Aspects of Mathematics Vol. E 10
Edited by Klas Diederich

Anthony J. Tromba

Seminar on New Results in Nonlinear Partial Differential Equations

A Publication of the Max-Planck-Institut für Mathematik, Bonn

Reprint

Nonlinear Stability in Fluids and Plasmas
Jerrold E. Marsden and *Tudor Ratiu*

Friedr. Vieweg & Sohn
Braunschweig / Wiesbaden

Nonlinear Stability in Fluids and Plasmas

Jerrold E. Marsden[1]
and
Tudor Ratiu[2]

Arnold's geometric methods are used to establish physically meaningful stability criteria for the Kelvin-Stuart cat's eye solution for two dimensional ideal fluids, vortex patches, reduced magnetohydrodynamics, and oceanographically interesting stratified shear flows.

1. Introduction

This paper discusses some recent progress in the field of stability of fluid and plasma equilibria. The objective is to derive explicit criteria which guarantee the nonlinear stability of specific equilibria. Most of the work described was done by H. Abarbanel, V. Arnold, R. Hazeltine, D. Holm, P. Morrison, M. Pulvirente, T. Ratiu, Y. Tang, Y.H. Wan, A. Weinstein and the author, although others have been involved in related work cited in the paper.

There are various meanings that can be given to the word "stability." Section 2 uses ideas from the theory of dynamical systems to clarify the several guises carried by this fundamental concept. Intuitively, stability means that small disturbances do not experience large growths as time passes. Being more precise about this notion is not just mathematical nitpicking — indeed, different interpretations of the word stability can lead to different stability criteria, as we shall mention later on in connection with the stability of stratified shear flows that are often used to model oceanographic phenomena.

The basis of our method was originally given by Arnold [1966a,b] and applied to two dimensional ideal fluid flow, substantially augmenting some

1. Research Group in Nonlinear Systems and Dynamics and Department of Mathematics, University of California, Berkeley, California 94720. Research partially supported by DOE contract AT-3-82ER-12097.

2. Department of Mathematics, University of Arizona, Tucson, AZ 85721. Research supported by an NSF postdoctoral fellowship at the University of California at Berkeley.

pioneering work of Lord Rayleigh [1889]. Related methods were found somewhat earlier in the plasma physics literature, notably by Newcomb [1958], Fowler [1963] and Rosenbluth [1964]. However, these works did not provide some key estimates needed to deal with the nonlinear nature of the problem. In retrospect, we now view other stability results, such as that for solitons in the Korteweg-de Vries equation, due to Benjamin [1972] and Bona [1975] as being instances of the same method used by Arnold. A crucial part of the method exploits the fact that the basic equations of nondissipative fluid and plasma dynamics are Hamiltonian in character. Section 3 recalls some facts about Hamiltonian mechanics and explains the nature of the recently discovered Hamiltonian structures that are used in the stability analysis. The last four sections of the paper discusses the results which can be obtained when the method is applied to four specific problems. These are: first, the Kelvin-Stuart cat's eye solutions of the planar ideal fluid equations; second, vortex patches for planar ideal flow (these solutions are reminiscent of Jupiter's red spot); third, the equations of reduced magnetohydrodynamics (RMHD) that are sometimes used to study plasma confinement for fusion reactions in tokamaks; and, finally, the equations for an ideal, stratified fluid in a velocity and density regime of oceanographic relevance.

2. The Meaning of Stability

Stability is a dynamical concept. To explain it, we shall use some fundamental notions from the theory of dynamical systems (see, for example, Hirsch and Smale [1974]). The laws of dynamics are usually presented as equations of motion which we write in the abstract form

$$\dot{u} = X(u). \tag{2.1}$$

Here, u is a variable describing the state of the system under study, X is a system-specific function of u and $\dot{u} = \frac{du}{dt}$, where t is time. The set of all allowed u's forms the <u>state space</u> P. For classical mechanical systems,

u is often a 2n-tuple $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ of positions and momenta and for fluids, u is a velocity field in physical space. As time evolves, the state of the system changes; the state follows a curve $u(t)$ in P. The trajectory $u(t)$ is assumed to be uniquely determined if its initial condition $u_0 = u(0)$ is specified. An equilibrium state is a state u_e such that $X(u_e) = 0$. The unique trajectory starting at u_e is u_e itself; that is, u_e does not move in time.

The language of dynamics has been an extraordinarily useful tool in the physical and biological sciences, especially during the last few decades. The study of systems which develop spontaneous oscillations through a mechanism called the Poincaré-Andronov-Hopf bifurcation is an example of such a tool (see Marsden and McCracken [1976] and Chow and Hale [1982], for example). More recently, the concept of "chaotic dynamics" has sparked a resurgence of interest in dynamical systems. This occurs when deterministic systems such as (2.1) possess trajectories that are so complex that they behave as if they were random. Some believe that the theory of turbulence will use such notions in its further development. (See, for example, Ruelle [1980] for a popular account.) We are not concerned with chaos directly, although it can play a role in some of what follows. In particular, we remark that in the definition of stability below, stability does not preclude chaos. In other words, the trajectories near a stable point can still be very complex -- stability just prevents them from moving very far from equilibrium.

To define stability, we need to choose a measure of nearness in P. This is done in terms of a "metric" d. For two points u_1 and u_2 in P, d determines a positive number denoted $d(u_1,u_2)$ which is called the distance from u_1 to u_2. In the course of a stability analysis, it is necessary to specify, or construct a metric appropriate for the problem at hand. In this setting, one says that an equilibrium state u_e is stable when trajectories which start near u_e remain near u_e for all $t \geq 0$. (Technically, given any number $\varepsilon > 0$, there is a $\delta > 0$ such that if $d(u_0,u_e) < \delta$, then $d(u(t),u_e) < \varepsilon$ for all $t \geq 0$). Figure 1 shows examples of stable and unstable equilibria for dynamical systems whose state space is the plane.

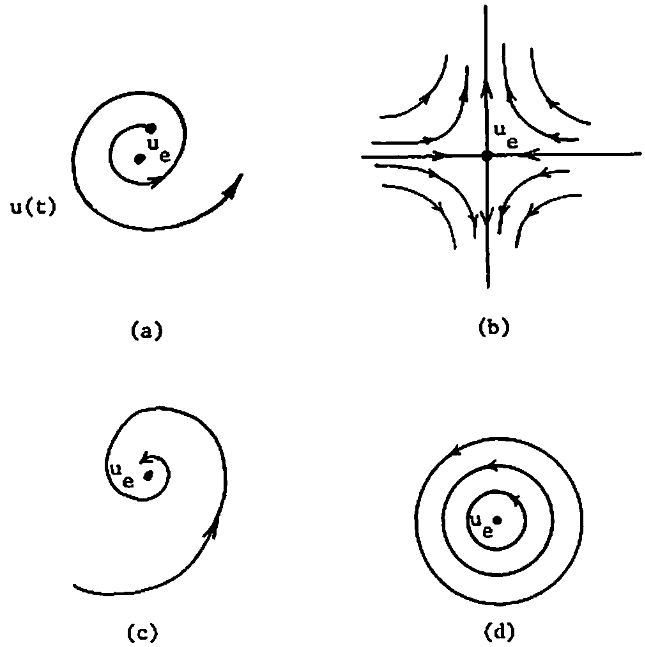


Figure 1. The equilibrium point (a) is unstable because the trajectory $u(t)$ does not remain near u_e. Similarly (b) is unstable since most trajectories (eventually) move away from u_e. The equilibria in (c) and (d) are stable because all trajectories near u_e stay near u_e.

As we shall see in Section 6, fluid systems can be stable relative to one distance measure and, simultaneously, unstable relative to another. This seeming pathology actually reflects important physical processes.

A well-known physical example illustrating the definition of stability is the motion of a free rigid body. This system can be simulated by tossing a book, held shut with a rubber band, into the air. It rotates stably when spun about its longest and shortest axes, but unstably when spun about the middle axis (Figure 2). (The distance measure defining stability in this example is a metric in body angular momentum space, which becomes indefinite in case (c) of Figure 2, see, e.g. Arnold [1978] and Holm, Marsden, Ratiu, and Weinstein [1984].)

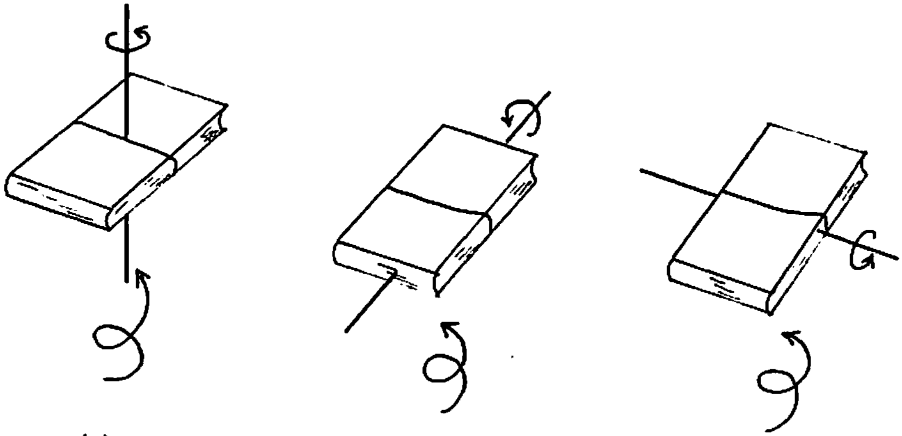


Figure 2. If you toss a book into the air, you can make it spin stably about its shortest (a) and longest (b) axis, but it is unstable when it rotates about the middle axis (c).

There are two other ways of treating stability. First of all, one can linearize equation (2.1); if δu denotes a variation in u and $DX(u_e)$ denotes the linearization of X at u_e (the matrix of partial derivatives, in the case of finitely many degrees of freedom), the linearized equations describe the time evolution of "infinitesimal" disturbances of u_e:

$$\frac{d}{dt}(\delta u) = DX(u_e)\cdot\delta u. \tag{2.2}$$

Equation (2.1) on the other hand, describes the nonlinear evolution of finite disturbances $\Delta u = u - u_e$. We say u_e is linearly stable if (2.2) is stable at $\delta u = 0$, in the sense defined above. Intuitively, this means that there are no infinitesimal disturbances which are growing in time. If $(\delta u)_0$ is an eigenfunction of $DX(u_e)$, that is, if

$$DX(u_e)\cdot(\delta u)_0 = \lambda(\delta u)_0 \tag{2.3}$$

for a complex number λ, then the corresponding solution of (2.2) is

$$\delta u = e^{t\lambda}(\delta u)_0. \tag{2.4}$$

This is growing when λ has positive real part. This leads us to the third notion of stability: we say that equations are spectrally stable if the eigenvalues (more precisely points in the spectrum) all have non positive real parts.

Under technical hypotheses, one has the following logical implications

stability $\Longrightarrow$ linear stability $\Longrightarrow$ spectral stability

If the eigenvalues all lie strictly in the left half plane, then a classical result of Liapunov guarantees stability. (See, for instance, Hirsch and Smale [1974] for the finite dimensional case and Marsden and McCracken [1976] for the infinite dimensional case.) However, in systems of interest to us, the dissipation is very small -- our systems are essentially conservative and, in an appropriate sense, Hamiltonian. For such systems it is known that the eigenvalues must be symmetrically distributed under reflection in the real and imaginary axis. This implies that the only possibility for spectral stability is when the eigenvalues lie exactly on the imaginary axis. Thus, the Liapunov theorem is of no help in this case.

In fact, spectral stability typically does not imply stability; instabilities can be generated by the nonlinear terms through a mechanism called Arnold diffusion. (See, for example, Lichtenberg and Lieberman [1983] for an account of much of what is known, both theoretical and numerical.) Thus, to obtain general stability results, one must use other techniques to augment or replace the linearized theory. We give this technique in Section 4.

Here is a simple planar example of a system which is spectrally stable at the origin, but which is unstable there. (This is not a conservative system -- to get such an example requires more work.) In polar coordinates (r,θ), consider the evolution of $u = (r,\theta)$ given by

$$\begin{aligned} \dot{r} &= r^2(1-r^2) \\ \dot{\theta} &= 1 \end{aligned} \tag{2.5}$$

The eigenvalues at the origin are readily verified to be $\pm\sqrt{-1}$, so the origin is spectrally stable; however the phase portrait, shown in Figure 3 shows that the origin is unstable. (We included the factor $1-r^2$ to give the system an attractive periodic orbit -- this is merely to enrich the example and show how a stable periodic orbit can attract the orbits expelled by an unstable equilibrium.)

Despite the above situation relating the linear and nonlinear theories, there has been much effort devoted to the development of spectral stability

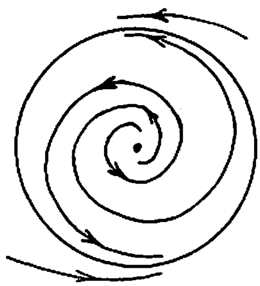


Figure 3. The origin is a spectrally stable, but unstable point for this dynamical system.

methods, including expensive numerical codes. When instabilities are present, spectral estimates do give important information on growth rates. As far as stability goes, spectral stability gives necessary, but not sufficient conditions for stability. In other words, for the nonlinear problem spectral stability can predict instability, but not stability. Our purpose is the opposite: to develop sufficient conditions for stability.

3. Hamiltonian Systems

The traditional view of Hamiltonian mechanics, as it is found in the classical treatises such as Whittaker [1917], is that the dynamics is governed by a special form of (2.1) called Hamilton's equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n. \tag{3.1}$$

Here $q^1, \ldots, q^n$, are configuration variables and $p_1, \ldots, p_n$ are their conjugate momenta. These 2n equations describe the evolution of the phase point $u = (q^1, \ldots, q^n, p_1, \ldots, p_n)$. Hamiltonian theory is remarkably successful in describing many situations in dynamics such as satellite motion. For our purposes, however, a generalization is needed. For instance, consider the classical Euler equations for rigid body motion: these are

$$\dot{m}_1 = \frac{I_2 - I_3}{I_2 I_3} m_2 m_3, \qquad \dot{m}_2 = \frac{I_3 - I_1}{I_3 I_1} m_3 m_1, \qquad \dot{m}_3 = \frac{I_1 - I_2}{I_1 I_2} m_1 m_2 \tag{3.2}$$

where $\underline{m} = (m_1, m_2, m_3)$ is the angular momentum of the body as seen by an observer moving with the body and I_1, I_2, I_3 are the fixed moments of inertia; the body angular velocity $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ is related to the angular momentum by $I_1\omega_1 = m_1$, $I_2\omega_2 = m_2$, $I_3\omega_3 = m_3$. It is clear that (3.2) are not in the form (3.1) because for one thing, (3.1) consist of an even number (2n) of equations, while (3.2) has an odd number (3) of equations. In what sense are (3.2) Hamiltonian? One way to answer this is to formulate the equations in terms of Euler angles (ϕ, θ, ψ) and their conjugate momenta $(p_\phi, p_\theta, p_\psi)$. Then the classical texts (such as Goldstein [1980], Chapter 4) show that the corresponding equations have the form (3.1). However this requires six equations, while (3.2) needs only three. If one wishes to deal with (3.2) directly, which is useful for a stability analysis, a new idea is needed. The crucial step is to concentrate on Poisson brackets.

Given two scalar valued functions F and G of $q^1, \ldots, q^n, p_1, \ldots, p_n$, their Poisson bracket is defined by

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i} \frac{\partial F}{\partial p_i} \right) \tag{3.3}$$

Using this notation, the chain rule shows that Hamilton's equations (3.1) can be equivalently written in Poisson bracket form:

$$\dot{F} = \{F,H\} \tag{3.4}$$

holds for any function $F(q^1, \ldots, q^n, p_1, \ldots, p_n)$.

The Poisson bracket (3.3) enjoys four properties which turn out to be crucial. Let F, G, K be functions of $q^1, \ldots, q^n, p_1, \ldots, p_n$ and a be a constant. Then

1. $\{F + aG, K\} = \{F,K\} + a\{G,K\}$
2. $\{F,G\} = -\{G,F\}$
3. $\{\{F,G\},K\} + \{\{K,F\},G\} + \{\{G,K\},F\} = 0$ (Jacobi's identity)
4. $\{FG,K\} = F\{G,K\} + \{F,K\}G$ (Leibniz' rule)

(These four properties have been isolated by many authors, such as Dirac [1964], p. 10.) Abstracting this situation, we say that a phase space P is a Poisson space (or Poisson manifold) when there is an operation $\{\cdot\ ,\ \cdot\}$ on pairs of functions on P satisfying properties 1.-4.

This abstraction would of course be useless unless it included interesting examples of non-canonical brackets; that is, brackets not of the form (3.3). The rigid body provides one. Let P be the space of all $\underline{m}$'s and let the Hamiltonian be the kinetic energy, known to be simply

$$H = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right), \tag{3.5}$$

and let the bracket be defined by

$$\{F,G\} = -\underline{m}\cdot(\underline{\nabla}F \times \underline{\nabla}G), \tag{3.6}$$

where "$\cdot$" is the vector dot product, "$\times$" is the vector cross product and $\underline{\nabla}F$, $\underline{\nabla}G$ denote the gradients of F and G (understood to be evaluated at $\underline{m}$ in (3.6)). It is straightforward to check that the bracket (3.6) makes

P into a Poisson space and that the equations of motion (3.2) can be recast in the form $\dot{F} = \{F,H\}$.

The general notion of a Poisson manifold goes back to Sophus Lie [1890] (see Weinstein [1983]). The first textbook we know of that discusses generalized Poisson brackets in this sense and their applications to mechanics is Sudarshan and Mukunda [1974]. (The bracket (3.6) appears there on page 318.)

To deepen our understanding of where (3.6) comes from, we consider the relation between the angular momentum vector $\underline{m}$ and the Euler angles and their conjugate momenta $(\phi,\theta,\psi,p_\phi,p_\theta,p_\psi)$. This relation is given by classical mechanics texts (such as Goldstein [1980]) to be

$$\begin{aligned} m_1 &= [(p_\phi - p_\psi \cos\theta)\sin\psi + p_\theta \sin\theta\cos\psi]/\sin\theta, \\ m_2 &= [(p_\phi - p_\psi \cos\theta)\cos\psi - p_\theta \sin\theta\sin\psi]/\sin\theta, \\ m_3 &= p_\psi . \end{aligned} \tag{3.7}$$

Given functions F and G of (m_1,m_2,m_3), substitution of (3.7) expresses them in terms of $(\phi,\theta,\psi,p_\phi,p_\theta,p_\psi)$. Now compute the canonical bracket. A lengthy, but straightforward calculation shows that the answer is precisely (3.6). Thus, in a certain sense, (3.6) is a transformation of a canonical bracket.

The procedure described for the rigid body turns out to hold for other systems as well. An important one is the Euler equations for homogeneous incompressible flow. We state the result for planar flow in a region D for simplicity, although there is an analogous result for three dimensional flows. The equations for the velocity $\underline{v}$ are

$$\frac{\partial v}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\underline{\nabla}p, \tag{3.8}$$

where the pressure p is determined implicitly from the condition div $\underline{v} = 0$ together with the boundary condition that $\underline{v}$ be tangent to the boundary of D. If $\underline{v} = (U,V)$, the <u>vorticity</u> is the third component of the curl of $\underline{v}$:

$$\omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}. \tag{3.9}$$

Taking the curl of (3.8) produces the vorticity equation,

$$\frac{\partial \omega}{\partial t} + \underline{v}\cdot\nabla\omega = 0 . \tag{3.10}$$

Let P be the space of ω's (with $\underline{v}$ determined by (3.9), div $\underline{v} = 0$ and the boundary conditions -- such a $\underline{v}$ is uniquely determined if D has no holes). The Hamiltonian is the kinetic energy, given by

$$H = \frac{1}{2}\int_D |\underline{v}|^2 \, dx \, dy,$$

assuming the density is unity. The bracket of two scalar functions of the vorticity is defined by

$$\{F,G\} = \int_D \omega\left\{\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}\right\}_{xy} dx \, dy, \tag{3.11}$$

where $\frac{\delta F}{\delta \omega}$ is the functional derivative (defined as in Goldstein [1980]) and $\{\ ,\ \}_{xy}$ denotes the canonical bracket (3.3) with $q = x$, $p = y$. Again, (3.11) makes P into a Poisson space and Euler's equations (3.10) (or equivalently (3.8)) can be recast as $\dot{F} = \{F,H\}$. (In writing (3.11) we have, for simplicity, ignored certain boundary terms; see Lewis et. al. [1985] for a complete treatment.

The brackets (3.6) and (3.11) share a common structure. Both are examples of what Marsden and Weinstein [1983] call a <u>Lie-Poisson bracket</u>, which is a special type of bracket associated with a group. If $\mathfrak{g}^*$ is the dual of the associated Lie algebra, these brackets are given by the formula

$$\{F,G\}_{\pm} = \pm\mu\cdot\left[\frac{\delta F}{\delta \mu}, \frac{\delta F}{\delta \mu}\right] \tag{3.12}$$

where "·" denotes an "inner product" and [,] is the Lie algebra bracket. Formula (3.12) is due to Sophus Lie [1890]. We won't define Lie brackets in general here, but just point out that it corresponds to the cross product in (3.6) and to $\{\ ,\ \}_{xy}$ in (3.11).

The group associated with the rigid body is SO(3), the rotation group and we choose "-" in (3.12), while that for fluids is the group of area preserving, invertible, transformations of D to D (also called the particle relabeling group, or the area-preserving diffeomorphism group) and we choose "+" in (3.12).

The procedure described earlier for deriving the rigid body bracket has an analogue for fluids. This involves the passage from the material to the spatial description of a fluid: canonical brackets in material representation get mapped to the non-canonical brackets (3.11) in spatial representation. Figure 4 summarizes the situation for a general continuum theory.

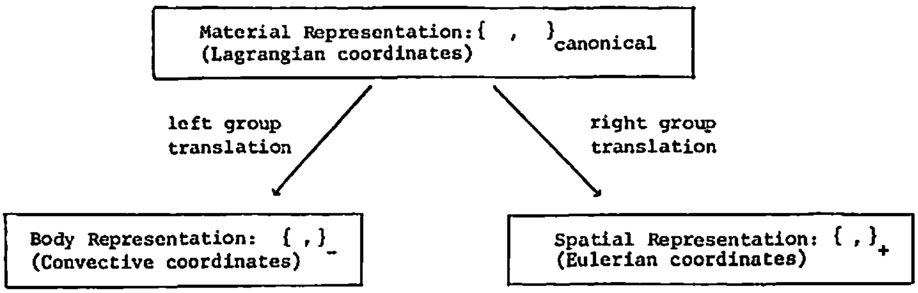


Figure 4. The passage from the material to the body and spatial representations takes canonical brackets to Lie-Poisson brackets: "+" for spatial, "-" for body.

This picture was painted by Arnold [1966a]; he does not express things in terms of Poisson brackets, but in equivalent terms. The two passages in Figure 4 are special cases of a general procedure called <u>reduction</u> which was developed by, amongst others, Smale [1970] and Marsden and Weinstein [1974]; this general theory is described in books which give the geometric approach to mechanics, such as Arnold [1978] and Abraham and Marsden [1978].

A resurgence of interest in Poisson structures began with the infusion of examples from plasma physics. Notable amongst these are the papers of Bialynicki-Birula and Iwinski [1973], Iwinski and Turski [1976], Dyzhaloshinski and Volovick [1980], and Morrison [1980]. Brackets were found, using trial and error and quantum limiting procedures, for, amongst others, MHD and the Maxwell-Vlasov equations — these are basic sets of equations for fluids and plasmas moving in magnetic and electromagnetic fields. The relativistic case and charged fluids were also treated by Iwinski and Turski. The Maxwell-Vlasov bracket was derived by the reduction method (and one term of Morrison's bracket corrected) by Marsden and Weinstein [1982]. A similar derivation for the charged fluid bracket was given by Spencer and Kaufmann [1982]. We shall give the bracket for RMHD in Section 7, referring the reader to Marsden, Weinstein, Ratiu, Schmidt and Spencer [1983] for a survey of the theory and other literature, including how the alternative methods of Lin constraits and Clebsch representations fit into the theory of Poisson manifolds in a natural way. (See Seliger and Whitham [1968] and Holm and Kupershmidt [1983], respectively.)

In many examples (such as RMHD) the Lie-Poisson bracket involved comes from a special Lie algebra structure called a semi direct product. These are typified by the Euclidean group: the group of rotations and translations in space. The first time (known to us) this was shown to occur in mechanics was for a rigid body with a fixed point under gravity by Sudarshan and Mukunda [1974], p. 366. (See also Holmes and Marsden [1983] and references therein.) For compressible fluids the bracket of Morrison and Greene [1980] is readily checked to be Lie-Poisson for a semi direct product (see Marsden [1982]). The Poisson bracket for MHD is also of this same type, as was shown by Holm and Kupershmidt [1983]. The general theory for this, based on the ideas of Figure 4 was developed by Marsden et. al. [1983] and Marsden, Ratiu and Weinstein [1984a,b], building on some key theory of Guillemin and Sternberg [1980] and Ratiu [1980, 1982]. More recently, general relativistic systems of this kind have also been treated; see Bao et. al. [1984], Holm [1984] and references therein. Various other examples (including extensions to nonabelian fields and generalized two-cocycles) in physics of semi-direct product Lie-Poisson brackets appear in Holm, Kupershmidt, and Levermore [1983].

It is reasonable to expect that the stability method described in Section 4 is just one of several possible applications of the new Hamiltonian formalism. (See for example, Simelon, Kaufman and Holm [1984], and Lewis, Marsden, Montgomery, and Ratiu [1984]). It should also prove useful for plasma-wave interactions model building (such as for guiding center plasmas), perturbation theory and for understanding the relationship between the classical and quantum theories (for example, Littlejohn [1979], Kaufman and Bogosian [1984] and Goldin [1984]).

4. The Energy-Casimir Method

Non-canonical brackets have another interesting feature: they can admit large classes of conserved quantities. There is, besides the energy, conserved quantities associated with group symmetries such as linear and angular momentum. Some of these may be termed "reduction remnants" since they are associated with the group that underlies the passage from material to spatial or body coordinates. These are called Casimirs; such a quantity, denoted C, is characterized by the fact that it Poisson commutes with every function; that is,

$$\{C,F\} = 0 \tag{4.1}$$

for all functions F on phase space P.

For example, if Φ is any function of one variable, the quantity

$$C(\underline{m}) = \Phi(|\underline{m}|^2) \tag{4.2}$$

is a Casimir for the rigid body bracket, (3.6), as is easily seen by using the chain rule. Likewise,

$$C(\omega) = \int_D \Phi(\omega)\, dx\, dy \tag{4.3}$$

is a Casimir for the two dimensional fluid bracket (3.11). (As in (3.11), this calculation ignores boundary terms that arise in an integration by parts).

Casimirs are conserved by the dynamics associated with any Hamiltonian H since $\dot{C} = \{C,H\} = 0$ by (4.1). Conservation of (4.2) corresponds to conservation of total angular momentum for the rigid body, while conservation of (4.3) represents Kelvin's circulation theorem for the Euler equations. (Note that (4.3) provides infinitely many constants of the motion which Poisson commute; i.e. $\{C_1, C_2\} = 0$, but this does not imply that these equations are integrable.)

For Hamiltonian systems in canonical form (3.1), there is a classical stability criterion due to Lagrange and Dirichlet. First of all, notice that an equilibrium point $(\underline{q}_e, \underline{p}_e)$ is a point at which the partial derivatives of H vanish, i.e. it is a critical point of H. If the $2n \times 2n$ matrix D^2H of second partial derivatives evaluated at $(\underline{q}_e, \underline{p}_e)$ is positive or negative definite (i.e. all the eigenvalues have the same sign), then $(\underline{q}_e, \underline{p}_e)$ is stable. This follows from conservation of energy and the fact, proven in advanced calculus, that the level sets of H near $(\underline{q}_e, \underline{p}_e)$ are approximately ellipsoids. This condition implies, but is not implied by, spectral stability. Apart from KAM (Kolmogorov, Arnold and Moser) theory, which gives stability of periodic solutions for two degree of freedom systems, the Lagrange-Dirichlet theorem is the only known general stability theorem for canonical systems.

The energy-Casimir method is a generalization of the Lagrange-Dirichlet method. Given an equilibrium u_e for $\dot{u} = X(u)$, it proceeds in the following steps:

<u>Energy-Casimir Method</u>

<u>Step A</u>. Write the equations in Hamiltonian form $\dot{F} = \{F, H\}$.

<u>Step B</u>. Find a family of conserved quantities C, such as a family of Casimirs.

<u>Step C</u>. Select C such that $H + C$ has a critical point at u_e.

<u>Step D</u>. Check to see if $D^2(H + C)(u_e)$, the matrix of second partial derivatives of $H + C$ at u_e, is positive or negative definite.

With regard to step C, we point out that an equilibrium solution need not be a critical point of H alone; in general $DH(u_e) \neq 0$. An example where this occurs is a rigid body spinning about one of its principal axes of inertia. In this case, a critical point of H alone would have zero angular velocity; but a critical point of $H + C$ is a (nontrivial) stationary rotation about one of the zero axes.

In principle, the same argument used to establish the Lagrange-Dirichlet test also works here. Unfortunately, for systems with infinitely many degrees of freedom (like fluids and plasmas), there is a technical snag. The calculus argument used before simply runs into problems; one might think these are just technical and that we just need to improve the methods. In fact there is widespread belief in this "energy criterion" (see, for instance, the discussion and references in Marsden and Hughes [1983], Chapter 6). However, Ball and Marsden [1984] have shown by means of a realistic example from elasticity theory that the difficulty is genuine. To overcome this difficulty we must modify step D using a convexity argument of Arnold [1966b].

Convexity Analysis

Modified Step D

(a) Let $\Delta u = u - u_e$ denote a finite variation in phase space.

(b) Find quadratic functions Q_1 and Q_2 such that

$$Q_1(\Delta u) \leq H(u_e + \Delta u) - H(u_e) - DH(u_e)\cdot\Delta u$$

$$Q_2(\Delta u) \leq C(u_e + \Delta u) - C(u_e) - DC(u_e)\cdot\Delta u.$$

(c) Require that $Q_1(\Delta u) + Q_2(\Delta u) > 0$ for all $\Delta u \neq 0$.

(d) Introduce the <u>norm</u> $\|\Delta u\|$ by

$$\|\Delta u\|^2 = Q_1(\Delta u) + Q_2(\Delta u),$$

so $\|\Delta u\|$ as a measure of the distance from u to u_e;

$d(u, u_e) = \|\Delta u\|$.

(c). Require that

$$|H(u_e + \Delta u) - H(u_e)| \le C_1 \|\Delta u\|$$

and

$$|C(u_e + \Delta u) - C(u_e)| \le C_2 \|\Delta u\|$$

for constants C_1 and C_2, and $\|\Delta u\|$ sufficiently small.

These conditions guarantee stability of u_e and provide the distance measure relative to which stability is defined. The key part of the proof is simply the observation that if we add the two inequalities in (b), we get

$$\|\Delta u\|^2 \le H(u_e + \Delta u) + C(u_e + \Delta u) - H(u_e) - C(u_e);$$

here, $DH(u_e)\cdot\Delta u$ and $DC(u_e)\cdot\Delta u$ have added up to zero by step C. But H and C are constant in time so

$$\|(\Delta u)_{\text{time } t}\|^2 \le [H(u_e + \Delta u) + C(u_e + \Delta u) - H(u_e) - C(u_e)]\big|_{\text{time } 0}.$$

Now employ the inequalities in (c) to get

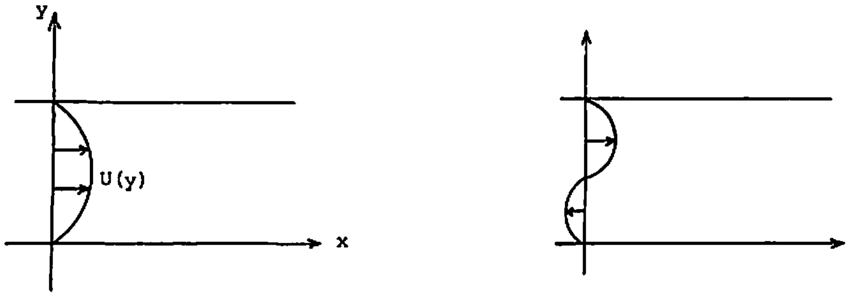
$$\|(\Delta u)_{\text{time } t}\|^2 \le (C_1 + C_2)\|(\Delta u)_{\text{time } 0}\|$$

This estimate bounds the temporal growth of finite perturbations in terms of initial perturbations, which is exactly what is needed for stability.

In the ensuing sections we will give examples of how this technique applies in concrete examples. We shall only discuss the results and their significance, leaving the technical details to the research literature cited. The examples we have chosen are only a fraction of those to which the method applies. We refer to Holm, Marsden, Ratiu and Weinstein [1985] for a more extensive survey.

5. Stability of the Kelvin-Stuart Cats Eyes

The energy-Casimir method was used by Arnold [1966a,b] to establish the nonlinear stability of a shear flow solution to ideal incompressible flow in two dimensions. The condition needed for stability is satisfied when, for example, the velocity profile has no point of inflection; this situation was established for the linearized equations by Rayleigh [1880]; see Fig. 5.



(a) Stable (b) could be unstable

Figure 5. For equilibrium flows of the form $u = (U(y), 0)$, stability holds if U has no inflection point as a function of y.

The "cats eyes" solution of the Euler equations is given by the stream function

$$\psi_e = \log[a \cosh y + \sqrt{a^2-1} \cos x] \tag{5.1}$$

where x is a 2π-periodic variable and $a \geq 1$. The corresponding velocity field is $v_e = \left(\frac{\partial\psi_e}{\partial y}, -\frac{\partial\psi_e}{\partial x}\right)$. For $a = 1$ we recover a shear flow. The solution (5.1) was found by Stuart [1967] and was known to Kelvin for the linearized equations; it has the interesting flow pattern shown in Figure 6 which gives rise to the name "cats eyes." This solution is believed to be important in many fluid pehnomena such as the roll patterns one sees in clouds. For a linearized analysis, see Drazin and Reid [1981), p. 141.

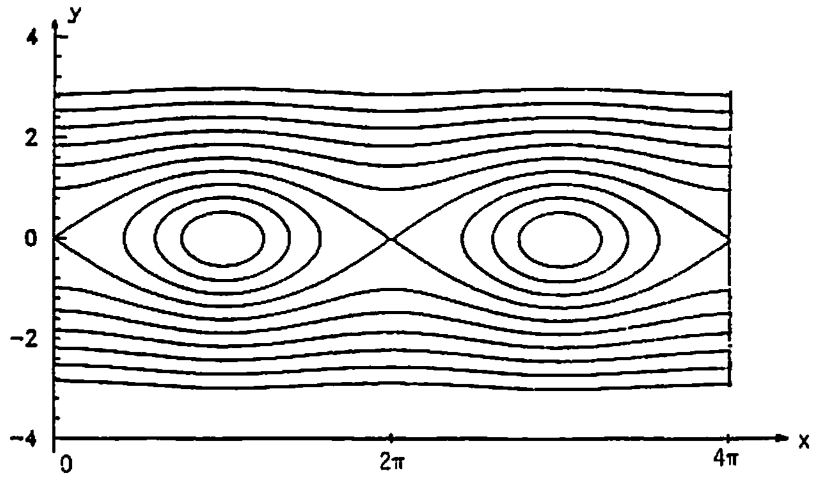


Figure 6. Computer plot of the cat's eyes streamlines for the stream function $\psi(x,y)$ in (5.1) with a = 1.175.

One now goes through the steps of the energy-Casimir method to test (5.1) for stability. The energy and Casimir functions for this situation have already been discussed. For step C one computes directly that H + C has a critical point at the cats eye solution if Φ (see equation (4.3)) is chosen to be

$$\Phi(\lambda) = \frac{1}{2}\lambda(\log(-\lambda) - 1), \quad \text{for } \lambda < 0 \qquad (5.2)$$

With this choice, one can now test step D, (or more precisely, the modified step D.) An interesting complication which is not encountered for shear flows without an inflection point (but which Arnold noticed could occur for certain situations with inflection points) is that the second variations of H and C do not have the same sign. One has to show that, nevertheless, the second variation of H + C does have a definite sign. To do this requires the use of Poincaré's inequality, which has the form

$$\iint |f|^2 \, dx\, dy \leq \text{Constant} \iint |\nabla f|^2 \, dx\, dy,$$

where the constant can be determined by the eigenvalues of the Laplacian.

The results of the calculations, which are done in Holm, Marsden and Ratiu [1984] are that if the range of y is limited to a region containing the eyes (shaded in Figure 6), one has stability if

$$a - \sqrt{a^2-1})^2 > 2\ell^2/(\pi^2 + \ell^2), \tag{5.3}$$

where

$$\ell = \cosh^{-1}\left(1 + \frac{2\sqrt{a^2-1}}{a}\right).$$

Numerically, one finds (5.3) to hold for

$$1 \leq a \leq 1.175\ldots \tag{5.4}$$

The method also produces an estimate on the growth of perturbations. Here it shows that the square of the L^2 norm of the vorticity perturbations, defined by

$$\|\Delta\omega\|^2 = \iint |\Delta\omega|^2 dx\, dy \tag{5.5}$$

remains small in time if it starts out small. We will come back to the stability of the cats eyes solution in Section 7.

6. Vortex Patches

A vortex patch is a vorticity distribution which is constant in a region D of the xy plane and is zero outside the plane. The vorticity evolves in time according to the Euler equations and this causes the region D to move, giving a new vortex patch. The motion of these patches is believed to be basic in the understanding of fluid phenomena and could even be related to the motion of Jupiter's red spot. The problem has been extensively discussed by Zabusky and his co-workers (see Zabusky [1984] and references therein).

The stability of vortex patches is interesting for these physical reasons and has a number of curious mathematical features that are relevant to the problem. As a subclass of solutions of the Euler equations, vortex patches have their own Hamiltonian structure. This structure, worked out by Marsden and Weinstein [1983] has features in common with the KdV (Korteweg-deVries) equation, famous for its solitons. However, an asymptotic analysis suggested that ripples on the vortex patch boundary do not retain permanent form, but rather form cusps or break like waves. That the boundary of a smooth patch need not remain smooth is consistent with the existence theorems that apply to this situation (Iudovich [1964]).

The energy-Casimir method does not literally apply to this situation. For one thing, the fact that the vorticity is not a smooth function on the plane causes technical difficulties. Nevertheless, inspired by the method, Wan [1984], Wan and Pulvirente [1984] and Tang [1984], were able to prove stability theorems.

There are two solutions that are fundamental. One is the circular patch and the other is the Kirchhoff rotating ellipse (Figure 7). The circular patch is a stationary solution, while the ellipse is a steadily rotating solution. (The latter becomes stationary in a rotating reference frame). The circular case can be modified to an annular distribution of vorticity or to a circular patch in a circular container. Calculations for

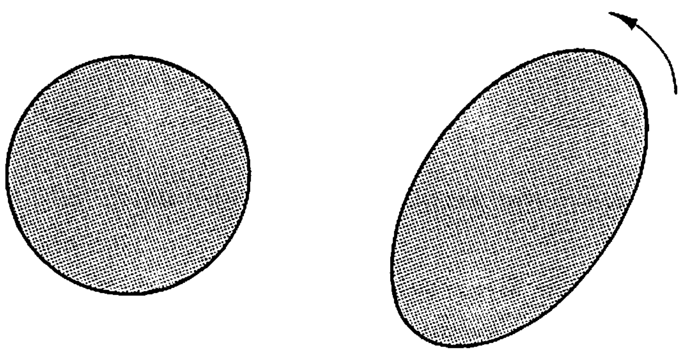
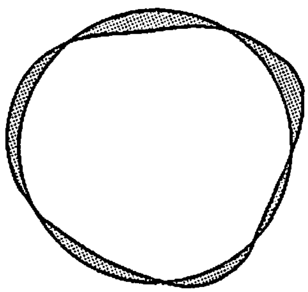


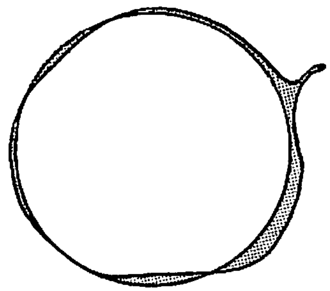
Figure 7. Two exact solutions of two dimentional Euler flow.

the linearized case by Kelvin [1880] show that the circular type solutions ought to be stable, while calculations of Love [1893] show that the Kirchhoff solution is linearly stable if the ratio of the lengths of the axes of the ellipse does not exceed 3.

These results have been proved in the nonlinear case in the references cited. However the exact interpretation requires care. The measure used for the distance from the equilibrium solution is the shaded area in Figure 8. This result does not say that the absolute distance between the boundaries of the patches remains small. Nor does it say that the boundaries remain smooth. In fact, computational work indicates that such possibilities really do occur. Figures 9 and 10 show some numerical studies of Dritschel [1984]. Figure 9 shows the evolution of a nearly annular distribution of vorticity and the development of steep, small waves. Figure 10 shows the evolution of a perturbation of an ellipse with axis ratio near 1/3 and the development of a long thin tail.



(a) Initial, nearly circular patch



(b) The patch evolves in time

Figure 8. If the shaded area starts small, it remains small.

Thus, in this context we have stability with respect to one distance measure, but not with respect to another. The distinction is reflecting important physical mechanisms.

For unstable patches it is very interesting to understand how they tend to split apart, merge, or form other curious patterns. We refer to Dritschel [1984] for some examples. The theory on this aspect requires much development.

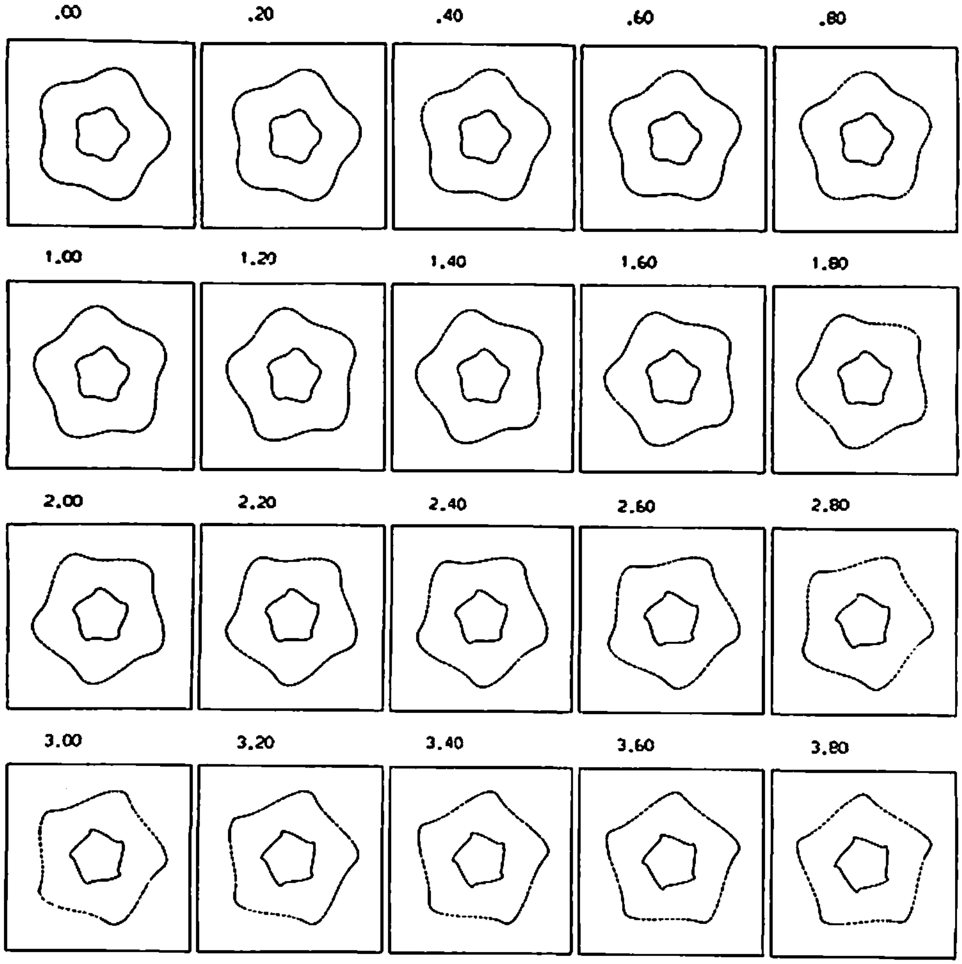


Figure 9. The evolution of a nearly annular vortex patch. (Dritschel [1984]).

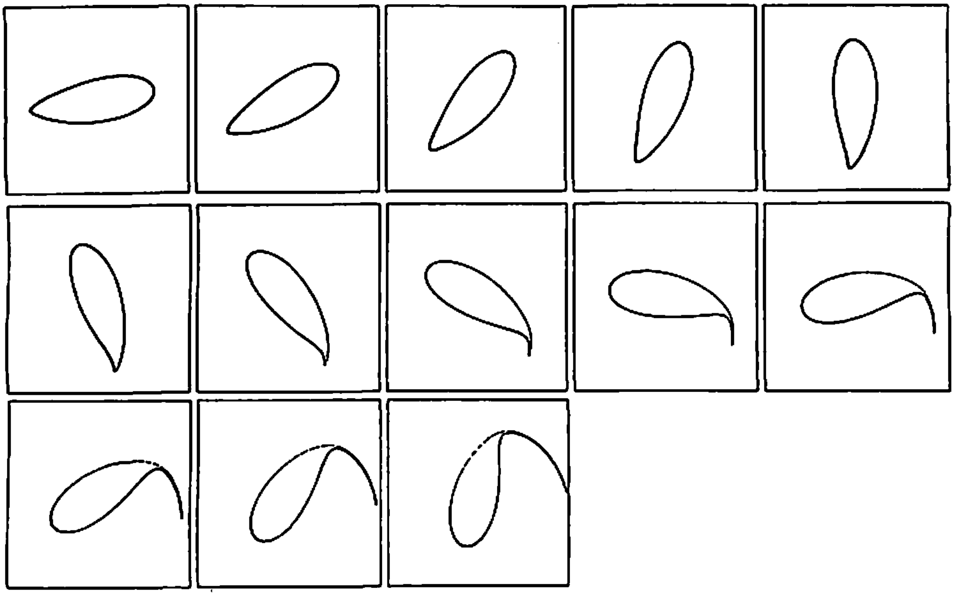


Figure 10. The evolution of a nearly elliptical vortex patch. (Dritschel [1984]).

7. Reduced Magnetohydrodynamics (RMHD)

Here we describe a result of Hazeltime, Holm, Marsden and Morrison [1984]. RMHD is a simplified model deriving from three dimensional MHD. It is a model which is contemplated for use in describing a plasma in a tokamak configuration (Figure 11).

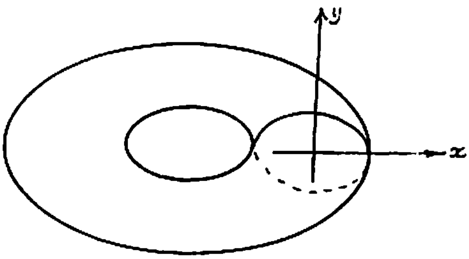


Figure 11. RMHD deals with plasma in a toroidal cavity.

In this approximation the fields that are singled out are the components of the fluid velocity $\underline{v}$ and magnetic field $\underline{B}$ parallel to a cross-sectional plane (shaded in Figure 11). We assume $\underline{v}$ and $\underline{B}$ are parallel to the boundary of this planar region and are divergence free. We introduce a stream function ψ and magnetic scalar potential A by writing

$$\underline{v} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right), \quad \underline{B} = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial z}\right). \tag{7.1}$$

Let

$$\omega = -\nabla^2\psi \quad \text{and} \quad J = -\nabla^2 A \tag{7.2}$$

be the current and vorticity. These variables evolve according to

$$\begin{aligned} \frac{\partial\omega}{\partial t} &= \{\psi,\omega\}_{xy} + \{J,A\}_{xy} , \\ \frac{\partial A}{\partial t} &= \{\psi,A\}_{xy} , \end{aligned} \tag{7.3}$$

where $\{f,g\}_{xy} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$ is the Poisson bracket with x and y playing the role of conjugate variables. Here one can choose either ψ or ω as the basic dynamic variable for the fluid; one is determined by the other via (7.2) and suitable boundary conditions. As an example of an equilibrium solution of (7.3), we consider the Grad-Shafranov equilibria for which the equilibrium values satisfy

$$\psi_e = 0, \quad J_e = G(A_e) \tag{7.4}$$

for some function G. When substituted into the right hand side of (7.3), one gets zero, so we have an equilibrium.

The equations (7.3) are Hamiltonian. The energy and Poisson brackets are:

$$H = \frac{1}{2} \int (|\underline{v}|^2 + |\underline{B}|^2) \, dx \, dy \tag{7.5}$$

(regarded as a function of ω and A) and

$$\{F,G\} = \int\!\!\int \omega\left\{\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta\omega}\right\}_{xy} + A\left(\left\{\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta A}\right\}_{xy} - \left\{\frac{\delta G}{\delta\omega}, \frac{\delta F}{\delta A}\right\}_{xy}\right) dx \, dy \tag{7.6}$$

This bracket is due to Morrison and Hazeltine [1984]; its group theoretic meaning and its derivation from a Lagrangian description is found in Marsden and Morrison [1984].

A class of Casimirs for the bracket (7.6) are checked to be

$$C(\omega,A) = \int (\omega\Phi(A) + \Psi(A))\, dx\, dy \tag{7.7}$$

where Φ and Ψ are arbitrary functions of one variable. Steps A and B in the energy Casimir method being complete, we turn to step C. One must show that for suitable Φ, Ψ, the sum $H + C$ has a critical point at (6.4). The condition $D(H + C)(\omega_e, A_e) = 0$ is found to hold if $\Phi = 0$ and $\Psi(a) = -\int G(a)\,da$. The convexity estimates in step D hold if Ψ is a convex function; this amounts to the assumption

$$0 < s \leq -G'(a) \leq S < \infty \tag{7.8}$$

for constants s, S. The procedures in step D also yield the norm for measuring the size of perturbations:

$$\|(\Delta\psi,\Delta A)\|^2 = \frac{1}{2}\int (|\underline{\nabla}(\Delta\psi)|^2 + |\underline{\nabla}(\Delta A)|^2 + s|\Delta A|^2\, dx\, dy\ . \tag{7.9}$$

Thus, in this case the method shows that <u>if (7.8) holds, then the Grad-Shafranov equilibrium (7.4) is nonlinearly stable in the norm (7.9); that is, if</u> (ψ,A) <u>starts close to</u> (ψ_e,A_e) <u>at</u> $t = 0$ <u>in the sense that (7.9) is small, where</u> $\Delta\psi = \psi-\psi_e$, <u>and</u> $\Delta A = A-A_e$, <u>then (7.9) remains small for all time</u>. One can treat cats eye type equilibria (also called magnetic islands) where A_e is given by equation (5.1). In fact the methods mentioned in that section show that the same conditions on the parameter $\underline{a}$ imply stability in the RMHD setting rather than the fluid setting. In the literature (Finn and Kaw [1977], Pritchett and Wu [1979], and Bondeson [1983]) these magnetic island solutions are seen to be unstable; as D. Holm pointed out to us, this can happen if one allows arbitrary disturbances in the y direction--transverse to the eyes. Our approach gives stability since our disturbances are confined to a finite extent in that direction.

We mention that the Grad-Shafranov equilibrium is just one member of a class of equilibria that can be treated by the same method. Furthermore, more sophisticated models and even the full three dimensional MHD equations can be treated (see Holm, Marsden, Ratiu and Weinstein [1984]). Also there is a very beautiful application of this method to RF stabilization of plasma oscillations in Similon, Kaufman and Holm [1984]. It is obvious that stability results of this sort are crucial to the design of fusion reactors.

8. Stratified Fluids

Our final example, taken from Abarbanel, Holm, Marsden and Ratiu [1984] concerns a situation of oceanographic interest. In this context, one is interested in incompressible fluids with density variations. The equations we shall use comprise the Boussinesq approximation to the inhomogeneous Euler equations (we do not include the Earth's rotation to simplify the exposition):

$$\begin{aligned} \rho_0\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) &= -\underline{\nabla} p - \rho g \hat{\underline{z}} \\ \frac{\partial \rho}{\partial t} + \underline{v}\cdot\underline{\nabla}\rho &= 0 \end{aligned} \tag{8.1}$$

Here, $\hat{\underline{z}}$ is a unit upward pointing vector (the z-direction), g is the acceleration due to gravity, ρ_0 is a reference density, which we can choose to be one, $\underline{v}$ is a divergence free, three dimensional, velocity field (parallel to the boundary of a given region in space), and ρ is the density (constant on the boundary). The equations (8.1) are Hamiltonian; the energy is

$$H = \int [\tfrac{1}{2}\,|\underline{v}|^2 + \rho g z]\, dx\, dy\, dz\ . \tag{8.2}$$

The bracket is the one associated to the Lie algebra which is the semidirect product of divergence-free vector fields and functions. (This Hamiltonian structure was also found by Benjamin [1984]).

If we let $\underline{\omega} = \underline{\nabla} \times \underline{v}$ be the vorticity and $q = \underline{\omega} \cdot \underline{\nabla} \rho$ be the potential vorticity, then a family of Casimirs is

$$C = \int \Phi(q, \rho)\ dx\ dy\ dz \tag{8.3}$$

One can also check directly that C is conserved by computing dC/dt. Using (8.1) let $(\underline{v}_e, \rho_e)$ be an equilibrium solution. In particular, we are especially interested in shear flows of the form $\underline{v}_e(x,y,z) = (U(z) + f(y), 0, 0)$, $\rho_e = \rho_e(z)$ (Figure 12).

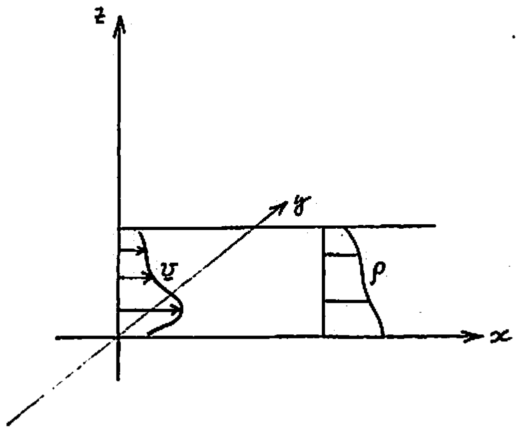


Figure 12. An equilibrium soluiton of (8.1) with a shearing velocity field and a density gradient.

For step C in the energy Casimir method, one computes that $H + C$ has a critical point at $\underline{v}_e$, ρ_e provided

$$\underline{v}_e = \Phi_{qq} \underline{\nabla} \rho_e \times \underline{\nabla} q_e \tag{8.4a}$$

and

$$gz + \Phi_\rho = \underline{\omega}_e \cdot \underline{\nabla} \Phi_q \tag{8.4b}$$

where $\Phi_q = \frac{\partial\Phi}{\partial q}$, etc., and are evaluated at equilibrium. Equation (8.4b) is referred to as Long's equation. Equilibrium solutions can be independently characterized by a Bernoulli function $K(q_e, \rho_e)$, which satisfies

$$p_e + \rho_e gz + \frac{1}{2}|\underline{v}_e|^2 = K(q_e, \rho_e), \tag{8.5}$$

which is a form of the well-known Bernoulli law for stationary flows. Equations (8.4) and (8.5) can be connected by choosing

$$\Phi(q_e, \rho_e) = q_e \int \frac{K(q_e, \rho_e)}{q_e^2}\, dq_e \tag{8.6}$$

which is analogous to the way we chose the Casimir (7.7) depending on the equilibrium (7.4) in RMHD.

One can now proceed to examine second variations and do the convexity analysis for the shear flow equilibrium $\underline{v}_e$. If it is stable then one can assert that nearby solutions remain nearby. These nearby solutions are related to internal waves in the ocean that are approximately described by the Benjamin-Ono equation.

The computations required to ascertain stability are a bit lengthy; here is roughly what comes out. Define the generalized Richardson number by

$$\tilde{R}i = - \frac{g}{\left(\frac{d\rho}{dz}\right) \frac{d^2}{d\rho^2} \left(\frac{U^2}{2}\right)} \tag{8.7}$$

which, to linear approximation in U, agrees with the standard definition:

$$Ri = - \frac{g\, d\rho/dz}{\left(\frac{dU}{dz}\right)^2} \tag{8.8}$$

(see, for instance, Drazin and Reid [1981]). Our result states that

$$\tilde{R}i > 1 \ \text{implies nonlinear stability}. \tag{8.9}$$

There are some important comments to be made:

(a) The result depends on limiting the values of $\underline{\nabla}\rho$ to a range consistent with the Boussinesq approximation.

(b) The result mentioned is three dimensional: some genuine three dimensionality through a small but non-zero f in $\underline{v}_e = (U(z) + f(y), 0, 0)$ is used. There is also a two dimension result, but the stability criteria are different. The reason is, basically, because the rich family of Casimirs given by (8.3) is not present in two dimensions.

(c) The result for linearized theory, due to Synge, Miles and Howard (see Drazin and Reid [1981] for a complete account) is:

$$Ri > \frac{1}{4} \quad \underline{\text{implies spectral stability}} \qquad (8.10)$$

The results (8.9) and (8.10) are consistent. But here we see a case where nonlinear stability requires more stringent conditions for its validity. It is not outrageous to conjecture that nonlinear instability can occur if $\tilde{R}i < 1$; indeed, the mechanism of Arnold diffusion mentioned earlier will generally occur in the absence of any nonlinear saturation mechanisms -- the only such mechanisms known are nonlinear bounds of the sort provided by the energy-Casimir method. Of course only a closer examination of the theory and experiments will settle this issue for sure. The fact that Ri for the Earth's ocean is often in the range between 1/4 and 1 makes the whole issue of linear versus nonlinear stability especially interesting.

References

H. Abarbanel, D. Holm, J. Marsden and T. Ratiu [1984]. Nonlinear stability of stratified flow, Phys. Rev. Lett. 52, 2352-2355.

H. Abarbanel, D. Holm, J. Marsden, and T. Ratiu [1984]. Nonlinear stability analysis of stratified fluid equilibria. (preprint).

R. Abraham and J. Marsden [1978]. Foundations of Mechanics, Addison-Wesley.

V. Arnold [1965]. Conditions for nonlinear stability of the stationary plane curvilinear flows of an ideal fluid, Doklady Mat. Nauk. 162 (5), 773-777.

V. Arnold [1966a]. Sur la géometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits, Ann. Inst. Fourier 16, 319-361.

V. Arnold [1966b]. An a priori estimate in the the theory of hydrodynamic stability, Izv. Vyssh. Uchebn. Zved. Mat. 54, 3-5. English Transl.: Am. Math. Soc. Transl. 19 [1969].

V. Arnold [1978]. Mathematical Methods of Classical Mechanics, Springer-Verlag.

J.M. Ball and J.E. Marsden [1984]. Quasiconvexity at the boundary, positivity of the second cariation and elastic stability, Arch. Rat. Mech. An. 86, 251-277.

D. Bao, J. Marsden and R. Walton [1984]. The Hamiltonian structure of general relativistic perfect fluids, Comm. Math. Phys. (to appear).

T.B. Benjamin [1972]. The stability of solitary waves, Proc. Roy. Soc. London 328A, 153-183.

T.B. Benjamin [1984]. Impulse, flow, force, and variational principles, IMA Journal of Applied Math 32, 3-68.

I.B. Bernstein [1958]. Waves in a plasma in a magnetic field, Phys. Rev. 109, 10-21.

I. Bialynieki-Birula and Z. Iwinski [1973]. Canonical formulation of relativistic hydrodynamics, Reports Math. Phys. 4, 139-151.

J. Bona [1975]. On the stability theory of solitary waves, Proc. Roy. Soc. London 344A, 363-374.

A. Bondeson [1983]. Three dimensional instability of magnetic islands, Phys. Rev. Lett. 51, 1668-1671.

S.N. Chow and J.K. Hale [1982]. Methods of Bifurcation Theory, Springer-Verlag.

P.A.M. Dirac [1964]. Lectures on Quantum Mechanics, Belfer Graduate School of Sci., Monograph Series, Vol. 2, Yeshiva University.

P.G. Drazin and W.H. Reid [1981]. Hydrodynamic Stability, Cambridge.

D.G. Dritschel [1984]. The nonlinear stability of co-rotating uniform vortices. (preprint).

I.E. Dzyaloshinskii and G.E. Volovich [1980]. Poisson brackets in condensed matter physics, Ann. Phys. 125, 67-97.

J.M. Finn and P.K. Kaw [1977]. Coalescence instability of magnetic islands, Phys. of Fluids 20, 72-78.

T.K. Fowler [1963]. Liapunov's stability criteria for plasma, J. Math Phys. 4, 559-569.

G.A. Goldin [1984]. Diffeomorphism groups, semidirect products, and quantum theory, Cont. Math. AMS 28, 189-210.

H. Goldstein [1980]. Classical Mechanics, Second Edition. Addison-Wesley.

V. Guillemin and S. Sternberg [1980]. The moment map and collective motion, Ann. of Phys. 127, 220-253.

R.D. Hazeltine, D.D. Holm, J.E. Marsden and P.J. Morrison [1984]. Generalized Poisson Brackets and nonlinear Liapunov stability--Application to reduced MHD. Proc. Lausanne Plasma Phys. Conf., June, 1984.

M.W. Hirsch and S. Smale [1974]. Differential Equations, Dynamical Systems and Linear Algebra, Academic Press.

D.D. Holm [1984]. Hamiltonian formalism for general-relativistic adiabatic fluids. (preprint).

D.D. Holm and B.A. Kuperschmidt [1983]. Poisson Brackets and Clebsch representations for magnetohydrodynamics, multifluid plasmas, and elasticity, Physica D 6D, 347-363.

D.D. Holm, B.A. Kupershmidt, and C.D. Levermore [1984]. Canonical maps between Poisson brackets in Eulerian and Lagrangian descriptions of continuum mechanics, Phys. Lett. 98A, 389-395.

D.D. Holm, J. Marsden and T. Ratiu [1984]. Nonlinear stability of the Kelvin-Stuart cats eye solutions. Proc. AMS-SIAM Summer Seminar, Santa Fe, July, 1984.

D.D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein [1983]. Nonlinear stability conditions and a priori estimates for barotropic hydrodynamics, Physics Letters 98A, 15-21.

D.D. Holm, J. E. Marsden, T. Ratiu and A. Weinstein [1984]. Stability of rigid body motion using the energy-Casimir method, Cont. Math. AMS 28, 15-23.

D.D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein [1985]. Nonlinear stability of equilibria in fluid and plasma systems, Physics Reports (to appear).

P.J. Holmes and J.E. Marsden [1983]. Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups. Indiana Univ. Math. J. 32, 273-310.

V.I. Iudovich [1964]. Two-dimensional nonstationary problem of the flow of an ideal incompressible fluid in a given region, Mat. Sb. 64, 562-588.

ZR. Iwinski and L.A. Turski [1974]. Canonical theories of systems interacting electromagnetically, Lett. Appl. Eng. Sci. 4, 179-191.

A. N. Kaufman and B.M. Boghosian [1984]. Lie-transform derivation of the gyrokinetic Hamiltonian system, Cont. Math. AMS 28, 169-176.

Lord Kelvin [1880]. On a disturbing infinity in Lord Rayleigh's solution for waves in a plane vortex stratum, Nature 23, 45-46.

D. Lewis, J. Marsden, R. Montgomergy and T. Ratiu [1985]. The Hamiltoniain structure for free boundary problems. (preprint).

A.J. Lichtenberg and M.A. Lieberman [1983]. Regular and Stochastic Motion, Springer-Verlag, Applied Math. Sciences, Vol. 38.

S. Lie [1890]. Theorie der transformationsgruppen, Zweiter Abshnitt, B.G. Teupner, Leipzig.

R.G. Littlejohn [1979]. A guiding center Hamiltonian: A new approach, J. Math. Phys. 20, 2445-2458.

A.E.H. Love [1893]. On the stability of certain vortex motions, Proc. London Math Soc. 25, 18-42.

J.E. Marsden [1982]. A group theoretic approach to the equations of plasma physics, Can. Math. Bull. 25, 129-142.

J.E. Marsden and T.J.R. Hughes [1983]. Mathematical Foundations of Elasticity, Prentice-Hall.

J.E. Marsden and M. McCracken [1976]. The Hopf Bifurcation and Its Applications, Appl. Math. Sciences Vol. 19, Springer-Verlag.

J.E. Marsden and P.J. Morrison [1984]. Noncanonical Hamiltonian field theory and reduced MHD, Cont. Math. AMS 28, 133-150.

J.E. Marsden, P.J. Morrison, and A. Weinstein [1984]. The Hamiltonian structure of the BBGKY hierarchy equations, Cont. Math. AMS 28, 115-124.

J. Marsden, T. Ratiu and A. Weinstein [1984a]. Semi direct products and reduction in mechanics, Trans. Am. Math. Soc. 281, 147-177.

J.E. Marsden, T. Ratiu and A. Weinstein [1984b]. Reduction and Hamiltonian structures on duals of semidirect product Lie algebras, Cont. Math. AMS 28, 55-100.

J.E. Marsden and A. Weinstein [1974]. Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5, 121-130.

J.E. Marsden and A. Weinstein [1982]. The Hamiltonian structure of the Maxwell-Vlasov equations, Physica D 4, 394-406.

J.E. Marsden and A. Weinstein [1983]. Coadjoint orbits, vortices and Clebsch variables for incompressible fluids, Physica D, 305-323.

J.E. Marsden, A. Weinstein, T. Ratiu, R. Schmid and R.G. Spencer [1983]. Hamiltonian systems with symmetry, coadjoint orbits and plasma physics, Proc. IUTAM-ISIMM Symposium on "Modern Developments in Analytical Mechanics," Torino (June 7-11, 1982), Atti della Academia della Scienze di Torino 117, 289-340.

P.J. Morrison [1980]. The Maxwell-Vlasov equations as a continuous Hamiltonian system, Phys. Lett. 80A, 383-386.

P.J. Morrison and J.M. Greene [1980]. Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics, Phys. Rev. Letters 45, 790-794.

P.J. Morrison and R.D. Hazeltine [1984]. Hamiltonian formulation of reduced magnetohydrodynamics, Phys. Fluids 27, 886-897.

W. Newcomb [1958]. Appendix in Bernstein [1958].

P.L. Pritechett and C.C. Wu [1979]. Coalescence of magnetic islands, Phys. of Fluids 22, 2140-2146.

T. Ratiu [1980]. Euler-Poisson Equations on Lie Algebras, Thesis. Berkeley.

T. Ratiu [1982]. Euler-Poisson equations on Lie algebras and the N-dimensional Lagrange top, Am. J. Math. 104, 409-448.

Lord Rayleigh [1880]. On the stability, or instability of certain fluid motions, Proc. London Soc. 11, 57-70.

M.N. Rosenbluth [1964]. Topics in microinstabilities, Advanced Plasma Physics, Academic Press, p. 137.

D. Ruelle [1980]. Strange attractors, Math. Intelligencer 2, 126-137.

R.D. Seliger and G.B. Whitham [1968]. Variational principles in continuum mechanics, Proc. Roy. Soc. Lon. 305, 1-25.

R.G. Spencer and A.N. Kaufman [1982]. Hamiltonian structure of two fluid plasma dynamics, Phys. Rev. A 25, 2437-2439.

S. Smale [1970]. Topology and mechanics, Inv. Math. 10, 305-331; 11, 45-64.

P.L. Similon, A.N. Kaufman, and D.D. Holm [1984]. Ponderomotive Hamiltonian and Lyapunov stability for magnetically confined plasma in the presence of r.f. field. (Preprint).

J.T. Stuart [1967]. On finite amplitude oscillations in laminar mixing layers, J. Fluid Mech. 29, 417-440.

E.C.G. Sudarshan and N. Mukunda [1974]. Classical Mechanics-A Modern Perspective, Wiley. Second edition, Krieger, 1983.

Y. Tang [1984]. Nonlinear Stability of Vortex Patches, Thesis, SUNY, Buffalo.

Y.H. Wan [1984]. On the nonlinear stability of circular vortex patches, Cont. Math. 28 AMS, 215-220.

Y.H. Wan and M. Pulvirente [1984]. Nonlinear stability of circular vortex patches, Comm. Math. Phys. (to appear).

A. Weinstein [1983]. Sophus Lie and symplectic geometry, Expo. Math. 1, 95-96.

E.T. Whittaker [1917]. A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. Fourth Edition, Cambridge.

N. Zabusky [1984]. Contour dynamics for two dimensional flow, Cont. Math. 28, 211-214.