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Controlling Homoclinic Orbits[1]

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Abstract. In this paper we analyze various control-theoretic aspects of a nonlinear control system possessing homoclinic or heteroclinic orbits. In particular, we show that for a certain class of nonlinear control system possessing homoclinic orbits, a control can be found such that the system exhibits arbitrarily long periods in a neighborhood of the homoclinic. We then apply these ideas to bursting phenomena in the near wall region of a turbulent boundary layer. Our analysis is based on a recently developed finite-dimensional model of this region due to Aubry, Holmes, Lumley, and Stone.

1. Introduction

The goal of this paper is to provide a method for using control theory specifically in certain areas of fluid dynamics, but also in mechanics rather generally. The technique is based on the control of orbits near homoclinic or heteroclinic trajectories of dynamical systems.

In fluid dynamics we show how the techniques of modern control theory can be brought to bear on the model by Aubry *et al.* (1988) of the turbulent boundary-layer system as a finite-dimensional dynamical system. With this in mind we sketch both the essence of the modeling procedure of Aubry *et al.* and the essentials of the nonlinear control theory that we use. Further, as an illustration of both the key features of the dynamical system under discussion and the control-theoretic techniques, we discuss first the control of a simple model problem—the inverted nonlinear pendulum. This model acts as a bridge between the control theory and the fluid dynamics.

One of our motivations is eventually to understand how to control the dynamics in the near wall region of a turbulent boundary layer. The dynamics of this region has recently been analyzed by Aubry *et al.* (1988). In this paper the instantaneous field is expanded in a basis of eigenfunctions using the proper orthogonal decomposition of Lumley (1967, 1970, 1981). This expansion turns out to be particularly suitable for flows in which large coherent structures contain a major fraction of the

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energy. The wall region of a turbulent boundary layer exhibits such structures, called large eddies. These large eddies undergo intermittent jumps between fixed points. These jumps are called *bursting events* and are thought to be an important mechanism for the transfer of energy between the inner and outer layer.

The proper orthogonal decomposition used in conjunction with Fourier analysis and the Galerkin projection yields a truncated set of ODEs which captures the maximum amount of kinetic energy among all possible truncations of the same order. Aubry *et al.* (1988) have examined sets of ODEs of various orders which appear to capture some of the essential dynamics of the bursting process.

In particular a model of two complex dimensions or four real dimensions turns out to be very useful to analyze. While this model is of too low an order for good physical representation, it does exhibit many of the essential features of the higher-order models. This model has $O(2)$ symmetry and exhibits asymptotically stable and structurally stable heteroclinic cycles in certain regions of the parameter space. Further analysis of this system has been carried out by Armbruster *et al.* (1988). The key idea is that bursting corresponds to passage close to the heteroclinic cycle, while no bursting corresponds to remaining close to a given hyperbolic fixed point. Another important ingredient in this model is the presence of pressure fluctuations in the outer layer which can trigger a bursting event; noise can be used to model these fluctuations. Such an analysis has been given by Stone and Holmes (1988).

Our main purpose in controlling such a system is to control the frequency of bursting events, which hopefully can be used to control the amount of turbulence in the boundary layer. We might in general wish to reduce the frequency of bursting, but in other instances it might be advisable to encourage a burst or to regularize its period. Possible mechanisms for control are the use of heatable patches on the boundary (combined with hot film sensors) or welts raised by piezoelectric effects by which we could feed back selected eigenfunctions. (Welts appear more likely to have a singificant effect on the eigenfunctions.) We also consider classical drag reduction by polymer addition (Kubo and Lumley, 1980; Lumley and Kubo, 1984; Aubry *et al.*, 1989), which can be viewed from a control standpoint. These approaches alter the equations and affect our consequent ability to control the dynamics in rather different ways as we shall see later.

Our main purpose is to analyze the control of the four-dimensional system discussed above, but we begin by analyzing some simpler systems. The simplest prototype system with a homoclinic orbit is the simple pendulum. Thus we begin by considering methods for controlling the inverted pendulum and in particular for keeping it near its unstable hyperbolic fixed point. More generally we consider the control of arbitrary n-dimensional systems possessing a homoclinic orbit to a hyperbolic fixed point. Using the linear dynamics near the fixed point we show that such a system in the *idealized* situation of no noise and infinite control accuracy can be given an arbitrary period and we give an *explicit* method for achieving this. As far as the boundary-layer model is concerned, this corresponds to arranging an arbitrary length of time between bursts.

We can of course *feedback stabilize* a system about a fixed point if it is locally controllable there, and we discuss some standard methods for doing this. However, our general philosophy is that this is not always possible because it requires too much control energy. Hence we let the hyperbolic point remain hyperbolic rather than adding so much control that it turns into a stable fixed point.

Another important realistic consideration is noise and we discuss control in the presence of noise using the model of Stone and Holmes (1988). We show in this situation how feedback control can increase the expected mean passage time of the system in the presence of noise. In this model we assume the homoclinic orbit is asymptotically stable and structurally stable. These assumptions allow robust control.

We thus assume the noise is reasonably small and/or respects the symmetry of the system, allowing the homoclinic structure to persist, at least approximately. This means in effect that we assume that symmetry breaking terms that might lead to horseshoe chaos are small compared with the noise and control effects. At the same time the control energy is also assumed to be too small to overwhelm the homoclinic structure.

Finally, we discuss control of the four-dimensional model of the wall region of the turbulent boundary layer with similar considerations in mind. Heatable patches, piezoelectric welts, and polymer injection are suggested as possible control mechanisms.

We remark also that Armbruster *et al.* (1988) have pointed out that structurally stable cycles will occur in many partial differential equations with translational and reflectional invariance. This invariance induces $O(2)$ symmetry in the problem. Such stable cycles were found, for example, by Jones and Proctor (1987) in convection models and have also been observed for the Kuramoto–Sivashinsky equations by Armbruster *et al.* and Nicolaenko (1986). Thus our techniques may be applicable to fluid systems other than the wall region of the boundary layer discussed here.

2. Controlling the Period

Our aim in this section is to show how to control the period of a system possessing a homoclinic orbit under the assumption that there is no noise in the system and that our controls have infinite accuracy. We relax these assumptions later. More specifically, we show that we can keep the system near its hyperbolic fixed point for an arbitrarily long time by driving it to a suitable point in the phase space and then letting it evolve under its own free dynamics.

We begin by presenting firstly a discussion of the nonlinear dynamics of a system possessing a homoclinic orbit and secondly a description of some key ideas from nonlinear control theory.

A prototype system is the nonlinear pendulum whose free dynamics is given by

$$\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0.$$

For convenience we normalize, letting $g/\ell = 1$. The controlled system is taken to be

$$\ddot{\theta} + \sin \theta = u,$$

which is a Hamiltonian control system with Hamiltonian

$$H = \frac{\dot{\theta}^2}{2} - \cos \theta + u\theta.$$

The free system has a hyperbolic fixed point at $\theta = \pi$. A slightly more complex (and realistic) example that we discuss is the inverted pendulum with moving base, where the base is controlled. This system has a nonhyperbolic fixed point at $\theta = \pi$.

Consider first (as in Silnikov (1967) and Wiggins (1988)) an ordinary differential equation

$$\dot{z} = F(z), \tag{2.1}$$

where $z \in \mathbb{R}^{s+u}$, and $F: U \to \mathbb{R}^{s+u}$ is C^r ($r \geq 2$) on an open set $U \subset \mathbb{R}^{s+u}$. We assume z_0 is a hyperbolic fixed point and that $DF(z_0)$ has s eigenvalues with negative real parts and u eigenvalues with positive real parts. Furthermore, assume there is a homoclinic orbit connecting z_0 to itself. (Recall that a homoclinic orbit is an orbit from a fixed point to itself, while a heteroclinic orbit links two distinct points.)

Transforming the fixed point to the origin and utilizing the stable and unstable manifolds as coordinates in a neighborhood of the saddle point, (2.1) may be transformed to the system

$$\begin{aligned} \dot{x} &= Ax + f_1(x, y), \\ \dot{y} &= By + f_2(x, y), \end{aligned} \tag{2.2}$$

where $(x, y) \in \mathbb{R}^s \times \mathbb{R}^u$, A is an $s \times s$ Jordan block with all diagonal entries having negative real parts, and B is a $u \times u$ block with diagonal entries having positive real parts. Here, f_1 and f_2 are of second order and (locally) satisfy

$$f_1(0, y) = f_2(x, 0) = 0. \tag{2.3}$$

Consider the following neighborhood of the origin:

$$N = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^u \,|\, |x| \leq \varepsilon, |y| \leq \varepsilon\} \tag{2.4}$$

whose boundary is given by the closures of the following sets which give cross sections to the vector field (2.2):

$$C_\varepsilon^s = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^u \mid |x| = \varepsilon, |y| < \varepsilon\},$$
$$C_\varepsilon^u = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^u \mid |x| < \varepsilon, |y| = \varepsilon\}. \tag{2.5}$$

Note that (2.2) points strictly to the interior of N on C_ε^s and strictly to the exterior of N on C_ε^u. Let S_ε^s and S_ε^u denote the intersection of the stable manifold with C_ε^s and the intersection of the unstable manifold with C_ε^u, respectively. Consider the flow in a neighborhood of the origin. The key, as in Silnikov (1967) and Wiggins (1988), is to divide the time t-map into two parts, one restricted to the interior of N, which we call P_0 and the other restricted to the exterior which we call P_1. We are mainly concerned with P_0 and the corresponding approximate Poincaré map P_0^L, which is the Poincaré map for the vector field linearized about the origin. Denote the flow generated by (2.2) by

$$\varphi(t, x_0, y_0) = (x(t, x_0, y_0), y(t, x_0, y_0)). \tag{2.6}$$

If $(x_0, y_0) \in C_\varepsilon^s \setminus S_\varepsilon^s$, then (x_0, y_0) reaches $C_\varepsilon^u \setminus S_\varepsilon^u$ in a time $T = T(x_0, y_0)$ that is a solution of the equation

$$|y(T, x_0, y_0)| = \varepsilon. \tag{2.7}$$

Define the map P_0: $C_\varepsilon^s \setminus S_\varepsilon^s \to C_\varepsilon^u \setminus S_\varepsilon^u$ by

$$(x_0, y_0) \mapsto ((x(T(x_0, y_0), x_0, y_0)), y(T(x_0, y_0), x_0, y_0))$$

with T defined by (2.7). Then P_0 may be defined, and, subject to a technical condition (see Wiggins, 1988), we may define the composition $P = P_1 \circ P_0$. The approximate map P_0^L is given by the map P_0^L: $C_\varepsilon^s \setminus S_\varepsilon^s \to C_\varepsilon^u \setminus S_\varepsilon^u$ defined by

$$(x_0, y_0) \mapsto (e^{AT} x_0, e^{BT} y_0), \tag{2.8}$$

where T solves

$$|e^{BT} y_0| = \varepsilon. \tag{2.9}$$

For our purposes, there are two key results (see Wiggins, 1988) that we need. The first is that near the origin the system behaves to within an error $O(\varepsilon^2)$ like its linear approximation. More precisely,

Proposition 2.1.

$$|P_0 - P_0^L| = O(\varepsilon^2) \quad and \quad |DP_0 - DP_0^L| = O(\varepsilon^2).$$

Secondly, the time the system spends in the neighborhood N goes to ∞ as $|y_0| \to 0$. More precisely,

Proposition 2.2. $T(x_0, y_0) \to +\infty$ *logarithmically as* $y_0 \to 0$.

This follows from (2.9).

We now wish to consider a system of the type given in (2.1) with additional control vector fields. We then want to ask when a control u may be found such that the system spends an (arbitrarily long) prescribed time in a neighborhood of the fixed point.

We begin by considering some basic ideas from nonlinear control theory.

We consider the following description of a nonlinear control system (see, e.g., Brockett, 1972; Sussman and Jurdjevic, 1972; Hermann and Krener, 1977):

$$\dot{x} = f(x, u), \tag{2.10}$$

where $x \in \mathbb{R}^n$, $u \in \Omega$, a subset of $\mathbb{R}^m$, u is bounded and measurable, and f is a C^∞ function.

More generally, we can consider $x \in M$, a C^∞ connected manifold. Here x represents the so-called "state-space" of the system, and u represents the input or control variables.

In practice, we may not be able to observe the entire state, and we append to (2.10) an algebraic output equation $y = g(x)$, $y \in \mathbb{R}^l$, say, where g is C^∞ and where y represents the observed or output variables. For the purposes of this paper we assume the state is observable, although this is not a crucial assumption.

Often we assume a special form of (2.10), namely that it is affine in the controls. Explicitly, (2.10) then takes the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i(t)g_i(x), \tag{2.11}$$

$x \in \mathbb{R}^n$, where f and the g_i are C^∞, and the u_i are bounded and measurable.

This is a nonlinear generalization of the well-known linear control system

$$\dot{x} = Ax + Bu, \tag{2.12}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, where A and B are $n \times n$ and $n \times m$ matrices, respectively. Such a system is said to be *completely controllable* if for any t_0, any initial state $x(t_0) = x_0$, and any final state x_f, there exists a finite time $t_1 > t_0$ and a control $u(t)$, $t_0 \le t \le t_1$, such that $x(t_1) = x_f$. This system can be shown (see, e.g., Brockett, 1970) to be completely controllable if and only if the $n \times nm$ controllability matrix

$$[B, AB, A^2B, \ldots, A^{n-1}B] \tag{2.13}$$

has rank n.

Definitions and conditions for controllability are rather more subtle in the nonlinear case and we give a brief and very incomplete description of some key ideas here.

Consider again (2.10) with $x(0) = x_0$ and denote a solution of this system by $\phi(x_0, u, t)$. Assume that this is defined for all $t \in [0, \infty)$.

We make the following definitions (see Sussman and Jurdjevic, 1972). A somewhat different approach is given in Hermann and Krener (1977). $y \in \mathbb{R}^n$ is said to be *attainable* from $x \in \mathbb{R}^n$ at time $t \ge 0$ if there exists a u such that $\phi(x, u, t) = y$. For each $x \in \mathbb{R}^n$, let $\mathbf{A}(x, t)$ denote the set of all points attainable from x at time t, and for $0 \le t < \infty$ let $\mathbf{A}(x, t) = \bigcup_{s \le t} \mathbf{A}(x, s)$ and $\mathbf{A}(x) = \bigcup_{t \ge 0} \mathbf{A}(x, t)$. The system is said to be *controllable from* x if $\mathbf{A}(x) = \mathbb{R}^n$ and *controllable* if it is controllable from every x in $\mathbb{R}^n$. The system is said to have the *accessibility property* from x if $\mathbf{A}(x)$ has nonempty interior and it has the accessibility property if it has the accessibility property from every $x \in \mathbb{R}^n$. The system has the *strong accessibility property* from x if $\mathbf{A}(x, t)$ has a nonempty interior for some $t > 0$ and it has the *strong accessibility property* if it has the strong accessibility property from x for every $x \in M$.

A number of results are available on controllability and accessibility for nonlinear systems under various assumptions. Of interest to us is generalizing the rank condition (2.13) for linear systems. The affine control system (2.10) is said to satisfy the *ad* condition if the dimension of $\mathscr{L}_0 = \operatorname{span}\{g_i, [f, g_i], [f, [f, g_i]], \ldots, i = 1, \ldots, m\}$ equals n for all x in $\mathbb{R}^n$. Here $[f, g]$ denotes the Lie bracket of the vector fields f and g, i.e., $[f, g] = (\partial g/\partial x)f - (\partial f/\partial x)g$. Also, let $\mathscr{L}$ denote the smallest Lie algebra of vector fields containing f and the g_i (that is, $\mathscr{L}$ is the smallest set of vector fields containing f and the g_i which is closed under the Lie bracketing operation). Then the control system (2.10) is said to satisfy the accessibility rank condition if $\dim \mathscr{L} = n$. Note that $\mathscr{L}_0 \subset \mathscr{L}$ in general.

We can show that an affine control system of the form (2.10) has the strong accessibility property if $\mathscr{L}$ has dimension n. Intuitively, satisfaction of the accessibility rank condition means that the vector fields at the given point span the tangent space at that point. Further, suppose the function f in (2.10) has a fixed point at $x = 0$. Then we can define the linearized control system about $x = 0$, $\dot{x} = Ax + Bu$, where here $A = (\partial F/\partial x)(0, 0)$ and $B = (\partial F/\partial u)(0, 0)$ and where $F(x, u) = f(x) + \sum_i u_i g_i(x)$. Then, if $\dim \mathscr{L}_0(0) = n$, this linearized system is controllable.

With these control-theoretic ideas in mind we are ready to state our first result.

Theorem 2.3. *Consider the C^r ($r \ge 2$) affine nonlinear control system given by*

$$\dot{z} = f(z) + \sum_{i=1}^{m} u_i(t)g_i(z), \qquad z \in \mathbb{R}^n,$$

where the u_i are piecewise continuous scalar functions and f and the g_i are C^r functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assume that the free system $\dot{z} = f(z)$ has a hyperbolic fixed point at $z = z_0$ and that z_0 has a homoclinic orbit connecting z_0 to itself. Let $\mathscr{L}_0(x)$ be given by

$$\mathscr{L}_0 = \operatorname{span}\{g_i, [f, g_i], [f, [f, g_i]], \ldots, i = 1, \ldots, m\}.$$

If $\dim \mathscr{L}_0(0) = n$, *then a control* u *may be found such that the system spends an arbitrarily long time in a neighborhood* N *of the fixed point* z_0 *after the control force is removed. In particular, if all trajectories of the free system near the homoclinic orbit are periodic, a control may be found such that the system exhibits arbitrarily long periods when the control force is removed.*

Proof. The proof depends on Propositions 2.1 and 2.2. Choose a neighborhood N of z_0 as described above. By Proposition 2.1 we know the system is approximated to $O(\varepsilon^2)$ by the linearized system in N. By the condition on $\mathscr{L}_0$ we know the system is accessible and, in particular, the linearized system at z_0 is controllable. Hence we may find (in fact explicitly—see the following example) a control that takes the system to a point on $C_\varepsilon^s \backslash S_\varepsilon^s$ as described above, choosing the point such that y_0 is as close to zero as we wish. As $y_0 \to 0$, the time spent in N by the free system goes to ∞. An estimate of this time, accurate to $O(\varepsilon^2)$ is given by Proposition 2.2. Hence we remove the control at the desired point on the boundary of N, and then use the free nonlinear dynamics. □

Example 2.4 (The Nonlinear Pendulum). The equation

$$\ddot{\theta} + \sin\theta = u(t)$$

has free dynamics having an orbit homoclinic to the hyperbolic equilibrium point at $\theta = \pi$. The orbits inside the homoclinic loop are periodic. Denote the linearized system at $\theta = \pi$ by $\dot{x} = Ax + Bu$. This system is controllable as a simple check of the rank of the controllability matrix $[B, AB]$ reveals. Here, the matrices A and B are

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For the linear system, it is a standard result for controllable systems (see, e.g., Barnett, 1975) that a control driving the system from state x_0 to x_f in time $t_1 - t_0$ is given by

$$u(t) = -B^T(t)\Phi^T(t_0, t)U^{-1}(t_0, t_1)[x_0 - \Phi(t_0, t_1)x_f],$$

where

$$U(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)\, d\tau$$

and Φ is the transition matrix for the free system.

Example 2.5 (The Inverted Pendulum on a Movable Cart). Here we have an inverted pendulum of length ℓ and mass m attached by a hinge to a cart with mass M moving in a line (with coordinate s) and a control u which acts on the cart (Figure 1). The Lagrangian of the free system is

$$L = \tfrac{1}{2}(M + m)\dot{s}^2 + \frac{m}{2}(\ell^2\dot{\varphi}^2 + 2\dot{s}\ell \cos\varphi\dot{\varphi}) - mg\ell\cos\varphi$$

yielding the equations of motion

$$(M + m)\ddot{s} + m\ell\cos\varphi\ddot{\varphi} - m\ell\sin\varphi\dot{\varphi}^2 = 0$$

and

$$m\ell^2\ddot{\varphi} + m\ell\ddot{s}\cos\varphi - m\ell\sin\varphi\dot{\varphi} - mg\ell\sin\varphi = u.$$

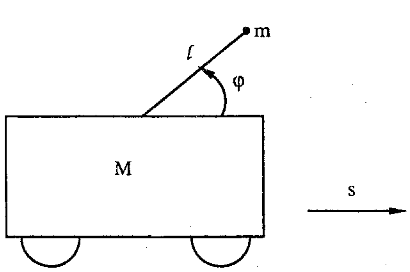

Figure 1

Linearizing and making a suitable change of coordinates ($s + \ell\varphi \mapsto x_3$) and assuming $m/M \to 0$ (see Kwakernaak and Sivan, 1972), we get the equations

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \frac{u}{M},$$

$$\dot{x}_3 = x_4,$$

$$\dot{x}_4 = \frac{g}{\ell}(x_3 - x_1).$$

Writing the system as $\dot{x} = Ax + Bu$ as before, we find that the controllability matrix (2.13) has rank four and thus the linearized equations are controllable. Regarding the free dynamics, we note that the full system has a *nonhyperbolic* fixed point at $\varphi = s = 0$. However, the free system also has an integral of motion—the linear momentum $(M + m)\dot{s} + (m\ell \cos\varphi)\dot{\varphi}$. This enables us to reduce the system to a one degree of freedom system in the configuration variable φ. This system does have a hyperbolic fixed point at $\varphi = 0$. Hence Theorem 2.4 applies.

The above scheme is clearly not robust in the sense that we would need infinite accuracy to get infinitely close to the stable manifold, and, further, the trajectories are sensitive to outside perturbations. Therefore, a practical controller would be one which drives the system to within a small distance, say δ, of the stable manifold, and, when it senses that the system has moved a certain distance from the stable and unstable manifolds, it reactivates the control and returns the system to a point close to the stable manifold. The key is to consider such an analysis when noise is present and we discuss this topic below. Further, for a realistic analysis we really need the homoclinic orbit to be structurally stable.

3. Stabilization

If we wish to stabilize the system via feedback—essentially by transforming the hyperbolic point into a stable fixed point, there are a number of approaches available—see, for example, van der Schaft (1986). Firstly, recall that a system can be made locally asymptotically stable by state feedback if the linearized system about the given fixed point is controllable (see, e.g., Jurdjevic and Quinn, 1978). We have (see van der Schaft, 1986).

Theorem 3.1. *Consider the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \qquad x \in \mathbb{R}^n,$$

with fixed point $x = 0$. *Let* $\mathscr{L}_0 = \operatorname{span}\{g_i, [f, g_i], [f, [f, g_i]], \ldots, i = 1, \ldots, m\}$. *If* $\dim \mathscr{L}_0(0)$ *equals* n, *there exists a state feedback* $u = \alpha(x)$ *such that the system can be locally asymptotically stabilized about* $x = 0$.

For Hamiltonian systems, we have a natural candidate for use as a Lyapunov function for ensuring stability, namely the Hamiltonian. If H has a strict local minimum at $(q, p) = (0, 0)$, the system is already stable. If not, we can try to adjust H via feedback such that this is true.

If we have a Hamiltonian control system where the number of controls and outputs is less than the number of configuration variables, there are some conditions which need to be satisfied before this method can be applied (see van der Schaft, 1986). In the case, however, where we have n controls and n configuration variables, and the system is in the following natural form,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + u_i, \tag{3.1}$$

where $i = 1, \ldots, n$ and

$$H(q, p) = \tfrac{1}{2} \sum_{i,j=1}^{n} g^{ij}(q) p_i p_j + V(q),$$

then this can always be done. Here g^{ij} is a positive definite metric on $\mathbb{R}^n$.

Theorem 3.2. *The system* (3.1) *can be made globally stable about* $q = 0$ *if* $(\partial^2 V/\partial q_i \, \partial q_j)(q)$ *is bounded, by a control of the form*

$$u_i = -\frac{\partial V}{\partial q_i}(0) - k_i q_i, \qquad i = 1, \ldots, n,$$

and can be made globally asymptotically stable about $q = 0$ *if* $(\partial^2 V/\partial q_i \, \partial q_j)(q)$ *is bounded by a control of the form*

$$u_i = \frac{\partial V}{\partial q_i} - k_i q_i - c_i \dot{q}_i. \tag{3.2}$$

Example 3.3. For the simple pendulum we have the equation

$$\ddot{\theta} + \sin\theta = u(t).$$

If we wish to stabilize about $\theta = \pi$ we choose the control $u = -k(\theta - \pi)$, $k > 1$. For asymptotic stability we set $u = -k(\theta - \pi) - c\dot{\theta}$.

An alternative method of stabilization for the inverted pendulum is to apply a vertical oscillation at its base (see Stoker, 1950; Arnold, 1978).

The linearized equation of motion can then be written

$$\ddot{\theta} + \left(-\frac{g}{\ell} + \frac{1}{\ell} p(t) \right) \theta = 0, \tag{3.3}$$

where $p(t)$ is periodic in t. Setting $(1/\ell)p(t) = \varepsilon \cos t$ and $-g/\ell = \delta$, we have the classical Mathieu equation

$$\frac{d^2\theta}{dt^2} + (\delta + \varepsilon \cos t)\theta = 0. \tag{3.4}$$

Then we can show

Theorem 3.4 ([see Stoker, 1950). *There are values of* ε *for which* (3.2) *has only stable solutions.*

Finally, we make some remarks on the energy required for the various types of control we have discussed. We restrict ourselves to a discussion of the simple pendulum, but the results generalize in an obvious fashion.

For saddle-point control, we do not stablize the system, but rather, each time there is a perturbation, we bring the system back as close as possible to the stable manifold. This involves the expenditure of a certain amount of energy equal to the change in potential energy. This will be approximately $-mg\ell \cos\theta$ where θ is the displacement from the saddle. For small θ this is $mg\ell(\theta^2/2)$.

Similarly, using the feedback $-k\theta$, the amount of energy used in driving the system from a distance $\bar{\theta}$ back to the saddle will be approximately $\int_0^{\bar{\theta}} k\theta \, d\theta = k\bar{\theta}^2/2$. On the other hand, to stabilize the system via feedback, i.e., change the saddle point to an elliptic point, we will need a larger k (the larger k the larger the stability margin) and hence will expand more energy in dealing with each perturbation.

Thus, as remarked earlier, our techniques make it possible in theory to stabilize the two-mode model of the boundary layer, but we assume that in general this will not be possible as we simply will not have sufficient control force to achieve this.

4. Control of the Mean Passage Time in the Presence of Noise for a System with an Asymptotically Stable Homoclinic Orbit

A situation which appears to be a reasonable model for the dynamics in the near boundary layer is a system with an asymptotically stable homoclinic orbit which is perturbed by (small) white noise. In addition, we assume structural stability of the orbit (a property that is due to symmetry considerations). An analysis of the expected mean passage time around the fixed point was carried out by Stone and Holmes (1988).

Consider the system

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) + \delta w(t),$$

where $w(t)$ is a vector white noise process and δ is a small parameter. We assume that for $(\delta, u_i) = (0, 0)$ this system has an asymptotically stable homoclinic orbit to a hyperbolic saddle point p. As discussed in Stone and Holmes (1988) this can be guaranteed by two properties:

(1) $W^u(p) \subset W^s(p)$ where W^s and W^u are the stable and unstable manifolds of p, respectively, and
(2) $\lambda_s > \lambda_u$ where the eigenvalues of $Df(p)$ are given by

$$\lambda_u \equiv \mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2) \geq \cdots \geq \mathrm{Re}(\lambda_{k-1}) > 0 > \mathrm{Re}(\lambda_k) \equiv -\lambda_s \geq \cdots \geq \mathrm{Re}(\lambda_n).$$

The behavior of the general n-dimensional system is captured essentially by the two-dimensional system

$$dx = -\lambda_s x\, dt + \delta\, dw_x,$$

$$dy = \lambda_u y\, dt + \delta\, dw_y + u,$$

where w_x and w_y are zero mean, independent Wiener processes.

With $u = 0$, Stone and Holmes (1988) show that the expected mean passage time τ across the region N defined in (2.4) is given by

$$\tau \sim \frac{1}{\lambda_u} \ln\left(\frac{\varepsilon}{\delta}\right) + O(1).$$

Thus, changing u so that λ_u *decreases* will increase the passage time. Hence we set $u = -ky\, dt$ in this case, where $0 < k < \lambda_u$.

The system is not even controllable in this case, but we can control the magnitude of λ_u. In general, if the system is controllable we can certainly control the magnitude of the real part of the largest unstable eigenvalue, and hence increase the expected mean passage time.

5. Two-Mode Model of the Boundary Layer

We now consider the two-mode model of the near wall region of the turbulent boundary layer. As mentioned in the introduction, the smallest really physical model has three modes, but the two-mode model captures many of the essential features of the bigger models, and, hopefully, of the true dynamics. Initially (see Aubry *et al.*, 1988) we assume a three-dimensional flow, approximately homogeneous in the streamwise and spanwise directions, approximately stationary in t, and inhomogeneous in the normal direction. An expansion of the random field is then made via the proper orthogonal decomposition in the normal direction and which is harmonic orthogonal in the other two directions. (The flow is assumed to be periodic in these latter two directions.) Substitution in the Navier–Stokes equations, and use of the Fourier transformation and Galerkin projection yields a set of nonlinear ordinary differential equations. The minimal reasonable truncation of the model is then one which has one eigenmode (in the normal direction), one streamwise wavenumber (thus neglecting streamwise variations), and three spanwise wavenumbers.

The equations for such a model may be found in Aubrey *et al.* These equations have important symmetry properties which are reflected in the $O(2)$ equivariant two-mode "model" for the boundary layer which we now describe.

The two-mode model of the wall region of the turbulent boundary layer may be written (see Aubry et al., 1988; Armbruster et al., 1988)

$$\dot{z}_1 = z_1(\mu_1 + d_{11}|z_1|^2 + d_{12}|z_2|^2) + c_{12}\bar{z}_1 z_2 + O(4), \tag{5.1a}$$

$$\dot{z}_2 = z_2(\mu_2 + d_{21}|z_1|^2 + d_{22}|z_2|^2) + c_{11}\bar{z}_1^2 + O(4), \tag{5.1b}$$

where the z_i are complex variables and μ_i, d_{ij}, and c_{ij} are parameters. The vector field here is $O(2)$ equivariant.

Assuming $c_{12}, c_{11} \neq 0$, we can rescale (5.1) to

$$\dot{z}_1 = \bar{z}_1 z_2 + z_1(\mu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2), \tag{5.2a}$$

$$\dot{z}_2 = \pm z_1^2 + z_2(\mu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2), \tag{5.2b}$$

where $e_{11} = d_{11}/|c_{11}c_{12}|$, $e_{12} = d_{12}/c_{12}^2$, $e_{21} = d_{21}/|c_{11}c_{12}|$, and $e_{22} = d_{22}/c_{12}^2$. The real Cartesian form of (5.2) is

$$\dot{x}_1 = x_1 x_2 + y_1 y_2 + x_1(\mu_1 + e_{11}r_1^2 + e_{12}r_2^2), \tag{5.3a}$$

$$\dot{y}_1 = x_1 y_2 - y_1 x_2 + y_1(\mu_1 + e_{11}r_1^2 + e_{12}r_2^2), \tag{5.3b}$$

$$\dot{x}_2 = \pm(x_1^2 - y_1^2) + x_2(\mu_2 e_{21}r_1^2 + e_{22}r_2^2), \tag{5.3c}$$

$$\dot{y}_2 = \pm 2x_1 y_1 + y_2(\mu_2 + e_{21}r_1^2 + e_{22}r_2^2), \tag{5.3d}$$

where $r_i^2 = x_i^2 + y_i^2$.

We consider here (in rough outline) two mechanisms for implementing control. A more detailed control mechanism will be formulated as the model develops. We consider primarily control via a "checkerboard" *of heating elements or piezoelectric welts on the boundary. We also consider polymer* addition, a classical drag reduction technique, from a control perspective. Through the use of heating elements or welts, we assume we can essentially change the magnitude of all eigenvalues via a choice of a suitable pattern on the checkerboard (feeding back selected eigenfunctions); through sensors we can monitor the amplitude of the modes, thus allowing feedback. Polymer injection, on the other hand, changes the magnitude of certain coefficients in the equations. As we see below, this can be viewed as a type of control, although it is slightly more restrictive.

In Cartesian form, the equations with heating element or piezoelectric controls are thus

$$\dot{x}_1 = x_1 x_2 + y_1 y_2 + x_1(\mu_1 + e_{11}r_1^2 + e_{12}r_2^2) + u_1, \tag{5.4a}$$

$$\dot{y}_1 = x_1 y_2 - y_1 x_2 + y_1(\mu_1 + e_{11}r_1^2 + e_{12}r_2^2) + u_2, \tag{5.4b}$$

$$\dot{x}_2 = \pm(x_1^2 - y_1^2) + x_2(\mu_2 e_{21}r_1^2 + e_{22}r_2^2) + u_3, \tag{5.4c}$$

$$\dot{y}_2 = \pm 2x_1 y_1 + y_2(\mu_2 + e_{21}r_1^2 + e_{22}r_2^2) + u_4, \tag{5.4d}$$

where $r_i^2 = x_i^2 + y_i^2$.

We are interested in the "$-$" case, as this is when a heteroclinic cycle exists in certain regions of the parameter space. In fact we have fixed points at $(x_1, y_1, x_2, y_2) = (0, 0, \pm(-\mu_2/e_{22})^{1/2}, 0)$ and (providing no "mixed modes" exist (see Armbruster *et al.* (1988), i.e., no steady-state solutions with x_1 and x_2 nonzero) we have a heteroclinic cycle connecting these fixed points if $e_{11}, e_{22} < 0$, $e_{12} + e_{21} < 2(e_{11}e_{22})^{1/2}$, $\mu_1, \mu_2 > 0$, and

$$\mu_1 - \mu_2\frac{e_{12}}{e_{22}} - \left(-\frac{\mu_2}{e_{22}}\right)^{1/2} < 0 < \mu_1 - \mu_2\frac{e_{12}}{e_{22}} + \left(-\frac{\mu_2}{e_{22}}\right)^{1/2}.$$

If, in addition,

$$\min\left\{2\mu_2, -\left[\mu_1 - \mu_2\frac{e_{12}}{e_{22}} - \left(-\frac{\mu_2}{e_{22}}\right)^{1/2}\right]\right\} > \mu_1 - \mu_2\frac{e_{12}}{e_{22}} + \left(-\frac{\mu_2}{e_{22}}\right)^{1/2},$$

then the cycle is locally asymptotically stable.

Now the linearized equations about the fixed points are of the form

$$\dot{x} = Ax + Bu, \tag{5.5a}$$

where

$$A = \begin{bmatrix} \mu_1 - \mu_2 \frac{e_{12}}{e_{22}} \pm \left(-\frac{\mu_2}{e_{22}}\right)^{1/2} & 0 & 0 & 0 \\ 0 & \mu_1 \mp \left(-\frac{\mu_2}{e_{22}}\right)^{1/2} - \mu_2 \frac{e_{12}}{e_{22}} & 0 & 0 \\ 0 & 0 & -2\mu_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.5b}$$

B is the identity matrix, and $u = [u_1, u_2, u_3, u_4]^T$. Thus of course the linearized systems are completely controllable and all our earlier theorems apply.

If we view polymer drag reduction as a control problem, we can control the linear coefficients μ_1 and μ_2 and the coefficients c_{12} and c_{11} and their equivalents.

We concentrate here on the control of μ_1 and μ_2. These take the form $\mu_i = \mu_i^1 + (1 + \alpha/K)\mu_i^2$ where $\mu_i^2 < 0$, K is a constant, and we have control over the parameter α (see Aubry *et al.*, 1989). In the region of interest, $(1 + \alpha/K)$ is negative, so we write

$$\mu_i = \mu_i^1 - \beta(t)\mu_i^2, \tag{5.6}$$

where $\beta(t)$ is an adjustable positive constant, representing changes in the value of α.

Then it is clear that the controlled equations with variable α are equivalent to (5.4) with the u_i restricted to be of the form

$$\begin{aligned} u_1 &= -\beta\mu_1^2 x_1, \qquad & u_3 &= -\beta\mu_2^2 x_2, \\ u_2 &= -\beta\mu_1^2 y_1, \qquad & u_4 &= -\beta\mu_2^2 y_2. \end{aligned} \tag{5.7}$$

Increasing α is thus equivalent to exercising control as before.

The linearized control system is then also as before except that

$$Bu = -\beta \operatorname{diag}(\mu_1^2 x_1, \mu_1^2 y_1, \mu_1^2 x_2, \mu_1^2 y_2). \tag{5.8}$$

While we do not have complete freedom in choosing the u_i, we do have negative feedback and the ability to decrease the magnitude of the largest unstable eigenvalue. This finding is consistent with the findings of Aubry *et al.* (1989) regarding the changes in the wall region induced by stretching. There it was found that a stretched boundary layer corresponded to drag reduction (as observed), and that in a stretched boundary layer bifurcations occurred at higher values of α, suggesting decreased stability of the stretched layer, requiring greater α for stabilization. Although α was initially conceived of as a measure of the loss to unresolved modes, it could be equally well a viscous loss, since the simplicity of the model cannot distinguish the two (and the way in which the stretching was done confines this to the turbulent fluid). An increase in viscous loss in the turbulent fluid due to polymers has been suggested as the mechanism primarily responsible for the stretching of the wall region, bringing about drag reduction.

6. Concluding Remarks

We have discussed a model that may be useful for the control of turbulence in the wall region of a turbulent boundary layer, based on the existence of stable heteroclinic or homoclinic cycles. Our basic aim is to decrease the frequency of bursting events, which correspond to passage close to the cycle away from a given hyperbolic point. We noted that in general we may not have sufficient control energy to overwhelm the heteroclinic structure and change a given hyperbolic fixed point to a stable fixed point. The basic control strategy we have developed for heating element or piezoelectric welt controls thus consists of

(1) decreasing the magnitude of the largest unstable eigenvalue of the system linearized about the given hyperbolic point, thus increasing the time the system is expected to remain near this point, and

(2) if a perturbation drives the system a sufficiently large distance from the fixed point, returning it to a point as close as possible to the stable manifold.

We also observed that (1) could be carried out by polymer drag reduction. The finding that for increasing the expected passage time, classical polymer drag reduction is equivalent to control by heating patches or piezoelectric control, is an important one, and supports a speculation by Lumley (Lumley *et al.*, 1988) that such a controlled boundary layer would structurally resemble a polymer-drag-reduced boundary layer.

We have given explicit strategies for carrying out these procedures, but of course the details will depend on the specifics of any given problem—the structure of the heating pad controls, the estimators, and so on. Research on these matters is expected to be carried out in the near future.

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