omega}$ to the vector field $\boldsymbol{\omega}_Q$ on Q is a Lie-algebra anti-homomorphism, we find that

$$\begin{aligned}
\boldsymbol{\omega} \cdot (D\mathscr{I}(q) \cdot \boldsymbol{\eta}_Q(q))\, \boldsymbol{\nu} &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \langle \boldsymbol{\omega}_Q(q_\varepsilon), \boldsymbol{\nu}_Q(q_\varepsilon) \rangle_g \\
&= \langle \mathscr{L}_{\eta_Q} \boldsymbol{\omega}_Q(q), \boldsymbol{\nu}_Q(q) \rangle_g + \langle \boldsymbol{\omega}_Q(q), \mathscr{L}_{\eta_Q} \boldsymbol{\nu}_Q(q) \rangle_g \\
&= \langle [\boldsymbol{\omega}, \boldsymbol{\eta}]_Q\,(q), \boldsymbol{\nu}_Q(q) \rangle_g + \langle \boldsymbol{\eta}_Q(q), [\boldsymbol{\nu}, \boldsymbol{\eta}]_Q\,(q) \rangle_g \\
&= [\boldsymbol{\omega}, \boldsymbol{\eta}] \cdot \mathscr{I}(q)\, \boldsymbol{\nu} + [\boldsymbol{\nu}, \boldsymbol{\eta}] \cdot \mathscr{I}(q)\, \boldsymbol{\omega},
\end{aligned} \tag{2.40}$$

which proves (2.37). Here $\mathscr{L}$ denotes Lie differentiation.

According to definition (2.26), we have

$$\begin{aligned}
\text{ident}_{\xi_e}(\boldsymbol{\eta}_Q(q_e)) \cdot \boldsymbol{\nu} &= -[D\mathscr{I}(q_e) \cdot \delta q]\; \xi_e \\
&= -[\xi_e, \boldsymbol{\eta}] \cdot \mathscr{I}(q_e)\, \boldsymbol{\nu} - \mathscr{I}^{-1}(q_e)\; \xi_e \cdot [\boldsymbol{\nu}, \boldsymbol{\eta}] \\
&= (\mathscr{I}(q_e)\, [\boldsymbol{\eta}, \xi_e] + \text{ad}^*_{\eta}\, \boldsymbol{\mu}_e) \cdot \boldsymbol{\nu}.
\end{aligned} \tag{2.41}$$

Since $\boldsymbol{\nu} \in \mathscr{G}$ is arbitrary, expression (2.38) follows.

To prove (2.39), observe that setting $\boldsymbol{\nu} = \xi_e$ in (2.41) gives

$$\operatorname{ident}_{\xi_e}(\boldsymbol{\eta}_Q(q_e)) \cdot \xi_e = 2\operatorname{ad}^*_\eta \boldsymbol{\mu}_e \cdot \xi_e = 0 \tag{2.42}$$

according to (1.14) and Proposition 1.2.ii. □

The preceding relations along with the explicit characterization of the space $\mathscr{V}_{INT}$ given in (2.36) lead to the following block-diagonalization result:

Theorem 2.4. *Let* $\mathscr{V}_{RIG}, \mathscr{V}_{INT} \subset \mathscr{V}$ *be defined as above. Then*

$$D^2 V_{\mu_e}(q_e)\,(\boldsymbol{\eta}_Q(q_e), \delta q) = 0 \quad \textit{for } (\boldsymbol{\eta}_Q(q_e), \delta q) \in \mathscr{V}_{RIG} \times \mathscr{V}_{INT}. \tag{2.43}$$

Proof. First, we compute $D^2 V_{\xi_e}(q_e)|_{\mathscr{V}_{RIG} \times \mathscr{V}_{INT}}$. From (1.26), G-invariance of V, and (2.14), we obtain

$$\begin{aligned} D^2 V_{\xi_e}(q_e)\,(\boldsymbol{\eta}_Q(q_e), \delta q) &= -\tfrac{1}{2}\mathscr{L}_{\delta q}(\xi_e \cdot (D\mathscr{I}(q) \cdot \boldsymbol{\eta}_Q(q))\,\xi_e)|_{q=q_e} \\ &= -[\xi_e, \boldsymbol{\eta}] \cdot (D\mathscr{I}(q_e) \cdot \delta q)\,\xi_e \\ &= [\xi_e, \boldsymbol{\eta}] \cdot \operatorname{ident}_{\xi_e}(\delta q). \end{aligned} \tag{2.44}$$

Substitution of (2.44) and (2.38) into (2.28) then yields

$$\begin{aligned} D^2 V_{\mu_e}(q_e)\,(\boldsymbol{\eta}_Q(q_e), \delta q) &= [\xi_e, \boldsymbol{\eta}] \cdot \operatorname{ident}_{\mu_e}(\delta q) \\ &\quad + (\operatorname{ad}^*_\eta \boldsymbol{\mu}_e + \mathscr{I}(q_e)\,[\boldsymbol{\eta}, \xi_e]) \cdot \mathscr{I}^{-1}(q_e)\operatorname{ident}_{\xi_e}(\delta q) \\ &= \operatorname{ad}^*_\eta \boldsymbol{\mu}_e \cdot \mathscr{I}^{-1}(q_e)\operatorname{ident}_{\xi_e}(\delta q) = 0, \end{aligned} \tag{2.45}$$

since $\delta q \in \mathscr{V}_{INT}$ and (2.36) imply that $\mathscr{I}^{-1}(q_e)\operatorname{ident}_{\xi_e}(\delta q) \in \mathscr{G}_{\mu_e}$. □

2.D.2. The amended potential test: Stability conditions. In view of Theorem 2.4 the second variation of the amended potential block-diagonalizes relative to the decomposition (2.32) and the test (2.31) for formal stability of a relative equilibrium is equivalent to the *uncoupled* conditions

$$D^2 V_{\mu_e}(q_e)|_{\mathscr{V}_{RIG} \times \mathscr{V}_{RIG}} > (<)\, 0 \quad \text{and} \quad D^2 V_{\mu_e}(q_e)|_{\mathscr{V}_{INT} \times \mathscr{V}_{INT}} > 0. \tag{2.46}$$

As discussed in 2.C.4, if $\dim Q > \dim G$, i.e., if the space $\mathscr{V}_{INT}$ is nontrivial, then formal stability is possible only if $D^2 V_{\mu_e}(q_e)$ is positive-definite. However, if $\dim Q = \dim G$, then $\mathscr{V}_{RIG} = \mathscr{V}$ and $D^2 V_{\mu_e}(q_e)$ may be either positive- or negative-definite on $\mathscr{V}$. A familiar example of this latter situation is afforded by the classical stability conditions of a rotating rigid body (for which $Q = G = SO(3)$).

Next, we focus our attention on the explicit implementation of the two uncoupled conditions in (2.46), which ensure formal stability.

i. *Stability conditions in* $\mathscr{V}_{RIG}$. Expression (2.45) implies that the second variation of V_{μ_e} restricted to $\mathscr{V}_{RIG}$ is

$$D^2 V_{\mu_e}(q_e)\,(\boldsymbol{\eta}_Q(q_e), \boldsymbol{\nu}_Q(q_e)) = -\operatorname{ad}^*_\eta \boldsymbol{\mu}_e \cdot \mathscr{I}^{-1}(q_e)\operatorname{ident}_{\xi_e}(\boldsymbol{\nu}_Q(q_e)). \tag{2.47}$$

The use of expression (2.38) for $\mathrm{ident}_{\xi_e}(\cdot)$ yields

$$D^2 V_{\mu_e}(q_e)\,(\eta_Q(q_e), \nu_Q(q_e)) = \mathrm{ad}^*_\eta\, \mu_e \cdot \mathscr{I}^{-1}(q_e)\,\mathrm{ad}^*_\nu\, \mu_e + \mathrm{ad}^*_\eta\, \mu_e \cdot [\nu, \xi_e] \tag{2.48}$$

for any $\eta, \nu \in \mathscr{G}^\perp_{\mu_e}$. Expression (2.48), introduced by ARNOLD [1966, eq. (2.44)], will be referred to as the *Arnold form*. In the case that $Q \approx G$, definiteness of the Arnold form is a sufficient condition for formal stability; this condition is the generalization to an arbitrary Lie group G of the rigid body stability conditions for $G = SO(3)$. (See Remark 2. for the derivation of the rigid body stability conditions from (2.48)).

Remarks 2.5. 1. The bilinear form $D^2 V_{\mu_e}(q_e)|_{\mathscr{V}_{RIG}\times\mathscr{V}_{RIG}}$, as given by (2.48) is, of course, symmetric. A direct check of this property rests on Jacobi's identity; e.g.,

$$\begin{aligned}\mathrm{ad}_\eta\,[\nu, \xi_e] &= -[\xi_e, [\eta, \nu]] - [\nu, [\xi_e, \eta]]\\ &= -\mathrm{ad}_{\xi_e}\,[\eta, \nu] + \mathrm{ad}_\nu\,[\eta, \xi_e],\end{aligned} \tag{2.49}$$

which leads to symmetry of the second term in (2.48).

2. As indicated above, for $Q = G = \mathrm{SO}(3)$, definiteness of the Arnold form (2.48) reproduces the classical rigid body stability conditions. To see this, recall that the equilibrium condition $\mu_e = \mathscr{I}(q_e)\,\xi_e$, along with the invariance condition $\mathrm{ad}^*_{\xi_e}\,\mu_e = 0$, implies that

$$\xi_e \times \mu_e = \xi_e \times \mathscr{I}(q_e)\,\xi_e = 0. \tag{2.50}$$

Hence, the angular velocity at equilibrium is an eigenvector of the locked inertia tensor $\mathscr{I}(q_e)$ at the relative equilibrium configuration. Expression (2.48) then reduces to the stability condition

$$\begin{aligned}D^2 V_{\mu_e}(q_e)\,(\eta_Q(q_e), \nu_Q(q_e)) &= (\mu_e \times \eta)\cdot[\mathscr{I}^{-1}(q_e)\,(\mu_e \times \nu) + \nu \times \xi_e]\\ &= (\mu_e \times \eta)\cdot[\mathscr{I}^{-1}(q_e) - \lambda_e^{-1}\,\mathbb{1}]\,(\mu_e \times \nu),\end{aligned} \tag{2.51}$$

where $\mu_e = \mathscr{I}(q_e)\,\xi_e = \lambda_e\,\xi_e$. Definiteness of (2.51) reproduces the classical result that rotations about the smallest or largest axis of $\mathscr{I}(q_e)$ are formally stable.

ii. *Stability conditions on* $\mathscr{V}_{INT}$. It follows from (2.28) and (2.46) that the positive-definiteness of the second variation of V_{μ_e} on $\mathscr{V}$ requires that

$$D^2 V_{\mu_e}(q_e)\,(\delta q, \delta q) = D^2 V_{\xi_e}(q_e)\,(\delta q, \delta q) + \mathrm{ident}_{\xi_e}(\delta q)\cdot\mathscr{I}^{-1}(q_e)\,\mathrm{ident}_{\xi_e}(\delta q) > 0, \tag{2.52}$$

for all $\delta q \in \mathscr{V}_{INT}$. As explained in detail in Part II, in the context of elasticity, condition (2.52) can be precisely estimated in terms of a classical eigenvalue problem closely related to that defining the natural frequencies of the system at the relative equilibrium $q_e \in Q$. We remark that the crucial difference between the *sharp condition* (2.52) and the estimate (1.37), which arises in the context of the energy-momentum method, is the presence of the additional term

$$\mathrm{ident}_{\xi_e}(\delta q) \cdot \mathscr{I}^{-1}(q_e)\, \mathrm{ident}_{\xi_e}(\delta q) \geqq 0 . \tag{2.53}$$

To complete our analysis of the reduced energy-momentum method, we establish the precise conditions under which the split $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$ holds. The result is given in the following

Proposition 2.6. *Let* $D^2 V_{\mu_e}(q_e)|_{\mathscr{V}_{RIG} \times \mathscr{V}_{RIG}}$ *be definite. If either* $\mathscr{G}_{\mu_e}$ *is finite-dimensional or the map* $\mathscr{A} : \mathscr{G} \to \mathscr{G}$ *given by*

$$\mathscr{A}\boldsymbol{\eta} := \mathscr{I}^{-1}(q_e)\, \mathrm{ad}^*_\eta\, \boldsymbol{\mu}_e \tag{2.54}$$

is elliptic (so that the Fredholm alternative holds), then $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$.

Proof. We first show that $\mathscr{V}_{INT} \cap \mathscr{V}_{RIG} = \{\mathbf{0}\}$ if $D^2 V_{\mu_e}(q_e)|_{\mathscr{V}_{RIG} \times \mathscr{V}_{RIG}} > 0$. Assume that there exists a $\boldsymbol{\nu} \in \mathscr{G}^\perp_{\mu_e}$ such that $\boldsymbol{\nu}_Q(q_e) \in \mathscr{V}_{INT}$. According to (2.36), $\boldsymbol{\nu}_Q(q_e) \in \mathscr{V}_{INT}$ implies that $\mathrm{ident}_{\xi_e}(\boldsymbol{\nu}_Q(q_e)) \in \mathscr{G}_{\mu_e}$ and hence

$$0 = \boldsymbol{\eta} \cdot \mathrm{ident}_{\xi_e}(\boldsymbol{\nu}_Q(q_e)) = \boldsymbol{\eta} \cdot (\mathrm{ad}^*_\nu\, \boldsymbol{\mu}_e + \mathscr{I}(q_e)\, [\boldsymbol{\nu}, \boldsymbol{\xi}_e]) \tag{2.55}$$

for all $\boldsymbol{\eta} \in \mathscr{G}^\perp_{\mu_e}$. In particular, (2.48) implies that

$$D^2 V_{\mu_e}(q_e)\, (\boldsymbol{\nu}_Q(q_e), \boldsymbol{\nu}_Q(q_e)) = \mathrm{ad}^*_\nu\, \boldsymbol{\mu} \cdot \mathscr{I}^{-1}(q_e)\, \mathrm{ad}^*_\nu\, \boldsymbol{\mu}_e + \mathrm{ad}^*_\nu\, \boldsymbol{\mu}_e \cdot \mathrm{ad}_\nu\, \boldsymbol{\xi}_e = 0 . \tag{2.56}$$

Thus, we have proved that

$$\boldsymbol{\nu}_Q(q_e) \in \mathscr{V}_{RIG} \cap \mathscr{V}_{INT} \Rightarrow D^2 V_{\mu_e}(q_e)\, (\boldsymbol{\nu}_Q(q_e), \boldsymbol{\nu}_Q(q_e)) = 0 . \tag{2.57}$$

In the case that $\mathscr{G}^\perp_{\mu_e}$ is finite-dimensional, the result now follows from a straightforward argument based on a dimension count. In fact, $\dim \mathscr{V}_{RIG} = \dim \mathscr{G}^\perp_{\mu_e}$, since $\boldsymbol{\eta}_Q(q_e) \neq \mathbf{0}$ for all $\boldsymbol{\eta} \in \mathscr{G}$. Since the subspace $\mathscr{V}_{INT}$ of $\mathscr{V}$ is defined by a set of d equations where $d = \dim \mathscr{G}^\perp_{\mu_e}$, it follows that $\mathrm{codim}_{\mathscr{V}}\, \mathscr{V}_{INT} \leqq \dim \mathscr{G}^\perp_{\mu_e}$. Thus

$$\dim \mathscr{V}_{RIG} + \dim \mathscr{V}_{INT} \geqq \dim \mathscr{V} - \dim (\mathscr{V}_{RIG} \cap \mathscr{V}_{INT}) = \dim \mathscr{V} \tag{2.58}$$

and $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$. If $\mathscr{G}_{\mu_e}$ is infinite-dimensional, the more technical argument given below is required to show that $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$.

Our first step is to show that if the map $\mathscr{A}$ is elliptic with respect to the inner product induced on $\mathscr{G}$ by $\mathscr{I}(q_e)$, then $\mathscr{A}$ maps $\mathscr{G}$ onto $\mathscr{G}^\perp_{\mu_e}$. From (1.14) we have

$$(\mathscr{A}\boldsymbol{\eta}) \cdot \mathscr{I}^{-1}(q_e)\, \boldsymbol{\zeta} = \mathrm{ad}^*_\eta\, \boldsymbol{\mu}_e \cdot \boldsymbol{\zeta} = 0 \tag{2.59}$$

for all $\boldsymbol{\eta} \in \mathscr{G}$ and $\boldsymbol{\zeta} \in \mathscr{G}_{\mu_e}$; hence $\mathscr{A}$ maps $\mathscr{G}$ into $\mathscr{G}^\perp_{\mu_e}$. $\mathscr{A}$ is skew-adjoint with respect to the metric induced by $\mathscr{I}(q_e)$, so ellipticity of $\mathscr{A}$ implies that

$$\mathscr{G}_{\mu_e} \oplus \mathscr{G}^\perp_{\mu_e} = \mathscr{G} = \ker \mathscr{A} \oplus \mathrm{range}\, \mathscr{A} . \tag{2.60}$$

Since $\mathscr{I}(q_e)$ is invertible, $\ker \mathscr{A} = \mathscr{G}_{\mu_e}$ and, hence, range $\mathscr{A} = \mathscr{G}_{\mu_e}^{\perp}$. Now define the map $\vartheta : \mathscr{G}_{\mu_e}^{\perp} \to \mathscr{G}_{\mu_e}^{\perp}$ by

$$\begin{aligned}\vartheta(\boldsymbol{\eta}) &:= \mathbf{P}_{\mu_e}(\mathscr{I}^{-1}(q_e)\,\mathrm{ident}_{\xi_e}(\boldsymbol{\eta}_Q(q_e))) \\ &= \mathscr{I}^{-1}(q_e)\,\mathrm{ad}_\eta^*\,\boldsymbol{\mu}_e + \mathbf{P}_{\mu_e}[\boldsymbol{\eta}, \xi_e],\end{aligned} \tag{2.61}$$

where $\mathbf{P}_{\mu_e}$ denotes the orthogonal projection $\mathbf{P}_{\mu_e} : \mathscr{G} \to \mathscr{G}_{\mu_e}^{\perp}$. To show that the map ϑ is, in fact, an isomorphism assume that there exists $\boldsymbol{\eta} \in \mathscr{G}_{\mu_e}^{\perp}$ such that $\vartheta(\boldsymbol{\eta}) = \mathbf{0}$. From (2.45) and (1.14) we then conclude that

$$\begin{aligned}D^2 V_{\mu_e}(q_e)\,(\boldsymbol{\eta}_Q(q_e), \boldsymbol{\mu}_Q(q_e)) &= -\mathrm{ad}_\eta^*\,\boldsymbol{\mu}_e \cdot \mathscr{I}^{-1}(q_e)\,\mathrm{ident}_{\xi_e}(\boldsymbol{\eta}_Q(q_e)) \\ &= -\mathrm{ad}_\eta^*\,\boldsymbol{\mu}_e \cdot \vartheta(\boldsymbol{\eta}_Q(q_e)) = 0.\end{aligned} \tag{2.62}$$

Hence definiteness of $D^2 V_{\mu_e}(q_e)|_{\mathscr{V}_{RIG} \times \mathscr{V}_{RIG}}$ implies that ϑ is one-to-one. To show that ϑ is surjective, assume that there exists $\boldsymbol{\eta} \in \mathscr{G}_{\mu_e}^{\perp}$ such that $\boldsymbol{\eta} \notin \mathrm{Im}\,\vartheta$. By renormalizing if necessary, we can assume that $\boldsymbol{\eta} \cdot \mathscr{I}(q_e)\,\vartheta(\boldsymbol{\eta}) = 0$. Since $\mathscr{A}$ is surjective and $\ker \mathscr{A} = \mathscr{G}_{\mu_e}$, there exists a $\boldsymbol{\nu} \in \mathscr{G}_{\mu_e}^{\perp}$ such that $\boldsymbol{\eta} := \mathscr{A}\boldsymbol{\nu}$. The calculation given in (2.62) shows that $D^2 V_{\mu_e}(q_e)\,(\boldsymbol{\nu}_Q(q_e)\,(\boldsymbol{\nu}_Q(q_e), \boldsymbol{\nu}_Q(q_e)) = 0$; thus if $D^2 V_{\mu_e}(q_e)|_{\mathscr{V}_{RIG} \times \mathscr{V}_{RIG}}$ is definite, then ϑ is an isomorphism.

Next, we now show that $\mathscr{V}_{RIG} \oplus \mathscr{V}_{INT} = \mathscr{V}$. Given $\delta q \in \mathscr{V}$, define

$$\boldsymbol{\eta} := \vartheta^{-1}(\mathbf{P}_{\mu_e}(\mathscr{I}^{-1}(q_e)\,\mathrm{ident}_{\xi_e}(\delta q))). \tag{2.63}$$

By construction, $\boldsymbol{\eta} \in \mathscr{G}_{\mu_e}^{\perp}$, so $\boldsymbol{\eta}_Q(q_e) \in \mathscr{V}_{RIG}$. Now define $\widetilde{\delta q} := \delta q - \boldsymbol{\eta}_Q(q_e)$; using (2.61), we see that

$$\mathbf{P}_{\mu_e}(\mathscr{I}^{-1}(q_e)\,\mathrm{ident}_{\xi_e}(\widetilde{\delta q})) = \mathbf{P}_{\mu_e}(\mathscr{I}^{-1}(q_e)\,\mathrm{ident}_{\xi_e}(\delta q)) - \vartheta(\boldsymbol{\eta}) = \mathbf{0}. \tag{2.64}$$

Thus $\mathscr{I}^{-1}(q_e)\,\mathrm{ident}_{\xi_e}(\widetilde{\delta q}) \in \mathscr{G}_{\mu_e}$ and hence $\widetilde{\delta q} \in \mathscr{V}_{INT}$. It follows that $\mathscr{V}_{RIG} + \mathscr{V}_{INT} = \mathscr{V}$, which implies the result. □

§ 2.E. The split of $\mathscr{S}$: Block-diagonalization of H_{μ_e}

The decomposition described in the preceding sections is closely related to a decomposition of the space $\mathscr{S} \subset T_{z_e}P$, defined by (1.19), which appears in the original energy-momentum method. In fact, the rigid-internal decomposition was originally developed in this context.

The constrained subspace $\mathscr{S}$ can be explicitly realized as the following subspace of $\ker [T_{z_e}J]$:

$$\mathscr{S} := \{\delta z = (\delta q, \delta p) \in \ker [T_{z_e}J] \mid \delta q \in \mathscr{V}\}. \tag{2.65}$$

We shall show that the decomposition $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$ induces decompositions $\mathscr{S} = \mathscr{S}_{RIG} \oplus \mathscr{S}_{INT}$ and $\mathscr{S}_{INT} = \mathscr{W}_{INT} \oplus \mathscr{W}_{INT}^*$ relative to which the second variation $D^2 H_{\mu_e}(z_e, \xi_e)$ block-diagonalizes. For $G = SO(3)$, $\mathscr{S}_{RIG}$ is interpreted as the space of superposed rigid body variations (modulo motions about $\boldsymbol{\mu}_e$) which satisfy the linearized constant angular momentum condition. $\mathscr{S}_{INT}$ is interpreted as the space of internal or 'deformation' variations; $\mathscr{W}_{INT}$ and $\mathscr{W}^*_{INT}$ correspond to the spaces of internal configuration and momentum variations, respectively.

We first decompose the space $\mathscr{S}_0$ of admissible variations of the reduced Hamiltonian h_{μ_e}. We define the spaces $\mathscr{S}_{0RIG}$ and $\mathscr{W}_{0INT}$ of pure configuration variations by lifting the elements of the spaces $\mathscr{V}_{RIG}$ and $\mathscr{V}_{INT}$ to $\mathscr{S}_0$:

$$\mathscr{S}_{0RIG} := \{(\Delta q, \mathbf{0}) \mid \Delta q \in \mathscr{V}_{RIG}\}, \tag{2.66}$$

$$\mathscr{W}_{0INT} := \{(\delta q, \mathbf{0}) \mid \delta q \in \mathscr{V}_{INT}\}. \tag{2.67}$$

The space $\mathscr{W}^*_{0INT}$ of pure momentum variations is defined by lifting the element of the annihilator $(\mathscr{G} \cdot q_e)^A$ to $\mathscr{S}_0$:

$$\mathscr{W}^*_{0INT} := \{(\mathbf{0}, \widetilde{\delta \mathbf{p}}) \mid \widetilde{\delta \mathbf{p}} \in (\mathscr{G} \cdot q_e)^A\}. \tag{2.68}$$

Using (2.16) and Theorem 2.4, we see immediately that $D^2 h_{\mu_e}(\tilde{z}_e)$ block-diagonalizes with respect to the spaces $\mathscr{S}_{0RIG}$, $\mathscr{W}_{0INT}$, and $\mathscr{W}^*_{0INT}$.

The space $\mathscr{S}$ of admissible variations satisfies $\mathscr{S} = T_{\tilde{z}_e}\Sigma^{-1}_{\mu_e} \cdot \mathscr{S}_0$, i.e.,

$$\mathscr{S} = \{T_{\tilde{z}_e}\Sigma^{-1}_{\mu_e} \cdot \widetilde{\delta z} \mid \widetilde{\delta z} \in \mathscr{S}_0\}. \tag{2.69}$$

Hence we can define a decomposition of $\mathscr{S}$ by taking the images of the subspaces of $\mathscr{S}_0$ under the mapping $T_{\tilde{z}_e}\Sigma^{-1}_{\mu_e}$. Specifically,

$$\mathscr{S}_{RIG} := T_{\tilde{z}_e}\Sigma^{-1}_{\mu_e} \cdot \mathscr{S}_{0RIG} = \{T_{\tilde{z}_e}\Sigma^{-1}_{\mu_e} \cdot (\Delta q, \mathbf{0}) \mid \Delta q \in \mathscr{V}_{RIG}\}, \tag{2.70}$$

$$\mathscr{W}_{INT} := T_{\tilde{z}_e}\Sigma^{-1}_{\mu_e} \cdot \mathscr{W}_{0INT} = \{T_{\tilde{z}_e}\Sigma^{-1}_{z_e} \cdot (\delta q, \mathbf{0}) \mid \delta q \in \mathscr{V}_{INT}\}, \tag{2.71}$$

$$\mathscr{W}^*_{INT} := T_{\tilde{z}_e}\Sigma^{-1}_{\mu_e} \cdot \mathscr{W}^*_{0INT} = \mathscr{W}^*_{0INT}. \tag{2.72}$$

Note that the subspace $\mathscr{S}_{RIG}$ is parametrized solely in terms of elements of $\mathscr{V}_{RIG}$. Put in mechanical terms: If the configuration variations are restricted to be group variations (e.g., rigid body variations), then the linearized constant angular momentum condition completely defines the corresponding momenta variations. Consequently, we have

$$\mathscr{S}_{RIG} \approx \mathscr{V}_{RIG} \approx \mathscr{G}^{\perp}_{\mu_e}. \tag{2.73}$$

The space $\mathscr{S}_{INT}$ of all internal variations consists of elements of $\mathscr{S}$ with configuration component in $\mathscr{V}_{INT}$:

$$\mathscr{S}_{INT} := \mathscr{W}_{INT} \oplus \mathscr{W}^*_{INT} = \{\delta z = (\delta q, \boldsymbol{\delta p}) \in \mathscr{S} \mid \delta q \in \mathscr{V}_{INT}\}. \tag{2.74}$$

Conceptually, $\mathscr{S}_{INT} \approx T^*(Q/G)$, where Q/G is often referred to as the *shape space*. If the Arnold form (2.48) is nonsingular, then $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$ implies that

$$\mathscr{S} = \mathscr{S}_{RIG} \oplus \mathscr{S}_{INT}. \tag{2.75}$$

We now show that the second variation of the energy-momentum functional block-diagonalizes with respect to the subspaces $\mathscr{S}_{RIG}$, $\mathscr{W}_{INT}$, and $\mathscr{W}^*_{INT}$. This result is an immediate consequence of the block-diagonalization result for the second variation of the reduced Hamiltonian $D^2h_{\mu_e}(\tilde{z}_e)$.

Theorem 2.7. (Block-diagonalization theorem). *Let $z_e \in P$ be a relative equilibrium and let $\mathscr{S}_{RIG}$, $\mathscr{S}_{INT}$, $\mathscr{W}_{INT}$, and $\mathscr{W}^*_{INT}$ be constructed as above. Then*

$$D^2H_{\mu_e}(z_e, \xi_e)\,((\Delta z, \mathbf{0}), (\delta z, \mathbf{0})) = 0 \tag{2.76}$$

for all pairs $(\Delta z, \delta z) \in \mathscr{S}_{RIG} \times \mathscr{S}_{INT}$ or $\mathscr{W}_{INT} \times \mathscr{W}^*_{INT}$. Equation (2.76) implies that

$$\begin{array}{cccc} & \mathscr{S}_{RIG} & \mathscr{W}_{INT} & \mathscr{W}^*_{INT} \\ & \mathscr{V}_{RIG} \approx \mathscr{G}/\mathscr{G}_{\mu_e} & \mathscr{V}_{INT} & (\mathscr{G}\cdot q_e)^{\Delta} \\ \delta^2 H_{\mu_e}(z_e, \xi_e)|_{\mathscr{S}\times 0} = & \left[\begin{matrix} \left[\begin{smallmatrix}\text{Arnold}\\ \text{Form}\end{smallmatrix}\right] \\ O \\ O \end{matrix}\right. & \begin{matrix} O \\ \delta^2 V_{\mu_e}(q_e) \\ O \end{matrix} & \left.\begin{matrix} O \\ O \\ \langle\, , \rangle_{\mathbb{g}^{-1}} \end{matrix}\right]. \end{array} \tag{2.77}$$

Each column of the matrix of (2.77) belongs to the space above the column in the first row of spaces. These spaces can be respectively identified with the model spaces in the second row.

Proof. Given variations Δz and $\delta z \in \mathscr{S}$, define $\widetilde{\Delta z} := T_{z_e}\Sigma \cdot \Delta z$ and $\widetilde{\delta z} := T_{z_e}\Sigma \cdot \delta z$. Then $h_{\mu_e} = H_{\mu_e} \circ \Sigma^{-1}_{\mu_e}$ and $DH_{\mu_e}(z_e, \xi_e) = \mathbf{0}$ imply that

$$\begin{aligned} D^2H_{\mu_e}&(z_e, \xi_e)\,((\Delta z, \mathbf{0}), (\delta z, \mathbf{0})) \\ &= D^2H_{\mu_e}((\Sigma^{-1}_{\mu_e}(\tilde{z}_e)), \xi_e)\,((T_{\tilde{z}_e}\Sigma^{-1}_{\mu_e} \cdot \Delta z, \mathbf{0}), (T_{\tilde{z}e}\Sigma^{-1}_{\mu_e} \cdot \delta z, \mathbf{0})) \\ &\quad + DH_{\mu_e}(\Sigma^{-1}_{\mu_e}(\tilde{z}_e), \xi_e) \cdot (T^2_{\tilde{z}_e}\Sigma^{-1}_{\mu_e}(\widetilde{\Delta z}, \widetilde{\delta z}), \mathbf{0}) \\ &= D^2h_{\mu_e}(\tilde{z}_e)\,(\widetilde{\Delta z}, \widetilde{\delta z}). \end{aligned} \tag{2.78}$$

In particular, the block-diagonalization of $D^2h_{\mu_e}(\tilde{z}_e)$ with respect to $\mathscr{S}_{0RIG}$, $\mathscr{W}_{0INT}$, and $\mathscr{W}^*_{0INT}$ implies the block-diagonalization of $D^2H_{\mu_e}(z_e)$ with respect to $\mathscr{S}_{RIG}$, $\mathscr{W}_{INT}$, and $\mathscr{W}^*_{INT}$. □

As a consequence of the block-diagonalization theorem, the test (1.20) for formal G_{μ_e}*-orbital* stability of a relative equilibrium z_e reduces to two *uncoupled* conditions

$$D^2H_{\mu_e}(z_e, \xi_e)|_{\mathscr{S}_{RIG}\times\mathscr{S}_{RIG}} > (<)\, 0, \quad \text{and} \quad D^2H_{\mu_e}(z_e, \xi_e)|_{\mathscr{W}_{INT}\times\mathscr{W}_{INT}} > 0, \tag{2.79}$$

It follows from equation (2.78) that these conditions are identical to those given in (2.46). As before, if $Q \approx G$, then $\mathscr{S}_{INT}$ is trivial and the condition for formal stability reduces to the condition that $D^2 H_{\mu_e}(z_e, \xi_e)|_{\mathscr{S}_{RIG} \times \mathscr{S}_{RIG}}$ be either positive- or negative-definite; if $\mathscr{S}_{INT}$ is nontrivial, then both blocks must be positive-definite.

Remark 2.8. The linearized dynamics at a relative equilibrium z_e are determined by the second variation of the energy-momentum functional and the symplectic two-form at z_e. The decompositions used in our formal stability analysis can be incorporated into the symplectic structure by identifying the canonical symplectic structure restricted to the subspace $\mathscr{S}$ with the Poisson structure on the bundle $P_{\mu_e} \to Q/G$. We refer to MARSDEN, MONTGOMERY, & RATIU [1984] and references cited therein for the abstract computation of the bracket and to SIMO, LEWIS, & MARSDEN [1989] for the derivation of the bracket in this specific case. The derivation makes use of the following chain of identifications:

$$\mathscr{S} \approx \mathscr{G}_{\mu_e} \oplus \mathscr{V}_{INT} \oplus (\mathscr{G} \cdot q_e)^A \approx \mathscr{O}_{\mu_e} \oplus T_{[q_e]}(Q/G) \oplus T^*_{[q_e]}(Q/G), \tag{2.80}$$

where $\mathscr{O}_{\mu_e}$ is the coadjoint orbit of $\boldsymbol{\mu}_e$. If we define the map $\varrho : \mathscr{G}^\perp_{\mu_e} \times \mathscr{V}_{INT} \times (\mathscr{G} \cdot q_e)^A \to \mathscr{S}$ by

$$\varrho(\boldsymbol{\eta}, \delta q, \boldsymbol{\delta p}) := T_{\tilde{z}_e} \Sigma^{-1}_{\mu_e} \cdot (\boldsymbol{\eta}_Q(q) + \delta q, \boldsymbol{\delta p}) \tag{2.81}$$

and define

$$\Omega_\varrho((\boldsymbol{\eta}, \delta q, \boldsymbol{\delta p}), (\tilde{\boldsymbol{\eta}}, \widetilde{\delta q}, \widetilde{\boldsymbol{\delta p}})) := \Omega(z_e)\,(\varrho(\boldsymbol{\eta}, \delta q, \boldsymbol{\delta p}), \varrho(\tilde{\boldsymbol{\eta}}, \widetilde{\delta q}, \widetilde{\boldsymbol{\delta p}})), \tag{2.82}$$

then Ω_ϱ takes the form

$$\begin{aligned}\Omega_\varrho((\boldsymbol{\eta}, \delta q, \boldsymbol{\delta p}), (\tilde{\boldsymbol{\eta}}, \widetilde{\delta q}, \widetilde{\boldsymbol{\delta p}})) = \boldsymbol{\mu}_e \cdot ([\tilde{\boldsymbol{\eta}}, \boldsymbol{\eta}] + [\tilde{\boldsymbol{\eta}}, \boldsymbol{\alpha}(\delta q)] - [\boldsymbol{\eta}, \boldsymbol{\alpha}(\widetilde{\delta q})]) \\ + \langle \widetilde{\boldsymbol{\delta p}}, \delta q \rangle - \langle \boldsymbol{\delta p}, \widetilde{\delta q} \rangle + \boldsymbol{d}\boldsymbol{\alpha}_{\xi_e}(\widetilde{\delta q}, \delta q),\end{aligned} \tag{2.83}$$

where $\boldsymbol{\alpha}_{\xi_e} : Q \to T^*Q$ is given by $\boldsymbol{\alpha}_{\xi_e}(q) := (q, \mathbf{FL}[\xi_{eQ}(q)])$ and $\boldsymbol{\alpha} : TQ \to \mathscr{G}$ is the *simple mechanical connection* given by

$$\boldsymbol{\alpha}(\delta q) := \mathscr{I}^{-1}(q)\, \boldsymbol{J}(\mathbf{FL}(\delta q)). \tag{2.84}$$

In the schematic notation of (2.77) we have

$$\begin{array}{cccc} & \mathscr{G}^\perp_{\mu_e} & \mathscr{V}_{INT} & (\mathscr{G} \cdot q_e)^A \\ & \mathscr{O}_{\mu_e} & T_{[q_e]}(Q/G) & T^*_{[q_e]}(Q/G) \end{array}$$

$$\Omega_\varrho = \left[\begin{array}{ccc} \left[\begin{array}{c}\text{Coadjoint Orbit} \\ \text{Symplectic Form}\end{array}\right] & \left[\begin{array}{c}\text{Internal-Rigid} \\ \text{Coupling}\end{array}\right] & O \\ -\left[\begin{array}{c}\text{Internal-Rigid} \\ \text{Coupling}\end{array}\right] & \multicolumn{2}{c}{\left[\begin{array}{c}\text{Canonical symplectic} \\ \text{form plus a} \\ \text{`magnetic' terms}\end{array}\right]} \\ O & & \end{array}\right]. \tag{2.85}$$

§ 3. Homogeneous elasticity and pseudo-rigid bodies

As an application of the reduced energy-momentum method, we present here a treatment of the stability of the relative equilibria for a mechanical system that can be regarded as a particular case of three-dimensional nonlinear elasticity. The model in question is an extension to general elastic materials of the affine fluid model studied by JACOBI, MACLAURIN, and RIEMANN, among others. We begin with a motivation and a brief summary of the essential terminology, then examine the Hamiltonian structure of the model and finally discuss the stability of relative equilibria. The main objective of this presentation is to provide a detailed illustration of the abstract ideas discussed in the preceding section in the concrete setting of a simple, yet nontrivial example.

§ 3.A. *Homogeneous elasticity: Governing equations*

Let $\mathscr{B} \subset \mathbb{R}^3$ be the reference placement of an elastic body with smooth boundary $\partial\mathscr{B}$. Its particles are denoted by $X \in \mathscr{B}$ and labeled by the position vector $\boldsymbol{X} \in \mathbb{R}^3$ relative to some fixed orthonormal frame. Further, let $\boldsymbol{b} : \mathscr{B} \times [0, T] \to \mathbb{R}^3$ and $\boldsymbol{t} : \partial\mathscr{B} \times [0, T] \to \mathbb{R}^3$ be the body force per unit of mass and the nominal traction vector on the boundary, respectively, with associated *astatic load* tensor given by (see, e.g., TRUESDELL & NOLL [1970, p. 127])

$$\operatorname{vol}[\mathscr{B}]\, A(t) := \int_{\mathscr{B}} \varrho_0 \boldsymbol{b} \otimes \boldsymbol{X}\, d\mathscr{B} + \int_{\partial\mathscr{B}} \boldsymbol{t} \otimes \boldsymbol{X}\, d\mathscr{B}, \tag{3.1}$$

where $\varrho_0 : \mathscr{B} \to \mathbb{R}$ is the reference density, and $[0, T]$ is the time interval of interest. We assume that the elastic body is homogeneous with frame-invariant stored energy function $\overline{W} : GL^+(3) \to \mathbb{R}$, so that $\overline{W}(\Lambda F) = \overline{W}(F)$ for all $\Lambda \in SO(3)$. Here $GL^+(3)$ is the subgroup of the general linear group consisting of real 3×3 matrices with positive determinant. Frame-invariance implies that $\overline{W}$ depends on $\boldsymbol{F}$ through the right Cauchy-Green tensor $\boldsymbol{C} := \boldsymbol{F}^T\boldsymbol{F}$. We write $\overline{W}(\boldsymbol{F}) = W(\boldsymbol{C})$.

The mechanical system of interest here can be approached from different perspectives, all of them leading to essentially the same governing equations. For our purposes, it suffices to regard the present model as a model of *homogeneous elasticity* obtained from general three-dimensional elasticity by restricting attention to *affine deformations*. Under this (strong) assumption the deformation gradient $\boldsymbol{F}$ is independent of $X \in \mathscr{B}$, and the quasi-linear hyperbolic system of nonlinear elastodynamics reduces to the following system of nonlinear ordinary differential equations

$$\left.\begin{aligned} \dot{\boldsymbol{F}} &= \boldsymbol{\Pi}\mathbb{E}^{-1} \\ \dot{\boldsymbol{\Pi}} &= -\boldsymbol{F}[2\partial_C W(\boldsymbol{F}^T\boldsymbol{F})] + A \end{aligned}\right\} \quad \text{in } [0, T], \tag{3.2}$$

where $\mathbb{E}$ is the *convected* inertia dyadic associated with the reference placement $\mathscr{B}$. Denoting by $\mathbf{e}$ the *spatial* inertia dyadic (associated with current placement of the body), we have the expressions

$$\mathrm{E} := \int_{\mathscr{B}} \boldsymbol{X} \otimes \boldsymbol{X} \varrho_0(\boldsymbol{X})\, d\mathscr{B}, \quad \text{and} \quad \mathbf{e} := \boldsymbol{F}\mathrm{E}\boldsymbol{F}^T. \tag{3.3}$$

One refers to $\boldsymbol{\Pi} = \dot{\boldsymbol{F}}\mathrm{E} \in L(3)$ as the momentum associated with a configuration $\boldsymbol{F} \in GL^+(3)$, where $L(3)$ denotes the vector space of 3×3 matrices. The evolution equations (3.2) can also be regarded as an extension of those governing classical rigid dynamics; see e.g., SLAWIANOWSKI [1988], and the comprehensive exposition in COHEN & MUNCASTER [1988], who coined the denomination of *pseudo-rigid bodies.*

With the preceding notation in hand, the kinetic energy K can be expressed as

$$K := \tfrac{1}{2} \int_{\mathscr{B}} \dot{\boldsymbol{F}}\boldsymbol{X} \cdot \dot{\boldsymbol{F}}\boldsymbol{X} \varrho_0(\boldsymbol{X})\, d\mathscr{B} = \tfrac{1}{2} \operatorname{tr} [\dot{\boldsymbol{F}}\mathrm{E}\dot{\boldsymbol{F}}^T] = \tfrac{1}{2} \operatorname{tr} [\boldsymbol{\Pi}\mathrm{E}^{-1}\boldsymbol{\Pi}^T]. \tag{3.4}$$

In view of this relation, we regard K as a function of $\boldsymbol{\Pi} \in L(3)$ and use the notation $K(\boldsymbol{\Pi}) = \frac{1}{2} \langle \boldsymbol{\Pi}, \boldsymbol{\Pi} \rangle_{\mathrm{E}^{-1}}$, where $\langle \cdot, \cdot \rangle_{\mathrm{E}^{-1}}$ defined by (3.4) is referred to as the kinetic energy inner product. In the absence of external loading the astatic tensor $\boldsymbol{A} = \boldsymbol{0}$ and the total energy function takes the form

$$H(\boldsymbol{F}, \boldsymbol{\Pi}) := \tfrac{1}{2} \langle \boldsymbol{\Pi}, \boldsymbol{\Pi} \rangle_{\mathrm{E}^{-1}} + W(\boldsymbol{F}^T\boldsymbol{F}). \tag{3.5}$$

This completes our summary of the governing equations for the present model problem.

§ 3.B. *Hamiltonian structure of homogeneous elasticity*

Equations (3.2) define the evolution of a dynamical system with basic variables the deformation gradient $\boldsymbol{F} \in GL^+(3)$ and the momenta $\boldsymbol{\Pi} \in L(3)$. From the perspective of § 1, homogeneous elasticity then becomes a simple mechanical system with the following characteristics:

i. *Configuration manifold*: $Q = GL^+(3) := \{\boldsymbol{F} : \det [\boldsymbol{F}] > 0\}$. That is, Q is the subgroup of the general linear group, $GL(3)$, consisting of matrices with positive determinant, $(\dim Q = 9)$.
ii. *Canonical phase space*: The phase space is the cotangent bundle $P = T^*GL^+(3)$, realized as $P = \{(\boldsymbol{F}, \boldsymbol{\Pi}) : \boldsymbol{F} \in Q, \text{ and } \boldsymbol{\Pi} \in L(3)\}$, $(\dim P = 18)$.
iii. *Duality pairing*: The pairing between $T^*GL^+(3)$ and $TGL^+(3)$ is denoted by $\langle \cdot, \cdot \rangle$ and defined by the standard matrix inner product $\langle \boldsymbol{\Pi}, \boldsymbol{V} \rangle := \operatorname{tr}[\boldsymbol{\Pi}^T\boldsymbol{V}]$ for all $(\boldsymbol{F}, \boldsymbol{\Pi}) \in T^*_F GL^+(3)$ and $(\boldsymbol{F}, \boldsymbol{V}) \in T_F GL^+(3)$.
iv. *Symplectic two-form*: The canonical symplectic form $\Omega : TP \times TP \to \mathbb{R}$ is defined by the usual skew-symmetric form induced by the duality pairing. Note that Ω does not depend on the base point $(\boldsymbol{F}, \boldsymbol{\Pi}) \in P$.
v. *Canonical Hamiltonian*: The Hamiltonian $H : P \to \mathbb{R}$ is defined by expression (3.5), which is the sum of the kinetic energy and potential energy.
vi. *Symmetry group*: $G = SO(3)$. The Hamiltonian H is invariant under the action of the rotation group; i.e., $H(\boldsymbol{\Lambda F}, \boldsymbol{\Lambda \Pi}) = H(\boldsymbol{F}, \boldsymbol{\Pi})$ for all $(\boldsymbol{F}, \boldsymbol{\Pi}) \in P$ and all $\boldsymbol{\Lambda} \in SO(3)$.

One can readily show that for the simple mechanical system outlined above, the abstract form (1.2) of Hamilton's equations yields the evolution equations (3.2) of homogeneous elasticity.

3.B.1. Momentum map. The symmetry group $G = SO(3)$ induces a momentum map which coincides with the total angular momentum of the system and is computed, in the abstract setting of § 1, as follows. Recall that the Lie algebra $\mathscr{G} = so(3)$ of $SO(3)$ is the linear space of skew-symmetric matrices, which is identified with $\mathbb{R}^3$ via the isomorphism

$$\hat{\xi} \in so(3) \mapsto \xi \in \mathbb{R}^3 \Leftrightarrow \hat{\xi}\boldsymbol{v} = \xi \times \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \mathbb{R}^3, \tag{3.6}$$

where $\times$ denotes the ordinary cross product. Under this isomorphism the Lie bracket becomes the cross product:

$$[\hat{\xi}, \hat{\eta}\,] \in so(3) \mapsto \xi \times \boldsymbol{\eta} \subset \mathbb{R}^3. \tag{3.7}$$

Moreover, the dual of the Lie algebra, $\mathscr{G}^* = so^*(3)$, can also be identified with $\mathbb{R}^3$, and the co-adjoint action of $\mathscr{G}^*$ on $\mathscr{G}$ becomes

$$\mathrm{ad}^*_\mu\, \xi = \boldsymbol{\mu} \times \xi, \qquad \forall\, (\boldsymbol{\mu}, \xi) \in so^*(3) \times so(3) \approx \mathbb{R}^3 \times \mathbb{R}^3. \tag{3.8}$$

With these standard conventions in hand, we note that in the present context $\xi_Q(F) = \hat{\xi}\boldsymbol{F}$. The momentum map $\boldsymbol{J}: P \to so^*(3) \approx \mathbb{R}^3$ given by the abstract formula (1.6) takes the following concrete form:

$$J(\boldsymbol{F}, \boldsymbol{\Pi}) \cdot \xi = \langle \boldsymbol{\Pi}, \hat{\xi}\boldsymbol{F} \rangle = \langle \boldsymbol{\Pi}\boldsymbol{F}^T, \hat{\xi} \rangle = \langle \mathrm{skew}\, [\boldsymbol{\Pi}\boldsymbol{F}^T], \hat{\xi} \rangle, \tag{3.9}$$

for all $\xi \in \mathbb{R}^3$, where $\mathrm{skew}\,[A] := \frac{1}{2}\,[A - A^T]$ denotes the skew-symmetric part of $A \in L(3)$. Therefore, the angular momentum vector is defined by

$$\hat{J}(\boldsymbol{F}, \boldsymbol{\Pi}) = \mathrm{skew}\, [\boldsymbol{\Pi}\boldsymbol{F}^T]. \tag{3.10}$$

This result agrees with the standard expression obtained by a direct computation of the total angular momentum for homogeneous elasticity.

3.B.2. The locked inertia tensor. The $\mathscr{G}$-orbit associated with a configuration $\boldsymbol{F} \in Q$ is given by

$$so(3) \cdot \boldsymbol{F} = \{\hat{\xi}\boldsymbol{F} : \hat{\xi} \in so(3)\}. \tag{3.11}$$

From a mechanical point of view, (3.11) defines all possible infinitesimal rigid motions superposed onto a configuration with deformation gradient $\boldsymbol{F}$. The abstract formula (1.27) then defines the locked inertia tensor $\mathscr{I}(\boldsymbol{F}): \mathbb{R}^3 \to \mathbb{R}^3$ associated with a configuration $\boldsymbol{F} \in Q$ by

$$\xi \cdot \mathscr{I}(\boldsymbol{F})\, \boldsymbol{\eta} = \langle \hat{\xi}\boldsymbol{F}, \hat{\eta}\boldsymbol{F} \rangle_{\mathbb{E}} := -\,\mathrm{tr}\, [\hat{\xi}\boldsymbol{F}\mathbb{E}\boldsymbol{F}^T\hat{\eta}] = -\mathrm{tr}\, [\hat{\eta}\, \hat{\xi}(\boldsymbol{F}\mathbb{E}\boldsymbol{F}^T)]. \tag{3.12}$$

Use of the identity $\hat{\eta}\hat{\xi} = -[(\boldsymbol{\eta} \cdot \xi)\, \mathbf{1} - \xi \otimes \boldsymbol{\eta}]$ along with $(3.3)_2$ yields the result

$$\mathscr{I}(\boldsymbol{F}) = [\mathrm{tr}\, [\mathbf{e}]\, \mathbf{1} - \mathbf{e}] = \boldsymbol{F}[\mathrm{tr}\, [\mathbb{E}C]\, C^{-1} - \mathbb{E}]\, \boldsymbol{F}^T, \tag{3.13}$$

It is clear from this expression that $\mathscr{I}(F)$ is a *spatial* tensor. Observe that its counterpart in the convected description, namely $F^{-1}\mathscr{I}(F)\,F^{-T}$, depends on the convected metric $C = F^T F$, which coincides with the right Cauchy-Green tensor. This result is in agreement with the geometric setting of nonlinear elasticity; see SIMO, MARSDEN & KRISHNAPRASAD [1988]. In keeping with the interpretation of $\mathscr{I}$ as a spatial tensor, we can realize the tangent space of admissible variations at a configuration $F \in Q$ by right-translation of $L(3)$ as

$$T_F Q = \{\delta F = \delta f\, F\colon \delta f \in L(3)\}. \tag{3.14}$$

We shall refer to δF as a material variation, and call $\delta f = \delta F F^{-1}$ a *spatial* variation of the configuration $F \in Q$.

3.B.3. The augmented and amended potentials, and the shifted momenta. The specializations of expressions (1.26) and (2.8) for the augmented and amended potentials to homogeneous elasticity are

$$\begin{aligned} V_\xi(F) &:= W(F^T F) - \tfrac{1}{2}\, \xi \cdot \mathscr{I}(F)\, \xi, \\ V_\mu(F) &:= W(F^T F) + \tfrac{1}{2}\, \mu \cdot \mathscr{I}^{-1}(F)\, \mu, \end{aligned} \tag{3.15}$$

where $\mathscr{I}(F)$ is defined by (3.13). The Legendre transformation $\mathsf{FL}: T_F Q \to T_F^* Q$ defined by formula (1.22) becomes

$$\mathsf{FL}(\delta F) = \delta F \mathbb{E} = [\delta f\, \mathrm{e}]\, F^{-T}. \tag{3.16}$$

Consequently, the momentum component Π_{μ_e} of the map $\alpha_{\mu_e}: Q \to J^{-1}(\mu_e)$ defined by the abstract formula (2.3) is

$$\Pi_{\mu_e}(F) := (\widehat{\mathscr{I}(F)^{-1}\, \mu_e})\, F\mathbb{E} = [(\widehat{\mathscr{I}(F)^{-1}\, \mu_e})\, \mathrm{e}]\, F^{-T}. \tag{3.17}$$

In view of (3.17), the shifted momentum $(F, \tilde{\Pi}) \in J^{-1}(0)$ defined by (2.9) now becomes

$$\tilde{\Pi} := \Pi - \Pi_{\mu_e} = [\Pi F^T - (\widehat{\mathscr{I}(F)^{-1}\, \mu_e})\, \mathrm{e}]\, F^{-T}. \tag{3.18}$$

This expression is consistent with the following alternative characterization of $\tilde{\Pi}$. According to the general theory, $(F, \tilde{\Pi})$ must lie in the level set of zero momentum. This condition is ensured by setting $\tilde{\Pi} = sF^{-T}$, for some symmetric matrix $s = s^T$, since then $\hat{J}(F, \tilde{\Pi}) = \operatorname{skew}[\tilde{\Pi} F^T] = \operatorname{skew}[s] = \mathbf{0}$.

§ 3.C. *Relative equilibria and the rigid-internal decomposition*

According to Proposition 1.4, relative equilibria are critical points of the augmented potential V_ξ which, as shown in § 2, coincide with the critical points of the amended potential V_{μ_e}. Therefore, from $(3.15)_1$ and (3.13) we conclude that $F_e \in Q$ is a critical point of V_{ξ_e} if and only if

$$DV_{\xi_e}(F_e) \cdot \delta F = \langle F_e [2\partial_C W(F_e^T F_e)]\, F_e^T - (|\xi_e|^2\, \mathbf{1} - \xi_e \otimes \xi_e)\, \mathrm{e}_e, \delta f \rangle = 0 \tag{3.19}$$

for all $\delta F = \delta f\, F_e \in T_{F_e}Q$. This relation yields the following local form of the relative equilibrium conditions

$$\boldsymbol{\tau}_e := \boldsymbol{F}_e[2\partial_C W(\boldsymbol{C}_e)]\,\boldsymbol{F}_e^T = [|\xi_e|^2\,\mathbf{1} - \xi_e \otimes \xi_e]\,\mathbf{e}_e, \tag{3.20}$$

where $\boldsymbol{C}_e := \boldsymbol{F}_e^T\boldsymbol{F}_e$, and $\boldsymbol{\tau}_e$ is the (symmetric) spatial Kirchhoff stress tensor. Use of the symmetry condition $\boldsymbol{\tau}_e = \boldsymbol{\tau}_e^T$ in (3.20) then yields the result that

$$\mathbf{e}_e\xi_e = \bar{\lambda}_e\,\xi_e, \qquad \text{where} \quad \bar{\lambda}_e = \frac{\xi_e \cdot \mathbf{e}_e\,\xi_e}{|\xi_e|^2} \geqq 0. \tag{3.21}$$

Thus, for homogeneous elasticity the angular velocity ξ_e at a relative equilibrium $\boldsymbol{F}_e \in Q$ *must be an eigenvector of the spatial inertia dyadic* $\mathbf{e}_e := \boldsymbol{F}_e\mathbb{E}\boldsymbol{F}_e^T$ *and, hence, of the locked inertia dyadic* $\mathscr{I}(\boldsymbol{F}_e)$ *defined by* (3.13).

3.C.1. Decomposition of $\mathscr{V}$. First, we implement the abstract definition (2.21) of the space $\mathscr{V}$ of admissible configuration variations by providing an explicit characterization of the Lie algebra $\mathscr{G}_{\mu_e}$ and its metric-orthogonal complement $\mathscr{G}_{\mu_e}^{\perp}$. Recall that the subalgebra $so(3)_{\mu_e}$ associated to the isotropy subgroup $SO(3)_{\mu_e}$ consists of the elements $\zeta \in so(3)$ such that $\mathrm{ad}_\zeta^*\,\boldsymbol{\mu}_e = \boldsymbol{\mu}_e \times \zeta = \mathbf{0}$. Note that result (3.21), which requires that ξ_e be an eigenvector of $\mathscr{I}(\boldsymbol{F}_e)$ and, hence, parallel to $\boldsymbol{\mu}_e$, is a particular case of this general equilibrium condition. Thus $so(3)_{\mu_e}$ is simply the one-dimensional space spanned by $\boldsymbol{\mu}_e$. Using this fact, we have the characterization

$$so(3)_{\mu_e}^{\perp} := \{\boldsymbol{\eta} \in so(3) \mid \boldsymbol{\mu}_e \cdot \boldsymbol{\eta} = \xi_e \cdot \boldsymbol{\eta} = 0\}. \tag{3.22}$$

Note that $so(3)/so(3)_{\mu_e} \approx so(3)_{\mu_e}^{\perp}$. Now recall that the space $\mathscr{V} \subset T_{F_e}Q$ of admissible variations is specified by (2.21) as the metric-orthogonal complement to the space $so(3)_{\mu_e} \cdot \boldsymbol{F}_e$. Thus

$$\mathscr{V} = \{\boldsymbol{\delta F} = \boldsymbol{\delta f}\,\boldsymbol{F}_e \in T_{F_e}Q \mid \langle \boldsymbol{\delta f}\,\mathbf{e}_e, \hat{\xi}_e\rangle = 0\}. \tag{3.23}$$

From (3.11), (3.22) and the abstract definition (2.34), the space $\mathscr{V}_{RIG}$ of rigid variations becomes

$$\mathscr{V}_{RIG} = \{\hat{\boldsymbol{\eta}}\boldsymbol{F}_e \mid \boldsymbol{\eta} \cdot \boldsymbol{\mu}_e = 0\}. \tag{3.24}$$

Finally, to determine the space $\mathscr{V}_{INT}$ of internal variations, we first compute the first variation of the locked inertia tensor. From (3.13) and $(3.3)_2$ we obtain

$$D\mathscr{I}(\boldsymbol{F}_e) \cdot \boldsymbol{\delta F} = 2\,(\mathrm{tr}\,[\boldsymbol{\delta f}\,\mathbf{e}_e]\,\mathbf{1} - \mathrm{sym}\,[\boldsymbol{\delta f}\,\mathbf{e}_e]). \tag{3.25}$$

Therefore the mapping $\mathrm{ident}_{\xi_e}(\cdot) : \mathscr{V} \to \mathscr{G}^*$ defined by (2.26) now becomes

$$\mathrm{ident}_{\xi_e}(\boldsymbol{\delta F}) = -2\,(\mathrm{tr}\,[\boldsymbol{\delta f}\,\mathbf{e}_e]\,\mathbf{1} - \mathrm{sym}\,[\boldsymbol{\delta f}\,\mathbf{e}_e])\,\xi_e \tag{3.26}$$

and the abstract definition (2.35) of the space of internal variations then specializes to

$$\mathscr{V}_{INT} = \{\boldsymbol{\delta f}\,\boldsymbol{F}_e \in \mathscr{V} \mid \boldsymbol{\eta} \cdot \mathrm{sym}\,[\boldsymbol{\delta f}\,\mathbf{e}_e]\,\xi_e = 0 \quad \forall\, \boldsymbol{\eta} \in so(3)_{\mu_e}^{\perp}\}. \tag{3.27}$$

Since $\dim so(3)_{\mu_e} = 1$, we have $\dim \mathscr{V} - 9 - 1 = 8$. Furthermore, $\dim \mathscr{V}_{RIG} = \dim so(3)^{\perp}_{\mu_e} = 2$ and $\dim \mathscr{V}_{INT} = 6$. Thus, consistent with the abstract setting, we have $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$, provided that the second variation $\delta^2 V_{\mu_e}$ is definite.

§ 3.D. *The second variation of* V_{μ_e}

Let F_e be a critical point of the augmented potential V_{ξ_e}. The test (2.46) for stability of the relative equilibrium associated to F_e^1 requires the computation of the second variation $D^2 V_{\mu_e}(F_e)(\cdot, \cdot): \mathscr{V} \times \mathscr{V} \to \mathbb{R}$ by means of formula (2.52). In the present context, we have

$$D^2 V_{\mu_e}(F_e)(\cdot, \cdot) = D^2 W(C_e)(\cdot, \cdot) + D^2 L_{\xi_e}(F_e)(\cdot, \cdot) + \text{ident}_{\xi_e}(\cdot) \cdot \mathscr{I}^{-1}(F_e)\, \text{ident}_{\xi_e}(\cdot), \tag{3.28}$$

where $L_{\xi_e}(F) := -\frac{1}{2}\, \xi_e \cdot \mathscr{I}(F)\, \xi_e$ can be interpreted as the potential associated with the centrifugal force. The first two terms in (3.28) comprise the second variation of the augmented potential V_{ξ_e} defined by $(3.15)_1$. For convenience, with a slight abuse in notation, we shall often identify a variation δF (or ΔF) with its right translation $\delta f := \delta F F^{-1}$ (or $\Delta f := \Delta F F^{-1}$) in the computations that follow.

i. *Second variation of the stored energy function.* Let C denote the spatial elasticity tensor with components defined by the standard expression

$$\mathsf{C}^{ijkl} = 4 F^i{}_I F^j{}_J F^k{}_K F^l{}_L \frac{\partial^2 W(C)}{\partial C_{IJ}\, \partial C_{KL}}. \tag{3.29}$$

A straightforward computation from (3.19) then gives the following result

$$D^2 W(F) \cdot (\delta F, \Delta F) = \langle F[4\partial^2_{CC} W]\, F^T \Delta F, \delta F\rangle + \langle \Delta F[2\partial_C W], \delta F\rangle = \langle \text{sym}\,[\delta f], \mathsf{C}[\text{sym}\,[\Delta f]]\rangle + \langle \boldsymbol{\tau}, \delta f^T \Delta f\rangle, \tag{3.30}$$

where $\boldsymbol{\tau} := F[2\partial_C W] F^T$ is the spatial Kirchhoff stress tensor. Particularizing the preceding result at a relative equilibrium configuration and using (3.20) yields

$$D^2 W(F_e) \cdot (\delta F, \Delta F) = \langle \text{sym}\,[\delta f], \mathsf{C}_e[\text{sym}\,[\Delta f]]\rangle + \langle |\xi_e|^2\, \mathbf{1} - \xi_e \otimes \xi_e, \delta f^T \Delta f\, \mathbf{e}_e\rangle. \tag{3.31}$$

ii. *Second variation of the augmented potential.* The second variation of the centrifugal potential $L_{\xi_e}(F)$ is readily computed from expression (3.13) for the locked inertia tensor. The result, expressed in the spatial description, takes the form

$$D^2 L_{\xi_e}(F_e)(\delta F, \Delta F) = -\langle |\xi_e|^2\; \mathbf{1} - \xi_e \otimes \xi_e, \delta f\, \mathbf{e}_e\, \Delta f^T\rangle. \tag{3.32}$$

Combining expressions (3.31) and (3.32), we can write the second variation of the augmented potential $V_{\xi_e} = W + L_{\xi_e}$ as

$$D^2 V_{\xi_e}(\boldsymbol{F}_e) \cdot (\delta \boldsymbol{F}, \Delta \boldsymbol{F}) = \langle \text{sym}\,[\delta \boldsymbol{f}], \mathsf{C}_e[\text{sym}\,[\Delta \boldsymbol{f}]]\rangle + G(\Delta \boldsymbol{f}, \delta \boldsymbol{f}), \tag{3.33}$$

where the 'geometric' term $G(\Delta \boldsymbol{f}, \delta \boldsymbol{f})$ is defined by

$$G(\Delta \boldsymbol{f}, \delta \boldsymbol{f}) := \langle |\xi_e|^2 \mathbf{1} - \xi_e \otimes \xi_e, \delta \boldsymbol{f}^T \Delta \boldsymbol{f}\, \mathbf{e}_e - \delta \boldsymbol{f}\, \mathbf{e}_e\, \Delta \boldsymbol{f}^T \rangle. \tag{3.34}$$

This expression can be further simplified by noting that

$$\langle \mathbf{1}, \delta \boldsymbol{f}^T \Delta \boldsymbol{f}\, \mathbf{e}_e - \delta \boldsymbol{f}\, \mathbf{e}_e\, \Delta \boldsymbol{f}^T \rangle = 2\,\text{tr}\,[\text{skew}\,[\delta \boldsymbol{f}^T \Delta \boldsymbol{f}]\, \mathbf{e}_e] = 0. \tag{3.35}$$

Furthermore, the spatial inertia dyadic, $\mathbf{e}_e$ can be replaced by (minus) the locked inertia tensor since, from (3.13), $\mathbf{e} = \text{tr}\,[\mathbf{e}]\,\mathbf{1} - \mathscr{I}_e$ and

$$\langle \xi_e \otimes \xi_e, \delta \boldsymbol{f}^T \Delta \boldsymbol{f}\, \mathbf{1} - \delta \boldsymbol{f}\, \mathbf{1}\, \Delta \boldsymbol{f}^T \rangle = 2\langle \xi_e \otimes \xi_e, \text{skew}\,[\delta \boldsymbol{f}^T \Delta \boldsymbol{f}]\rangle = 0, \tag{3.36}$$

By combining (3.34), (3.35) and (3.36) we arrive at the final result

$$G(\Delta \boldsymbol{f}, \delta \boldsymbol{f}) = \lambda_e(\delta \boldsymbol{f}\, \xi_e) \cdot (\Delta \boldsymbol{f}\, \xi_e) - (\delta \boldsymbol{f}^T \xi_e) \cdot \mathscr{I}_e(\Delta \boldsymbol{f}^T \xi_e), \tag{3.37}$$

where

$$\lambda_e := \frac{\xi_e \cdot \mathscr{I}_e \xi_e}{|\xi_e|^2}. \tag{3.38}$$

According to Theorem 2.4 and the discussion in § 2.D.2, the test for formal stability reduces to an independent test for positive-definiteness of the second variation $D^2 V_{\mu_e}$ on the subspaces $\mathscr{V}_{RIG}$ and $\mathscr{V}_{INT}$. For homogeneous elasticity, this test gives the following conditions.

3.D.1. Stability conditions on $\mathscr{V}_{RIG}$. Consider arbitrary variations in $\mathscr{V}_{RIG}$, which, according to (3.24), are of the form $\delta \boldsymbol{F} = \hat{\boldsymbol{\eta}} \boldsymbol{F}_e$ and $\Delta \boldsymbol{F} = \hat{\zeta} \boldsymbol{F}_e$ for some $\boldsymbol{\eta}, \zeta \in \mathscr{G}_{\mu_e}^{\perp}$. Inserting these expressions into (3.37), (3.33) yields

$$D^2 V_{\xi_e}(\boldsymbol{F}_e)\,(\delta \boldsymbol{F}, \Delta \boldsymbol{F}) = \lambda_e(\boldsymbol{\eta} \times \xi_e) \cdot (\zeta \times \xi_e) - (\boldsymbol{\eta} \times \xi_e) \cdot \mathscr{I}_e(\zeta \times \xi_e). \tag{3.39}$$

To compute the last term in (3.28), we restrict expression (3.26) to $\mathscr{V}_{RIG}$ to obtain

$$\begin{aligned} \text{ident}_{\xi_e}(\delta \boldsymbol{F}) &= (\hat{\boldsymbol{\eta}} \mathbf{e}_e - \mathbf{e}_e \hat{\boldsymbol{\eta}})\, \xi_e = (\mathscr{I}_e \hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\eta}} \mathscr{I}_e)\, \xi_e \\ &= \mathscr{I}_e(\boldsymbol{\eta} \times \xi_e) + \boldsymbol{\mu}_e \times \boldsymbol{\eta}. \end{aligned} \tag{3.40}$$

This result agrees with the general expression (2.38) given in Proposition 2.3. Using (3.40), we obtain

$$\begin{aligned} \text{ident}_{\xi_e}(\delta \boldsymbol{F}) \cdot \mathscr{I}_e^{-1}\, \text{ident}_{\xi_e}(\Delta \boldsymbol{F}) = (\boldsymbol{\mu}_e \times \boldsymbol{\eta}) \cdot \mathscr{I}_e^{-1}(\boldsymbol{\mu}_e \times \zeta) - 2\lambda_e(\boldsymbol{\eta} \times \xi_e) \cdot (\zeta \times \xi_e) \\ + (\boldsymbol{\eta} \times \xi_e) \cdot \mathscr{I}_e(\zeta \times \xi_e). \end{aligned} \tag{3.41}$$

Combining (3.39) and (3.41) according to (3.28) yields

$$D^2 V_{\mu_e}(F_e)\,(\delta F,\,\Delta F) = (\mu_e \times \eta)\cdot(\mathscr{I}_e^{-1} - \lambda_e^{-1}\,\mathbf{1})\,(\mu_e \times \zeta). \tag{3.42}$$

It can be easily shown that (3.42) agrees with the result obtained by particularizing the abstract expression (2.48) for the Arnold form to homogeneous elasticity. In view of (3.42) we conclude that: *A necessary condition for formal stability of a relative equilibrium in homogeneous elasticity is that the rotation axis coincide with the axis of maximal inertia of the equilibrium configuration.*

3.D.2. Stability conditions on $\mathscr{V}_{INT}$. For the analysis that follows, it proves convenient to define $\bar{\delta f} := \delta f\,\mathbf{e}_e$ so that admissible variations in $\mathscr{V}_{INT}$, defined by (3.27), can be written as

$$\delta F = \bar{\delta f}\,\mathbf{e}_e^{-1}\,F_e \in \mathscr{V}_{INT} \quad \text{where} \quad \bar{\delta f} = \hat{\eta} + s, \quad \text{with} \quad \eta \in \mathscr{G}_{\mu_e}^{\perp}. \tag{3.43}$$

Here $s := \operatorname{sym}[\bar{\delta f}]$ and $\hat{\eta} := \operatorname{skew}[\bar{\delta f}]$ denote the (unique) symmetric and skew-symmetric parts of $\bar{\delta f}$. The constraint condition in definition (3.27) of $\mathscr{V}_{INT}$ is then automatically satisfied by requiring that $s\,\xi_e = \lambda_s \xi_e$ for some $\lambda_s \in \mathbb{R}$. With this reparametrization of the space $\mathscr{V}_{INT}$, the second variation of the amended potential takes the following convenient form.

i. We assert that the geometric term $G(\Delta f, \delta f)$ defined by (3.37) depends only on the skew-symmetric parts skew $[\bar{\delta f}]$ and skew $[\bar{\Delta f}]$. In fact, a direct calculation gives

$$\begin{aligned} G(\Delta f, \delta f) &= -\langle \xi_e \otimes \xi_e,\ \mathbf{e}_e^{-1}\,\bar{\delta f}^T\,\bar{\Delta f} - \bar{\delta f}\,\mathbf{e}_e^{-1}\,\Delta f^T \rangle \\ &= -\bar{\lambda}_e^{-1}(\bar{\delta f}\,\xi_e)\cdot(\bar{\Delta f}\,\xi_e) + (\bar{\delta f}^T\,\xi_e)\cdot\mathbf{e}_e^{-1}(\bar{\Delta f}^T\,\xi_e), \end{aligned} \tag{3.44}$$

where $\bar{\lambda}_e$ is the eigenvalue of $\mathbf{e}_e$ associated to the eigenvector ξ_e. Thus, choosing $\delta F = s\mathbf{e}_e^{-1}\,F_e$ with $s = s^T$ (and $s\xi_e = \lambda_s \xi_e$ for some $\lambda_s \in \mathbb{R}$), we obtain

$$G(\Delta f, \delta f) = -\bar{\lambda}_e^{-1}(\lambda_s\,\xi_e)\cdot(\Delta f\,\xi_e) + (\lambda_s\,\xi_e)\cdot\mathbf{e}_e^{-1}(\Delta f^T\,\xi_e) = 0, \tag{3.45}$$

which proves the assertion. Thus, setting $\hat{\eta} = \operatorname{skew}[\bar{\delta f}]$ and $\hat{\zeta} = \operatorname{skew}[\bar{\Delta f}]$, we find that

$$\begin{aligned} G(\Delta f, \delta f) &= -(\eta \times \xi_e)\cdot(\bar{\lambda}_e^{-1}\ \mathbf{1} - \mathbf{e}_e^{-1})\,(\zeta \times \xi_e) \\ &= \eta \cdot \hat{\xi}_e(\bar{\lambda}_e^{-1}\,\mathbf{1} - \mathbf{e}_e^{-1})\,\hat{\xi}_e\,\zeta. \end{aligned} \tag{3.46}$$

ii. We assert that $\operatorname{ident}_{\xi_e}(\delta F)$ restricted to $\mathscr{V}_{INT}$ depends only on the symmetric part sym $[\bar{\delta f}]$ of $\bar{\delta f}$. To see this, note that $\delta F \in \mathscr{V}_{INT}$ implies that $\operatorname{ident}_{\xi_e}(\delta F) = \lambda\,\xi_e$ for some $\lambda \in \mathbb{R}$. Hence, using (3.26), we obtain

$$\begin{aligned} \operatorname{ident}_{\xi_e}(\delta F) &= |\xi_e|^{-2}\,(\xi_e \cdot \operatorname{ident}_{\xi_e}(\delta F))\,\xi_e \\ &= -2|\xi_e|^{-2}\,\langle |\xi_e|^2\,\mathbf{1} - \xi_e \otimes \xi_e,\ \bar{\delta f}\rangle\,\xi_e \\ &= -2\,\langle \mathbf{P}_{\xi_e},\ \operatorname{sym}[\bar{\delta f}]\rangle\,\xi_e, \end{aligned} \tag{3.47}$$

where $\mathbf{P}_{\xi_e} := \mathbf{1} - |\xi_e|^{-2} (\xi_e \otimes \xi_e)$ is the orthogonal projection matrix along the direction defined by $\xi_e \in \mathbb{R}^3$. Therefore, the last term in (3.28) can be written as

$$\begin{aligned} &\operatorname{ident}_{\xi_e}(\boldsymbol{\delta F}) \cdot \mathscr{I}_e^{-1} \operatorname{ident}_{\xi_e}(\boldsymbol{\delta F}) \\ &\quad = 4\langle \mathbf{P}_{\xi_e}, \operatorname{sym}[\bar{\boldsymbol{\delta}}\boldsymbol{f}]\rangle \langle \mathbf{P}_{\xi_e}, \operatorname{sym}[\bar{\boldsymbol{\Delta}}\boldsymbol{f}]\rangle\, \xi_e \cdot \mathscr{I}_e^{-1} \xi_e \\ &\quad = |\xi_e|^2 \frac{4}{\lambda_e} \langle \mathbf{P}_{\xi_e}, \operatorname{sym}[\bar{\boldsymbol{\delta}}\boldsymbol{f}]\rangle \langle \mathbf{P}_{\xi_e}, \operatorname{sym}[\bar{\boldsymbol{\Delta}}\boldsymbol{f}]\rangle . \end{aligned} \tag{3.48}$$

iii. In order to derive an *explicit* matrix expression for $D^2 V_{\mu_e}$ restricted to $\mathscr{V}_{INT}$ we assume, without loss of generality, that ξ_e is parallel to $\boldsymbol{k}$ and introduce the mapping $S_3 : \mathbb{R}^4 \to \operatorname{Sym}(3)$ defined by the matrix expression

$$S_3(\boldsymbol{x}) := \begin{pmatrix} x_1 & x_0 & 0 \\ x_0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, \tag{3.49}$$

relative to the standard basis in $\mathbb{R}^3$. The condition that ξ_e be an eigenvector of $\mathscr{I}(\boldsymbol{F}_e)$ implies that there exist vectors $\boldsymbol{l}$ and $\boldsymbol{I}$ in $\mathbb{R}^4$ such that $\mathbf{e}_e = S_3(\boldsymbol{l})$ and $\mathscr{I}(\boldsymbol{F}_e) = S_3(\boldsymbol{I}) := \operatorname{tr}[\mathbf{e}_e]\,\mathbf{1} - \mathbf{e}_e$. Furthermore, the constraint condition on variations in $\mathscr{V}_{INT}$, which requires that ξ_e be an eigenvector of $\operatorname{sym}[\bar{\boldsymbol{\delta}}\boldsymbol{f}]$, can be explicitly enforced at the outset by setting

$$\bar{\boldsymbol{\delta}}\boldsymbol{f} = (\widehat{(\sigma_1, \sigma_2, 0)} + S_3(\sigma_3, \tfrac{1}{2}(\delta_1 + \delta_2), \tfrac{1}{2}(\delta_1 - \delta_2), \delta_3)) . \tag{3.50}$$

For convenience, we identify $\bar{\boldsymbol{\delta}}\boldsymbol{f}$ defined by (3.50) with a vector $\boldsymbol{v} \in \mathbb{R}^6$ defined as $\boldsymbol{v} = (\sigma_1, \sigma_2, \sigma_3, \delta_1, \delta_2, \delta_3)$. Similarly, we identify $\bar{\boldsymbol{\Delta}}\boldsymbol{f}$ with a vector $\boldsymbol{w} \in \mathbb{R}^6$ defined by the same convention. In view of these results, the second variation of the amended potential can be written in matrix form as

$$D^2 V_{\mu_e}(\boldsymbol{F}_e)(\boldsymbol{\delta F}, \boldsymbol{\Delta F}) = \langle \operatorname{sym}[\boldsymbol{\delta f}], \mathsf{C}_e[\operatorname{sym}[\boldsymbol{\Delta f}]]\rangle + |\xi_e|^2\, \boldsymbol{v} \cdot M\boldsymbol{w}, \tag{3.51}$$

where

$$M = \operatorname{diag}[\hat{\boldsymbol{k}}(\mathbf{e}_e^{-1} \quad \bar{\lambda}_e^{-1}\mathbf{1})\,\hat{\boldsymbol{k}}, 4\bar{\lambda}_e^{-1}, 0, 0] . \tag{3.52}$$

Note that if a basis has been chosen in which $\mathscr{I}(\boldsymbol{F}_e)$ is diagonal, then M can be completely diagonalized. The stability condition that $D^2 V_{\mu_e}(\boldsymbol{F}_e)|_{\mathscr{V}_{INT}}$ be positive-definite can be easily implemented by solving the 6×6 standard eigenvalue problem associated with (3.51). Specific examples are discussed next in the context of isotropic elasticity.

§ 3.E. *Isotropic equilibria*

A homogeneous elastic material is *isotropic* if the stored energy function is invariant under both the left and the *right* $SO(3)$ action, i.e., if

$$\bar{W}(\boldsymbol{FQ}) = \bar{W}(\boldsymbol{F}) = \bar{W}(\boldsymbol{QF}), \quad \forall\, Q \in SO(3) . \tag{3.53}$$

In this case, there exists a function $\Phi : \mathbb{R}^3 \to \mathbb{R}$ such that $W(C) = \Phi(\mathrm{I}, \mathrm{II}, \mathrm{III})$, where $\mathrm{I} = \mathrm{tr}\,(C)$, $\mathrm{II} = \mathrm{cof}\,(C)$, and $\mathrm{III} = \det\,(C)$ are the principal invariants of C. Note that the Hamiltonian $H(F, \boldsymbol{\Pi}) = W(F^T F) + \frac{1}{2} \langle \boldsymbol{\Pi}, \boldsymbol{\Pi} \rangle_{\mathbb{E}^{-1}}$ need *not* be right-$SO(3)$-invariant; the right symmetry of the kinetic energy term is determined by the symmetry of the inertia matrix $\mathbb{E}$. However, we shall see that isotropy introduces additional structure even when the Hamiltonian is not right-invariant. Without loss of generality, we can assume that the equilibrium inertia dyadic is diagonal, i.e., that $\mathbf{e}_e = \mathrm{diag}\,[e_1, e_2, e_3]$ for some positive constants e_i, and that $\boldsymbol{\mu}_e$ is parallel to $\boldsymbol{k}$. The equilibrium conditions (3.20) then take the form $\boldsymbol{\tau}_e = |\xi_e|^2\, \mathrm{diag}\,[e_1, e_2, 0]$. The assumption of isotropy leads to the following consequences:

i. The relative equilibrium conditions imply that $\boldsymbol{\tau}_e$ is diagonal; since the principal directions of the *left* Cauchy-Green tensor $\boldsymbol{B}_e := \boldsymbol{F}_e \boldsymbol{F}_e^T$ and $\boldsymbol{\tau}_e$ coincide by isotropy, it follows that $\boldsymbol{B}_e$ is *diagonal.* Hence $\boldsymbol{B}_e = \mathrm{diag}\,[\boldsymbol{\Lambda}]$, where the Λ_i are the squares of the principal stretches of $\boldsymbol{F}_e$.

ii. By the polar decomposition theorem, there is a unique rotation matrix $\boldsymbol{Q}_e \in SO(3)$ such that $\boldsymbol{F}_e = \sqrt{\boldsymbol{B}_e}\,\boldsymbol{Q}_e$. Recalling expression $(3.3)_1$, we conclude that $\boldsymbol{C}_e$ and $\mathbb{E}$ are simultaneously diagonalizable, since

$$\boldsymbol{C}_e = \boldsymbol{Q}_e^T\, \mathrm{diag}\,[\boldsymbol{\Lambda}]\, \boldsymbol{Q}_e \quad \text{and} \quad \mathbb{E} = \boldsymbol{Q}_e^T\, \mathrm{diag}\,[E]\, \boldsymbol{Q}_e, \tag{3.54}$$

where $E_i = \dfrac{e_i}{\Lambda_i}$. Hence, a condition of equilibrium is that the left (or right) Cauchy-Green tensor and spatial (or convected) inertia matrix share a common eigenbasis.

We now express the equilibrium conditions in a form which explicitly determines the equilibrium values of the derivatives of the stored energy Φ with respect to the principal invariants in terms of the angular velocity ξ_e and the eigenvalues Λ_i and E_i of $\boldsymbol{C}_e$ and $\mathbb{E}$. By substituting this form of the equilibrium conditions into the second variation, we shall obtain an expression for the second variation with the following properties: First, the second variation possesses additional block-diagonal structure beyond that guaranteed by the energy-momentum method; in fact, with the exception of the 3×3 three block associated to variations of the principal stretches, i.e., diagonal variations of $\boldsymbol{F}$, the second variation can be completely diagonalized a priori. Second, the explicit dependence of the second variation on the stored energy function is minimized; the derivatives of the stored energy appear only in the 3×3 block mentioned above. Thus, we obtain simple, general stability conditions with natural physical interpretations.

The equilibrium conditions for a diagonal equilibrium deformation of an isotropic material may be expressed as follows: Let

$$\varkappa_{\xi_e} := \frac{\xi_e^2}{2(\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_3)(\Lambda_2 - \Lambda_3)}, \tag{3.55}$$

$$\varkappa_1 := E_1(\Lambda_2 - \Lambda_3), \quad \varkappa_2 := E_2(\Lambda_1 - \Lambda_3). \tag{3.56}$$

Then F_e is a critical point of the augmented potential V_{ξ_e} if the diagonalization conditions given above are satisfied and

$$\begin{aligned} \Phi_{\mathrm{I}} &= \varkappa_{\xi_e}(\Lambda_1^2\varkappa_1 - \Lambda_2^2\varkappa_2), \\ \Phi_{\mathrm{II}} &= -\varkappa_{\xi_e}(\Lambda_1\varkappa_1 - \Lambda_2\varkappa_2), \\ \Phi_{\mathrm{III}} &= \varkappa_{\xi_e}(\varkappa_1 - \varkappa_2). \end{aligned} \tag{3.57}$$

Using equations (3.57), we can express much of the first term in (3.51) in terms of the angular velocity ξ_e and the reference and current configurations. Specifically, let Ψ express the stored energy as a function of the squares of the principal stretches, i.e., define $\Psi: \mathbb{R}^3 \to \mathbb{R}$ by

$$\Psi(\Lambda) := W(\mathrm{diag}\,[\Lambda]), \tag{3.58}$$

identify the variation $\delta F \in \mathscr{V}_{INT}$ given by

$$\delta F := ((\widehat{\sigma_1, \sigma_2}, 0) + S_3(\sigma_3, \delta_1, \delta_2, \delta_3))\, \mathbf{e}_e^{-1}\, F_e \tag{3.59}$$

with the vectors σ and $\delta \in \mathbb{R}^3$, and analogously identify $\widetilde{\delta F} \in \mathscr{V}_{INT}$ with the vectors $\tilde{\sigma}$ and $\tilde{\delta}$. Note that to first order the principal invariants depend only on the variations determined by the δ_i. Let $I_i = \Sigma_{j \neq i}\, e_j = \Sigma_{j \neq i}\, \Lambda_j E_j$ denote the diagonal entries of the equilibrium inertia dyadic $\mathscr{I}_e$. Then the second variation takes the form

$$D^2 V_{\mu_e}(F_e)\,(\delta F, \widetilde{\delta F}) = |\xi_e|^2\, \sigma \cdot \chi \tilde{\sigma} + 4 \delta \cdot \Delta \tilde{\delta}, \tag{3.60}$$

where

$$\chi = \mathrm{diag}\left[\frac{\Lambda_2\Lambda_3(I_3 - I_2)(E_2 - E_3)}{I_2(\Lambda_2 - \Lambda_3)}, \quad \frac{\Lambda_1\Lambda_3(I_3 - I_1)(E_1 - E_3)}{I_1(\Lambda_1 - \Lambda_3)}, \quad \frac{I_3^2(E_1 - E_2)}{E_1E_2(\Lambda_1 - \Lambda_2)}\right] \tag{3.61}$$

and Δ is a symmetric 3×3 matrix with entries

$$\Delta_{ij} = \frac{\partial^2 \Psi(\Lambda_e)}{\partial \Lambda_i\, \partial \Lambda_j} \frac{1}{E_i E_j} + |\xi_e|^2 R_{ij}, \tag{3.62}$$

where

$$\begin{aligned} &R_{11} = \frac{1}{I_3}, \quad R_{12} = \frac{1}{I_3} - \frac{(E_1 - E_2)}{2E_1(E_2(\Lambda_1 - \Lambda_2)}, \quad R_{22} = \frac{1}{I_3}, \\ &R_{13} = \frac{1}{2E_3(\Lambda_3 - \Lambda_1)}, \quad R_{23} = \frac{1}{2E_3(\Lambda_3 - \Lambda_2)}, \quad R_{33} = 0. \end{aligned} \tag{3.63}$$

Thus, in this case, the second variation is diagonalized up to the 3×3 matrix $\boldsymbol{\Delta}$; this degree of structure enables us to essentially 'read off' the necessary conditions for definiteness of the second variation. Since the system is finite dimensional, formal stability implies nonlinear stability modulo G_{μ_e}. Since Λ_i, E_i, and I_i are all positive, the stability conditions can be summarized as follows:

A rigidly rotating diagonal relative equilibrium of an isotropic material is nonlinearly stable modulo G_{μ_e} *if*

i. *The body is in rotation about the axis 3 of maximal inertia of the equilibrium configuration, i.e., if* $I_3 > I_1$ *and* $I_3 > I_2$.
ii. *The ordering of the principal stretches and the principal axes of the reference configuration agree, i.e., if*

$$\frac{E_i - E_j}{\Lambda_i - \Lambda_j} > 0 \quad \textit{for } i < j \tag{3.64}$$

iii. $\boldsymbol{\Delta}$ *is positive-definite.*

The first set of conditions are simply the usual rigid-body stability conditions associated to $\mathscr{V}_{RIG}$; the second set are obtained from the matrix $\boldsymbol{\chi}$ associated to the off-diagonal internal variations. The equilibrium condition that the convected metric C_e and the convected inertia matrix $\mathbb{E}$ share a common eigenbasis is strengthened to the stability condition that C_e and $\mathbb{E}$ have a common *ordered* eigenbasis, where the ordering is given by the magnitude of the eigenvalues. This condition, which relates the principal stretches to the principal axes of the reference configuration, is strongly reminiscent of the Baker-Ericksen [B-E] inequalities, which require agreement of the ordering of the principal stretches and the force per current area. In fact, for a rotating relative equilibrium, the first two sets of stability conditions imply the [B-E] inequalities. This can be seen as follows: Assume that $I_3 > I_1$ and $I_3 > I_2$. Then

$$\frac{E_i - E_j}{\Lambda_i - \Lambda_j} > 0 \quad \Leftrightarrow \quad \frac{\Psi_i - \Psi_j}{\Lambda_i - \Lambda_j} > 0 \quad \Rightarrow \quad \frac{\Lambda_i \Psi_i - \Lambda_j \Psi_j}{\Lambda_i - \Lambda_j} > 0. \tag{3.65}$$

The inequalities (3.65) are the [B-E] inequalities.

For two relatively simple isotropic materials, the Ciarlet-Geymonat material, with stored energy function

$$\Phi(\mathrm{I}, \mathrm{II}, \mathrm{III}) := \frac{\mu}{2}(\mathrm{I} - \log \mathrm{III}) + \frac{\lambda}{2}(\mathrm{III} - \log \mathrm{III}), \tag{3.66}$$

and the St. Venant-Kirchhoff material, one can show that if the rigid body stability conditions are satisfied, then the inequalities (3.64) hold and $\boldsymbol{\Delta}$ is positive definite. Hence, for these materials. any relative equilibrium rotating about its axis of maximal inertia is nonlinearly stable. For a compressible Mooney-Rivlin material, with stored energy function

$$\Phi(\mathrm{I}, \mathrm{II}, \mathrm{III}) := \frac{\mu}{2}\left(\left(\tfrac{1}{2} + \beta\right)(\mathrm{I} - \log \mathrm{III}) + \left(\tfrac{1}{2} - \beta\right)(\mathrm{II} - 2\log \mathrm{III})\right)$$
$$+ \frac{\lambda}{2}(\mathrm{III} - \log \mathrm{III}), \tag{3.67}$$

both *stable* and *unstable* relative equilibria may exist; see LEWIS & SIMO [1990].

§ 4. Concluding remarks

We have presented a general approach to the rigorous nonlinear stability analysis of relative equilibria in Hamiltonian systems. The present approach, referred to as the *reduced energy-momentum method*, constitutes a substantial extension of results of ARNOLD [1966, 1968], SMALE [1970a, b], HOLM et al. [1985], and others. In particular, the method involves only the configuration space and not the full phase space, enforces automatically the constraint of conservation of momentum without introducing Lagrange multipliers, and does not require explicit knowledge of the conserved quantities in the reduced space (Casimirs).

We have introduced in a general context and have proved and exploited in a crucial manner a new block-diagonalization procedure, which, for rotating systems, decouples '*rotational*' from '*internal*' (deformation) modes in the stability analysis. In fact, the stability conditions associated with the rotational modes are explicit, and reduce to the stability conditions of ARNOLD [1966] for the case in which the configuration space coincides with the symmetric group. In particular, for the rotation group, these are the classical rigid-body stability conditions.

In Part II of this work we shall discuss, apply and further develop the techniques introduced above in the concrete setting of nonlinear elasticity.

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