Stability and bifurcation of a rotating planar liquid drop

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The stability and symmetry breaking bifurcation of a planar liquid drop is studied using the energy-Casimir method and singularity theory. It is shown that a rigidly rotating circular drop of radius r with surface tension coefficient τ and angular velocity $\Omega/2$ is stable if $(\Omega/2)^2 < 3\tau/r^3$. A new branch of stable rigidly rotating relative equilibria invariant under rotation through π and reflection across two axes bifurcates from the branch of circular solutions when $(\Omega/2)^2 = 3\tau/r^3$.

I. INTRODUCTION

Bifurcation of systems with symmetry has been a subject of much interest in recent years. Symmetric systems are common in nature and even more common in the literature, as multidimensional bifurcation problems possessing symmetry are typically more tractable than asymmetric problems of comparable dimensions. The requirement that the bifurcation equation be equivariant under the action of a given group G, i.e., that $f(g\cdot\mathbf{x},\lambda) = g\cdot f(\mathbf{x},\lambda)$ for all $g\in G$, can force the bifurcation equation to take on a relatively simple form. For example, if one considers a function f on $\mathbb{R}$ which is equivariant with respect to the $\mathbf{Z}_2$ action $x \to -x$ it is clear that f can be written as $\tilde{f}(x^2)x$ for some function $\tilde{f}$. [See Golubitsky and Schaeffer[1] for a thorough presentation of the singularity theory approach to bifurcations with (and without) symmetry.]

The class of bifurcation equations with which we are particularly concerned here arise in Hamiltonian systems with symmetry. Using the energy-Casimir method (cf. Holm *et al.*[2]), one can typically find a combination C of conserved quantities such that a given (relative) equilibrium of a Hamiltonian system is a critical point of $H + C$, where H is the usual Hamiltonian of the system. The bifurcation parameter may appear in either the Hamiltonian itself or in the added conserved quantities; if we denote the parameter-dependent modified Hamiltonian by $(H + C)_\lambda$, then the appropriate bifurcation equation is $D_{\mathbf{x}}(H + C)_\lambda(\mathbf{x}) = \mathbf{0}$.

Invariance of the Hamiltonian under a given group action usually induces constraints on the form of its differential. In the analysis of a symmetric bifurcation problem it is important to exploit these constraints as fully as possible; behavior exceptional in an asymmetric context may be typical or even necessary if all existing symmetry is taken into account. Several important generic properties of bifurcations of Hamiltonian systems are presented in Golubitsky and Stewart.[3] The present paper is largely the result of discussions with Golubitsky and Stewart; the lemma presented here is a variation on results due to Cicogna[4] and Golubitsky *et al.*[5]

There are a number of well known, but as yet incompletely understood, examples of bifurcation with symmetry breaking in hydrodynamics, including Taylor–Couette flow and the vortex breakdown. The energy-Casimir method has been applied to a wide variety of hydrodynamic problems with a great deal of success in recent years (see Holm *et al.*[2] for a generous selection of applications of the energy-Casimir method). In earlier works we have determined the Hamiltonian structure for free boundary fluid problems (see Lewis *et al.*[6]) and formal stability for the two-dimensional circular liquid drop (see Lewis *et al.*[7]); in Lewis,[8] conditional nonlinear stability under the same hypotheses is established. The method is readily applicable to analytic solutions (e.g., the Kelvin–Stuart cat's eye, cf. Holm *et al.*[9]) and should be implementable for approximate numerical solutions.

Our basic approach is to determine the stability of a relatively simple equilibrium flow by applying the energy-Casimir method and then, at the point at which this flow loses formal stability, apply the techniques of symmetric bifurcation theory to gain information about the new, typically more complicated, solution branch. The techniques and general results discussed here are not, however, restricted to problems in fluid dynamics; another class of examples currently being studied is the stability of coupled rigid bodies and spacecraft with flexible attachments; see Krishnaprasad and Marsden.[10]

The paper consists of three sections. Section II gives a brief (and incomplete) summary of existing results in this area. Section III contains a lemma outlining conditions under which bifurcation of the critical manifold of an SO(2) invariant function on $\mathbb{R}^2$ can be shown to occur. Section IV discusses, as an application of the lemma, the bifurcation of a two-dimensional rotating liquid drop with surface tension from a rigidly rotating circular configuration. In future publications we hope to present some numerical studies of the drop configurations and possibly search for boundary bifurcations from the "flip" symmetric two-lobed branch.

II. BACKGROUND

Rotating liquid drops have been the object of intense study, both in the nineteenth century and in the last twenty years. While the original research was necessarily restricted to the study of approximate theoretical and experimental models, recent work has benefitted greatly from the availability of computer simulation and elaborate and accurate experimental configurations. Swiatecki[11] provides a thorough review of research in this area up to the early seventies.

The principal analytic approach to the study of the equilibrium configurations and their stability has been to analyze linearized models and low-order approximations of the actual drop shapes. Analytic linear stability results for axisymmetric drops held together by surface tension have been found by Chandrasekhar[12] using the method of virials. Second-order expansions for the evolution of a perturbed spherical drop have been developed by Tsamopoulos and Brown.[13]

Several thorough numerical studies of rotating liquid drops have been made. Brown and Scriven[14] use a finite element code to trace the bifurcations of an initially spherical rotating drop held together by surface tension; they analyze the linear stability of the solution branches and show general agreement with Chandrasekhar's analytic results. Benner[15] has performed numerical studies of cylindrical (i.e., planar) drops under the effect of surface tension and traced the evolution of small potential flow perturbations of the stationary circular solution. The results of his simulations indicate that these perturbations remain bounded for at least a short period of time. Both the calculations of Brown and Scriven and Benner assume that the drop possesses reflectional symmetry across some axis; equilibria lacking this symmetry could conceivably appear through subsequent secondary bifurcations.

Experimental research regarding rotating liquid drops with surface tension dates back to Plateau's study of fat globules suspended in a liquid of nearly equal density. The most dramatic recent research is that of Wang *et al.*[16]; these experiments, which involved free floating, acoustically accelerated droplets, were conducted in near zero gravity in Spacelab. The observed bifurcation of a family of two-lobed drops from a family of oblate, axisymmetric drops agrees qualitatively with both the analytic and numerical predictions, although there are some unresolved quantitative discrepancies. (In particular, the bifurcation from the axisymmetric to the two-lobed branch appears to have occurred somewhat earlier than predicted.)

III. BIFURCATION LEMMA

The initial step in the analysis of a given bifurcation is to establish that a bifurcation has, in fact, taken place. It is typically the case that if a known solution loses stability as a given parameter is varied, then a "transfer of stability" occurs and another stable solution exists for nearby parameter values. This supposition must, however, be checked in each case. In complicated examples, e.g., those obtained from large or even infinite-dimensional systems by Liapunov–Schmidt reduction, the task of determining points of bifurcation need not be trivial.

At a point of bifurcation one typically expects to see a new one-dimensional solution branch emerge; a typical nondegeneracy condition for bifurcation results is that only one eigenvalue of the system pass through zero at the point of bifurcation. In problems without symmetry, or with discrete symmetry, this is an entirely reasonable assumption, but if the symmetry group is continuous, it may be impossible to satisfy. If a map f is equivariant under the linear action of a group G, then the following situation occurs. If $\mathbf{x}$ is a zero of f then, for any $g\in G$, $g\cdot\mathbf{x}$ must be a zero as well, since

$$f(g\cdot\mathbf{x}) = g\cdot f(\mathbf{x}) = \mathbf{0}$$

if $f(\mathbf{x}) = \mathbf{0}$. Thus the solution branches are made up of orbits of the group action. If $\mathscr{G}$ acts freely on a given solution branch, then the dimension of that branch cannot be less than the dimension of G. Even if the action is not free, it may still force the solution branch to be multidimensional, implying that at the point of bifurcation multiple eigenvalues pass through zero simultaneously. In this case many standard bifurcation theorems may not be applicable.

If analyzed strictly with regard to dimension, the study of bifurcation problems with continuous symmetry groups may appear to be extremely difficult. In fact, the multidimensional solution branches are usually redundant; all essential information about the bifurcation may be obtained by studying a representative point in the orbit swept out by the group action. In some cases it is feasible to explicitly reduce the original manifold by the group action, but there are circumstances under which this reduction can be somewhat complicated. For example, if one considers a linear group action on a vector space, the action at the origin is not free and the reduced space may fail to be a manifold at that point. Thus, if one is considering a bifurcation from the "trivial" solution $(\mathbf{0},\lambda)$, analytic difficulties arise exactly at the point of interest. In such cases it seems preferable to leave the state space unaltered and instead generalize the usual criteria for bifurcation to account for the redundancy induced by the group action. The central result of this section is a simple generalization to the case of the group SO(2) acting on $\mathbb{R}^2$. (In this case both eigenvalues pass through zero simultaneously at a point of bifurcation never leaving the imaginary axis.)

The following lemma is a modification of results of Cicogna[4] and Golubitsky *et al.*[5] The idea behind the lemma is to split the bifurcation map into a scalar function that depends on the bifurcation parameter and a multidimensional map that is independent of the parameter and equal to zero at the bifurcation point; one then applies the implicit function theorem to the scalar equation to establish the existence of a new solution branch. The second result in this section is an application of the lemma to the differential of an SO(2) invariant function on $\mathbb{R}^2$, where the restrictions imposed on the function by SO(2) invariance guarantee that the decomposition of the differential into scalar and vector-valued components is possible.

Lemma 1: Let V be a vector bundle over a manifold M and $\lambda\in\mathbb{R}$. Let F be a λ-dependent section of V. Assume $F(\mathbf{x},\lambda) = g(\mathbf{x},\lambda)\cdot\mathbf{h}(\mathbf{x})$ for some (smooth) maps g: $M\times\mathbb{R}\to\mathbb{R}$ and $\mathbf{h}$: $M\to V$. Let $S_0 = \{\mathbf{x}: \mathbf{h}(\mathbf{x}) = \mathbf{0}\}$. If for some point $(\mathbf{x}_0,\lambda_0)$ with $\mathbf{x}_0\in S_0$ we have

(i) $D_{\mathbf{x}}F(\mathbf{x}_0,\lambda_0) = \mathbf{0}$;
(ii) $D_{\mathbf{x}}\mathbf{h}(\mathbf{x}_0) \neq \mathbf{0}$;
(iii) $D_{\mathbf{x}\lambda}F(\mathbf{x}_0,\lambda_0) \neq \mathbf{0}$,

then a branch (or possibly family) of solutions (i.e., points mapped into 0) bifurcates from the trivial solution manifold S_0 at $(\mathbf{x}_0,\lambda_0)$.

Proof:

$$\mathbf{0} = D_\mathbf{x} F(\mathbf{x}_0,\lambda_0)$$
$$= D_\mathbf{x}\, g(\mathbf{x}_0,\lambda_0)\mathbf{h}(\mathbf{x}_0) + g(\mathbf{x}_0,\lambda_0) D_\mathbf{x}\mathbf{h}(\mathbf{x}_0)$$

implies $g(\mathbf{x}_0,\lambda_0)=0$, since $\mathbf{x}\in S_0$ implies $\mathbf{h}(\mathbf{x}_0)=\mathbf{0}$ and, by (ii), $D_\mathbf{x}\mathbf{h}(\mathbf{x}_0)\neq\mathbf{0}$. Similarly,

$$\mathbf{0}\neq D_{\mathbf{x}\lambda} F(\mathbf{x}_0,\lambda_0)$$
$$= D_\lambda g(\mathbf{x}_0,\lambda_0) D_\mathbf{x}\mathbf{h}(\mathbf{x}_0)$$

implies $D_\lambda g(\mathbf{x}_0,\lambda_0)\neq 0$. Thus we can apply the implicit function theorem to g and find a function Λ: $M\rightarrow\mathbb{R}$ such that $g(\mathbf{x},\Lambda(\mathbf{x}))=0$ for all $\mathbf{x}$ in a neighborhood of $\mathbf{x}_0$. It follows that there must be a set of solutions of $F=\mathbf{0}$ passing through S_0 at $(\mathbf{x}_0,\lambda_0)$. ■

We now specialize the above result to the study of critical points of an SO(2) invariant function on $\mathbb{R}^2$.

Corollary 1: If

(i) f: $\mathbb{R}^2\times\mathbb{R}\rightarrow\mathbb{R}$ is (smooth and) invariant under the standard SO(2) action on $\mathbb{R}^2$;

(ii) $D_{\mathbf{xx}} f(0,0,\lambda_0)=\mathbf{0}$ for some λ_0;

(iii) $D_{\mathbf{xx}\lambda} f(0,0,\lambda_0)\neq\mathbf{0}$,

then a branch of critical points of f emanates from the trivial critical point branch $(0,0,\lambda)$ at λ_0.

Proof: The invariance of f: $\mathbb{R}^2\times\mathbb{R}\rightarrow\mathbb{R}$ under the SO(2) action implies the existence of a function $\tilde f$: $\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ such that $f(x,y,\lambda)=\tilde f(x^2+y^2,\lambda)$. (For smoothness of $\tilde f$, see Golubitsky and Shaeffer.[1]) Identifying $T^*\mathbb{R}^2$ with $\mathbb{R}^2\times\mathbb{R}^2$, it follows that

$$D_\mathbf{x} f(x,y,\lambda)=\frac{\partial\tilde f}{\partial r}(x^2+y^2,\lambda)(2x,2y).$$

Thus, letting $g(x,y,\lambda)=(\partial\tilde f/\partial r)(x^2+y^2,\lambda)$ and $\mathbf{h}(x,y)=(2x,2y)$, we have

$$F(x,y,\lambda)=D_\mathbf{x} f(x,y,\lambda)$$
$$=g(x,y,\lambda)\cdot\mathbf{h}(x,y).$$

Conditions (ii) and (iii) imply that

$$D_\mathbf{x} F(0,0,\lambda_0)=D_{\mathbf{xx}} f(0,0,\lambda_0)$$
$$=\mathbf{0}$$

and

$$D_{\mathbf{x}\lambda} F(0,0,\lambda_0)=D_{\mathbf{xx}\lambda} f(0,0,\lambda_0)$$
$$\neq\mathbf{0}.$$

Differentiating the linear map $\mathbf{h}$ gives

$$D_\mathbf{x}\mathbf{h}(0,0)=\begin{pmatrix}2&0\\0&2\end{pmatrix}.$$

Thus the conditions of the lemma are satisfied and a branch of nonzero solutions of $F=\mathbf{0}$, i.e., critical points of f, must branch from $(0,0,\lambda_0)$. ■

Remark: The above result for SO(2) acting on $\mathbb{R}^2$ can be generalized to the case of an n dimensional Lie group G acting on an $n+1$ dimensional manifold $\mathscr{M}$. If a function f: $\mathscr{M}\times\mathbb{R}\rightarrow\mathbb{R}$ is G invariant, then Df typically lies in a one-dimensional subspace of the cotangent bundle of $\mathscr{M}$; thus, if the appropriate nondegeneracy conditions are satisfied, the lemma can be applied. More precisely, let f: $\mathscr{M}\rightarrow\mathbb{R}$ be a G invariant function, $x\in\mathscr{M}$ and $\Theta(s)$ be a curve in G tangent to a vector $\xi\in\mathscr{G}$, the Lie algebra of G, at $s=0$. Differentiating the equality $f(\Theta(s)\cdot\mathbf{x})=f(\mathbf{x})$, one sees that $D_\mathbf{x} f(\mathbf{x})\cdot\xi_\mathscr{M}(x)=0$. Here $\xi_\mathscr{M}(x)$ denotes the infinitesimal generator of ξ, defined by $\xi_\mathscr{M}(x)=(d/ds)|_{s=0}\,\Theta(s)\cdot\mathbf{x}$. [For example, in the case of $\mathbb{R}^2$ with the usual SO(2) action, $1_\mathscr{M}(\mathbf{x})=\hat{\mathbf{z}}\times\mathbf{x}$.] Let $\xi^1,\ldots,\xi^n$ be a basis of $\mathscr{G}$. At any point $\mathbf{x}$ in $\mathscr{M}$ at which G acts freely, $\xi^1_\mathscr{M}(\mathbf{x}),\ldots,\xi^n_\mathscr{M}(\mathbf{x})$ span an n-dimensional subspace $\Xi_\mathbf{x}$ of $T_\mathbf{x}\mathscr{M}$. Then $Df(\mathbf{x})$ must lie in the one-dimensional subspace $\Xi_\mathbf{x}^\perp$ of $T_\mathbf{x}^*\mathscr{M}$ consisting of one forms annihilating $\Xi_\mathbf{x}$. Any nondegenerate local section of $\Xi_\mathbf{x}^\perp$ will serve as $\mathbf{h}$, so that the lemma may be applied.

IV. ROTATING PLANAR LIQUID DROP

As an application of the preceding results, we consider a planar liquid drop consisting of an incompressible, inviscid fluid with a free boundary and forces of surface tension on the boundary. The dynamic variables are the free boundary Σ and the spatial velocity field $\mathbf{v}$, a divergence-free vector field on the region D_Σ bounded by Σ. The surface Σ is an element of the set $\mathscr{S}$ of closed curves in $\mathbb{R}^2$ diffeomorphic to the boundary of a reference region D and enclosing the same area as D. We let $\mathscr{N}$ denote the space of all such pairs $(\Sigma,\mathbf{v})$. The Hamiltonian approach to hydrodynamic problems was introduced in the fixed boundary case by Arnold[17] and developed by Marsden and Weinstein.[18] The free boundary case has also been studied by Sedenko and Iudovich.[19]

The equations of motion for an ideal fluid with a free boundary Σ with surface tension τ are

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p,\quad \frac{\partial\Sigma}{\partial t}=\langle\mathbf{v},\nu\rangle,\qquad(1)$$
$$\operatorname{div}\mathbf{v}=0\quad\text{and}\quad p|\Sigma=\tau\kappa,$$

where ν is the unit normal to the surface, Σ, κ is the mean curvature of Σ, and τ is the surface tension coefficient, a numerical constant.

The Poisson bracket will be defined for functions F,G: $\mathscr{N}\rightarrow\mathbb{R}$, which possess *functional derivatives* defined as follows.

(i) $\delta F/\delta\mathbf{v}$ is a divergence-free vector field on D_Σ such that

$$D_\mathbf{v} F(\Sigma,\mathbf{v})\cdot\delta\mathbf{v}=\int_{D_\Sigma}\left\langle\frac{\delta F}{\delta\mathbf{v}},\delta\mathbf{v}\right\rangle dA,$$

where the partial (Fréchet) derivative $D_\mathbf{v}F$ is computed with Σ fixed.

(ii) $\delta F/\delta\varphi$ is the function on Σ with integral zero given by

$$\frac{\delta F}{\delta\varphi}=\left\langle\frac{\delta F}{\delta\mathbf{v}},\nu\right\rangle.$$

(The symbol φ represents the potential for the gradient part of $\mathbf{v}$ in the Helmholtz, or Hodge, decomposition.)

(iii) $\delta F/\delta\Sigma$ is a function on Σ determined up to an additive constant as follows. A variation $\delta\Sigma$ of Σ is identified with a function on Σ representing the infinitesimal variation of Σ in its normal direction. It follows from the incompressibility assumption that $\delta\Sigma$ has integral zero. Let $\delta F/\delta\Sigma$ be the function determined up to an additive constant by

$$\int_\Sigma \frac{\delta F}{\delta \Sigma}\,\delta\Sigma\, ds = D_\Sigma\, F(\Sigma,\mathbf{v})\cdot\delta\Sigma.$$

We now define a Poisson bracket on $\mathscr{N}$ as follows. For functions F and G mapping $\mathscr{N}$ to $\mathbb{R}$ and possessing functional derivatives as defined above, set

$$\{F,G\} = \int_{D_\Sigma} \left\langle \omega, \frac{\delta F}{\delta \mathbf{v}} \times \frac{\delta G}{\delta \mathbf{v}} \right\rangle dA + \int_\Sigma \left(\frac{\delta F}{\delta \Sigma}\frac{\delta G}{\delta \varphi} - \frac{\delta G}{\delta \Sigma}\frac{\delta F}{\delta \varphi} \right) ds, \tag{2}$$

where $\omega = \operatorname{curl} \mathbf{v}$. This Poisson bracket on $\mathscr{N}$ is derived from the canonical cotangent bracket on $T^*\mathscr{C}$, where, in the two-dimensional case, $\mathscr{C} = \mathrm{Emb}_{\mathrm{vol}}(D,\mathbb{R}^2)$ is the manifold of volume-preserving embeddings of a two-dimensional reference manifold D into $\mathbb{R}^2$, by reduction by the group $G = \mathrm{Diff}_{\mathrm{vol}}(D)$, the group of volume-preserving diffeomorphisms of D (i.e., the group of particle relabeling transformations). (See Lewis *et al.*[6] for details.)

We take our Hamiltonian to be

$$H(\Sigma,\mathbf{v}) = \int_{D_\Sigma} \frac{1}{2}|\mathbf{v}|^2\, dA + \tau \int_\Sigma ds. \tag{3}$$

The functional derivatives of H are computed to be

$$\frac{\delta H}{\delta \mathbf{v}} = \mathbf{v}, \quad \frac{\delta H}{\delta \varphi} = \left\langle \frac{\delta H}{\delta \mathbf{v}}, \nu \right\rangle = \langle \mathbf{v}, \nu \rangle,$$

$$\frac{\delta H}{\delta \Sigma} = \frac{1}{2}|\mathbf{v}|^2 + \tau\kappa,$$

where $\delta H/\delta\Sigma$ is taken modulo constants. For this H and the Poisson bracket (2), the equations of motion (1) for the free boundary fluid with surface tension are equivalent to the relation $\partial F/\partial t = \{F,H\}$ for all functions F on $\mathscr{N}$ possessing functional derivatives.

We consider the stability of the planar incompressible fluid flow such that the boundary Σ_e is a circle of radius r and the fluid is rigidly rotating with angular velocity Ω. We shall apply the energy-Casimir method as follows. For the circular equilibrium solution of the equations of motion, we shall find a conserved quantity C such that $H_C = H + C$ has a critical point at the equilibrium. We shall then test for definiteness of the second variation of H_C at the equilibrium point. If it is definite, then the equilibrium is said to be formally stable. (See Holm *et al.*[2] for a thorough description and applications of the energy-Casimir method. For details of the following stability analysis, see Lewis *et al.*[7])

One class of conserved quantities consists of the Casimirs of the Poisson manifold $\mathscr{N}$, i.e., functions C on $\mathscr{N}$ satisfying $\{C,F\} = 0$ for all functions F for which the bracket is defined. We will make use of Casimirs of the form

$$C_1(\Sigma,\mathbf{v}) = \int_{D_\Sigma} \Phi(\omega)\, dA,$$

where Φ is a C^2 function on $\mathbb{R}^2$ and $\omega = \langle \operatorname{curl} \mathbf{v}, \hat{\mathbf{z}} \rangle$. We will also include the angular momentum

$$J(\Sigma,v) = \int_{D_\Sigma} \langle \mathbf{x}\times\mathbf{v}, \hat{\mathbf{z}} \rangle\, dA.$$

Here J is the momentum map associated to the left action of the group $O(2)$ on $\mathscr{N}$. The conservation of J is a consequence of the invariance of the Hamiltonian H under the $O(2)$ action, which implies $\partial J/\partial t = \{J,H\} = 0$. The inclusion of J in the modified Hamiltonian H_C allows us, roughly speaking, to view the fluid from a rotating frame with arbitrary angular velocity.

We take our total conserved quantity to be

$$H_C(\Sigma,v) = \int_{D_\Sigma} \left(\frac{1}{2}|\mathbf{v}|^2 - \mu\langle \mathbf{x}\times\mathbf{v}, \hat{\mathbf{z}} \rangle + \Phi(\omega) \right) dA + \tau \int_\Sigma ds,$$

where μ is a constant, as yet undetermined. Using elementary vector identities, we can rewrite H_C as

$$H_C(\Sigma,\mathbf{v}) = \int_{D_\Sigma} \left(\frac{1}{2}|\tilde{\mathbf{v}}|^2 - \frac{1}{2}\mu^2|\mathbf{x}|^2 + \Phi(\omega) \right) dA + \tau \int_\Sigma ds,$$

where $\tilde{\mathbf{v}} = \mathbf{v} - \mu\hat{\mathbf{z}}\times\mathbf{x}$. This rephrasing corresponds to viewing the fluid from a flame rotating with constant angular velocity μ; $\tilde{\mathbf{v}}$ is the fluid velocity in the rotating flame.

The first variation of H_C is computed to be

$$DH_C(\Sigma,\mathbf{v})\cdot(\delta\Sigma,\delta\mathbf{v}) \tag{4}$$

$$= \int_{D_\Sigma} (\langle \tilde{\mathbf{v}}, \delta\mathbf{v} \rangle + \Phi'(\omega)\cdot\langle \operatorname{curl} \delta\mathbf{v}, \hat{\mathbf{z}} \rangle) dA \tag{5}$$

$$+ \int_\Sigma \left(\frac{1}{2}|\tilde{\mathbf{v}}|^2 - \frac{1}{2}\mu^2|\mathbf{x}|^2 + \tau\kappa + \Phi(\omega) \right) \delta\Sigma\, ds. \tag{6}$$

We now consider the case where Σ_e is a circle of radius r and $v_e = (\Omega/2)\hat{\mathbf{z}}\times\mathbf{x}$ for some constant Ω, i.e., the equilibrium flow is rigid rotation with angular velocity Ω. The circle Σ_e has constant mean curvature $\kappa = 1/r$. We require DH_C to vanish at this equilibrium. Since $\omega_e = \langle \operatorname{curl} \mathbf{v}_e, \hat{\mathbf{z}} \rangle = \Omega$, DH_C depends on Φ only through the constants $\Phi(\Omega)$ and $\Phi'(\Omega)$. If we set $\mu = \Omega/2$, corresponding to choosing a frame moving with the rigidly rotating fluid, then $\tilde{\mathbf{v}}_e = \mathbf{0}$, so

$$DH_C(\Sigma_e,\mathbf{v}_e)\cdot(\delta\Sigma,\delta\mathbf{v})$$

$$= \int_{D_\Sigma} \Phi'(\Omega)\cdot\langle \operatorname{curl} \delta\mathbf{v}, \hat{\mathbf{z}} \rangle dA + \left(-\frac{1}{2}\left(\frac{\Omega}{2}\right)^2 r^2 + \frac{\tau}{r} + \Phi(\Omega) \right) \int_\Sigma \delta\Sigma\, ds$$

$$= \int_{D_\Sigma} \Phi'(\Omega)\cdot\langle \operatorname{curl} \delta\mathbf{v}, \hat{\mathbf{z}} \rangle dA,$$

since $\delta\Sigma$ satisfies $\int_\Sigma \delta\Sigma\, ds = 0$. Thus $DH_C(\Sigma_e,\mathbf{v}_e) = 0$ iff $\Phi'(\Omega) = 0$. For convenience we choose $\Phi = 0$. (Other choices of Φ will give better stability estimates.)

The second variation of H_C at a general point $(\Sigma,\mathbf{v})$ is calculated to be

$$D^2H_C(\Sigma,\mathbf{v})\cdot(\delta\Sigma,\delta\mathbf{v})^2$$
$$= \int_{D_\Sigma} (|\delta\mathbf{v}|^2 + \Phi''(\omega)\cdot|\mathrm{curl}\,\delta\mathbf{v}|^2)dA$$
$$+ \int_\Sigma \Big[2(\langle\tilde{\mathbf{v}},\delta\mathbf{v}\rangle + \Phi'(\omega)\cdot\langle\mathrm{curl}\,\delta\mathbf{v},\hat{\mathbf{z}}\rangle)\delta\Sigma$$
$$+ (\tfrac{1}{2}|\tilde{\mathbf{v}}|^2 - \tfrac{1}{2}\mu^2|\mathbf{x}|^2 + \tau\kappa + \Phi(\omega))(\delta^2\Sigma + \kappa\delta\Sigma^2)$$
$$+ \frac{\partial}{\partial\nu}\Big(\frac{1}{2}|\tilde{\mathbf{v}}|^2 - \frac{1}{2}\mu^2|\mathbf{x}|^2 + \Phi(\omega)\Big)\delta\Sigma^2$$
$$- \tau(\Delta\delta\Sigma)\delta\Sigma - \tau\kappa^2\delta\Sigma^2\Big]\,ds,$$

where Δ is the Laplacian on Σ and $\delta^2\Sigma$ is the variation of $\delta\Sigma$ with respect to Σ. (The presence of the terms involving $\delta^2\Sigma$ is due to the constraints on the variations of Σ arising from the fact that the manifold $\mathscr{S}$ of boundary curves is not a linear space; for fixed Σ the space of $\mathbf{v}$'s on Σ is linear, so no such $\delta^2\mathbf{v}$ term arises.)

For the circular flow described above the second variation reduces to

$$D^2H_C(\Sigma_e,\mathbf{v}_e)\cdot(\delta\Sigma,\delta\mathbf{v})^2$$
$$= \int_{D_\Sigma} |\delta\mathbf{v}|^2\,dA$$
$$- \int_\Sigma \Big[\Big(\frac{\Omega}{2}\Big)^2 r\delta\Sigma^2 + \tau(\Delta\delta\Sigma)\delta\Sigma + \frac{\tau}{r^2}\delta\Sigma^2\Big]\,ds.$$

It follows that $D^2H_C(\Sigma_e,\mathbf{v}_e)$ is positive definite iff

$$\tau\int_\Sigma\Big(-\frac{1}{r^2}\delta\Sigma^2 - (\Delta\delta\Sigma)\delta\Sigma\Big)\,ds > \Big(\frac{\Omega}{2}\Big)^2 r\int_\Sigma \delta\Sigma^2\,ds \tag{7}$$

for all area preserving variations $\delta\Sigma$.

We simplify the expression of this condition by estimating $-(\Delta\delta\Sigma)\delta\Sigma$ using eigenvalues of the negative of the Laplacian on the circle of radius r. The eigenfunctions are $\delta\Sigma_{k,\phi}(\theta) = \cos k(\theta-\phi)$ with eigenvalues $\lambda_{k,\phi} = (k/r)^2$ for all positive integers k. The eigenfunction $\delta\Sigma_{1,\phi} = \cos(\theta-\phi)$ corresponds to an infinitesimal translation in the ϕ direction. If we wish to consider our system modulo position, regarding two configurations as equivalent if one can be obtained from the other by a Euclidean motion, then we can simply ignore the perturbations generated by the lowest eigenfunctions $\delta\Sigma_{1,\phi}$ and test for the definiteness of D^2H_C only with respect to perturbations which actually distort the drop shape. In this case, taking $\lambda_{2,\phi} = 4/r^2$ as the lowest admissible eigenvalue, D^2H_C is positive definite iff

$$3\tau/r^3 > (\Omega/2)^2. \tag{8}$$

It follows from the stability analysis above that the rigidly rotating circular drop $(\Sigma_e,\mathbf{v}_e)$ is formally stable iff (8) holds. If we fix values for τ and r and consider the rotation rate Ω as a variable parameter, then the above statement may be interpreted as saying that the circular solution loses (formal) stability as the parameter Ω increases through the critical value $\Omega_2 = \sqrt{12\tau/r^3}$. Typically, one expects that at a point where a known curve of solutions loses stability (in this case, when the second variation of the Hamiltonian loses definiteness) a "new" branch of solutions bifurcates from the known curve. Thus we look for a bifurcation of critical points of $H - (\Omega/2)J$ at $(\Sigma_e,\mathbf{v}_e)$ when $\Omega = \Omega_2$.

We now consider the O(2) action on the manifold $\mathscr{N}$. This action is induced by the O(2) action on $\mathbb{R}^2$ as follows: Let R_γ: $\mathbb{R}^2\to\mathbb{R}^2$ denote the action of $\gamma\in O(2)$ on $\mathbb{R}^2$. Then $\gamma\cdot\Sigma = \{R_\gamma(\mathbf{x}): \mathbf{x}\in\Sigma\}$ and $\gamma\cdot(\Sigma,\mathbf{v}) = (\gamma\cdot\Sigma, R_{\gamma_*}\mathbf{v})$. We are concerned here primarily with relative equilibria; in particular, we are seeking equilibria whose motion is given by the action of some curve in the group O(2). Since the motion of our configurations must be continuous, we do not allow a sudden flip; hence the motion must be given by a smooth rotation. We choose to work with the group O(2) so as to be able to capture any reflectional symmetries of the equilibrium configurations, although this is not the appropriate group for a study of the dynamics of the problem. While the Hamiltonian is invariant under the O(2) action, the dynamics are not invariant under reflection; hence, if one wishes to consider the time-dependent behavior of solutions near the bifurcating equilibria, it is necessary to take SO(2), rather than O(2), as the appropriate symmetry group. The SO(2) action preserves both the bracket and the Hamiltonian; thus the theory of bifurcations of Hamiltonian systems with symmetry may be applied in this case.

When discussing the symmetries of a given configuration it is convenient to do so within a given rotating frame. This is motivated as follows: consider a drop moving in rigid rotation with angular velocity Ω; if the drop shape is fixed at some time t_0 by a reflection across an axis $\bar{\mathbf{x}}$, then at time t it must be fixed by reflection across $R_{\Omega(t-t_0)/2}\bar{x}$, where $R_{\Omega(t-t_0)/2}$ denotes rotation through the angle $\Omega(t-t_0)/2$, while in general it will not continue to be fixed by reflection across $\bar{\mathbf{x}}$. Thus, while the conjugacy class of the isotropy subgroup of the drop is fixed, the actual axes of symmetry of the drop vary in time. Shifting the problem to a rotating frame eliminates this complication; a rigidly rotating drop is stationary in the appropriately chosen frame and hence has a constant isotropy subgroup.

Another advantage of viewing drop symmetries from within a rotating frame is that in this context one can have nontrivial velocity fields which are fixed by orientation reversing actions. More specifically, if one considers rigidly rotating equilibrium configurations, then such drops are fixed points of some subgroup of the O(2) action in the sense that the drop shape is preserved by the subgroup, although the velocity field is reversed. [If one incorporates a time reversal as part of the flip action, then rigid rotation is fixed by the O(2) action.] Within an appropriately chosen rotating frame the velocity field of a rigidly rotating drop is equal to zero; thus, if we consider the action of O(2) within this frame, the drops described above are actual fixed points under the action. For these reasons we shall now shift the problem to a rotating frame and work with triples $(\Sigma,\tilde{\mathbf{v}},\Omega)$, where Σ denotes as usual the drop boundary, Ω is the rotation rate of the rotating frame, and $\tilde{\mathbf{v}}$ is the velocity field in the rotating frame. For an arbitrary pair $(\Sigma,\mathbf{v})$, we take Ω to be the average angular velocity of the velocity field, i.e.,

$$\Omega = \frac{1}{\mathrm{volume}\,D_\Sigma}\int_{D_\Sigma}\langle\mathrm{curl}\,\mathbf{v},\hat{\mathbf{z}}\rangle dA;$$

for a rigidly rotating drop, this sets the frame rotation rate equal to the rotation rate of the drop. For example, the configuration $(\Sigma,(\Omega/2)\hat{\mathbf{z}}\times\mathbf{x})$ is identified with the triple $(\Sigma,\mathbf{0},\Omega)$. The dynamics in the rotating frame are determined by the bracket

$$\{F,G\} = \int_{D_\Sigma} \left\langle \widetilde{\omega} + \Omega\hat{\mathbf{z}}, \frac{\delta F}{\delta \tilde{\mathbf{v}}} \times \frac{\delta G}{\delta \tilde{\mathbf{v}}} \right\rangle dA + \int_\Sigma \left(\frac{\delta F}{\delta \Sigma} \frac{\delta G}{\delta \widetilde{\varphi}} - \frac{\delta G}{\delta \Sigma} \frac{\delta F}{\delta \widetilde{\varphi}} \right) ds,$$

where $\widetilde{\omega}$ (respectively, $\delta F/\delta\tilde{\mathbf{v}}$ and $\delta F/\delta\widetilde{\varphi}$) is the vorticity (respectively, functional derivatives of F with respect to $\tilde{\mathbf{v}}$ and $\widetilde{\varphi}$), and the Hamiltonian $\widetilde{H}$: $\mathscr{N}\times\mathbb{R}\to\mathbb{R}$ is given by

$$\widetilde{H}(\Sigma,\tilde{\mathbf{v}},\Omega) = \frac{1}{2}\int_{D_\Sigma} \left(|\tilde{\mathbf{v}}|^2 - \left(\frac{\Omega}{2}\right)^2 |\mathbf{x}|^2 \right) dA + \tau \int_\Sigma ds = \left(H - \frac{\Omega}{2} J \right)\left(\Sigma, \tilde{\mathbf{v}} + \frac{\Omega}{2}\hat{\mathbf{z}}\times\mathbf{x} \right).$$

The trivial solution $(\Sigma_e,\mathbf{v}_e) = (\Sigma_e,\mathbf{0},\Omega)$ is a fixed point of the O(2) action in the rotating frame; we expect that the new solution branch bifurcating from $(\Sigma_e,\mathbf{v}_e)$ should be fixed by some subgroup of O(2). We find, in fact, that the new solutions have isotropy subgroup conjugate to the subgroup $\mathbf{Z}_2\times\mathbf{Z}_2$ of O(2) generated by rotation through π and reflection across the x axis. (For a discussion of the theory of bifurcation with symmetry relevant here, see Ihrig and Golubitsky[20] or Golubitsky *et al.*[5])

As we are concerned only with the immediate neighborhood of the point $(\Sigma_e,\mathbf{v}_e)$, it is convenient to work in normal coordinates centered at $(\Sigma_e,\mathbf{v}_e)$. We endow $\mathscr{N}$ with the O(2) invariant metric

$$\langle\langle(\delta\Sigma,\delta\mathbf{v}),(\tilde{\delta\Sigma},\tilde{\delta\mathbf{v}})\rangle\rangle = \int_\Sigma \delta\Sigma\tilde{\delta\Sigma}\, ds + \int_{D_\Sigma} \langle\delta\mathbf{v},\tilde{\delta\mathbf{v}}\rangle dA$$

and use the exponential map exp associated to the metric given above to map a neighborhood V of (0,0) in $T_{(\Sigma_e,\mathbf{v}_e)}\mathscr{N}$ diffeomorphically onto a neighborhood U of $(\Sigma_e,\mathbf{v}_e)$ in $\mathscr{N}$. We define the function $\widehat{H}$ on $V\times\mathbb{R}$ to be the pullback of the Hamiltonian plus conserved quantity;

$$\widehat{H}((\delta\Sigma,\delta\mathbf{v}),\Omega) = \widetilde{H}(\exp(\delta\Sigma,\delta\mathbf{v})).$$

It follows from the invariance of $\widetilde{H}$ and the equivariance of exp that $\widehat{H}$ is O(2) invariant.

We construct the bifurcation equation using the Liapunov–Schmidt procedure. First we construct the splitting $V = V_1\oplus V_2$, where $V_1 = \operatorname{Ker} D^2\widehat{H}(0,0,\Omega_2)$ and V_2 is the $\langle\langle\ ,\ \rangle\rangle$ orthogonal complement to V_1. We have

$$D^2\widehat{H}(0,0,\Omega_2) = D^2(H - (\Omega_2/2)J)(\Sigma_e,\mathbf{v}_e).$$

Thus

$$V_1 = \operatorname{Ker} D^2(H - (\Omega_2/2)J)(\Sigma_e,\mathbf{v}_e) = \{(\cos 2\theta,0),(\sin 2\theta,0)\}.$$

The pure rotation elements of O(2) act on the $\delta\Sigma$ component of $(\delta\Sigma,\delta\mathbf{v})$ by a negative phase shift, i.e.,

$$R^r_\varphi\cdot\delta\Sigma(\theta) = \delta\Sigma(\theta-\varphi);$$

a reflection across the axis at an angle φ to the x axis is given by

$$R^f_\varphi\cdot\delta\Sigma(\theta) = \delta\Sigma(2\varphi-\theta).$$

Let

$$F: V\times\mathbb{R}\to V,$$
$$(\delta\Sigma,\delta\mathbf{v},\Omega)\to\left(\frac{\delta\widehat{H}}{\delta\Sigma}(\delta\Sigma,\delta\mathbf{v},\Omega), \frac{\delta\widehat{H}}{\delta\mathbf{v}}(\delta\Sigma,\delta\mathbf{v},\Omega)\right)$$

denote the map determined by

$$\int_{D_\Sigma}\left\langle\frac{\delta\widehat{H}}{\delta\mathbf{v}}(\delta\Sigma,\delta\mathbf{v},\Omega),\delta\mathbf{v}\right\rangle dA + \int_\Sigma \frac{\delta\widehat{H}}{\delta\Sigma}(\delta\Sigma,\delta\mathbf{v},\Omega)\delta\Sigma\, ds = D\widehat{H}(\delta\Sigma,\delta\mathbf{v},\Omega)\cdot(\delta\Sigma,\delta\mathbf{v}),$$

where Σ denotes the first component of $\exp(\delta\Sigma,\delta\mathbf{v})$, for all $(\delta\Sigma,\delta\mathbf{v})\in V$. Let $\mathbf{P}$ denote the orthogonal projection $\mathbf{P}$: $V\to V_2$. The mapping

$$\mathbf{P}\circ F: V_1\times V_2\times\mathbb{R}\to V_2$$

is nonsingular at $(0,0,\Omega_2)$; hence, by the implicit function theorem, there exists an O(2) equivariant mapping $\mathbf{u}$: $V_1\times\mathbb{R}\to V_2$ such that

$$(\mathbf{P}\circ F)((\delta\Sigma,0) + \mathbf{u}((\delta\Sigma,0),\Omega),\Omega) = 0$$

for all $(\delta\Sigma,0)\in V_1$. The bifurcation equation is then given by

$$(\mathrm{Id}-\mathbf{P})\circ F((\delta\Sigma,0) + \mathbf{u}((\delta\Sigma,0),\Omega),\Omega) = 0.$$

We introduce the coordinate chart Ψ on a neighborhood $W\times Y$ in $\mathbb{R}^2\times\mathbb{R}$, given by

$$\Psi: W\times Y\to V_1\times Y,$$
$$(x,y,\Omega)\to((x\cos 2\theta + y\sin 2\theta,0) + \mathbf{u}((x\cos 2\theta + y\sin 2\theta,0),\Omega),\Omega).$$

We pull back $\widehat{H}$ by Ψ to obtain the bifurcation Hamiltonian $\check{H}$: $W\times Y\to\mathbb{R}$ given by $\check{H} = \widehat{H}\circ\Psi$. In summary, we have reduced the original problem to that of finding critical points of an O(2) invariant function on a two-dimensional space with an O(2) invariant metric.

The bifurcation space W possesses nontrivial symmetry. This symmetry is not artifically imposed on the system; it is a natural property of $\operatorname{Ker} D^2\widetilde{H}(\Sigma_e,\mathbf{0})$ which is inherited by the bifurcation space. The O(2) action on W induced by that on V_1 is simply twice the standard O(2) action on $\mathbb{R}^2$; i.e., for $\mathbf{x} = (x,y)$, $\theta\cdot\mathbf{x} = R_{2\theta}(\mathbf{x})$. In this action, rotation through π is equivalent to the identity action, thus the entire space W is fixed by the subgroup $\mathbf{Z}_2$ generated by rotation through π. We also note that any element (x,y) of W is fixed by reflection across the line through the angles $\arctan(x/y)$ and $\arctan(x/y)+\pi/2$. Thus any element of W has isotropy subgroup $O(2)_{\mathbf{x}}$ conjugate to $\mathbf{Z}_2\times\mathbf{Z}_2$. Since the mappings $\mathbf{u}$ and exp are equivariant, it follows that any solution $\mathbf{x}\in W\times Y$ of the bifurcation equation must be mapped to an $O(2)_{\mathbf{x}}$ invariant solution in $\mathscr{N}\times\mathbb{R}$ (the "rotating frame space") under $\exp\circ\Psi$.

There are two possible methods for demonstrating that a bifurcation does, in fact, occur. If we consider the group O(2) acting on the space W, then each isotropy subgroup $O(2)_{\mathbf{x}}$, for some nonzero element $\mathbf{x}$ of W, has a one-dimensional fixed point space consisting of the line spanned by $\mathbf{x}$. Thus, we can apply the equivariant branching lemma to show that there is a branch of relative equilibria with isotropy subgroup $O(2)_{\mathbf{x}}$ branching from the trivial solution branch at $\Omega = \Omega_2$. The equivariant branching lemma states that, given a Lie group G acting on a vector space V such that

(i) $\mathrm{Fix}(G) = \{\mathbf{0}\}$;

(ii) $\Gamma \subset G$ is an isotropy subgroup satisfying $\dim(\mathrm{Fix}(\Gamma)) = 1$;

(iii) $g: V \times \mathbb{R} \to V$ is a G-equivariant bifurcation problem satisfying $D_\lambda Dg(\mathbf{0},\lambda_0)\cdot\mathbf{v}_0 \neq 0$ for some λ_0 and some nonzero $\mathbf{v}_0 \in \mathrm{Fix}(\Gamma)$,

then there exists a branch of solutions $(t\mathbf{v}_0,\lambda(t))$ to the equation $g(\mathbf{v},\lambda) = \mathbf{0}$. (See Cicogna[4] or Golubitsky *et al.*[5] for a proof of the equivariant branching lemma.) The first two conditions are clearly satisfied for the O(2) action on W; we take, for example, the subgroup $\mathbf{Z}_2\times\mathbf{Z}_2$ corresponding to reflection across the x axis and rotation through π as our isotropy subgroup and let $\mathbf{v}_0 = (1,0)$. The equivariance of the map $F = D\check{H}$ follows from the O(2) invariance of $\check{H}$; the fact that $DF(0,0,\Omega_2) = D^2\check{H}(0,0,\Omega_2) = \mathbf{0}$ implies that the map F and the point $(0,0,\Omega_2)$ form a "bifurcation problem." Finally, we compute that

$$D_\Omega DF(0,0,\Omega_2)\cdot(1,0) = D_\Omega\, D^2\check{H}(0,0,\Omega_2)\cdot(1,0) = -\Omega_2\pi r^2/2 \neq 0,$$

thus the conditions of the equivariant branching lemma are fulfilled and a branch of solutions of $F(x,0,\Omega) = \mathbf{0}$ must exist. It follows from the equivariance of the equations that the existence of one solution branch implies the existence of an entire circle of solution branches swept out by the group action.

If we wish to consider only symplectic group actions, then we must restrict our attention to the group SO(2), which preserves the symplectic two-form on the space W. In this case, there are no one-dimensional fixed point spaces, so the equivariant branching lemma is not applicable. We can, however, apply the corollary given above to show that a bifurcation occurs. [The fact that the SO(2) action on W is twice the usual SO(2) action does not effect the applicability of the corollary.] The space W and function $\check{H}$ clearly satisfy condition (i) of the corollary; we shall show that the point $(0,0,\Omega_2)$ satisfies conditions (ii) and (iii):

(ii) $$D^2\check{H}(0,0,\Omega_2) = \begin{pmatrix} (3\tau/r^2 - (\Omega_2/2)^2 r)\pi r & 0 \\ 0 & (3\tau/r^2 - (\Omega_2/2)^2 r)\pi r \end{pmatrix} = 0;$$

(iii) $$D_\Omega\, D^2\check{H}(0,0,\Omega_2) = \begin{pmatrix} -\Omega_2\pi r^2/2 & 0 \\ 0 & -\Omega_2\pi r^2/2 \end{pmatrix} \neq 0;$$

provided that $\Omega_2 = \sqrt{12\tau/r^3} \neq 0$ (e.g., that the surface tension coefficient τ is nonzero).

Thus the corollary applies to W and $\check{H}$ and so there is a branch of critical points of $\check{H}$ bifurcating from $(0,0,\Omega)$ at $\Omega = \Omega_2$. Note: The matrices computed above are simply scalar multiples of the identity matrix; these scalars are the relevant quantities which must be computed when checking the conditions of the equivariant branching lemma in the O(2) case. Taking the image of the solution branch under the map $\exp\circ\Psi$, we obtain a curve in $\mathcal{N}$ of critical points of the original function $H + \mu J$. The elements in $\mathcal{N}$ thus obtained have the same isotropy subgroups as their preimages in W; in particular, the isotropy subgroups of elements along the new branch near the bifurcation point contain a subgroup conjugate to $\mathbf{Z}_2\times\mathbf{Z}_2$.

By computing higher-order derivatives of the bifurcation equation, it may be seen that the bifurcation equation has normal form $\mathbf{0} = -\nabla((x^2+y^2)(x^2+y^2+\Omega-\Omega_2))$ (see Lewis[8] for details). Thus, the bifurcation at Ω_2 is subcritical with respect to the bifurcation parameter Ω (i.e., locally the nontrivial solutions exist only for values of Ω less than Ω_2).

Remark 1: The bifurcation is supercritical with respect to angular momentum. Angular momentum is the "physically appropriate" bifurcation parameter in the sense that angular momentum is a physically meaningful conserved quantity for all isolated flows (whereas the bifurcation parameter Ω, which functions mathematically as a Lagrange multiplier, is related to angular velocity, a physical parameter which is only defined for rigidly rotating flows). In this case, the bifurcation equation has normal form $\mathbf{0} = \nabla((x^2+y^2)(x^2+y^2+\mu^2-\mu))$, where μ is the bifurcation parameter and μ_2 is the angular momentum at the bifurcation point; the energy-Casimir method shows the new branch is formally stable near the bifurcation point, which agrees with the general notion of transfer of stability if one views the bifurcation as supercritical. Despite the greater physical relevance of angular momentum, we have chosen the Lagrange multiplier Ω as the bifurcation parameter, since the necessary computations are straightforward in this context and it is easy to interpret the results with respect to angular momentum once the bifurcation branches have been determined.

Remark 2: The symplectic form induced on the reduced space W is a multiple of the standard symplectic two-form on $\mathbb{R}^2$, given by $\omega((x,y),(\tilde{x},\tilde{y})) = y\tilde{x} - \tilde{y}x$, which changes sign under the action of reflections; hence, as remarked above, the symplectic structure on the reduce space W is *not* preserved by the action of the orientation reversing elements of O(2). The symplectic form is, however, preserved under the action of S^1; hence the analysis of Golubitsky and Stewart[4] may be applied, viewing the drop as an S^1 invariant Hamiltonian system. We see that in this context the behavior of the drops near the point of bifurcation is generic.

Remark 3: It can be seen from the second variation of $H + \mu J$ (or $H + C$) that the variation will be indefinite in the direction of $(\delta\Sigma_{k,\phi},0) = (\cos k(\theta-\phi),0)$ when $\mu^2 = (k^2-1)\tau/r^3$. It may be shown as above that a subcritical bifurcation occurs at $\Omega_k = \sqrt{4(k^2-1)\tau/r^3}$. The solution branches intersecting the trivial solution branch are invariant under rotation through $2\pi/k$ and flips across lines conjugate to $n\pi/k$; thus their isotropy subgroups are conjugate to D_k, the dihedral group of symmetries of a k-gon. Note: D_k is the semidirect product $\mathbf{Z}_2 \circledS \mathbf{Z}_k$, where $\mathbf{Z}_2$ acts on $\mathbf{Z}_k$ by negation, i.e., by reversing the rotation associated with the elements of $\mathbf{Z}_k$.

Remark 4: The remark in Lewis *et al.*[7] regarding three-dimensional equilibria is incorrect; it will be corrected elsewhere.

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