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THE HAMILTONIAN STRUCTURE OF CONTINUUM MECHANICS IN MATERIAL, INVERSE MATERIAL, SPATIAL AND CONVECTIVE REPRESENTATIONS

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ABSTRACT

Ideal continuum models (fluids, plasmas, elasticity, etc.) can be studied using a variety of representations, each of which has a Hamiltonian structure. This paper shows how groups (typified by the group of particle relabelling symmetries) and the inversion operator which swaps the reference and current particle positions generate maps between the representations. These maps, derived using the theory of momentum maps and reduction, are all Poisson (or canonical) maps which carry the brackets in one representation to those in another. The results are developed abstractly in the framework of reduction of a pair of principal bundles by left and right group actions. Examples are given treating the motion of an incompressible fluid with surface tension, the heavy top, and ideal compressible (barotropic) flow.

1. INTRODUCTION

A Hamiltonian formulation of conservative continuum mechanics, such as fluid dynamics and elasticity, can be given in the material (sometimes called Lagrangian) representation using canonically conjugate variables. This certainly has been well known for a long time, going back in one form or another to the mid 1800's (see for example, Truesdell and Toupin [1960], pages 594 ff for an account and historical references). For an exposition of these ideas in the modern language of symplectic geometry, see Marsden and Hughes [1983], Chapter 5. We shall follow the latter's notation as far as possible.

The spatial (or Eulerian) representation of continuum mechanics also admits a Hamiltonian structure. A group theoretical framework for this and its relationship to the material formulation is given in Arnold [1966]. Of course, there have been many other contributions; however, we shall not attempt to review them systematically here. Those contributions directly relevant to our aims are, as follows. Marsden and Weinstein [1982, 1983] and Marsden et al. [1983] show that Arnold's idea of symmetry reduction from canonical material representation to non-canonical spatial representation holds for plasma physics. Holm and Kupershmidt [1983] derive Hamiltonian structures for a variety of continuum models using Clebsch representations and observe that these continuum mechanics brackets are of Lie-Poisson type for semidirect-product Lie algebras. Holm, Kupershmidt, and Levermore [1983], Marsden et al. [1983], and Marsden, Ratiu, and Weinstein [1984a,b] derive these semidirect-product structures by reduction from material to spatial representation. Marsden and Morrison [1984] derive the Poisson bracket for the RMHD (reduced magnetohydrodynamic) tokamak equations by material to spatial reduction. Montgomery, Marsden, and Ratiu [1984] abstract many of the previous works in terms of reduction of cotangent bundles of principal bundles, and Lewis et al. [1986] derive the Hamiltonian structure for the free-boundary problem of rotating fluid drops with surface tension by building on the preceding ideas.

There are a number of motivations for considering Hamiltonian structures, such as their use in finding nonlinear stability conditions for fluid equilibria. See, for example, Holm [1986] in this volume for the stability analysis of three-dimensional ideal incompressible and barotropic compressible fluid equilibria. See also Holm et al. [1985], Abarbanel et al. [1986], Abarbanel and Holm [1986], and references therein for additional applications of Hamiltonian stability analysis. This paper is concerned only with the theory of Hamiltonian structures. One of our main goals is to relate the material and spatial representations to two others: the inverse material, and convective representations. The motivation for this investigation came primarily from the work of Holm [1985] in which these latter two representations are used for the study of general relativistic adiabatic fluids. The inverse material representation is called the augmented Eulerian representation in Holm [1986]; since it consists of the usual Eulerian (or spatial) representation, augmented by the dynamics of the Lagrangian coordinate functions, or fluid labels. We note that the inverse material representation also appears in Ball's existence theory in elasticity (Ball [1977a,b]), that the spatial versus convective representations relate the Hamiltonian treatments of elasticity given by Holm and Kupershmidt [1983] and Marsden, Ratiu, and Weinstein [1984a] respectively, and that SO(3) reduction (treated in Section 5) puts the observations about Hamiltonian structures for elasticity given in Kupershmidt and Ratiu [1983] into a unified scheme. We hope that the present contribution will unify and deepen the understanding of the preceding works. It should also provide a setting in which other situations can be understood such as that of Simo and Marsden [1984] regarding the rotated stress tensor and the Doyle-Ericksen formula. We also note that the convective representation is useful in the stability analysis of the coupled rigid body-beam and plate models of Krishnaprasad and Marsden [1986] and Krishnaprasad, Marsden and Simo [1986].

To motivate some of the constructions in the body of the paper, we now make a few relevant general remarks before discussing briefly one of the key examples, an incompressible fluid with a free boundary, described in the material, spatial and convective representations.

Let P be a Poisson manifold and H_a be a family of Hamiltonians parameterized by a variable $a \in V^*$, where V is a vector space with dual space V^*. For instance, if one is describing a rigid body, the parameter a could be the inertia tensor; or if one is describing an inhomogeneous fluid in the Lagrangian (material) representation, the parameter a could be the reference density distribution. Let G be a group acting on the right on P and by a right representation on V. Thus, G also acts on V^* on the right.

Consider two conditions:

$$H_a(x\cdot g) = H_a(x) \quad , \qquad (C_1)$$

and

$$H_{a\cdot g}(x\cdot g) = H_a(x) \quad , \qquad (C_2)$$

for all $a \in V$, $x \in P$, $g \in G$. In both cases, extend H_a to $P \times T^*V = P \times V \times V^*$ by

$$H(x,v,a) = H_a(x) \quad ,$$

so $v \in V$ is a cyclic variable. Now the direct product group

$G \times V$ (V regarded as an abelian group)

acts on $P \times T^*V$ on the right by

$$(x,v,a) \cdot (g,u) = (x\cdot g, \; u + v, \; a)$$

and (C_1) implies H is invariant under the action of $G \times V$. The semidirect product

$G \circledS V$

with multiplication

$$(g_1,u_1) \cdot (g_2,u_2) = (g_1 g_2, \; u_2 + u_1 g_2)$$

acts on $P \times T^*V$ as well by

$$(x,v,a) \cdot (g,u) = (x\cdot g, \; -u + v\cdot g, \; a\cdot g) \quad ,$$

and condition (C_2) implies that H is invariant under the action of $G \circledS V$. Similar statements hold for left actions (using the left semidirect product).

Under condition (C_1) we can reduce the Poisson manifold $P \times T^*V$ by $G \times V$ (see Marsden and Ratiu [1986] for the general theory of Poisson reduction). Assuming the G action on P is regular so that P/G is a manifold, we get a Poisson isomorphism (denoted $\cong$)

$$(P \times T^*V)/(G \times V) \cong (P/G) \times V^* \quad ,$$

where $(P/G) \times V^*$ has the bracket structure of P/G alone, V^* having the trivial structure. Thus, under condition (C_1) the quantity a acts truly as a parameter with trivial dynamics on the reduced space.

Under condition (C_2), however, the manifold

$$(P \times T^*V)/(G \circledS V) \cong P/G \times V^*$$

has nontrivial structure. In fact, the equivalent manifolds

$$\begin{aligned}(P \times T^*V)/(G \circledS V) &\cong (P \times T^*V/V)/G \\ &\cong (P \times V^*)/G \\ &\cong (P/G) \times V^*\end{aligned}$$

have, in general, fairly complicated Poisson structures. For $P = T^*B$ with B a G-bundle, the structure is worked out in Montgomery, Marsden, and Ratiu [1984] and for $P = T^*G$, it is worked out in Krishnaprasad and Marsden [1986].

We will be considering the following general set up. We will start with our Hamiltonian H defined on material phase space P. The Hamiltonian will depend on parameters which will be linked with the group to be used for reduction. For example, for compressible flow we can regard H as depending on the material density and on the metric tensor on Eulerian space (the metric is used to form the kinetic energy from the velocity field). To pass to the spatial representation, we reduce by the group of diffeomorphisms of the fluid container; this group acts on the density by pull-back (i.e. composition and multiplication by the Jacobian determinant in this case)* but acts trivially on the metric tensor. Thus, if a is the density, the situation (C_2) holds, while if a is the metric tensor, (C_1) holds. To pass to the convective representation, we reduce by the group of spatial diffeomorphisms; these act trivially on the material density, and by pull-back on the spatial metrics, so now (C_1) and (C_2) hold, respectively, and the situation is reversed. The inclusion of the metric is crucial to obtain a covariant theory, just as it is in the fundamental aspects of elasticity (Marsden and Hughes [1983] and Simo and Marsden [1984]).

Since the body of the paper proceeds from the abstract to the specific, we will motivate the abstract theory by giving some more details about one of the main examples, namely incompressible free boundary problems for fluids (see Lewis, Marsden, Montgomery, and Ratiu [1986] and Lewis, Marsden, and Ratiu [1986]).

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* See Abraham and Marsden [1978] and Abraham, Marsden, and Ratiu [1983] for details about pull-backs and other geometric concepts used here.

Let $\mathcal{B} \subset \mathbb{R}^3$ denote a reference configuration, whose points are denoted $\underline{X} \in \mathcal{B}$, with coordinates X^A, $A = 1,2,3.$* The basic configuration space $\mathcal{C}$ consists of maps

$$\phi: \mathcal{B} \to \mathbb{R}^3 \quad ,$$

which map $\mathcal{B}$ diffeomorphically onto its image $\phi(\mathcal{B})$ and which are volume preserving. We write $\underline{x} = \phi(\underline{X})$ for the spatial point and its contravariant coordinates are denoted x^a. The cotangent bundle $T^*\mathcal{C}$, consisting of the maps ϕ and their conjugate momenta π, comprises the basic phase space for the material representation. In Section 2, the space denoted by B is an abstraction of $\mathcal{C}$.

For the spatial representation, one uses the spatial momentum density M_a, or velocity field v^a (which is chosen to be divergence free) and the boundary $\Sigma = \phi(\partial\mathcal{B})$. In Section 2 the space denoted by M is the space of all Σ's and the group G is the group of volume preserving diffeomorphisms of $\mathcal{B}$ to itself, denoted $\mathrm{Diff}_{vol}(\mathcal{B})$. This group is also called the <u>rearrangement</u>, or <u>particle relabelling group</u>. Note that $\mathrm{Diff}_{vol}(\mathcal{B})$ acts on $\mathcal{C}$ on the <u>right</u> by

$$\phi \cdot \eta = \phi \circ \eta \quad ,$$

where $\phi \circ \eta$ denotes composition of maps $\phi \in \mathcal{C}$ and $\eta \in \mathrm{Diff}_{vol}(\mathcal{B})$. As in Lewis, Marsden, Montgomery and Ratiu [1986], M is the quotient space B/G, while T^*B/G is identified with space of pairs (M_a, Σ) which is a gauged Lie-Poisson bundle over M (in the sense of Montgomery, Marsden, and Ratiu

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* denoted by q^A in Holm [1985].

[1984]) and which inherits a Poisson bracket structure from the canonical bracket on $T^*\mathcal{B}$, the phase space in material representation.

In terms of v^a and Σ, the bracket one obtains by this procedure is

$$\{F,G\} = \int_{\mathcal{B}_\Sigma} \underline{\omega} \cdot \left(\frac{\delta F}{\delta \underline{v}} \times \frac{\delta G}{\delta \underline{v}}\right) dA + \int_\Sigma \left(\frac{\delta F}{\delta \Sigma}\frac{\delta G}{\delta \phi} - \frac{\delta G}{\delta \Sigma}\frac{\delta F}{\delta \phi}\right) ds \quad , \tag{1.1}$$

where Σ bounds $\mathcal{B}_\Sigma = \phi(\mathcal{B})$, $\underline{\omega} = \nabla \times \underline{v}$ is the vorticity and $\underline{v} = \underline{w} + \nabla\phi$ where $\underline{w}$ is divergence free and parallel to Σ (so $\nabla\phi$ is the potential part of the flow). For $\underline{\omega} = 0$, the bracket reduces to that of Zakharov [1968] (see Lewis, Marsden, Montgomery and Ratiu [1986] for details of the bracket derivation).

If we were treating <u>inhomogeneous</u> incompressible flow, the space $\mathcal{C}$ would be unchanged, but the Hamiltonian (kinetic energy plus surface tension energy) would no longer be invariant under G. To accommodate this situation, we must also include the material density as a parameter; now condition (C_2) holds, so the reduction is by $\mathrm{Diff}_{vol}(\mathcal{B})$ Ⓢ (Functions on $\mathcal{B}$) one gets (1.1) plus the semidirect product piece

$$\int_{B_\Sigma} \rho\left(\frac{\delta G}{\delta \underline{v}} \cdot \nabla \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta \underline{v}} \cdot \nabla \frac{\delta G}{\delta \rho}\right) d^3x \quad ,$$

as in Abarbanel et al. [1986] for the fixed boundary case.

The inverse material description uses the space of maps $\bar{\phi}$ which take regions in $\mathbb{R}^3$ diffeomorphically to $\mathcal{B}$ and which are volume preserving. The phase space in the inverse material representation is $T^*\bar{\mathcal{C}}$, consisting of maps $\bar{\phi}$ and their conjugate momenta $\bar{\pi}$. In Section 2, $\bar{B}$ is an abstraction of $\bar{\mathcal{C}}$ and the group $\bar{G}$ is $\mathrm{Diff}_{vol}(\mathbb{R}^3)$ the group of volume-preserving diffeomorphisms of space. This group acts on $\bar{B}$ on the <u>right</u> by

$$\bar{\phi} \cdot \bar{\eta} = \bar{\phi} \circ \bar{\eta} \quad ,$$

where $\bar{\phi} \in \bar{C}$ and $\bar{\eta} \in \mathrm{Diff}_{vol}(\mathbb{R}^3)$. The Hamiltonian, however, is not invariant under this action of $\bar{G}$. To arrange this, we treat the metric g on space as a parameter and transform it too. Thus, we enlarge $\bar{B}$ to $\bar{B} \times \bar{V}$ where $\bar{V}^*$ is the space of spatial metrics, so now the Hamiltonian satisfies (C_2). The quotient $T^*(\bar{B} \times \bar{V})/(\bar{G} \circledS \bar{V})$ can be identified with the product of the space of convective momentum densities $\underline{M}$ (or velocities $\underline{V}$) and the space of Cauchy-Green tensors $C := \phi^*(g)$. One computes the bracket to be

$$\{F,G\} = \int_B M_A\left(\frac{\delta F}{\delta M_B} \frac{\partial}{\partial X^B} \frac{\delta G}{\delta M_A} - \frac{\delta G}{\delta M_B} \frac{\partial}{\partial X^B} \frac{\delta F}{\delta M_A}\right)d^3x$$

$$+ \int_B \left(\frac{\delta F}{\delta C_{AB}} (L_{\frac{\delta G}{\delta \underline{M}}} C)_{AB} - \frac{\delta G}{\delta C_{AB}} (L_{\frac{\delta F}{\delta \underline{M}}} C)_{AB}\right)d^3x \quad , \tag{1.2}$$

which is (up to boundary terms) a semidirect Lie-Poisson bracket for divergence free vector fields on B with the space of tensors C_{AB}. Here $L_{\underline{V}}C$ denotes Lie differentiation with respect to the convective velocity $\underline{V}$.

The inversion map $\phi \to \bar{\phi} = \phi^{-1}$ from C to $\bar{C}$ induces a canonical map between T^*C and $T^*\bar{C}$ and hence (with the metrics included) between the reduced spaces. In coordinates, the momenta are related by

$$M_a(\underline{x}) = - \frac{\partial X^A}{\partial x^a} M_A(\underline{X})$$

(Since this example is incompressible, we have dropped the volume elements); cf. Holm [1985], Eq. (1.7).

Two other group actions are also important, for they generate momentum maps which are Poisson maps implementing the maps from material to spatial, and from inverse material to convective representation. These are: first, $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{B})$ acts on $\bar{C}$ on the left by

$$\eta \cdot \bar{\phi} = \eta \circ \bar{\phi}$$

and $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{R}^3)$ acts on C on the left by

$$\bar{\eta} \cdot \phi = \bar{\eta} \circ \phi \quad .$$

The inversion map $\Psi : C \to \bar{C};\ \phi \to \phi^{-1} = \bar{\phi}$ is compatible with these actions; e.g.,

$$\Psi(\phi \cdot \eta) = \eta^{-1} \cdot \Psi(\phi) \quad .$$

These four actions of G and $\bar{G}$ on $\mathcal{B}$ and $\bar{\mathcal{B}}$ and their intertwining by the map Ψ are abstracted in §2. Associated with these four actions, we have four momentum maps (which are automatically Poisson maps, i.e., they preserve the values of Poisson brackets) that we denote

$$J_R^G:\ T^*C \to L(G)_-^* \quad ,$$

$$J_L^{\bar{G}}:\ T^*C \to L(\bar{G})_+^* \quad ,$$

$$J_L^G:\ T^*\bar{C} \to L(G)_+^* \quad ,$$

$$J_R^{\bar{G}}:\ T^*\bar{C} \to L(\bar{G})_-^* \quad ,$$

where $L(G)^*_+$ (resp. $L(G)^*_-$) is the dual of the Lie algebra of G with its + (resp. -) Lie-Poisson structure (see, for example, Marsden et al. [1983] for an exposition of Lie-Poisson structures and momentum maps). For fluids with a free boundary, we shall see that

J^G_R is an encoding of the basic conservation of circulation in material representation, i.e., the conservation law associated to particle relabelling (cf. Arnold [1966], Abarbanel and Holm [1986] and references therein);

J^G_L implements the material to spatial reduction;

$J^{\bar{G}}_R$ implements the inverse material to convective reduction;

$J^{\bar{G}}_L$ encodes the particle relabelling conservation law in the inverse material representation.

The map Ψ then interrelates these four maps as described in detail in Section 2.

The plan of the paper is as follows. Section 2 develops the abstract picture of reduction on dual bundles, following earlier work by Montgomery, Marsden, and Ratiu [1984]. Section 3 treats Hamiltonian systems that transform as semidirect products under the action of the symmetry group of the configuration space. Section 4 defines the continuum mechanical representations abstractly. Section 5 discusses an explicit example of the type of Hamiltonian systems treated in Section 3, namely the heavy top. Section 6 discusses ideal compressible flow in the present Hamiltonian set up for the material, inverse material, spatial, and convective representations. A dictionary of nomenclature is given below.

Dictionary

material point	=	Lagrangian point; points $\underline{X}$ in reference configuration $D \subset \mathbb{R}^3$;
spatial point	=	Eulerian point; points $\underline{x}$ in space;
material representation	=	Lagrangian representation; basic configuration variables are maps $\eta : D \to \mathbb{R}^3$; $\eta \in \mathcal{C}$;
inverse material representation	=	inverse Lagrangian representation; basic configuration variables are maps $\bar{\eta}: \mathbb{R}^3 \to D$, $\bar{\eta} \in \bar{\mathcal{C}}$;
spatial representation	=	Eulerian representation; basic variables include the spatial velocity $\underline{v}$;
convective representation	=	body representation; basic variables include the convective velocity $\underline{V}$.

Acknowledgments

We thank Debbie Lewis and Juan Simo for several valuable discussions of this material. In particular, the free boundary convective bracket (Formula 1.2) was calculated by Debbie Lewis, building on an analogous formula for convective elasticity due to Juan Simo.

2. REDUCTION OF DUAL BUNDLES

We consider two manifolds B and $\bar{B}$, and left and right actions of two Lie groups G and $\bar{G}$ on B and $\bar{B}$. Let the right actions be denoted by

$$b \cdot g = \phi^G(b,g) \text{ and } \bar{b} \cdot \bar{g} = \bar{\phi}^{\bar{G}}(\bar{b},\bar{g}),$$

so we have maps

$$\phi^G: \; B \times G \to B \quad \text{and} \quad \bar{\phi}^{\bar{G}}: \; \bar{B} \times \bar{G} \to \bar{B} \quad .$$

Similarly the left actions are written

$$\bar{g} \cdot b = \phi^{\bar{G}}(\bar{g},b) \quad \text{and} \quad g \cdot \bar{b} = \bar{\phi}^G(g,\bar{b}) \quad ,$$

with maps

$$\phi^{\bar{G}}: \; \bar{G} \times B \to B \quad \text{and} \quad \bar{\phi}^G: \; G \times \bar{B} \to \bar{B} \quad .$$

Assuming that the relevant quotients are manifolds, we obtain the fiber bundles

$$\pi^G: B \to M : = B/G \quad , \quad \bar{\pi}^{\bar{G}}: \bar{B} \to \bar{M} : = \bar{B}/\bar{G} \quad ,$$

$$\pi^{\bar{G}}: B \to N : = \bar{G}\backslash B \quad , \quad \bar{\pi}^G: \bar{B} \to \bar{N} : = G\backslash\bar{B} \quad .$$

In addition, assume that there are diffeomorphisms $\Psi: B \to \bar{B}$ and $\bar{\Psi}: B \to \bar{B}$ which are equivariant with respect to G and $\bar{G}$, i.e.,

$$\Psi(b \cdot g) = g^{-1} \cdot \Psi(b) \quad , \tag{2.1}$$

$$\bar{\Psi}(\bar{g} \cdot b) = \bar{\Psi}(b) \cdot \bar{g}^{-1} \quad , \tag{2.2}$$

for all $b \in B$, $g \in G$, and $\bar{g} \in \bar{G}$. Differentiating conditions (2.1) and (2.2) with respect to g and $\bar{g}$ yields

$$T\Psi \cdot \xi_B = - \xi_{\bar{B}} \cdot \Psi \quad , \tag{2.3}$$

$$T\bar{\Psi} \cdot \bar{\xi}_B = - \bar{\xi}_{\bar{B}} \cdot \bar{\Psi} \quad , \tag{2.4}$$

for all $\xi \in L(G)$ and $\bar{\xi} \in L(\bar{G})$, where ξ_B, $\xi_{\bar{B}}$, $\bar{\xi}_B$, $\bar{\xi}_{\bar{B}}$ are the infinitesimal generators of ξ and $\bar{\xi}$ respectively on the subscripted spaces and $L(G)$ denotes the Lie algebra of G.

For example, if G is a Lie group, and $B = \bar{B} = \bar{G} = G$, with the actions given by group multiplication, then $\Psi(g) = g^{-1}$ and $\bar{\Psi}(g) = g^{-1}$ satisfy (2.1) and (2.2).

Recall that a right action Φ of a Lie group K on a manifold N lifts naturally to a right symplectic action on T^*N via

$$(\gamma_n, k) \in T^*_n N \times K \to T^*_{n \cdot k} \Phi_{k^{-1}} (\gamma_n) \in T^*_{n \cdot k} N \quad .$$

Similarly, a left action of a Lie group H on a manifold Q induces a left symplectic action of H on the cotangent bundle T^*Q. From Abraham and

Marsden [1978, p. 283], we know that the equivariant momentum map for the lifted right action of K on T^*N is the map $J: T^*N \to L(K)^*_-$ given by

$$\langle J(\gamma_n), \zeta\rangle = \langle \gamma_n, \zeta_N(n)\rangle \quad ,$$

for any $\gamma_n \in T^*_nN$, $\zeta \in L(K)$. (For left actions, replace "-" by "+".) Here, $L(K)^*_\pm$ denotes the dual of the Lie algebra of K with the + (resp., -) Lie-Poisson structure; equivariance implies that J is a Poisson, or canonical map (i.e., preserves Poisson brackets; see, for example, Marsden et al. [1983] for a review of Lie-Poisson structures).

Applying these considerations to G, $\bar{G}$, T^*B and $T^*\bar{B}$ yields the following momentum maps:

$$J^G_R : T^*B \to L(G)^*_- \quad , \quad \langle J^G_R(\alpha_b), \xi\rangle = \langle \alpha_b, \xi_B(b)\rangle \quad , \tag{2.5}$$

$$\bar{J}^{\bar{G}}_R : T^*\bar{B} \to L(\bar{G})^*_- \quad , \quad \langle \bar{J}^{\bar{G}}_R(\bar{\alpha}_{\bar{b}}), \bar{\xi}\rangle = \langle \bar{\alpha}_{\bar{b}}, \bar{\xi}_{\bar{B}}(\bar{b})\rangle \quad , \tag{2.6}$$

$$J^{\bar{G}}_L : T^*B \to L(\bar{G})^*_+ \quad , \quad \langle J^{\bar{G}}_L (\alpha_b), \bar{\xi}\rangle = \langle \alpha_b, \bar{\xi}_B(b)\rangle \quad , \tag{2.7}$$

$$\bar{J}^G_L : T^*\bar{B} \to L(G)^*_+ \quad , \quad \langle \bar{J}^G_L(\bar{\alpha}_{\bar{b}}), \xi\rangle = \langle \bar{\alpha}_{\bar{b}}, \xi_{\bar{B}}(\bar{b})\rangle \quad , \tag{2.8}$$

for $\alpha_b \in T^*B$, $\bar{\alpha}_{\bar{b}} \in T^*_{\bar{b}}\bar{B}$, $\xi \in L(G)$ and $\bar{\xi} \in L(\bar{G})$. We next observe that it is sufficient to consider either right actions and right momentum maps, or left actions and left momentum maps. The other combinations follow by permutation of the symbols and bars.

Proposition 2.1. The Maps Ψ and $\bar{\Psi}$ induce Poisson isomorphisms

$$\psi: T^*B/G \to G\backslash T^*\bar{B} \text{ and } \bar{\psi}: \quad \bar{G}\backslash T^*B \to T^*\bar{B}/\bar{G} \quad ,$$

where the position of the group in the denominator indicates whether the action is on the right or left (as above, we assume the quotients are manifolds).

This proposition follows from the fact that the lifted maps $T^*\Psi^{-1}: T^*B \to T^*\bar{B}$ and $T^*\bar{\Psi}^{-1}: T^*B \to T^*\bar{B}$ are symplectic diffeomorphisms which, from (2.1) and (2.2), intertwine the right and left actions of G and $\bar{G}$ respectively. Thus, they induce Poisson diffeomorphisms of the quotient spaces.

Remark on Dual Pairs. When the right G action on B commutes with the left $\bar{G}$ action on B and the left G action on $\bar{B}$ commutes with the right $\bar{G}$ action on $\bar{B}$, then the momentum maps $J_R^{\bar{G}}$, $\bar{J}_R^{\bar{G}}$, J_L^G and $\bar{J}_L^G$ induce Poisson maps on the quotient spaces:

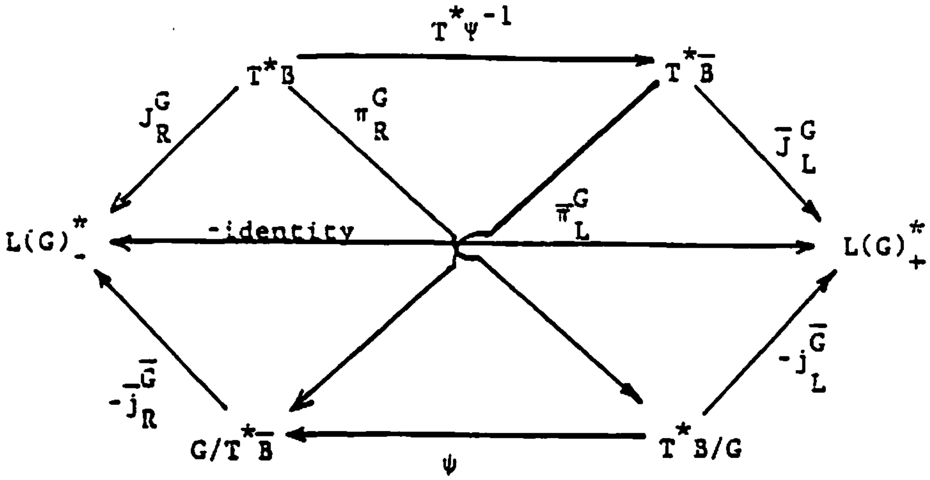
$$j_R^G : \bar{G}\backslash T^*B \to L(G)_-^*$$

$$\bar{j}_R^{\bar{G}} : G\backslash T^*\bar{B} \to L(\bar{G})_-^*$$

$$j_L^{\bar{G}} : T^*B/G \to L(\bar{G})_+^*$$

$$\bar{j}_L^G : T^*\bar{B}/\bar{G} \to L(G)_+^*$$

The maps Ψ and $\bar{\Psi}$ intertwine the actions and hence also intertwine the momentum maps, giving the following commutative diagrams of Poisson maps (the minus signs come from the minus signs in (2.3) and (2.4)):



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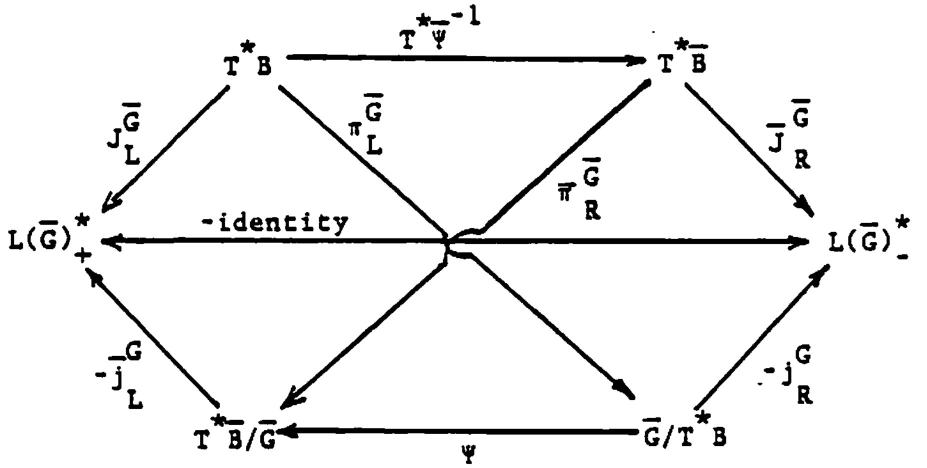
and



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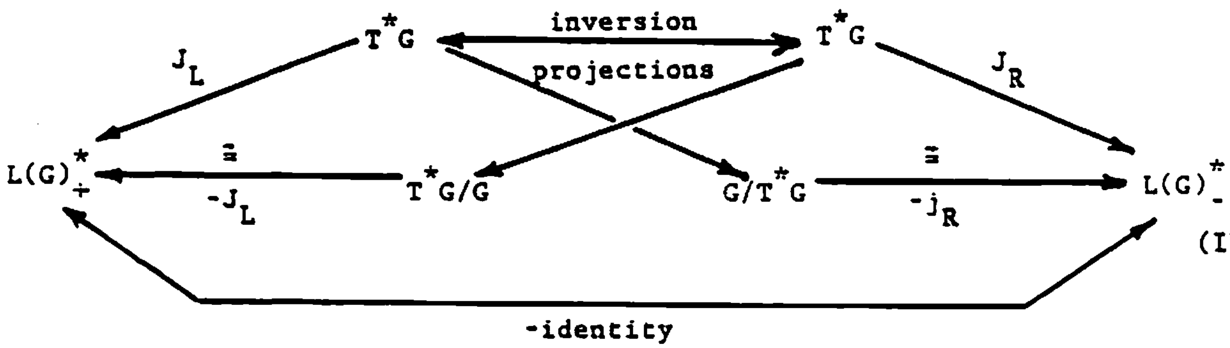
where π_R^G: $T^*B \to T^*B/G$ is the projection.

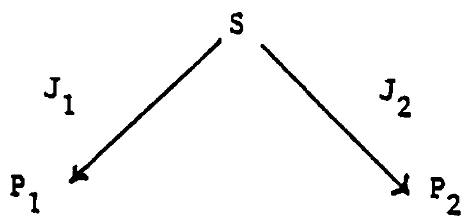
In case $B = \bar{B} = G = \bar{G}$, these two diagrams collapse to:



(II)

where $J_L(\alpha_g) = T_e^*R_g(\alpha_g)$, $J_R(\alpha_g) = T_e^*L_g(\alpha_g)$ and j_L, j_R are the induced Poisson diffeomorphisms on the indicated quotient spaces.

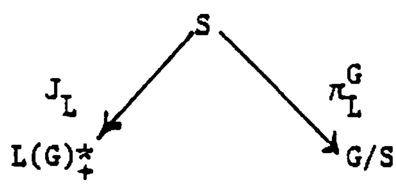
Let (S,Ω) be a symplectic manifold (space S, symplectic form Ω), P_1 and P_2 be Poisson manifolds, and



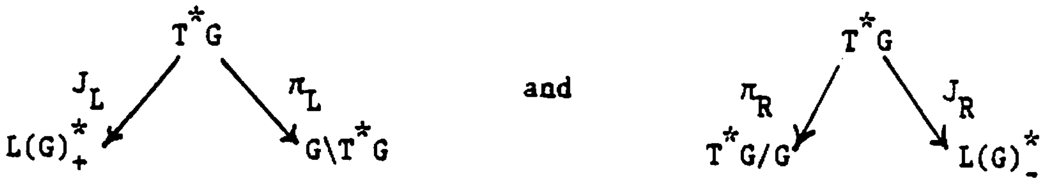
be Poisson maps. This diagram is called a <u>dual pair</u> if for (an open dense set of) $s \in S$,

$$\ker T_s J_1 = (\ker T_s J_2)^{\Omega} \quad ,$$

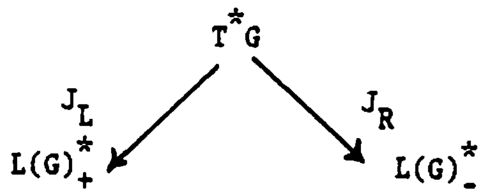
where the superscript Ω denotes the Ω-orthogonal complement. (See Marsden and Weinstein [1983] and Weinstein [1983] for further exposition on dual pairs.) For example, if $J_L\colon S \to L(G)^*_+$ is an equivariant momentum map for a left G action on S, then $\ker T_s J_L$ and $T_s(G\cdot s)$ are Ω-orthogonal complements (Abraham and Marsden [1978, p. 299]), so



is a dual pair (assuming $G\backslash S$ is a manifold). There is, of course, a similar remark for right actions. In particular, in each of the diagrams (IA) and (IB) there are two dual pairs. The maps $T^*\Psi^{-1}$ and $T^*\bar{\Psi}^{-1}$ implement isomorphisms between each of the sets of dual pairs. For example, inversion implements the isomorphism between

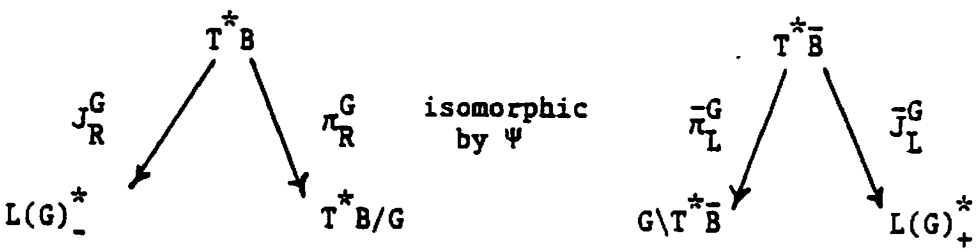


So there really is a single dual pair in this case:

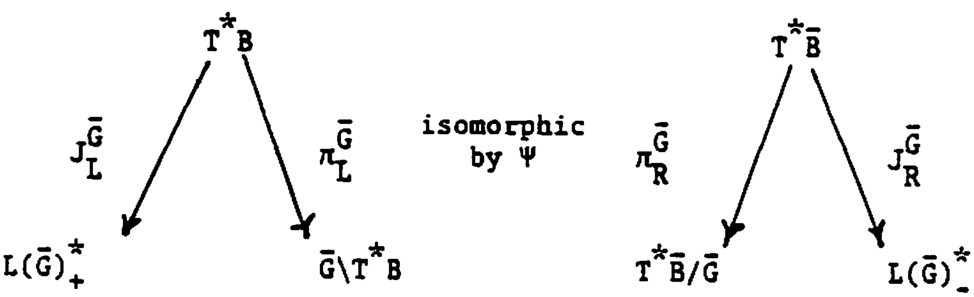


For the level of generality required in the examples considered in the later sections, the maps Ψ and $\bar{\Psi}$ need not be the same and $\bar{j}_L^G$ etc., need not be Poisson isomorphisms (as they are in the special case $B = \bar{B} = G = \bar{G}$).

We summarize by extracting the dual pairs in (IA) and (IB):



(IIIA)



(IIIB)

This ends our remark on dual pairs of Poisson maps.

Next we study dynamics. Let H_R: $T^*B \to \mathbb{R}$ be right G-invariant. By the intertwining properties of Ψ with respect to G, $\bar{H}_L := H_R \circ T^*\Psi : T^*\bar{B} \to \mathbb{R}$ is left G-invariant. Likewise, given $\bar{H}_L$, we can construct H_R by the same formula. The functions H_R and $\bar{H}_L$ determine smooth Hamiltonian functions

$$H_R^G : T^*B/G \to \mathbb{R} \quad \text{and} \quad \bar{H}_L^G : \quad G\backslash T^*\bar{B} \to \mathbb{R} \quad \text{via}$$

$$H_R^G \circ \pi_R = H_R \quad \text{and} \quad \bar{H}_L^G \circ \bar{\pi}_L = \bar{H}_L \quad .$$

On the respective quotients, the relationship between H_R^G and $\bar{H}_L^G$ takes the form of a functional composition,

$$H_R^G = \bar{H}_L^G \circ \psi \quad , \qquad (2.9)$$

where, as above, ψ: $T^*B/G \to G\backslash T^*\bar{B}$ is the Poisson isomorphism induced by Ψ. Equation (2.9) implies that the Hamiltonian vector fields corresponding to H_R^G on T^*B/G and $\bar{H}_L^G$ on $G\backslash T^*\bar{B}$ are ψ-related.

Similarly, a right $\bar{G}$-invariant Hamiltonian $\bar{H}_R$: $T^*\bar{B} \to \mathbb{R}$ determines H_L, a left $\bar{G}$ invariant function on T^*B (and vice-versa) and these functions induce

$$\bar{H}_R^{\bar{G}}: \quad T^*\bar{B}/\bar{G} \to \mathbb{R} \quad \text{and} \quad H_L^{\bar{G}}: \bar{G}\backslash T^*B \to \mathbb{R} \quad ,$$

which are related by

$$H_L^{\bar{G}} = \bar{H}_R^{\bar{G}} \circ \bar{\psi} \quad . \qquad (2.10)$$

We summarize, as follows.

Proposition 2.2. Under the hypotheses of Proposition 2.1, the Hamiltonian vector fields corresponding to the pairs $(H_R, \bar{H}_L)$ and $(H_R^G, \bar{H}_L^G)$ are $T^*\psi^{-1}$ and ψ-related and the Hamiltonian vector fields corresponding to the pairs $(H_L, \bar{H}_R)$ and $(H_L^G, \bar{H}_R^G)$ are $T^*\bar{\psi}^{-1}$ and $\bar{\psi}$ related.

Thus, not only the Poisson geometry, but also the dynamics on T^*B/G and $G\backslash T^*\bar{B}$ are equivalent. Likewise, $T^*\bar{B}/\bar{G}$ and $\bar{G}\backslash T^*B$ are equivalent in this sense. Consequently, in view of Propositions 2.1 and 2.2 we conclude that it is enough to work with the right [or left] actions and the reduced manifolds T^*B/G and $T^*\bar{B}/\bar{G}$ [or $\bar{G}/T^*B$ and $G\backslash T^*\bar{B}$], since the other pair may be recovered by isomorphisms.

Let P_1 and P_2 be Poisson manifolds and $f : P_1 \to P_2$ be a Poisson map. A Hamiltonian function $F_1 : P_1 \to \mathbb{R}$ is said to collectivize through f (see Guillemin and Sternberg [1980] and Holmes and Marsden [1983]) if there exists a function $F_2 : P_2 \to \mathbb{R}$ such that $F_2 \circ f = F_1$. Functions on P_1 of the form $F_2 \circ f$ for $F_2 : P_2 \to \mathbb{R}$ are called f-collective Hamiltonians. Two functions F and H on a Poisson manifold are said to be in involution if $\{F,H\} = 0$ on P. This is obviously equivalent to the fact that F is a conserved quantity for the flow of X_H or equivalently, H is a conserved quantity for the flow of X_F. With these notions we can relate involutivity on T^*B/G to involutivity on $\bar{G}\backslash T^*B$.

Proposition 2.3. Let F_1, F_2 be two functions in involution on T^*B. If F_1, F_2 collectivize through both $\pi_R : T^*B \to T^*B/G$ and the projection $T^*B \to \bar{G}\backslash T^*B$, then the induced functions are in involution on both T^*B/G and $\bar{G}\backslash T^*B$. In particular, if f_1, $f_2 : T^*B/G \to \mathbb{R}$ are in involution and $f_1 \circ \pi_R$

and $f_2 \circ \pi_R$ *are in involution and collectivize through the projection* $T^*B \to \bar{G}\backslash T^*B$, *then the induced functions on* $\bar{G}\backslash T^*B$ *are in involution. Similar statements hold for the pair* $(G\backslash T^*\bar{B}, T^*\bar{B}/\bar{G})$ *and the dual pairs in the diagrams* (IIIA) *and* (IIIB).

The proof of this Proposition is immediate since Poisson maps preserve involution.

3. SEMIDIRECT PRODUCTS AND HAMILTONIAN SYSTEMS WITH PARAMETERS

This Section treats Hamiltonian systems with parameters that transform in a special way under the action of the symmetry group of the configuration space. We start by recalling some relevant facts about semidirect products. In the continuum mechanical examples we shall treat later, all actions and representations are naturally on the right. That is why we adopt the conventions of right actions and right representations in this Section. In an Appendix at the end of this Section, we summarize the situation of a right principal bundle and a left representation. The formulas in the Appendix are used only in the example of the heavy rigid body. The reader should be warned that relative signs do change in the equations of motion when compared to the convention in which all actions are on the right.

3.1 Notation and Conventions Concerning Semidirect Products

Let G and K be Lie groups with the algebras L(G) and L(K) respectively and let $\Phi: G \to \mathrm{Aut}(K)$ be a smooth right Lie group action, i.e. the map:

$$(k,g) \in K \times G \to k \cdot g := \Phi(g)(k) \in K$$

is smooth and Φ is a Lie group antihomomorphism with values in Aut(K), the group of smooth automorphisms of K:

$$k \cdot g_1 g_2 = \Phi(g_1 g_2)(k) = (\Phi(g_2) \circ \Phi(g_1))(k) = (k \cdot g_1) \cdot g_2 \quad .$$

The semidirect product $S = G \circledS K$ of G with K is a Lie group with underlying manifold $G \times K$ and multiplication law

$$(g_1,k_1)(g_2,k_2) = (g_1g_2,k_2(k_1 \cdot g_2)) \quad . \tag{3.1}$$

If e_G and e_K denote the identity elements in G and K respectively, the identity element of $G \circledS K$ is (e_G,e_K) and $(g,k)^{-1} = (g^{-1},k^{-1} \cdot g^{-1})$.

Let $\mathrm{Der}(L(K))$ denote the Lie algebra of derivations of $L(K)$. The antihomomorphism Φ: $G \to \mathrm{Aut}(K)$ induces a Lie algebra antihomomorphism $\phi: L(G) \to \mathrm{Der}(L(K))$ in the following manner. For every $g \in G$, $\Phi(g): K \to K$ is a Lie group algebra automorphism $\tilde{\Phi}(g) := \Phi(g)' := T_e\Phi(g): L(K) \to L(K)$. In this way, one gets a Lie group antihomomorphism $\tilde{\Phi}: G \to \mathrm{Aut}(L(K))$, where $\mathrm{Aut}(L(K))$ is the group of Lie algebra automorphisms of $L(K)$. The induced Lie algebra antihomomorphism is defined via $\phi := \tilde{\Phi}' : L(G) \to \mathrm{Der}(L(K))$, which in turn defines the semidirect product Lie algebra $L(S) = L(G) \circledS L(K)$ as the Lie algebra with underlying vector space $L(G) \times L(K)$ and Lie algebra bracket

$$[(\xi_1,\eta_1),(\xi_2,\eta_2)] = ([\xi_1,\xi_2],\eta_1 \cdot \xi_2 - \eta_2 \cdot \xi_1 - [\eta_1,\eta_2]) \quad , \tag{3.2}$$

where $\eta \cdot \xi := \phi(\xi)(\eta)$, $\xi_i \in L(G)$, $\eta_i \in L(K)$, $i = 1,2$.

Remark. In infinite dimensions, the same definitions apply formally; rigorously, one introduces function space topologies and works with one of the Sobolev, Hölder or C^k spaces, or works in the ILB (inverse limit of Banach) or certain Fréchet categories. In this paper, we ignore such questions and apply the prior definitions and conventions to "Lie groups", such as the diffeomorphism groups of a compact manifold. See Ebin and Marsden [1970] and Adams, Ratiu and Schmid [1986] for more information.

The Lie algebra of $S = G \circledS K$ is $L(G) \circledS L(K)$ and the adjoint action of $G \circledS K$ on $L(G) \circledS L(K)$ is given by

$$Ad_{(g,k)}(\xi,\eta) = (Ad_g\xi, \tilde{\Phi}(g^{-1})[Ad_{k^{-1}}\eta + T_e(L_{k^{-1}} \circ \Phi_k)(\xi)]) \quad , \tag{3.3}$$

where $\Phi_k : G \to K$ is the map given by $\Phi_k(g) = k\cdot g$, where $k \in K$ and $g \in G$. Thus, the coadjoint action of S on $L(S)^*$ equals

$$Ad^*_{(g,k)^{-1}}(\mu,\nu) = (Ad^*_{g^{-1}}\mu + T^*_e(L_{k\cdot g^{-1}} + \Phi_{k^{-1}\cdot g^{-1}})\tilde{\Phi}(g)^*\nu \quad ,$$

$$Ad^*_{k^{-1}\cdot g^{-1}}\tilde{\Phi}(g)^*\nu) \quad , \tag{3.4}$$

where $\mu \in L(G)^*$, $\nu \in L(K)^*$, and the upper stars on different linear maps denote their duals with respect to natural pairings $\langle,\rangle : L(G)^* \times L(G) \to \mathbb{R}$ and $\langle,\rangle : L(K)^* \times L(K)) \to \mathbb{R}$. (In infinite dimensions, these pairings are usually weakly nondegenerate and have to be specified.) The natural pairings induce another pairing on $L(S)^* \times L(S)$, also denoted $\langle,\rangle$. In finite dimensions, $L(G)^*$ and $L(K)^*$ are the duals of $L(G)$ and $L(K)$, respectively, whereas in infinite dimensions they are vector spaces (with a given topology) "naturally" paired with $L(G)$ and $L(K)$. For example, if $L(G) = \mathcal{X}(M)$, the vector fields of a given Sobolev differentiability class on a compact manifold M, we choose $L(G)^* = \mathcal{X}(M)^*$ to be the one-form densities on M of the <u>same</u> differentiability class, and set $\langle\mu,X\rangle = \int_M \mu \cdot X$, where $\mu \in \mathcal{X}(M)^*$, $X \in \mathcal{X}(M)$, and the density $\mu \cdot X$ is the contraction of μ with X. The distributional dual $\mathcal{X}(M)'$ of $\mathcal{X}(M)$ is considerably larger than $\mathcal{X}(M)^*$ and consists of one-form densities on M of minus the Sobolev class of $\mathcal{X}(M)$. The pairing in both cases is the L^2-pairing, but only in the second case is it strongly nondegenerate.

The $\pm$ Lie-Poisson bracket of two functions $F,H : L(S)^*_\pm = L(G)^* \times L(K)^* \to \mathbb{R}$ equals, in veiw of (3.2),

$\{F,H\}_\pm(\mu,\nu) =$

$$\pm < \mu,[\frac{\delta F}{\delta\mu},\frac{\delta H}{\delta\mu}] > \mp < \nu,\phi(\frac{\delta F}{\delta\mu})\frac{\delta H}{\delta\nu} > \pm < -\nu,\phi(\frac{\delta H}{\delta\mu})\frac{\delta F}{\delta\nu} > \mp < \nu,[\frac{\delta F}{\delta\nu},\frac{\delta H}{\delta\nu}] > \quad , \tag{3.5}$$

where the partial functional derivatives belong to the following Lie algebras:

$$\frac{\delta F}{\delta\mu} \text{ and } \frac{\delta H}{\delta\mu} \in L(G) \quad , \quad \frac{\delta F}{\delta\nu} \text{ and } \frac{\delta H}{\delta\nu} \in L(K) \quad , \quad \text{for } \mu \in L(G)^* \quad , \quad \nu \in L(K)^* \quad .$$

The Hamiltonian vector field of $H : (L(G) \circledS L(K))^*_\pm \to \mathbb{R}$ equals

$$X_H(\mu,\nu) = \mp(\mathrm{ad}(\frac{\delta H}{\delta\mu})^*\mu + \phi^*_{\frac{\delta H}{\delta\nu}}\nu, \ - \phi(\frac{\delta H}{\delta\mu})^*\nu - \mathrm{ad}(\frac{\delta H}{\delta\nu})^*\nu) \quad , \tag{3.6}$$

where $\phi_\eta : L(G) \to L(K)$ is given by $\phi_\eta(\xi) = \phi(\xi)\eta$ for $\xi \in L(G)$ and $\eta \in L(K)$.

Let us specialize the foregoing definitions and formulas to the case that K = V, a vector space regarded as an abelian Lie group under addition, and $\Phi : G \to \mathrm{Aut}(V)$ is a right linear representation. Since the Lie algebra of V is V itself, it follows that $\tilde{\Phi} = \Phi$ and $\phi = \Phi' : G \to \mathrm{End}(V)$, where End(V) denotes the algebra of automorphisms of V. The composition law in $S = G \circledS V$ is given by

$$(g_1,v_1)(g_2,v_2) = (g_1g_2, \ v_2 + v_1 \cdot g_2) \tag{3.7}$$

and the Lie bracket on $L(S) = L(G) Ⓢ V$ equals

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2] \quad , \quad v_1 \cdot \xi_2 - v_2 \cdot \xi_1) \quad . \tag{3.8}$$

The adjoint and coadjoint actions become

$$\mathrm{Ad}_{(g,u)}(\xi, v) = (\mathrm{Ad}_g \xi, \ \Phi(g^{-1})(\ v + u \cdot \xi) \ ) \quad , \tag{3.9}$$

$$\mathrm{Ad}^*_{(g,u)^{-1}}(\mu, a) = (\mathrm{Ad}^*_{g^{-1}}\mu - \phi^*_{u \cdot g^{-1}} \Phi(g)^* a \quad , \quad \Phi(g)^* a) \quad , \tag{3.10}$$

and the Lie-Poisson bracket and Hamiltonian vector field have the expressions

$$\{F,H\}_{\pm}(\mu,a) = \pm < \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}] > \mp < a, \phi(\frac{\delta F}{\delta \mu})\frac{\delta H}{\delta a} > \pm < a, \phi(\frac{\delta H}{\delta \mu})\frac{\delta F}{\delta a} > \tag{3.11}$$

$$X_H(\mu, a) = \mp \ (\mathrm{ad}(\frac{\delta H}{\delta \mu})^* \mu + \phi^*_{\frac{\delta H}{\delta a}} a, \ - \ \phi(\frac{\delta H}{\delta \mu})^* a) \quad , \tag{3.12}$$

for $\mu \in L(G)^*$, $a \in V^*$, and $F,H : (L(G) Ⓢ V)^* \to \mathbb{R}$. These are the formulas that apply to the fluid dynamics examples in Section 5.

3.2 Semidirect Product Bundles

We turn next to the notion of a <u>semidirect</u> <u>product</u> <u>bundle</u>. As in Section 2, let G act on the right on B producing a surjective submersion $\pi: B \to M$, where $M = B/G$. Suppose the Lie Group G also acts on the right on another Lie group K via $\Phi: G \to \mathrm{Aut}(K)$. For example, $\pi: B \to M$ could be a principal bundle, but in some of the ensuing examples we will want to relax this condition to the prior hypotheses. Define an action of $S = G Ⓢ K$ on $B \times K$ by

$$(b,k) \cdot (g,h) = (b\cdot g,\ (k\cdot g)h^{-1}) \tag{3.13}$$

for $b \in B$; k and $h \in K$; and $g \in G$. The right S-bundle of $B \times K$ with the action (3.13) is called the *semidirect product bundle of* B *with* K.

Proposition 3.1. (i) *If* $M = B/G$ *is a manifold and* $\pi: B \to M$ *is a surjective submersion, then* $(B \times K)/S$ *is a manifold and* $\pi^S: B \times K \to M$, $\pi^S(b,k) = \pi(b)$, *is a surjective submersion.* (ii) *Let* $T^*B \times_G L(K)^*_+$ *denote the quotient of* $T^*B \times L(K)^*_+$ *by the right* G-*action. Then* $T^*B \times_G L(K)^*_+$ *is a manifold if and only if* $(T^*B \times T^*K)/S$ *is a manifold. In this case, these two manifolds are Poisson isomorphic and the following diagram commutes*

$$\begin{array}{ccc}
T^*B \times T^*K & \cong & T^*(B \times K) \\
\downarrow \rho & & \\
T^*B \times L(K)^*_+ & & \downarrow \pi^S_R \\
\downarrow \sigma & & \\
T^*B \times_G L(K)^*_+ & \cong & [T^*(B \times K)]/S
\end{array}$$

where $\rho(\alpha_b,\beta_k) = (\alpha_b, T^*_eL_k(\beta_k))$, σ, and π^S_R *denote the three projections associated to the reduction on the right by* $\{e_G\} \times K$, G, *and* S, *respectively.*

Proof. (i) The map $[b,k] \in (B \times K)/S \to \pi(b) \in M$ is easily seen to be well-defined and bijective, and so induces on $(B \times K)/S$ the differentiable structure of M.

(ii) The map $\phi:[\alpha_b,\nu] \in T^*B \times_G L(K)^*_+ \to [\alpha_b,\nu] \in (T^*B \times T^*K)/S$ has as its inverse the map $\phi^{-1}:[\alpha_b,\beta_k] \in (T^*B \times T^*K)/S \to [\alpha_b, T^*_e L_k(\beta_k)] \in T^*B \times_G L(K)^*_+$. Therefore, if one of these quotients is a manifold, so is the other and vice-versa. The commutativity of the diagram is obvious and the Poisson nature of all vertical maps implies that ϕ is a Poisson isomorphism.□

The action of S on $T^*B \times T^*K$ is given by

$$(\alpha_b,\beta_k) \cdot (g,h) = (\alpha_b \cdot g,\ T^*R_h(\beta \cdot g)) \quad . \tag{3.14}$$

Hence, the right G-action on $T^*B \times L(K)^*_+$ has the following expression:

$$(\alpha_b,\nu) \cdot g = (\alpha_b \cdot g,\ \nu \cdot g) \quad . \tag{3.15}$$

<u>Proposition 3.2</u>. <u>Let</u> $\pi:B \to B/G$ <u>be a surjective submersion</u>.

(i) <u>If</u> K <u>is another Lie group on which</u> G <u>acts on the right, then the semidirect product</u> S = G Ⓢ K <u>acts on</u> B × K <u>via</u> (3.13). <u>If</u> $(T^*B \times T^*K)/S$ <u>is a manifold and</u> $H : T^*B \times L(K)^*_+ \to \mathbb{R}$ <u>is a smooth function which is G-invariant</u>, <u>i.e.</u>,

$$H(\alpha_b \cdot g,\ \nu \cdot g) = H(\alpha_b,\nu) \quad , \tag{3.16}$$

<u>for all</u> g ∈ G, <u>then</u> H <u>induces a smooth function</u> $h : T^*B \times_G L(K)^*_+ \to \mathbb{R}$.

(ii) <u>Let</u> $\bar{K}$ <u>be another Lie group</u>. Then the direct product D = G × K <u>acts on</u> $B \times \bar{K}$ <u>via</u>

$$(b,\bar{k}) \cdot (g,\bar{h}) = (b \cdot g,\ \bar{k}\,\bar{h}) \quad , \tag{3.17}$$

so that G acts on $T^*B \times L(\bar{K})^*_+$ via

$$(\alpha_b,\bar{\nu}) \cdot g = (\alpha_b \cdot g,\ \bar{\nu}) \quad . \tag{3.18}$$

If $H : T^*B \times L(\bar{K})^*_+ \to \mathbb{R}$ is a smooth function, which is G-invariant, i.e.,

$$H(\alpha_b \cdot g, \bar{\nu}) = H(\alpha_b, \bar{\nu}) \tag{3.19}$$

for all $g \in G$, and if T^*B/G is a manifold, then H induces a smooth function $h : (T^*B/G) \times L(K)^* \to \mathbb{R}$.

3.3 Hamiltonians Depending on Parameters

In applications, one is usually given a Hamiltonian H depending on some parameters $(\nu,\bar{\nu}) \in L(K)^* \times L(\bar{K})^*$. Thus, if the symmetry group G of the configuration space B acts on $L(K)^* \times L(\bar{K})^*$ via:

$$(b,\nu,\bar{\nu}) \cdot g = (b \cdot g,\ \nu \cdot g,\ \bar{\nu}) \quad , \tag{3.20}$$

and the Hamiltonian function $H : T^*B \times L(K)^* \times L(\bar{K})^* \to \mathbb{R}$ is G-invariant, i.e.,

$$H(\alpha_b \cdot g,\ \nu \cdot g,\ \bar{\nu}) = H(\alpha_b,\nu,\bar{\nu}) \quad , \tag{3.21}$$

then H induces a Hamiltonian $h : (T^*B \times_G L(K)^*_+) \times L(\bar{K})^*_+ \to \mathbb{R}$. The Poisson manifold $(T^*B \times_G L(K)^*_+) \times L(\bar{K})^*_+$ is obtained by dividing $T^*(B \times K \times \bar{K})$ by $(G \circledS K) \times \bar{K}$.

Summarizing, we have the surjective submersion $\Pi^{\mathcal{G}} : \mathcal{B} \to \mathcal{M}$, where $\mathcal{B} = B \times K \times \bar{K}$, and the group $\mathcal{G} := (G \circledS K) \times \bar{K}$ acts on the right on $\mathcal{B}$ by

$$(b,k,\bar{k}) \cdot (g,h,\bar{h}) = (b \cdot g,\ (k \cdot g)h^{-1},\ \bar{k}\,\bar{h}) \quad . \tag{3.22}$$

It follows that $M = \mathcal{B}/\mathcal{G}$ from (3.22). Repeating the same construction under the same hypotheses for the triple $(\bar{B},\bar{G},\bar{K})$ instead of (B,G,K) gives another surjective submersion

$$\bar{\Pi} : \bar{\mathcal{B}} \rightarrow \bar{M} \quad , \quad \text{where} \quad \bar{M} = \bar{B}/\bar{G} \quad , \quad \bar{\mathcal{B}} = \bar{B} \times \bar{K} \times K \quad ,$$

with the group $\bar{\mathcal{G}} = (\bar{G} \circledS \bar{K}) \times K$ acting on the right on $\bar{\mathcal{B}}$ via the analog of (3.22). As before, it follows that $\bar{M} = \bar{\mathcal{B}}/\bar{\mathcal{G}}$. Thus, we have half of the hypotheses of Section 2. To get the other half, we define a <u>left</u> action of $\bar{\mathcal{G}} = (\bar{G} \circledS \bar{K}) \times K$ on $\mathcal{B} = B \times K \times \bar{K}$ by

$$(\bar{g},\bar{h},h) \cdot (b,k,\bar{k}) = (\bar{g}\cdot b,\ h\,k,\ (\bar{k}\,\bar{h})\cdot\bar{g}^{-1}) \tag{3.23}$$

and a left action of $\mathcal{G} = (G \circledS K) \times \bar{K}$ on $\bar{B} \times \bar{K} \times K$ by

$$(g,h,\bar{h}) \cdot (\bar{b},\bar{k},k) = (g \cdot \bar{b},\ \bar{k}\,\bar{k},\ (k\,h) \cdot g^{-1}) \quad . \tag{3.24}$$

Then, the maps $\Lambda,\bar{\Lambda}$: $\mathcal{B} \rightarrow \bar{\mathcal{B}}$ defined by

$$\Lambda(b,k,\bar{k}) = (\Psi(b),\overline{k}^{-1},k) \quad , \tag{3.25}$$

$$\bar{\Lambda}(b,k,\bar{k}) = (\bar{\Psi}(b),\bar{k},k^{-1}) \quad , \tag{3.26}$$

are $\mathcal{G}$ and $\bar{\mathcal{G}}$-equivariant respectively and so satisfy the hypotheses for Ψ and $\bar{\Psi}$ of Section 2. In particular, we have the dual pairs corresponding to diagrams (IIIA), (IIIB) of Section 2 and the dynamic statement of Proposition 2.2.

We close this subsection with explicit formulas for the case that $K = V$ and $\bar{K} = \bar{V}$ are vector spaces regarded as abelian Lie groups. This is the case most commonly encountered in practice, e.g., in Section 6 on fluid dynamics.

Formulas (3.13) - (3.15) become

$$(b,u)\cdot(g,v) = (b\cdot g,\ u\cdot g - v) \quad , \tag{3.27}$$

$$(\alpha_b,u,a)\cdot(g,v) = (\alpha_b\cdot g,\ u\cdot g - v,\ a\cdot g) \quad , \tag{3.28}$$

$$(\alpha_b,a)\cdot g = (\alpha_b\cdot g,\ a\cdot g) \quad , \tag{3.29}$$

for $g \in G$, $b \in B$, $u,v \in V$, $a \in V^*$ and $\alpha_b \in T_b^*B$. Formulas (3.16) - (3.21) remain unchanged after replacing ν by a and $\bar{\nu}$ by $\bar{a}$. The right action of $\mathcal{G} = (G \circledS V) \times \bar{V}$ on $\mathcal{B} = B \times V \times \bar{V}$ is given by the analog of (3.22). Namely,

$$(b,u,\bar{u}) \cdot (g,v,\bar{v}) = (b\cdot g,\ u\cdot g - v,\ \bar{u} + \bar{v}), \tag{3.30}$$

for $b \in B$, $u,v \in V$, and $\bar{u}$, $\bar{v} \in \bar{V}$. The right action of $\bar{\mathcal{G}} = (\bar{G} \circledS \bar{V}) \times V$ on $\bar{\mathcal{B}} = \bar{B} \times \bar{V} \times V$ is given by (3.30) with the obvious interchange of over-barred letters with unmarked letters. Finally, formulas (3.23) - (3.26) become

$$(\bar{g},\bar{v},v) \cdot (b,u,\bar{u}) = (\bar{g}\cdot b,\ u + v,\ (\bar{u} + \bar{v})\cdot\bar{g}^{-1}) \quad , \tag{3.31}$$

$$(g,v,\bar{v}) \cdot (\bar{b},\bar{u},u) = (g\cdot\bar{b},\ \bar{u} + \bar{v},\ (u + v)\cdot g^{-1}) \quad , \tag{3.32}$$

$$\Lambda(b,u,\bar{u}) = (\Psi(b), -\bar{u}, u) \quad , \tag{3.33}$$

$$\bar{\Lambda}(b,u,\bar{u}) = (\bar{\Psi}(b), \bar{u}, -u) \quad . \tag{3.34}$$

3.4 Lie-Poisson Structures for Semidirect Products

A special situation occurs when $K = V$ and $\bar{K} = \bar{V}$ are vector spaces and, in addition, $G = \bar{G} = B = \bar{B}$, i.e., when the configuration space is a group and the parameters of the Hamiltonian system are elements in the dual of a vector space. In this case, we take $\Psi = \bar{\Psi} : G \to G$ to be the inversion mapping $g \to g^{-1}$. We shall briefly review the main results of this theory following Marsden, Ratiu, Weinstein [1984a], [1984b]. We note that this specialization excludes free boundary problems for fluids (such as that outlined in Section 1) but includes problems with fixed boundaries.

We start with the right and left actions of the semidirect product $G = G \circledS V$ on its cotangent bundle T^*S:

$$(\alpha_k, v, a) \cdot (y,u) = (T^*_{kg} R_{g^{-1}}(\alpha_k),\ u + v\cdot g,\ a\cdot g) \quad , \tag{3.35}$$

$$(g,u) \cdot (\alpha_k, v, a) = (T^*_{gk} L_{g^{-1}}(\alpha_k) + T^*_{gk}\Phi_{-u\cdot g^{-1}}(a),\ v + u\cdot k,\ a) \quad , \tag{3.36}$$

where $\Phi_w : G \to V$ is defined by $\Phi_w(g) = w\cdot g$. The momentum map $J_R : T^*S \to L(S)^*$ of the right action (3.35) is given by

$$J_R(\alpha_k, v, a) = (k,v)^{-1} \cdot (\alpha_k, v, a) = (T^*_e L_k(\alpha_k) + \phi^*_v a,\ a) \quad , \tag{3.37}$$

and of the left action (3.36) by

$$J_L(\alpha_k,v,a) = (\alpha_k,v,a) \cdot (k,v)^{-1} = (T_e^*R_k(\alpha_b),\ a\cdot k^{-1}) \tag{3.38}$$

(see diagram (II) in Section 2). Since J_R is invariant under the left action (3.36), it is also invariant under the induced left action $u\cdot(\alpha_k,v,a) = (\alpha_k + T_k^*\phi_u(a),\ v + u\cdot k,a,$ of V on T^*S. Clearly T^*S/V is diffeomorphic to $T^*G \times V^*$. We search for a projection $P_L:T^*S \to T^*G \times V^*$ implementing this reduction, that in addition should have as its second component the momentum map of V on T^*S, i.e., the second component of J_L (see (3.38)). Such a projection is given by

$$(P_L(\alpha_k,v,a) = (\alpha_k + T_k^*L_{k^{-1}}\phi_v^*(a)\ ,\ a \cdot k^{-1})\quad . \tag{3.39}$$

The map P_L is Poisson, as the following argument shows. The second component of P_L is, as we already know, a momentum map and, thus, $(\alpha_k,v,a) \to a\cdot k^{-1}$ is a Poisson map. Since

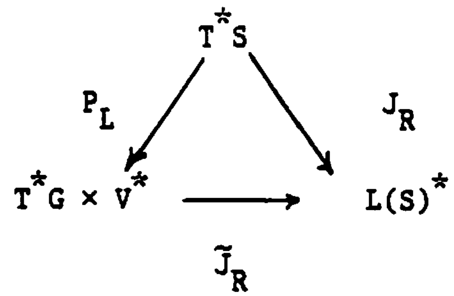
$$(T_h^*L_{h^{-1}} \circ \phi_v^*)(a) = df^a_{v\cdot h^{-1}}(h) \quad ,$$

where $f_u^a(g)$ denotes the "matrix element" <a, u·g>, we conclude that the map

$$\alpha_h \to \alpha_h + (T_h^*L_{h^{-1}} \circ \phi_v^*)(a)$$

is fiber translation by an exact differential; so it too, is Poisson. Therefore, the first component P_L is also Poisson and, consequently, P_L is a Poisson map.

It is easily seen that J_R in (3.37) factors through P_L, thus defining the map $\tilde{J}_R : T^*G \times V^* \to L(S)^*_-$, according to the commutative diagram,



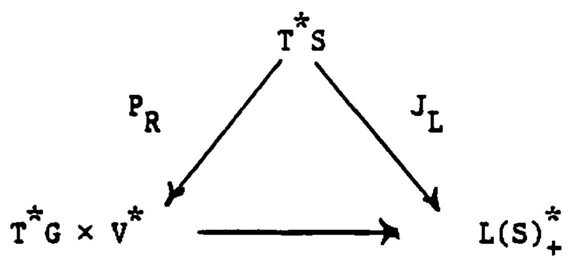
Explicitly,

$$\tilde{J}_R(\alpha_h, a) = (T_e^* L_h(\alpha_h);\ a \cdot h) \quad . \tag{3.40}$$

The situation for J_L in (3.38) is somewhat simpler. J_L is right invariant and thus is invariant under the right action $(\alpha_h, v, a) \cdot u = (\alpha_h,\ u+v,\ a)$ of V on T^*S. The projection $P_R : T^*S \to T^*G \times V^*$ implementing this reduction is given by

$$P_R(\alpha_h, v, a) = (\alpha_h, a) \quad , \tag{3.41}$$

and is, therefore, canonical and has as second component the second component of J_R (see (3.37)). In addition, J_L factors through P_R via



where

$$\tilde{J}_L(\alpha_h, v, a) = (T_e^* R_h(\alpha_h),\ a \cdot h^{-1}) \quad . \tag{3.42}$$

These observations are summarized as follows.

Theorem 3.3. The maps

$$\tilde{J}_L, \tilde{J}_R : T^*G \times V^* \to L(S)_\pm^* \quad ,$$

defined by

$$\tilde{J}_L(\alpha_h, a) = (T_e^* R_h(\alpha_h),\ a \cdot h^{-1}) \quad ,$$

$$\tilde{J}_R(\alpha_h, a) = (T_e^* L_h(\alpha_h),\ a \cdot h) \quad ,$$

are canonical, since they are reductions of the momentum maps J_L, J_R by the *actions of* V *on* T^*S. *They are themselves momentum maps for the left and right actions of* S *on* $T^*G \times V$.

The symplectic leaves of the quotient of $T^*G \times V^*$ *by the left and right* S-*actions are the coadjoint orbits of* $L(S)_\pm^*$ *and* $(T^*G \times V^*)/G \cong T^*S/S \cong L(S)_+^*$ (*and* $G\backslash(T^*G \times V^*) \cong S\backslash T^*S \cong L(S^*)_-$).

The last result is proved in Marsden, Ratiu and Weinstein [1984a]. The canonical nature of $\tilde{J}_L$ and $\tilde{J}_R$ is noted in Holm, Kupershmidt, and Levermore [1983] for a series of physical examples from continuum mechanics.

Remarks. Using the Cotangent Bundle Reduction Theorem due to Satzer and Marsden (see Abraham and Marsden [1978], p. 300, Theorem 4.3.3), Ratiu [1982] obtains the following additional result which we will find useful in our discussion of the heavy top in Section 5. Let $a \in V^*$ and let the isotropy group $G_a = \{g \in G | \ a \cdot g = a\}$ act on T^*G by the lift of right translation. The corresponding equivariant momentum map is J_R^a : $T^*G \to L(G)_a^*$, $J_R^a(\alpha_g) = T_e^*L_g(\alpha_g)|L(G)_a$, where $L(G)_a = \{\xi \in L(G)| \ a \cdot \xi = 0\}$ is the isotropy Lie subalgebra of $a \in V^*$, which is the Lie algebra of G_a. Let $\mu_a \in L(G)_a^*$ and $\text{Orb}(\mu_a)$ be the coadjoint orbit of G_a through μ_a in $L(G)_a^*$. Then the <u>reduced phase space</u> $(J_R^a)^{-1}(\text{Orb}(\mu_a))/G_a$ <u>is a smooth manifold symplectically diffeomorphic to the coadjoint orbit</u> $\text{Orb}(\nu, a)$ <u>of the semidirect product</u> $S = G \circledS V$ <u>on</u> $L(S)_-^*$, <u>where</u> $L(S) = L(G) \circledS V$, <u>for any</u> $\nu \in L(G)^*$ <u>whose restriction to</u> $L(G)_a$ <u>equals</u> μ_a, i.e., $\nu|L(G)_a = \mu_a$ (see also Marsden, Ratiu, Weinstein [1984a]). Note that $(J_R^a)^{-1}(\text{Orb}(\mu_a))/G_a$ is symplectically diffeomorphic to $(J_R^a)^{-1}(\mu_a)/(G_a)_{\mu_a}$, where $(G_a)_{\mu_a}$ is the coadjoint isotropy subgroup of G_a at μ_a. Denote by $J_{\mu_a}: T^*G \to L((G_a)_{\mu_a})^*$ the momentum map of the $(G_a)_{\mu_a}$-action on T^*G, i.e., J_{μ_a} is the restriction of J_R^a to $L((G_a)_{\mu_a})$. If $\mu \in L(G)^*$ is an arbitrary extension of $\mu_a \in L(G_a)^*$, the one-form on G defined by $\alpha_{\mu_a}(g) = \mu \circ T_g L_{g^{-1}}$ is G-left invariant and $(G_a)_{\mu_a}$-right invariant. Moreover, $J_{\mu_a}(\alpha_{\mu_a}(g)) = T_g^*L_g(\mu \circ T_g L_{g^{-1}})|((L(G_a))_{\mu_a} = \mu_a|L((G_a)_{\mu_a})$, i.e., α_{μ_a} has values in $J_{\mu_a}^{-1}(\mu_a|L((G_a)_{\mu_a})$. Under these hypotheses, the Cotangent Bundle Reduction Theorem guarantees that the <u>reduced manifold</u> $(J_R^a)^{-1}(\text{Orb}(\mu_a))/G_a \cong (J_R^a)^{-1}(\mu_a)/(G_a)_{\mu_a}$ <u>embeds symplectically onto a vector subbundle over</u> $G/(G_a)_{\mu_a}$ <u>in</u> $T^*(G/(G_a)_{\mu_a})$ <u>endowed with the symplectic structure</u> $\omega_0 - \hat{\beta}_{\mu_a}$, <u>where</u> ω_0 <u>is the canonical symplectic two-form on the cotangent bundle and</u> $\hat{\beta}_{\mu_a}$ <u>is the lift to</u> $T^*(G/(G_a)_{\mu_a})$ <u>of the</u>

closed two-form β_{μ_a} on $G/(G_a)_{\mu_a}$ given by $\pi^* \beta_{\mu_a} = d\alpha_{\mu_a}$ for $\pi: G \to G/(G_a)_{\mu_a}$. This embedding is onto iff $L((G_a)_{\mu_a}) = L(G_a)$. In the case of the rigid body, all conventions are on the left and in that case the definition of α_{μ_a} changes to $\alpha_{\mu_a}(g) = \mu \circ T_g R_{g^{-1}}$.

Next, we turn our attention to dynamics in the semidirect product context. Let $H: T^*G \times V^* \to \mathbb{R}$ be a Hamiltonian satisfying the following invariance property

$$H(T^*_{hg} R_{g^{-1}}(\alpha_h),\ a \cdot g) = H(\alpha_h, a) \quad , \tag{3.43}$$

for all $g \in G$, $\alpha_h \in T^*_h G$, $a \in V^*$; i.e. H is invariant under the right S-action on $T^*G \times V^*$ induced by (3.35) via P_R. Then, by right S-invariance of $\tilde{J}_L$, H induces a Hamiltonian $H_R: L(S)^*_+ \to \mathbb{R}$ by $H_R \circ \tilde{J}_L = H$, i.e.,

$$H_R(T^*_e R_g(\alpha_g),\ a \cdot g^{-1}) = H(\alpha_g,\ a) \quad . \tag{3.44}$$

Notice that the invariance property (3.43) implies that the Hamiltonian $H_a: T^*G \to \mathbb{R}$ given by $H_a(\alpha_g) = H(\alpha_g, a)$ is invariant under the lift of the right G_a-action on T^*G, where $G_a = \{g \in G |\ a \cdot g = a\}$ is the isotropy group of $a \in V^*$.

Let us investigate the evolution of a in $L(S)^*_+$. Let $c_a(t) \in T^*G$ denote an integral curve of the Hamiltonian vector field corresponding to H_a and let $g_a(t)$ be its projection on G. Then $t \to (c_a(t), a)$ is an integral curve of the Hamiltonian vector field on $T^*G \times V^*$ defined by H, so that the curve $\tilde{J}_L(c_a(t), a)$ is an integral curve for the Hamiltonian vector field on $L(S)^*_+$ defined by H_R. Its second component $t \to a \cdot g_a(t)^{-1}$ describes the evolution of a.

For left invariant Hamiltonians $H:T^*G \times V^* \to \mathbb{R}$, we assume that

$$H(T^*_{gh}L_{g^{-1}}(\alpha_h),\ a\cdot g^{-1}) = H(\alpha_h, a) \quad , \tag{3.45}$$

for all $g \in G$, i.e., H is invariant under the left S-action on $T^*G \times V^*$ induced from (3.36) via P_L. Then H induces a Hamiltonian $H_L:L(S)^*_- \to \mathbb{R}$ by $H_L \cdot \tilde{J}_R = H$, i.e.,

$$H_L(T^*_eL_g(\alpha_g),\ a\cdot g) = H(\alpha_g, a) \quad . \tag{3.46}$$

In this case, the evolution of a in $L(S)^*_+$ is given by the second component of $\tilde{J}_R(c_a(t),a)$ which is $t \to a \cdot g_a(t)$. We summarize what we have proved in the following.

Theorem 3.4. (i) *Let* $H:T^*G \times V^* \to \mathbb{R}$ *satisfy* (3.43), *i.e.*,

$$H(T^*_{hg}R_{g^{-1}}(\alpha_h),\ a\cdot g) = H(\alpha_h, a) \quad ,$$

for all $g \in G$. *Then* H *induces a Hamiltonian* $H_R:L(S)^*_+ \to \mathbb{R}$ by $H_R \circ \tilde{J}_L = H$, *i.e.*,

$$H_R(T^*_eR_g(\alpha_g),\ a\cdot g^{-1}) = H(\alpha_g, a) \quad .$$

The curve $c_a(t) \in T^*G$ *is a solution of the Hamiltonian vector field defined by* $H_a:T^*G \to \mathbb{R}$ *if and only if* $\tilde{J}_L(c_a(t),a)$ *is a solution curve for the Hamiltonian vector field given by* H_R *on* $L(S)^*_+$. *In particular, the* evolution of a in $L(S)^*_+$ is given by $t \to a \cdot g_a(t)^{-1}$, where $g_a(t)$ is the projection of $c_a(t)$ onto G.

(ii) Let $H: T^*G \times V^* \to \mathbb{R}$ satisfy (3.45), i.e.,

$$H(T^*_{gh} L_{g^{-1}}(\alpha_h),\ a \cdot g^{-1}) = H(\alpha_h, a)$$

for all $g \in G$. Then H induces a Hamiltonian $H_L : L(S)^*_- \to \mathbb{R}$ by $H_L \circ \tilde{J}_R = H$, i.e.,

$$H_L : (T^*_e L_g(\alpha_g),\ a \cdot g) = H(\alpha_g,\ a) \quad .$$

The curve $c_a(t) \in T^*G$ is an integral curve of the Hamiltonian vector field defined by $H_a: T^*G \to \mathbb{R}$ if and only if $\tilde{J}_R(c_a(t), a)$ is a solution of the Hamiltonian vector field given by H_L on $L(S)^*_-$. In particular, the evolution of a in $L(S)^*_-$ is given by $t \to a \cdot g_a(t)$, where $g_a(t)$ is the projection of $c_a(t)$ into G.

Finally, let us return to the set-up in part C of this Section. For the convective-spatial duality, we extend $T^*S = T^*G \times V \times V^*$ to the space $T^*(G \times V \times \bar{V})$. Recall that we write $\mathcal{G} = (G \circledS V) \times \bar{V}$ and $\bar{\mathcal{G}} = (\bar{G} \circledS \bar{V}) \times V = (G \circledS \bar{V}) \times V$. Hamiltonians are given on $T^*G \times V^* \times \bar{V}^*$. Thus, one should augment the preceding considerations by carrying along the trivial factors $\bar{V}^*$ or V^*, as the case may be. Since the $\mathcal{G}$(resp. $\bar{\mathcal{G}}$) actions on $T^*G \times V^* \times \bar{V}^*$ do not affect $\bar{V}^*$(resp., V^*), the evolution of $\bar{a} \in \bar{V}^*$ (resp., $a \in V^*$) is simply by the equation $d\bar{a}/dt = 0$ (resp., $da/dt = 0$). In this way, we get the following result.

Theorem 3.5. (i) *Let* $H: T^*G \times V^* \times \bar{V}^* \to \mathbb{R}$ *satisfy*

$$H(T^*_{hg}R_{g^{-1}}(\alpha_h),\ a\cdot g,\ \bar{a}) = H(\alpha_h, a, \bar{a}) \quad ,$$

for all $g \in G$. *Then* H *induces a Hamiltonian on* $[(g \circledS V) \times V]^*_-$ *by*

$$H_R(T^*_e R_g(\alpha_g),\ a\cdot g^{-1}, \bar{a}) = H(\alpha_g, a, \bar{a}) \quad .$$

The evolution of $(a,\bar{a}) \in V^* \times \bar{V}^*$ *for the dynamics defined by* H_R *is* $t \to ((a\cdot g_{a\bar{a}})(t)^{-1}, \bar{a})$, *where* $g_{a\bar{a}}(t)$ *is the projection of the integral curve* $c_{a\bar{a}}(t) \in T^*G$ *for the Hamiltonian* $H_{a\bar{a}}: T^*G \to \mathbb{R}$ *induced by* H.

(ii) With the same notations, let H *satisfy*

$$H(T^*_{gh}L_{g^{-1}}(\alpha_h),\ a\cdot g^{-1}, \bar{a}) = H(\alpha_h, a, \bar{a}) \quad ,$$

for all $g \in G$. *Then* H *induces a Hamiltonian* $H_L: [(g \circledS V) \times \bar{V}]^*_- \to \mathbb{R}$ *by*

$$H_L(T^*_e L_g(\alpha_g),\ a\cdot g, \bar{a}) = H(\alpha_g, a, \bar{a}) \quad .$$

The evolution of $(a,\bar{a}) \in V^* \times \bar{V}^*$ *is given by* $t \to (a\cdot g_a(t),\ \bar{a})$.

There are, of course, dual results involving $\bar{G} = (G \circledS \bar{V}) \times V$ which can be obtained by using Proposition 2.1 with (3.33) and (3.34) in this Theorem.

The rest of the paper is devoted to examples. We start with the simplest and most common one, the heavy top. The conventions for the representations that are the most useful in this case differs from the ones used for fluids and plasmas in which we have worked so far. Namely, we need to consider the standard conventions for left group representations and right bundles. All relevant formulas with these alternative conventions are summarized in the Appendix to this section and used in the next section. All the other examples in subsequent sections are infinite dimensional continuum mechanical cases that use exclusively the conventions of the present section.

Appendix

In this Appendix we summarize the key formulas of Section 3 when the actions of G on K and $\bar{G}$ on $\bar{K}$ are on the left. The correspondence is denoted by adding L to the equation number. The first twelve formulas are taken from Kupershmidt and Ratiu [1983] and (3.13L) from Montgomery, Marsden, and Ratiu [1984].

We denote by $\bar{\Phi}:\bar{G} \to Aut(K)$ and $\Phi:G \to Aut(\bar{K})$ left Lie group actions, and by $g \cdot k = \Phi(g)k$, $\bar{g} \cdot \bar{k} = \Phi(\bar{g})\bar{k}$, the associated maps $G \times K \to K$, $\bar{G} \times \bar{K} \to \bar{K}$. The multiplication law in G Ⓢ K is in this case

$$(g_1,k_1)(g_2,k_2) = (g_1 g_2,\ k_1(g_1 \cdot k_2)) \quad . \qquad (3.1L)$$

The identity is again formed by the pair (e_G, e_K) and $(g,k)^{-1} = (g^{-1},\ g^{-1} \cdot k^{-1})$. The maps $\tilde{\Phi}$, $\rho := \tilde{\Phi}'$ are homomorphisms and the Lie algebra L(G) Ⓢ L(K) of G Ⓢ K has the Lie algebra bracket

$$[(\xi_1,\eta_1),\ (\xi_2,\eta_2)] = ([\xi_1,\xi_1],\ \xi_1 \cdot \eta_2 - \xi_2 \cdot \eta_1 + [\eta_1,\eta_2]) \quad . \qquad (3.2L)$$

Let L(Aut(K)) denote the Lie algebra of Aut(K). To identify elements of L(Aut(K)), let $c:(-\varepsilon,\varepsilon) \to Aut(K)$ be a smooth curve with c(0) the identity map of K. For any $k \in K$,

$$\left.\frac{d}{dt}\right|_{t=0} c(t)(k) \in T_k K \quad ,$$

i.e., c'(0) defines a vector field on K by $k \to c'(0)(k)$. Thus, if $\mathcal{X}(K)$ denotes the space of smooth vector fields on K, $L(Aut(K)) \subset \mathcal{X}(K)$. If $\Phi':L(G) \to L(Aut(K))$ is the Lie algebra homomorphism induced by Φ, we have

$\Phi'(\xi) \in \mathcal{X}(K)$ for all $\xi \in L(G)$. With these notations, the adjoint action of $G \circledS K$ on $L(G) \circledS L(K)$ is given by

$$\mathrm{Ad}_{(g,k)}(\xi,\eta) = (\mathrm{Ad}_g\xi, (\mathrm{Ad}_k \circ \tilde{\Phi}(g))(\eta) + T_{k^{-1}}L_k([\Phi'(\mathrm{Ad}_g\xi)](k^{-1}))) \quad (3.3L)$$

where

$$\xi \in L(G) \quad , \quad \eta \in L(K) \quad , \quad g \in G \quad , \quad k \in K \quad ,$$

and

$$T_{k^{-1}}L_k \quad : \quad T_{k^{-1}}K \to L(K)$$

is the derivative of the left translation L_k on K at $k^{-1} \in K$. To compute the coadjoint action, some more notation is needed. For any Lie algebra homomorphism $F : L(G) \to L(\mathrm{Aut}(K))$ and any $k \in K$, let $F^{\vee}(k) : L(G) \to T_kK$ be the linear map defined by $F^{\vee}(k)(\xi) = F(\xi)(k)$ and let $F^{\vee}(k)^* : T_k^*K \to L(G)^*$ be the dual map. With this notation the coadjoint action is given by

$$\mathrm{Ad}^*_{(g,k)^{-1}}(\mu,\nu) = (\mathrm{Ad}^*_{g^{-1}}\mu + (\Phi' \circ \mathrm{Ad}_{g^{-1}})^{\vee}(\Phi(g^{-1})(k^{-1}))^*$$

$$(T_{\Phi(g^{-1})(k)}L_{\Phi(g^{-1})(k^{-1})})^*\nu \quad , \quad \tilde{\Phi}(g^{-1})^*\mathrm{Ad}^*_{\Phi(g^{-1})(k^{-1})}\nu) \quad . \quad (3.4L)$$

The Lie-Poisson bracket of $F,H : (L(G) \circledS L(K))^*_{\pm} \to \mathbb{R}$, becomes, with the use of (3.2L)

$$\{F,H\}(\mu,\nu) = \pm \langle\mu,[\frac{\delta F}{\delta\mu},\frac{\delta H}{\delta\mu}]\rangle \pm \langle\nu,\phi(\frac{\delta F}{\delta\mu})\frac{\delta H}{\delta\nu}\rangle$$

$$\mp \langle\nu,\phi(\frac{\delta H}{\delta\mu})\frac{\delta F}{\delta\nu}\rangle \pm \langle\nu,[\frac{\delta F}{\delta\nu},\frac{\delta H}{\delta\nu}]\rangle \quad , \qquad (3.5L)$$

where the partial functional derivatives belong to the following Lie algebras:

$$\frac{\delta F}{\delta\mu} \text{ and } \frac{\delta H}{\delta\mu} \in L(G) \quad , \quad \frac{\delta F}{\delta\nu} \text{ and } \frac{\delta H}{\delta\nu} \in L(K) \quad ,$$

for

$$\mu \in L(G)^* \text{ and } \nu \in L(K)^* \quad .$$

The Hamiltonian vector field of $H:(L(G) Ⓢ L(K))^*_{\pm} \to \mathbb{R}$ is given by

$$X_H(\mu,\nu) = \mp \left(ad(\frac{\delta H}{\delta\mu})^* \mu - \phi^*_{\frac{\delta H}{\delta\nu}} \nu \quad , \quad \phi(\frac{\delta H}{\delta\mu})^* \nu + ad(\frac{\delta H}{\delta\nu})^* \nu\right) \quad , \qquad (3.6L)$$

where $\mu \in L(G)^*$, $\nu \in L(G)^*$, and for $\eta \in L(K)$, $\phi_\eta : L(G) \to L(K)$ is given by $\phi_\eta(\xi) = \phi(\xi) \cdot \eta$.

Specializing the foregoing definitions and formulas to the case K = V, a vector space, we get

- the composition law in G Ⓢ V;

$$(g_1,v_1)(g_2,v_2) = (g_1g_2,\ v_1 + \Phi(g_1)v_2) \quad , \qquad (3.7L)$$

- the bracket in L(G) Ⓢ V:

$$[(\xi_1,v_1),\ (\xi_2,v_2)] = ([\xi_1,\xi_2],\ \phi(\xi_1)v_2 - \phi(\xi_2)v_1) \quad , \qquad (3.8L)$$

- the adjoint and coadjoint actions:

$$Ad_{(g,u)}(\xi,v) = (Ad_g\xi,\ \Phi(g)v - \phi(Ad_g\xi)u) \quad , \tag{3.9L}$$

$$Ad^*_{(g,u)^{-1}}(\mu,a) = (Ad^*_{g^{-1}}\ \mu + \phi^*_u\Phi(g^{-1})^*\ a,\ \Phi(g^{-1})^*a) \quad , \tag{3.10L}$$

- the Lie-Poisson bracket:

$$\{F,H\}_\pm\ (\mu,a) = \pm <\mu,[\frac{\delta F}{\delta\mu}, \frac{\delta H}{\delta\mu}]> \pm <a,\phi(\frac{\delta F}{\delta\mu})\ \frac{\delta H}{\delta a}> \mp <a,\phi(\frac{\delta H}{\delta\mu})\ \frac{\delta F}{\delta a}> \quad , \tag{3.11L}$$

- the Hamiltonian vector field of $H : L(G)\,\circledS\, V \to \mathbb{R}$

$$X_H(\mu,a) = \mp\ (ad(\frac{\delta H}{\delta\mu})^*\mu - \phi^*_{\frac{\delta H}{\delta a}}\ a,\ \phi(\frac{\delta H}{\delta\mu})^*a) \quad , \tag{3.12L}$$

where $\mu \in L(G)^*$ and $a \in V^*$.

The right action of $S = G\,\circledS\, K$ on $B \times K$ is given by

$$(b,k)\cdot(g,h) = (b\cdot g,\ g^{-1}\cdot(kh)) \cdot , \tag{3.13L}$$

and on $T^*B \times T^*K$ by

$$(\alpha_b,\beta_k)\cdot(g,h) = (\alpha_b\cdot g,\ g^{-1}\cdot T\ R^*_h(\beta_k)) \quad . \tag{3.14L}$$

The right G-action on $T^*B \times L(K)^*_+$ is

$$(\alpha_b,\nu)\cdot g = (\alpha_b\cdot g\ ,\ g^{-1}\cdot\nu) \quad , \tag{3.15L}$$

which is identical with (3.15). Therefore, formulas (3.16L) - (3.24L), coincide with the corresponding formulas for a right G-action on K and so Proposition 2.1 holds. Formulas (3.22) - (3.26) become:

$$(b,k,\bar{k}) \cdot (g,h,\bar{h}) = (b\cdot g,\ g^{-1}\cdot(kh),\ \overline{kh}) \quad , \tag{3.22L}$$

$$(\bar{g},\bar{h},h) \cdot (b,k,\bar{k}) = (\bar{g}\cdot b,\ hk,\ \bar{h}(\bar{g}\cdot\bar{k})) \quad , \tag{3.23L}$$

$$(g,h,\bar{h}) \cdot (\bar{b},\bar{k},k) = (g\cdot\bar{b},\ \bar{h}\ \bar{k},\ h(g\cdot k)) \quad , \tag{3.24L}$$

$$\Lambda(b,k,\bar{k}) = (\Psi(b),\ \bar{k}^{-1},\ k^{-1}) \quad , \tag{3.25L}$$

$$\bar{\Lambda}(b,k,\bar{k}) = (\bar{\Psi}(b),\ \bar{k}^{-1},\ k^{-1}) \quad . \tag{3.26L}$$

Note that unlike the situation of right semidirect products, the maps Λ and $\bar{\Lambda}$ are in this case identical. If $K = V$ and $\bar{K} = \bar{V}$ are vector spaces, the analogs of formulas (3.27) - (3.29) or (3.13L) - (3.15L) are

$$(b,u) \cdot (g,v) = (b\cdot g,\ g^{-1}\cdot(u + v)) \quad , \tag{3.27L}$$

$$(\alpha_b,u,a) \cdot (g,v) = (\alpha_b\cdot g,\ g^{-1}\cdot(u + v),\ g^{-1}\cdot a) \quad , \tag{3.28L}$$

$$(\alpha_b,a) \cdot g = (\alpha_b\cdot g,\ g^{-1}\cdot a) \quad , \tag{3.29L}$$

where $g \in G$, $b \in B$, u and $v \in V$, $a \in V^*$ and $\alpha_b \in T_b^*B$. Finally, the formulas corresponding to (3.30) - (3.34) or (3.22L) - (3.26L) take the forms:

$$(b,u,\bar{u}) \cdot (g,v,\bar{v}) = (b\cdot g,\ g^{-1}\cdot(u + v),\ \bar{u} + \bar{v}) \quad , \tag{3.30L}$$

$$(\bar{g},\bar{v},v) \cdot (b,u,\bar{u}) = (\bar{g}\cdot b,\ u + v,\ \bar{v} + \bar{g}\cdot\bar{u}) \quad , \tag{3.31L}$$

$$(g,v,\bar{v}) \cdot (\bar{b},\bar{u},u) = (g\cdot\bar{b},\ \bar{u} + \bar{v},\ v + g\cdot u) \quad , \tag{3.32L}$$

$$\Lambda(b,u,\bar{u}) = (\Psi(b),\ -\bar{u},\ -u) \quad , \tag{3.33L}$$

$$\bar{\Lambda}(b,u,\bar{u}) = (\bar{\Psi}(b),\ -\bar{u},\ -u) \quad . \tag{3.34L}$$

Denoting by $f_u^a(g)$ the "matrix element" $\langle a,g\cdot u\rangle$, where $a \in V^*$, $u \in V$ and $g \in G$, the right and left actions of S on T^*S have the expressions

$$(\alpha_k,v,a) \cdot (g,u) = (T^*_{hg}R_{g^{-1}}(\alpha_h) - df^a_{g^{-1}\cdot u}(hg),\ v + h\cdot u,\ a) \quad , \tag{3.35L}$$

$$(g,u) \cdot (\alpha_h,v,a) = (T^*_{gh}L_{g^{-1}}(\alpha_h),\ u + g\cdot v,\ g\cdot a) \quad . \tag{3.36L}$$

The momentum maps J_L, J_R, the projections P_L, P_R, and the induced momentum maps $\tilde{J}_L$ and $\tilde{J}_R$ take the form:

$$J_L(\alpha_g,v,a) = (T^*_eR_g(\alpha_g) + \phi^*_v a,\ a) \quad , \tag{3.37L}$$

$$J_R(\alpha_g,v,a) = (T^*_eL_g(\alpha_g),\ g^{-1}\cdot a) \quad , \tag{3.38L}$$

$$P_L(\alpha_g,v,a) = (\alpha_g,a) \quad , \tag{3.39L}$$

$$P_R(\alpha_g,v,a) = (\alpha_g + T_g^*R_{g^{-1}}\phi_v^*(a),\ g^{-1}\cdot a) \quad , \tag{3.41L}$$

$$\tilde{J}_L(\alpha_g,v,a) = (T_e^*R_g(\alpha_g),\ g\cdot a) \quad , \tag{3.42L}$$

$$\tilde{J}_R(\alpha_g,v,a) = (T_e^*L_g(\alpha_g),\ g^{-1}\cdot a) \quad . \tag{3.40L}$$

The conditions for right and left invariance of the Hamiltonians take the form

$$H(T_{hg}^*R_{g^{-1}}(\alpha_h),\ g^{-1}\cdot a) = H(\alpha_h,a) \quad , \tag{3.43L}$$

$$H(T_{gh}^*L_{g^{-1}}(\alpha_h),\ g\cdot a) = H(\alpha_h,a) \quad , \tag{3.45L}$$

for all $g \in G$ and $u \in V$. The induced Hamiltonians are given by

$$H_R : L(S)_+^* \to \mathbb{R} \quad , \quad H_L : L(S)_-^* \to \mathbb{R} \quad ,$$

defined by

$$H_L(T_e^*L_g(\alpha_g),\ g^{-1}\cdot a) = H(\alpha_g,a) \quad , \tag{3.46L}$$

$$H_R(T_e^*R_g(\alpha_g),\ g\cdot a) = H(\alpha_g,a) \quad . \tag{3.44L}$$

Finally, the evolution of $(a,\bar{a})$ is given by

$$t \to (g_a(t)^{-1}\cdot a,\ \bar{a}) \text{ for } H_L \text{ on } L(S)_-^* \tag{3.47L}$$

and

$$t \to (g_a(t) \cdot a, \bar{a}) \text{ for } H_R \text{ on } L(S)_+^* \quad . \tag{3.48L}$$

Of course, all the theorems from Section 3 have corresponding statements for the conventions in this Appendix.

4. CONTINUUM MECHANICAL CONSIDERATIONS

We now recall some continuum mechanics notation in accordance with Marsden and Hughes [1983], in preparation for the next sections.

The physical problem under consideration is the description of the motion of a body -- either a fluid or a solid. It is useful to think of the body abstractly as being separate from the position it occupies in space. A reference configuration D of the body is the closure of an open set in $\mathbb{R}^3$ with piecewise smooth boundary. We think of D as the position that the body occupies at some reference time and keep it separate from its subsequent shape during the time evolution. A configuration is an orientation-preserving embedding $\eta: D \rightarrow \mathbb{R}^3$ of a specific differentiability class. A motion of D is a time dependent family of configurations $\eta_t : D \rightarrow \mathbb{R}^3$, written as $\underline{x} = \underline{\eta}(\underline{X},t) = \underline{\eta}_t(\underline{X})$. We shall denote by $\underline{X} = (X^1, X^2, X^3)$ points in D and call them material, or Lagrangian points; X^i, $i = 1,2,3$ are called material, or Lagrangian coordinates. Points in the target space of a configuration are called spatial, or Eulerian points and are denoted by lower-case letters $\underline{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$; x^i, $i = 1,2,3$ are called spatial, or Eulerian coordinates. The relationship between a spatial vector quantity $\underline{z}$ and its corresponding material vector quantity $\underline{Z}$ is given by the relation

$$\underline{z}_t \circ \eta_t = \underline{Z}_t \quad . \tag{4.1}$$

Let us investigate some consequences of relationship (4.1). The diffeomorphism group Diff(D) of D acts on the space of configurations $C = \{\eta : D \to \mathbb{R}^3 \mid \eta$ is an orientation-preserving embedding $\}$ on the right via

$$(\eta,\phi) \in C \times \text{Diff}(D) \to \eta \circ \phi \in C \quad . \tag{4.2}$$

Let us consider the material quantities $\underline{Z}$ and $\underline{Z} \circ \phi$ and ask for the relationship between the corresponding spatial quantities. By (4.1) it follows that $\underline{z}_t \circ \eta_t \circ \rho = \underline{Z}_t \circ \phi$ which says that $\underline{z}_t$ is unchanged by the particle relabeling group G = Diff(D). Therefore, the passage (4.1) from material to spatial quantities factors through the projection $\pi^G : C \to C/\text{Diff}(D)$.

For example, the <u>Lagrangian</u> or <u>material</u> <u>velocity</u> is defined by

$$\underline{V}(\underline{X},t) = \partial\underline{x}(\underline{X},t)/\partial t \quad , \tag{4.3}$$

where we have written $\underline{x}(\underline{X},t)$ for $\eta(\underline{X},t)$, so that the corresponding <u>spatial</u> or <u>Eulerian</u> velocity is given by

$$\underline{v}(\underline{x},t) = \underline{V}(\underline{X},t) \quad , \text{ i.e. } \quad , \quad \underline{v}_t \circ \eta_t = \underline{V}_t \quad , \tag{4.4}$$

and, thus, $\underline{v}$ is invariant under particle relabeling.

There is another group acting on C, namely the diffeomorphism group $\text{Diff}(\mathbb{R}^3) = \bar{G}$ of space. The action is on the left and is given by

$$(\lambda,\eta) \in \text{Diff}(\mathbb{R}^3) \times C \to \lambda \circ \eta \in C \quad . \tag{4.5}$$

Note that while (4.1) was a free action (no fixed points), (4.5) is not free; one can alter the diffeomorphism λ outside the range of η and obtain the same embedding $\lambda \circ \eta$. What is invariant under Diff($\mathbb{R}^3$)? To answer this question, we review three formulations of continuum mechanics. First, one can follow each particle individually, this yields the Lagrangian or material picture. Second, one can keep the material point fixed and describe the evolution in space. Physically, in this case, one looks at the body from a fixed coordinate system in space. As we just saw, this has the effect of ignoring the particle relabelling group Diff(D). This is the Eulerian, or spatial picture. For example, in this description (4.4) shows that one takes derivatives with $\underline{X}$ fixed. The third way to describe the motion is in terms of quantities involving $\underline{X}$ as functions of $\underline{x}$. For example, one defines the body or convective velocity by (Note minus sign, which differs from the convention in Holm [1986].)

$$\underline{\mathcal{V}}(\underline{X},t) = -\,\partial\underline{X}(\underline{x},t)/\partial t \quad . \tag{4.6}$$

The relationship between $\underline{V}$, $\underline{v}$, and $\underline{\mathcal{V}}$ is obtained by the chain rule in the following way. Since $\underline{X}(\underline{x},t) = \eta_t^{-1}(\underline{x})$, we have

$$\begin{aligned}\underline{\mathcal{V}}_t(\underline{X}) &= T\eta_t^{-1}(\partial\eta_t(\eta_t^{-1}(\underline{x}))/\partial t)\\ &= T\eta_t^{-1}(\underline{V}_t(\underline{X}))\\ &= (T\eta_t^{-1} \circ \underline{v}_t \circ \eta_t)(\underline{X}) \quad .\end{aligned}$$

That is, [cf. Holm [1986] Eq. (2.17)]

$$\underline{\mathcal{V}}_t = T\eta_t^{-1} \circ \underline{V}_t = \eta_t^{*}\underline{v}_t \quad , \tag{4.7}$$

where $\eta_t^{*}\underline{v}_t = T\eta_t^{-1} \circ \underline{v}_t \circ \eta_t$ denotes the pull-back of the vector field $\underline{v}_t$.

Thus, we define convective quantities, in general, as pull-backs of the corresponding spatial quantities, i.e., if $\underline{Z}_t$ is a convective vector quantity on D, it is defined from the corresponding vector spatial quantity $\underline{z}_t$ in terms of the motion η_t by

$$\underline{Z}_t = \eta_t^* \underline{z}_t \quad . \tag{4.8}$$

The pull-back here is understood to be defined on the class of objects that $\underline{z}_t$ defines, e.g., if $\underline{z}_t$ is a vector field, η_t^* denotes pull-back of vector fields and if $\underline{z}_t$ is a tensor of a given type in D, then η_t^* denotes the pull-back operation on that type of tensor.

Now let us return to the invariance properties under the spatial diffeomorphism group Diff($\mathbb{R}^3$). If $\underline{z}_t$ is a tensor field on $\mathbb{R}^3$, the diffeomorphism $\lambda \in$ Diff($\mathbb{R}^3$) induces the push-forward action on $\underline{z}_t$. Therefore, the convective quantity corresponding to $\lambda_* \underline{z}_t$ is

$$(\lambda \circ \eta_t)^* \lambda_* \underline{z}_t = \eta_t^* \lambda^* \lambda_* \underline{z}_t = \eta_t^* z_t = \underline{Z}_t \quad ,$$

i.e., <u>convective quantities are invariant under the action of</u> Diff($\mathbb{R}^3$) <u>on tensor fields defined on</u> C.

Summarizing, in the language of Section 2 we have two groups G = Diff(D) and $\bar{G}$ = Diff($\mathbb{R}^3$), which act from the right and left, respectively, on the manifold $B = C$. The quotient C/Diff(D) is identified with the manifold M of unparameterized embedded boundaries of D in $\mathbb{R}^3$ because the Diff(D)-orbits in C coincide with the set of images of D in $\mathbb{R}^3$. The other quotient $N = \bar{G}\backslash B$ = Diff($\mathbb{R}^3$)$\backslash$C is a point. This is seen in the following manner. If η and η' are elements of C, then $\eta' \circ \eta^{-1}$ is a diffeomorphism of $\eta(D)$ with $\eta'(D)$. Now let $\lambda \in \text{Diff}(\mathbb{R}^3)$ be any diffeomorphism which on $\eta(D)$ coincides with η', so that $\eta' = \lambda \circ \eta$. This shows that C is a single $\text{Diff}(\mathbb{R}^3)$-orbit.

So far, we have half of the set-up of Section 2. To obtain the other half, let $\bar{C} = \{\bar{\eta} | \bar{\eta}$ is a diffeomorphism from a region in $\mathbb{R}^3$ to $D\}$ and set $\bar{B} = \bar{C}$. Then $\bar{G} = \text{Diff}(\mathbb{R}^3)$ acts on $\bar{C}$ by composition on the right and $G = \text{Diff}(D)$ action $\bar{C}$ by composition on the left. Therefore, $\bar{G}$ acts on $\bar{B}$ on the right and G acts on $\bar{B}$ on the left. The quotients $\bar{M} = \bar{B}/\bar{G} = \bar{C}/\text{Diff}(\mathbb{R}^3)$ and $\bar{N} = G\backslash\bar{B} = \text{Diff}(D)\backslash\bar{C}$ are, respectively, a point and the images of boundaries of D in $\mathbb{R}^3$. Finally, let's choose the diffeomorphism

$$\Psi = \bar{\Psi} : \eta \in C \to \eta^{-1} \in \bar{C} \quad , \tag{4.9}$$

and observe that the commutation relations (2.1) and (2.2) of Section 2 hold. Thus, we have now all the hypotheses of Section 2. However, even in the simplest examples, the Hamiltonians of interest depend on parameters; so we must apply the general theory of Section 3 to each specific example separately. As we shall see, all these examples have enough special structure to enable us to write down their Poisson brackets explicitly.

5. THE HEAVY TOP

In this section, we apply the results of Sections 3 and 4 to study the motion of a rigid body about a fixed point. We shall deduce here the equations of motion both in the convective (also called "body") and spatial pictures. The formulas we shall use are given in the Appendix to Section 3, since the nature of the problem summons left representations. As a by-product of the different formulations of the equations, we shall gain some insight into the complete integrability of various cases of the heavy top, including the cases of Lagrange and Kovalevski.

5.1 The Material Phase Space

A top is by definition a rigid body moving about a fixed point in three dimensional space. Rigidity of the top implies that the distances between points of the body are fixed as the body moves. This means that if the configuration $\underline{x}(\underline{X},t)$ represents the position of a particle that was at $\underline{X}$ when $t = 0$, then

$$\underline{x}(\underline{X},t) = A(t)\underline{X} \quad , \quad \text{i.e.} \; , \quad x^i = A^i_j(t)X^j \quad , \quad i,j = 1,2,3, \text{ sum on } j \quad , \qquad (5.1)$$

where $A(t) = A^i_j(t))$ is an orthogonal matrix. Since the motion is assumed to be at least continuous and $A(0)$ is the identity matrix, it follows that $\det(A(t)) = 1$ and thus $A(t) \in SO(3)$, the proper orthogonal group. Thus, <u>the configuration space of the heavy top may be identified with $SO(3)$</u>.

Consequently the phase space of the top is the cotangent bundle $T^*(SO(3))$, which will be described shortly.

In this example, convected, or body coordinates are easy to visualize. Let $\underline{E}_1$, $\underline{E}_2$, $\underline{E}_3$ be an orthonormal basis relative to which material coordinates $\underline{X} = (X^1, X^2, X^3)$ are defined and $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ be an orthonormal basis associated to spatial coordinates. Let the time dependent basis $\underline{\xi}_1$, $\underline{\xi}_2$, $\underline{\xi}_3$ be defined by

$$\underline{\xi}_i = A(t)\underline{E}_i \quad ,$$

so the $\underline{\xi}_i$ move attached to the body. The body coordinates of a vector in $\mathbb{R}^3$ are its components relative to $\underline{\xi}_i$. For $\underline{v} \in \mathbb{R}^3$, its spatial coordinates v^i are related to its body coordinates ν^j by

$$v^i = A^i_j \nu^j \quad ,$$

where A^i_j is the matrix of A relative to $\underline{E}_i$ and $\underline{e}_i$. Of course the components of a vector $\underline{V}$ relative to $\underline{E}_i$ are the same as the components of $A\underline{V}$ relative to $\underline{\xi}_i$. In particular, the body coordinates of $\underline{x}$ are X^i.

Euler angles are the traditional way to express the relationship between space and body coordinates, i.e., to parameterize SO(3). In what follows, we shall adopt the conventions of Arnold [1978] and Goldstein [1980], which are different from those of Whittaker [1917].

One can pass from the spatial basis $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ to the body basis $(\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_3)$, by means of three consecutive counterclockwise rotations performed in a specific order: first rotate by the angle ϕ around $\underline{e}_3$ and denote the new position of $\underline{e}_1$ by ON (line of nodes); then rotate by the angle θ around ON; and finally rotate by the angle ψ around ξ_3 (see Fig. 1). Consequently $0 \leqq \phi,\ \psi < 2\pi$ and $0 \leqq \theta < \pi$. Note that there is a bijective map between the (ϕ,ψ,θ) variables and SO(3). However, this bijective map does not define a chart, since its differential vanishes, for example, at $\phi = \psi = \theta = 0$. The differential is nonzero for $0 < \phi < 2\pi$, $0 < \psi < 2\pi$, $0 < \theta < \pi$ and on this domain, the Euler angles do form a chart. Explicitly this is given by $(\phi,\psi,\theta) \to A$, where A is uniquely determined by $\underline{x} = A\underline{X}$ and has the matrix relative to $\underline{\xi}_i$ and $\underline{e}_i$ given by

$$A = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}. \quad (5.2)$$

With the aid of the chart given by Euler angles we induce a natural chart $(\phi,\psi,\theta,\dot\phi,\dot\psi,\dot\theta)$ on the tangent bundle T(SO(3)) of the proper rotation group SO(3). Then, using a Legendre transformation given by a certain metric on SO(3) uniquely determined by the mass distribution of the top, we will define a mapping to the natural chart $(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta)$ on the cotangent bundle $T^*(SO(3))$ which is the canonical phase space. This will be done below.

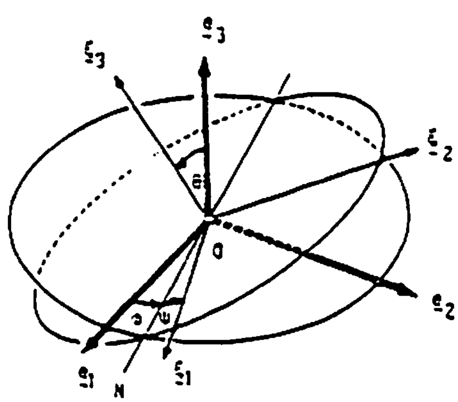


Fig. 1. Definition of Euler Angles.

5.2 The Lie Algebra so(3) and Its Dual

In order to simplify the computations and identify the geometric structure of the Hamiltonian of the heavy top, a summary of the Lie algebra so(3) and its dual is needed.

The proper rotation group SO(3) has as Lie algebra the 3×3 infinitesimal rotation matrices, i.e., the space so(3) of 3×3 skew-symmetric matrices; the bracket operation is the commutator of matrices. The Lie algebra so(3) is identified with $\mathbb{R}^3$ by associating to the vector $\underline{v} = (v^1, v^2, v^3) \in \mathbb{R}^3$, the matrix $\hat{\underline{v}} \in$ so(3) given by

$$\hat{\underline{v}} = \begin{vmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{vmatrix} \quad . \tag{5.3}$$

Then we have the following identities:

$$(\underline{u} \times \underline{v})\hat{} = [\hat{\underline{u}}, \hat{\underline{v}}] \quad , \tag{5.4}$$

$$\hat{\underline{u}} \cdot \underline{v} = \underline{u} \times \underline{v} \quad , \tag{5.5}$$

$$[\hat{\underline{u}}, \hat{\underline{v}}] \cdot \underline{w} = (\underline{u} \times \underline{v}) \times \underline{w} \quad , \tag{5.6}$$

$$\underline{u} \cdot \underline{v} = -\frac{1}{2} \operatorname{Tr}(\underline{\hat{u}\hat{v}}) = \frac{1}{2} \operatorname{Tr}(\hat{\underline{u}}^T \hat{\underline{v}}) \quad . \tag{5.7}$$

Moreover, if $A \in SO(3)$ and $\underline{v} \in \mathbb{R}^3$, then the adjoint action (conjugation) is given by

$$(A\underline{v})\hat{} = \mathrm{Ad}_A \hat{\underline{v}} := A\hat{\underline{v}}A^{-1} \quad . \tag{5.8}$$

Consequently, since the adjoint action is a Lie algebra homomorphism, for all $A \in SO(3)$, $\underline{u}$ and $\underline{v} \in \mathbb{R}^3$ we recover the vector algebra identity

$$A(\underline{u} \times \underline{v}) = A\underline{u} \times A\underline{v} \quad . \tag{5.9}$$

In what follows we shall identify the dual $so(3)^*$ with $\mathbb{R}^3$ by the inner product, i.e. $\tilde{\underline{m}} \in so(3)^*$ corresponds to $\underline{m} \in \mathbb{R}^3$ by $\tilde{\underline{m}}(\underline{v}) = \underline{m} \cdot \underline{v}$, for all $\underline{v} \in \mathbb{R}^3$. Then the coadjoint action of $SO(3)$ on $so(3)^*$ is represented by the usual action of $SO(3)$ on $\mathbb{R}^3$, i.e.

$$Ad^*_{A^{-1}} \, \underline{m} = A\underline{m} \quad , \tag{5.10}$$

since $(A^{-1})^T = A$.

5.3 The Hamiltonian

The material velocity at a point $\underline{X}$ in the body D is [by Eq. (5.1)]

$$\underline{V}(\underline{X},t) = \partial\underline{x}(\underline{X},t)/\partial t = \dot{A}(t)\underline{X} \quad , \tag{5.11}$$

so that the spatial and convective velocities have the following expressions, respectively

$$\underline{v}(\underline{x},t) = \partial\underline{x}(\underline{X},t)/\partial t = \underline{V}(\underline{X},t) = \dot{A}(t)A(t)^{-1}\underline{x} \quad , \tag{5.12}$$

$$\underline{\mathcal{V}}(\underline{X},t) = -\,\partial\underline{X}(\underline{x},t)/\partial t = A(t)^{-1}\dot{A}(t)A(t)^{-1}\underline{x}$$

$$= A(t)^{-1}\dot{A}(t)\underline{X} = A(t)^{-1}\underline{V}(\underline{X},t) = A(t)^{-1}\underline{v}(\underline{x},t) \quad . \tag{5.13}$$

Let D denote the reference configuration of the body, a compact region of $\mathbb{R}^3$ with piecewise smooth boundary. Let $\rho_0(\underline{X})$ denote the density of the body in the reference configuration. Then the kinetic energy at time t is, by (5.11), (5.12), (5.13), and the invariance of the Euclidean norm under SO(3),

$$K(t) = \frac{1}{2} \int_{D} \rho_0(\underline{X}) |\underline{V}(\underline{X},t)|^2 \, d^3\underline{X} \qquad \text{(material)} \qquad (5.14)$$

$$= \frac{1}{2} \int_{A(t)_D} \rho_0(A(t)^{-1}\underline{x}) |\underline{v}(\underline{x},t)|^2 \, d^3\underline{x} \qquad \text{(spatial)} \qquad (5.15)$$

$$= \frac{1}{2} \int_{D} \rho_0(\underline{X}) |V(\underline{X},t)|^2 \, d^3\underline{X} \ . \qquad \text{(convective)} \qquad (5.16)$$

Differentiating $A(t)^T A(t)$ = Identity and $A(t)\, A(t)^T$ = Identity, it follows that both $A(t)^{-1}\dot{A}(t)$ and $\dot{A}(t)\dot{A}(t)^{-1}$ are skew-symmetric. Moreover, by (5.5), (5.12), (5.13), and the classical definition of angular velocity, it follows that the vectors $\underline{\omega}_S(t)$ and $\underline{\omega}_B(t)$ in $\mathbb{R}^3$ defined by

$$\hat{\underline{\omega}}_S(t) = \dot{A}(t)A(t)^{-1} \quad , \qquad (5.17)$$

$$\hat{\underline{\omega}}_B(t) = A(t)^{-1}\dot{A}(t) \quad , \qquad (5.18)$$

are the <u>spatial</u> and <u>body angular velocities</u> of the top, respectively. Note that $\underline{\omega}_S(t) = A(t)\underline{\omega}_B(t)$, or as matrices, $\hat{\underline{\omega}}_S = \mathrm{Ad}_A \hat{\omega}_B = A\hat{\underline{\omega}}_B A^{-1}$. In the Euler angle parametrization (5.2) of SO(3), Eqs. (5.17), and (5.18) for $\underline{\omega}_S$ and $\underline{\omega}_B$ have the following expressions

$$\dot{\underline{\omega}}_S = \begin{vmatrix} \dot{\theta} \cos\phi + \dot{\psi} \sin\phi \sin\theta \\ \dot{\theta} \sin\phi - \dot{\psi} \cos\phi \sin\theta \\ \dot{\phi} + \dot{\psi} \cos\theta \end{vmatrix} \ , \ \underline{\omega}_B = \begin{vmatrix} \dot{\theta} \cos\psi + \dot{\phi} \sin\psi \sin\theta \\ -\dot{\theta} \sin\psi + \phi \cos\psi \sin\theta \\ \dot{\phi} \cos\theta + \dot{\psi} \end{vmatrix} \ . \qquad (5.19)$$

Since ρ_0 is independent of time in (5.14) and (5.16), the kinetic energy can be expressed in a simple manner in the material and reference configurations (convective representation). We have by (5.5) and (5.16),

$$K(t) = \frac{1}{2} \int_D \rho_0(\underline{X}) |\underline{\omega}_B(t) \times \underline{X}|^2 \, d^3\underline{X} \quad . \qquad \text{(convective)} \qquad (5.20)$$

Using (5.19), the kinetic energy of the body is a function of $(\phi,\psi,\theta,\dot{\phi},\dot{\psi},\dot{\theta})$ or of $\underline{\omega}_B$. To give it a more familiar expression, introduce the following inner product on $\mathbb{R}^3$,

$$\langle \underline{a},\underline{b} \rangle : = \int_D \rho_0(\underline{X})(\underline{a} \times \underline{X}) \cdot (\underline{b} \times \underline{X}) d^3\underline{X} \quad , \qquad (5.21)$$

completely determined by the density $\rho_0(\underline{X})$ of the body. Then (5.20) becomes

$$K(\underline{\omega}_B) = \frac{1}{2} \langle \underline{\omega}_B, \underline{\omega}_B \rangle \quad . \qquad (5.22)$$

Now define the linear isomorphism $I: \mathbb{R}^3 \to \mathbb{R}^3$ by $I\underline{a} \cdot \underline{b} = \langle \underline{a},\underline{b} \rangle$ for all $\underline{a},\underline{b} \in \mathbb{R}^3$; this uniquely determines I, since both the dot product and $\langle , \rangle$ are nondegenerate bilinear forms (assuming the rigid body is not concentrated on a line). It is clear that I is symmetric with respect to the dot product and is positive. To gain a physical interpretation of I we compute its matrix. Let $(\underline{E}_1,\underline{E}_2,\underline{E}_3)$ be an orthonormal basis for material coordinates. Thus,

$$I_{ij} = (I \underline{E}_j) \cdot \underline{E}_i = \langle \underline{E}_j, \underline{E}_i \rangle = \begin{cases} -\int_D \rho_0(\underline{X}) X^i X^j d^3\underline{X} \quad , & \text{if } i \neq j \\ \int_D \rho_0(\underline{X})(|\underline{X}|^2 - (X^i)^2) d^3\underline{X} \quad , & \text{if } i = j \end{cases} \qquad (5.23)$$

which are the expressions of the matrix of the inertia tensor from classical mechanics. Thus I is the physical inertia tensor. Since it is symmetric, it can be diagonalized; the basis in which it is diagonal is a principal axis body frame and the diagonal elements I_1, I_2, I_3 are the principal moments of intertia of the rigid body. In what follows we work in a principal axis body frame (convective representation).

To define the kinetic energy (5.22) as a function on the dual Lie algebra $so(3)^* \cong \mathbb{R}^3$, we must take into account that $so(3)^*$ and $\mathbb{R}^3$ are identified by the dot product and not by the pairing $\langle , \rangle$. Consequently, the linear functional $\langle \underline{\omega}_B, \cdot \rangle$ on $so(3) \cong \mathbb{R}^3$ is identified with $I\underline{\omega}_B := \underline{m} \in so(3)^* \cong \mathbb{R}^3$ since $\underline{m} \cdot \underline{a} = \langle \underline{\omega}_B, \underline{a} \rangle$ for all $\underline{a} \in \mathbb{R}^3$. Hence (5.22) becomes, for $I = \mathrm{diag}(I_1, I_2, I_3)$,

$$K(\underline{m}) = \frac{1}{2}\, \underline{m} \cdot I^{-1}\underline{m} = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right) , \qquad \text{(body)} \qquad (5.24)$$

which represents the expression of K on $so(3)^*$. Note that $\underline{m} = I\underline{\omega}_B$ is the angular momentum in the body frame.

By the second formula in (5.19) and the definition of $\underline{m}$ for $I = \mathrm{diag}(I_1, I_2, I_3)$, the angular momentum is expressible as

$$\underline{m} = \begin{bmatrix} I_1(\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi) \\ I_2(\dot{\phi} \sin\theta \cos\psi + \dot{\theta} \sin\psi) \\ I_3(\dot{\phi} \cos\theta + \dot{\psi}) \end{bmatrix} . \qquad (5.25)$$

Eq. (5.25) expresses $\underline{m}$ in terms of coordinates on T(SO(3)). Since T(SO(3)) and T^*(SO(3)) are to be identified by the metric defined as the left translate at every point of <,>, the canonically conjugate variables $(p_\phi, p_\psi, p_\theta)$ to (ϕ,ψ,θ) are given by the Legendre transformation $p_\phi = \partial K/\partial\dot\phi$, $p_\psi = \partial K/\partial\dot\psi$, $p_\theta = \partial K/\partial\dot\theta$ of the kinetic energy on T(SO(3)) which is obtained by plugging (5.25) into (5.24). This produces the standard formulas

$$p_\phi = I_1(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)\sin\theta \sin\psi$$

$$+ I_2(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)\sin\theta \cos\psi + I_3(\dot\phi \sin\theta + \dot\psi)\cos\theta \quad ,$$

$$p_\psi = I_3(\dot\phi \cos\theta + \dot\psi) \quad , \tag{5.26}$$

$$p_\theta = I_1(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)\cos\psi - I_2(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)\sin\psi \quad ,$$

whence, by (5.25),

$$\underline{m} = \begin{bmatrix} [(p_\phi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta \cos\psi]/\sin\theta] \\ [(p_\phi - p_\psi\cos\theta)\cos\psi - p_\theta \sin\theta \sin\psi]/\sin\theta] \\ p_\psi \end{bmatrix} . \tag{5.27}$$

Consequently, by (2.24) and (5.27) the coordinate expression of the kinetic energy in the material picture becomes

$$K(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta) = \frac{1}{2}\left\{\frac{[(p_\phi - p_\psi \cos\theta)\sin\psi - p_\theta \sin\theta \cos\psi]^2}{I_1\sin^2\theta}\right.$$

$$\left. + \frac{[(p_\phi - p_\psi \cos\theta)\cos\psi - p_\theta \sin\theta \sin\psi]^2}{I_2\sin^2\theta} + \frac{p_\psi^2}{I_3}\right\} \quad , \tag{5.28}$$

or invariantly, from either (5.24) via (5.18) or , (5.28)

$$K(A,\dot{A}) = -\frac{1}{4}\,\mathrm{Tr}(IA^{-1}\dot{A}\;A^{-1}\dot{A}) \quad . \qquad \text{(material)} \qquad (5.29)$$

To find the expression of K in spatial coordinates, observe · t by (5.8), (5.17), and (5.18) we have

$$\underline{\omega}_B(t) = A(t)^{-1}\underline{\omega}_S(t) \quad ,$$

so that defining

$$I_S(t) = A(t)IA(t)^{-1} \qquad (5.30)$$

yields the expression of the momentum in the Eulerian picture

$$\underline{m}_S(t) = I_S(t)\omega_S(t) = A(t)\underline{m}(t) \quad . \qquad (5.31)$$

Therefore, by (5.24), the expression of the kinetic energy K in the spatial picture takes the form

$$\bar{K}(\underline{m}_S,\, I_S) = \frac{1}{2}\,\underline{m}_S \cdot I_S^{-1}\underline{m}_S \quad . \qquad \text{(spatial)} \qquad (5.32)$$

The potential energy U for a heavy top is determined by the height of the center of mass over a horizontal plane in the spatial coordinate systems. Let $\ell\underline{\chi}$ denote the vector determining the center of mass in the reference configuration (i.e. the body frame at t = 0), where $\underline{\chi}$ is a unit vector along the straight line segment of length ℓ connecting the fixed point with the center of mass. Thus, if

$$M = \int_{IR^3} d\mu(x)$$

is the total mass of the body, g is the gravitational acceleration, and $\underline{k}$ denotes the unit vector along the spatial Oz axis, the potential energy at time t is

$$U(t) = Mg\underline{k} \cdot A(t)\ell\underline{\chi} = Mg\ell A^{-1}\underline{k} \cdot \underline{\chi} = Mg\ell\underline{\gamma} \cdot \underline{\chi} = Mg\ell\underline{k} \cdot \underline{\lambda} \quad ,$$

where $\underline{\gamma} = A^{-1}\underline{k}$ and $\underline{\lambda} = A\underline{\chi}$. Consequently,

$$U = Mg\ell\underline{k} \cdot A\underline{\chi} \qquad \text{(Lagrangian or material)} \tag{5.33}$$

$$= Mg\ell\underline{k} \cdot \underline{\lambda} \qquad \text{(Eulerian or spatial)} \tag{5.34}$$

$$= Mg\ell\underline{\gamma} \cdot \underline{\chi} \qquad \text{(convective or body)} \tag{5.35}$$

Summarizing, we have the following expressions of the Hamiltonian in the material, body, and spatial picture:

$$H(A,\dot{A}) = -\frac{1}{4}\,\mathrm{Tr}(IA^{-1}\dot{A}A^{-1}\dot{A}) + Mg\ell\underline{k} \cdot A\underline{\chi}) \quad , \quad \text{(material)} \tag{5.36}$$

or in the chart given by the Euler angles,

$$H(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta) = \frac{1}{2}\left\{\frac{[(p_\phi - p_\psi\cos\theta)\sin\Psi + p_\theta\sin\theta\cos\psi]^2}{I_1\sin^2\theta}\right.$$

$$\left. + \frac{[(p_\phi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\Psi]^2}{I_2\sin^2\theta} + \frac{p_\psi^2}{I_3}\right\} + Mg\ell\cos\theta \quad , \tag{5.37}$$

$$H(\underline{m},\underline{\gamma}) = \frac{1}{2}\sum_{j=1}^{3}\frac{m_j^2}{I_j} + Mg\ell\gamma_3 = \frac{1}{2}\,\underline{m}\cdot I^{-1}\underline{m} + Mg\ell\gamma_3 \quad , \text{(body)} \tag{5.38}$$

$$H(\underline{m}_S, I_S, \underline{\lambda}) = \frac{1}{2}\,\underline{m}_S \cdot I_S^{-1}\underline{m}_S + Mg\ell\underline{k}\cdot\underline{\lambda} \quad . \qquad \text{(spatial)} \tag{5.39}$$

The formulas below summarize all relationships between the variables $(\underline{m},\underline{\gamma})$ in the convective picture and the variables $(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta)$ in a chart given by the Euler angles in the material picture:

$$m_1 = [(p_\phi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi]/\sin\theta = I_1(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)$$

$$m_2 = [(p_\phi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi]/\sin\theta = I_2(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)$$

$$m_3 = p_\psi = I_3(\dot\phi\cos\theta + \dot\psi)$$

$$\gamma_1 = \sin\theta\sin\psi$$

$$\gamma_2 = \sin\theta\cos\psi$$

$$\gamma_3 = \cos\theta$$

$$p_\phi = \underline{m}\cdot\underline{\gamma} = I_1(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)\sin\theta\sin\psi +$$

$$I_2(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)\sin\theta\cos\psi + I_3(\dot\phi\sin\theta + \dot\psi)\cos\theta$$

$$p_\psi = m_3 = I_3(\dot\phi\cos\theta + \dot\psi)$$

$$p_\theta = (\gamma_2 m_1 - \gamma_1 m_2)/\sqrt{1-\gamma_3^2} = I_1(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)\cos\psi$$

$$- I_2(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)\sin\psi$$

$$\dot\phi = \frac{1}{I_1} \frac{m_1\gamma_1}{1-\gamma_3^2} + \frac{m_2\gamma_2}{1-\gamma_3^2}$$

$$\dot\psi = \frac{m_3}{I_3} - \frac{m_3 m_1 \gamma_1}{I_1(1-\gamma_3^2)} - \frac{m_3 m_2 \gamma_2}{I_2(1-\gamma_3^2)}$$

$$\dot\theta = \frac{m_1\gamma_2}{I_1\sqrt{1-\gamma_3^2}} - \frac{m_2\gamma_1}{I_2\sqrt{1-\gamma_3^2}}$$

There are similar relationships between $(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta)$ and $(\underline{\omega}_S,\ I_S,\ \underline{\lambda})$ which will not be used in this paper.

Note that the Hamiltonian H in the material picture (5.36) depends on three parameters: $(\underline{k},\ I,\ Mg\ell\underline{\chi}) \in \mathbb{R}^3 \times S_2(\mathbb{R}^3) \times \mathbb{R}^3$, where $S_2(\mathbb{R}^3)$ denotes the symmetric covariant two-tensors on $\mathbb{R}^3$. Therefore, in the spirit of Section 3, we take the material phase space of the rigid body motion to be the Poisson manifold

$$T^*SO(3) \times \mathbb{R}^3 \times S_2(\mathbb{R}^3) \times \mathbb{R}^3 \quad ,$$

where the last three factors are thought of as trivial Poisson manifolds.

5.4 Heavy Top in Convective Picture

We begin by studying the invariance of H in (5.36) under the left action of the group SO(3) on $T^*SO(3) \times \mathbb{R}^3 \times S_2(\mathbb{R}^3) \times \mathbb{R}^3$. Recall the action of SO(3) on $\mathbb{R}^3$ is given by the usual left multiplication $\underline{x} \to A\underline{x}$, $A \in SO(3)$, $\underline{x} \in \mathbb{R}^3$ and on $S_2(\mathbb{R}^3)$ by $S \to ASA^{-1}$, $S \in S_2(\mathbb{R}^3)$. If $B \in SO(3)$ is time independent, we have by (5.36)

$$H(BA, B\dot{A}, B\underline{k}, I, \underline{\chi}) = -\frac{1}{4}\,\mathrm{Tr}(IA^{-1}B^{-1}B\dot{A}A^{-1}B^{-1}B\dot{A}) + Mg\ell B\underline{k}\cdot BA\underline{\chi}$$

$$= -\frac{1}{4}\,\mathrm{Tr}(IA^{-1}\dot{A}A^{-1}\dot{A}) + Mg\ell\underline{k}\cdot A\underline{\chi}$$

$$= H(A,\dot{A},\underline{k},I,\underline{\chi}) \quad ,$$

i.e., H satisfies the left version of the hypothesis in Theorems 3.2 and 3.3(i). Therefore, the motion in body coordinates takes place on $(T^*SO(3) \times \mathbb{R}^3 \times S_2(\mathbb{R}^3) \times \mathbb{R}^3)/SO(3) \cong [(so(3), \circledS\, \mathbb{R}^3) \times S^2(\mathbb{R}^3) \times \mathbb{R}^3]^*_-$ where $(\mathbb{R}^3)^*$ is identified with $\mathbb{R}^3$ via the usual dot-product and $S^2(\mathbb{R}^3)$ denotes the contravariant symmetric two-tensors on $\mathbb{R}^3$ dual to $S_2(\mathbb{R}^3)$ via the pairing: $(I,T) \in S_2(\mathbb{R}^3) \times S^2(\mathbb{R}^3) \to -\frac{1}{2}\,\mathrm{Tr}(IT) \in \mathbb{R}$ (see Theorem 3.1(i)).

Consequently, the motion of the heavy top in the convective picture is given by the Hamiltonian (5.38) with respect to the Lie-Poisson bracket

$$\{F,G\}(\underline{m},\underline{\gamma},I,\underline{\chi}) = -\underline{m}\cdot(\nabla_{\underline{m}}F \times \nabla_{\underline{m}}G) - \underline{\gamma}\cdot(\nabla_{\underline{m}}F \times \nabla_{\underline{\gamma}}G + \nabla_{\underline{\gamma}}F \times \nabla_{\underline{m}}G) \quad , \qquad (5.40)$$

where $\nabla_{\underline{m}}$ and $\nabla_{\underline{\gamma}}$ denote the gradients with respect to $\underline{m}$ and $\underline{\gamma}$, respectively.

To write the equations of motions explicitly, we first note that $\phi : so(3) \to \mathrm{End}(\mathbb{R}^3)$ is given by $\phi(\hat{\underline{y}})\underline{x} = \hat{\underline{y}}\,\underline{x} = \underline{y} \times \underline{x}$, for $\underline{x}, \underline{y} \in \mathbb{R}^3$, so that via the identifications of $so(3)^*$ with IR^3 and $(IR^3)^*$ with itself, we have

$$\phi(\hat{\underline{x}})^* \underline{\gamma} = \underline{x} \times \underline{\gamma} = \phi^*_{\underline{x}} \underline{\gamma} \quad .$$

If $F:[(so(3) \circledS IR^3) \times S^2(IR^3) \times IR^3]^*_- \to IR$, we have:

$$\frac{\delta F}{\delta \underline{m}} = (\nabla_{\underline{m}} F)^{\wedge}, \qquad \frac{\delta F}{\delta \underline{\gamma}} = \nabla_{\underline{\gamma}} F \quad , \tag{5.41}$$

so that the equations of motions are obtained by (3.12L) and Theorem 3.3(i)

$$\left.\begin{aligned} \dot{\underline{m}} &= \underline{m} \times \underline{\omega} + Mg\ell\, \underline{\gamma} \times \underline{\chi} \quad , \\ \dot{\underline{\gamma}} &= \underline{\gamma} \times \underline{\omega} \quad , \\ \dot{I} &= 0 \quad , \\ \dot{\underline{\chi}} &= 0 \quad , \end{aligned}\right\} \tag{5.42}$$

since $\nabla_{\underline{m}} H = \underline{\omega}$, $\nabla_{\underline{\gamma}} H = Mg\ell\underline{\chi}$, $\underline{\gamma} = A^{-1}\underline{k}$, and H given by (5.38), i.e., $H(\underline{m},\underline{\gamma},\underline{I},\underline{k}) = \frac{1}{2}\, \underline{m} \cdot I^{-1}\underline{m} + Mg\ell\gamma_3$.

The Casimir functions of $[(so(3) \circledS IR^3) \times S^2(IR^3) \times IR^3]^*_-$ are given by the invariant functions on $(so(3) \circledS IR^3)^*$ under the coadjoint action

$$Ad^*_{(A,u)^{-1}}(\underline{m},\underline{\gamma}) = (A\underline{m} + \underline{u} \times A\underline{\gamma},\ A\underline{\gamma}) \quad , \tag{5.43}$$

plus all the functions that depend only on $(I,\underline{\chi})$. Therefore, these Casimir functions are given by $C_1(\underline{m},|\underline{\gamma}|,\underline{I},\underline{\chi}) = a(\underline{\gamma}^2)$, $C_2(\underline{m},\underline{\gamma},I,\underline{\chi}) = b(\underline{m \cdot \gamma})$ where $a,b: IR \to IR$ are arbitrary smooth functions, plus nine other functions depending respectively on the six invariants of I and the three coordinates of $\underline{\chi}$. The generic symplectic leaf is four dimensional and equals the generic four-dimensional coadjoint orbit $Orb(\underline{m},\underline{\gamma})$ of $(so(3) \circledS IR^3)^*_-$. By (5.43), this orbit $Orb(\underline{m},\underline{\gamma})$ contains a point of the form $(\underline{m}', |\underline{\gamma}|\underline{k})$, so we may think of $Orb(\underline{m},\underline{\gamma})$ as being obtained by reducing $T^*SO(3)$ by the circle S^1 of rotations leaving the spatial Oz-axis fixed. (See the Remark following Theorem 3.1.) The equivalence $SO(3)/S^1 \cong S^2$, implies that $Orb(\underline{m},\underline{\gamma})$ is symplectically diffeomorphic to $(T^*S^2, \omega_0 - \hat{\beta})$ where $\hat{\beta}$ is the lift to T^*S^2 of the following closed two-form β on S^2: $\pi^*\beta = d\alpha$, where $\alpha(A) = (A, |\underline{\gamma}|\underline{k})$ in the right trivialization of $T^*SO(3) \cong SO(3) \times IR^3$, and $\pi: SO(3) \to S^2$ is given in terms of Euler angles by $\pi(\phi,\psi,\theta) = (\phi,\theta)$. The degenerate two-dimensional leaves are characterized by $\underline{\gamma} = 0$ and they are spheres in IR^3. In fact when $\underline{\gamma} = 0$ for fixed $(I,\underline{X})$, one obtains the Lie-Poisson manifold $so(3)^*_- = IR^3$.

5.5 The Heavy Top in Space Coordinates

To study the equations of the heavy top in the Eulerian picture, we again apply the theorems of Section 3.4. First, we have to investigate the invariance properties of H in (5.36) under the right action of SO(3) on $T^*SO(3) \times IR^3 \times S_2(IR^3) \times IR^3$. We have for any $B \in SO(3)$

$$\begin{aligned} H(AB,\dot{A}B,\ \underline{k},\ B^{-1}IB,\ B^{-1}\underline{\chi}) &= -\tfrac{1}{4}\,Tr(B^{-1}IBB^{-1}A^{-1}\dot{A}BB^{-1}A^{-1}\dot{A}B) \\ &\quad + Mg\ell\ \underline{k}\cdot ABB^{-1}\underline{\chi} \\ &= H(A,\dot{A},\ \underline{k},\ I,\ \underline{\chi}) \quad , \end{aligned}$$

i.e., H satisfies (3.43L) and thus the motion in space coordinates takes place on $SO(3)\backslash(T^*SO(3) \times IR^3 \times S_2(IR^3) \times IR^3) \cong \{[so(3) \circledS (S^2(IR^3) \times IR^3)] \times IR^3\}^*_+$ by Theorems 3.1(ii) and 3.3(ii). Under the map $(\alpha_A,\ \underline{k},\ I,\ \underline{\chi})$

$\in SO(3)^* \times \mathbb{R}^3 \times S_2(\mathbb{R}^3) \times \mathbb{R}^3 \to (T^*R_A(\alpha_A),\ \underline{k},\ AIA^{-1}, A\underline{\chi}) \in so(3)^* \times \mathbb{R}^3 \times S^2(\mathbb{R}^3) \times \mathbb{R}^3$, the Hamiltonian (5.36) is easily seen to transform into its expression in space coordinates (5.39). The Lie-Poisson bracket on $[(so(3) \circledS (S^2(\mathbb{R}^3) \times \mathbb{R}^3)) \times \mathbb{R}^3]^*_+$ is given by

$$\{F,G\}(\underline{m}_S, I_S, \underline{\lambda}, \underline{k}) = \underline{m}_S \cdot (\nabla_{\underline{m}_S} F \times \underline{\nabla}_{\underline{m}_S} G) + \mathrm{Tr}(I_S([(\underline{\nabla}_{\underline{m}_S} F)^{\wedge} \cdot \frac{\delta G}{\delta I_S}]$$

$$- [(\underline{\nabla}_{\underline{m}_S} G)^{\wedge} \cdot \frac{\delta F}{\delta I_S}])) + \underline{\lambda} \cdot (\underline{\nabla}_{\underline{m}_S} F \times \nabla_{\underline{\lambda}} G + \underline{\nabla}_{\underline{m}_S} G \times \underline{\nabla}_{\underline{\lambda}} F) \ .$$

Since

$$\nabla_{\underline{m}_S} H = I_S^{-1} \underline{m}_S = \underline{\omega}_S \quad , \quad \nabla_{\underline{\lambda}} H = Mg\ell \underline{k} \quad , \quad \text{and} \quad \frac{\delta H}{\delta I_S} = \underline{\omega}_S \times \underline{\omega}_S \quad , \qquad (5.44)$$

where $\underline{a} \otimes \underline{b}$ represents the symmetric matrix whose entries are $a_i b_j$. Therefore, by (3.12L), the equations of motion are

$$\dot{\underline{m}}_S = \underline{\nabla}_{\underline{m}_S} H_S \times \underline{m}_S + [\frac{\delta H_S}{\delta I_S},\ I_S]^{\vee} + \underline{\nabla}_{\underline{\lambda}} H_S \times \underline{\lambda} \quad ,$$

$$\dot{I}_S = [(\nabla_{\underline{m}_S} H_S)^{\wedge}, I_S] \quad ,$$

$$(5.45)$$

$$\dot{\underline{\lambda}} = \underline{\nabla}_{\underline{m}_S} H_S \times \underline{\lambda} \quad ,$$

$$\dot{\underline{k}} = 0 \quad ,$$

and where $\vee : so(3) \to \mathbb{R}^3$ is the inverse of the Lie algebra isomorphism $\hat{}$. A direct computation shows that $[\underline{\omega}_S \times \underline{\omega}_S, I_S]^\vee = \underline{m}_S \times \underline{\omega}_S$ so that the first two terms in the top equation of (5.14) cancel by (5.44). Therefore, the equations of motion of the heavy top in the Eulerian picture have the expression

$$\begin{aligned} \dot{\underline{m}}_S &= Mg\ell\underline{k} \times \underline{\lambda} \quad , \\ \dot{I}_S &= [I_S, \hat{\underline{\omega}}_S] \quad , \\ \dot{\underline{\lambda}} &= \underline{\omega}_S \times \underline{\lambda} \quad , \\ \dot{\underline{k}}_S &= 0 \quad , \end{aligned} \tag{5.46}$$

where $\underline{m}_S = I_S\underline{\omega}_S$.

The Casimir functions on the Poisson manifold $\{[so(3) \circledS (S^2(\mathbb{R}^3) \times \mathbb{R}^3)] \times \mathbb{R}^3\}^*_+$ are given by the functions invariant under the coadjoint action of the Lie group $SO(3) \circledS (S^2(\mathbb{R}^3) \times \mathbb{R}^3)$ which is given by

$$Ad^*_{(A,J,\underline{u})^{-1}}(\underline{m}_S, I_S, \underline{\lambda}) = (A\underline{m}_S, \underline{u} \times A\underline{\lambda} + [J, AI_SA^{-1}]^\vee \ , \ AI_SA^{-1}, A\underline{\lambda}) \quad . \tag{5.47}$$

Let π_1, π_2, π_3 be the three invariants of the matrix I_S. Since they are invariant under conjugation, they are invariant under the above coadjoint action. Consequently, these give Casimirs. There are in fact six in all:

$$C_1(\underline{m}_S, I_S, \underline{\lambda}, \underline{k}) = \phi_1(\pi_1) \quad ,$$

$$C_2(\underline{m}_S, I_S, \underline{\lambda}, \underline{k}) = \phi_2(\pi_2) \quad ,$$

$$C_3(\underline{m}_S, I_S, \underline{\lambda}, \underline{k}) = \phi_3(\pi_3) \quad ,$$

$$C_4(\underline{m}_S, I_S, \underline{\lambda}, \underline{k}) = \phi_4(|\underline{\lambda}|^2) \quad , \tag{5.48}$$

$$C_5(\underline{m}_S, I_S, \underline{\lambda}, \underline{k}) = \phi_5((I_S\underline{\lambda}) \cdot \underline{\lambda}) \quad ,$$

$$C_6(\underline{m}_S, I_S, \underline{\lambda}, \underline{k}) = \phi_6(|I_S\underline{\lambda}|^2) \quad ,$$

where $\phi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, 6$ are arbitrary smooth functions. To these, one has to add another three additional Casimirs corresponding to the trivial Lie-Poisson structure of $\mathbb{R}^3$, namely three arbitrary functions of each coordinate of $\underline{k}$. Note that all these nine Casimirs are "nondynamic", i.e. they do not involve $\underline{m}_S$. For the specific Hamiltonian (5.39), there is, however, an additional integral $\underline{m}_S \cdot \underline{k}$, the momentum of the top about the Oz-axis. The generic symplectic leaf of $\{[so(3) Ⓢ (S^2(\mathbb{R}^3) \times \mathbb{R}^3)] \times \mathbb{R}^3\}_+^*$ is six dimensional and by the remark following Theorem 3.1 it is symplectically diffeomorphic to $T^*SO(3)$ with the canonical symplectic structure (because $AI_SA^{-1} = I_S$, $A\underline{\lambda} = \underline{\lambda}$ has generically only A = identity as a solution, i.e., $G_a = \{e\}$ in the general theory). Now reducing at $\underline{m}_S \cdot \underline{k}$ by the corresponding circle action yields again T^*S^2 with a noncanonical symplectic structure as in the study of the heavy top equations in the convective picture.

5.6 Completely Integrable Cases

We first investigate the complete integrability of the heavy top equations in the material picture. Via the Reconstruction Method (Abraham and Marsden [1978], p. 305), it suffices to study the complete integrability of the convective (5.42) or the Eulerian (5.46) equations of motion on the generic symplectic leaf of the corresponding Poisson manifold.

(a) Euler case: free rigid body, i.e. $\underline{\chi} = 0$. We already saw that if $\underline{\chi} = 0$, the degenerate leaf in $[(so(3) \circledS \mathbb{R}^3) \times S^2(\mathbb{R}^3) \times \mathbb{R}^3]^*_-$ (convective picture) is a sphere. Since the Hamiltonian (5.38) is conserved, this makes the free rigid body equations completely integrable.

(b) Lagrange case: $I_1 = I_2$, $\underline{\chi} = (0,0,1)$, a symmetric top whose center of mass lies on the axis of symmetry. One can deal with this problem in several ways. The symmetry of the top has as a consequence the existence of a second S^1-action, namely, rotations about the symmetry axis of the body. The associated momentum map is computed to be equal to m_3, the third component of $\underline{m}$. Thus, on the generic four-dimensional leaf T^*S^2 one has the Hamiltonian (5.38) and m_3 which are easily shown to Poisson-commute under the bracket (5.40). It turns out that the equations for the Lagrange top have a second Hamiltonian structure derivable from a Kac-Moody type extension of so(3) and that its complete integrability can be shown to follow from the bi-Hamiltonian character of the equations of motion. See Ratiu and van Moerbeke [1982] and Ratiu [1982] for this approach and its generalization to n dimensions.

Next, we shall show that the Lagrange top equations are integrable by using the Eulerian picture. The main idea is the following. Since in space the equations for the moment of inertia tensor I_S and the center of mass unit vector $\underline{\lambda}$ are nontrivial, one can ask whether the Lagrange top conditions define a degenerate four-dimensional symplectic leaf. If this occurs, one has Hamiltonian equations of motion on a four-dimensional symplectic manifold on which we have the Hamiltonian in space coordinates (5.39) and the momentum about the Oz-axis $\underline{m}_S \cdot \underline{k}$ as commuting, generically independent, conserved quantities. This would then prove the complete integrability of the Lagrange top.

Thus, consider the twelve dimensional Lie-Poisson submanifold $[so(3) \circledS (S^2(\mathbb{R}^3) \times \mathbb{R}^3)]^*_+$ defined by $\underline{k} = (0,0,1)$ of $\{[so(3) \circledS (S^2(\mathbb{R}^3) \times \mathbb{R}^3)] \times \mathbb{R}^3\}^*$. Compute the isotropy subgroup under the coadjoint action (5.47) of the point $(\underline{m}_S, I_S, \underline{\lambda})$, with I_S being a diagonal matrix of the form $I_S = \operatorname{diag}(\alpha,\alpha,\gamma)$ and $\underline{\lambda} = (0,0,1)$. Using the Euler angle formula (5.2), the relation $A\underline{\lambda} = \underline{\lambda}$ implies that $\theta = 0$, i.e., that A is a rotation by the angle $\phi + \psi$ in the plane defined by $\theta = 0$. But then automatically $AI_SA^{-1} = I_S$ since the diagonal matrix I_S has the (1,1) and (2,2)-entries equal. Therefore, the equations that define the isotropy subgroup of $(\underline{m}_S, I_S, \underline{\lambda})$ are

$$\left.\begin{aligned} &\theta = 0 \quad , \\ &m_1\cos(\phi + \psi) + m_2\sin(\phi + \psi) + u_1 + J_{23}(A - C) = 0 \quad , \\ &-m_1\cos(\phi + \psi) + m_2\sin(\phi + \psi) - u_1 + J_{13}(C - A) = 0 \quad , \end{aligned}\right\} \tag{5.49}$$

where

$$(A,\ J,\ \underline{u}) \in SO(3) \circledS (S^2(\mathbb{R}^3) \times \mathbb{R}^3)$$

and A is given by (5.2). The last two equations define a map $\mathbb{R}^6 \to \mathbb{R}^2$ whose Jacobian matrix has rank 2. Therefore, by the implicit function theorem, the set of $(\phi, \psi, u_1, u_2, J_{13}, J_{23}) \in \mathbb{R}^6$ satisfying the last two equations is a four-dimensional submanifold of $\mathbb{R}^6$. Thus, for $(A,J,\underline{u}) \in SO(3) \times S^2(\mathbb{R}^3) \times \mathbb{R}^3$ fixing $(\underline{m}_S, I_S, \Delta)$, the only other free parameters left are: $u_3, J_{11}, J_{22}, J_{33}, J_{12}$. Hence, the dimension of the isotropy group is 8, i.e., the orbit is four dimensional. This proves the Lagrange top is integrable.

To obtain the secondary Casimirs (Casimirs on the degenerate orbits), observe that by (5.47), the quantities

$$\underline{m}_S \cdot \underline{\lambda} \quad \text{and} \quad \underline{m}_S \cdot I_S\underline{\lambda} \tag{5.50}$$

would be invariant under the coadjoint action, if

$$[J, AI_SA^{-1}]^{\vee} \cdot A\underline{\lambda} = 0 \quad , \tag{5.51}$$

$$[J, AI_SA^{-1}]^{\vee} \cdot AI_S\underline{\lambda} = 0 \tag{5.52}$$

$$(\underline{u} \times A\underline{\lambda}) \cdot AI_S\underline{\lambda} = 0 \tag{5.53}$$

for all $J \in S^2(\mathbb{R}^3)$, $\underline{u} \in \mathbb{R}^3$, and $A \in SO(3)$. The first relation (5.50) can be written equivalently as

$$[\tilde{J}, I_S]^{\vee} \cdot \underline{\lambda} = 0$$

where $\tilde{J} = A^{-1}JA$. The computation leading to (5.49) shows that if $I_S = \mathrm{diag}(I_1, I_1, I_3)$, then $[\tilde{J}, I_S]^{\vee}$ has zero third component. Consequently, if in addition $\underline{\lambda} = (0,0,1)$, then (5.51) holds. One proceeds similarly to prove (5.52). Finally, (5.53) is equivalent to

$$(\underline{\lambda} \times I_S\underline{\lambda}) \cdot A^{-1}\underline{u} = 0 \quad ,$$

which clearly holds for our choice of I_S and $\underline{\lambda}$ since then already $\underline{\lambda} \times I_S\underline{\lambda} = 0$. To get from (5.50) quantities in the convective picture, recall that $\underline{m}_S = A\underline{m}$, $\underline{\lambda} = A\underline{\chi}$, so that

$$\underline{m}_S \cdot \underline{\lambda} = \underline{m} \cdot \underline{\chi} = m_3 \quad ,$$

in the case of the Lagrange top. Also $\underline{m}_S \cdot I_S\underline{\lambda} = I_3m_3$, since $I_S = AIA^{-1}$. Thus m_3, <u>the Lagrange integral, is a "secondary" Casimir in the spatial picture</u>.

(c) <u>Kovalevski Case</u>: $I_1 = I_2 = 2I_3$, $\underline{\chi} = (1,0,0)$, i.e., the top is symmetric with a very special shape along the third principal axis and the center of mass lies in the plane of the two principal moments of inertia. It is an outstanding current problem to explain the complete integrability of this top by symplectic means. The usual avenues of finding a second Hamiltonian structure using a Kac-Moody extension do not work here. In fact, the more general question of finding a second nonlinear Hamiltonian structure is still open. Let us apply the philosophy of the previous example to this case, i.e., let us compute the isotropy group of the coadjoint action of $SO(3)$ s $(S^2(\mathbb{R}^3) \times \mathbb{R}^3)$ at a point $(\underline{m}_S, I_S, \underline{\lambda})$ with $I_S = \text{diag}(2\alpha, 2\alpha, \alpha)$, $\underline{\lambda} = (1,0,0)$. A direct computation shows that the elements of this isotropy group are all of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} , \quad \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & 0 \\ J_{13} & 0 & J_{33} \end{bmatrix} , \quad \begin{bmatrix} u_1 \\ -(1-\varepsilon)m_{S3}/2 \\ \frac{\alpha}{2} J_{13} + \frac{(1-\varepsilon)}{2} m_{S2} \end{bmatrix} \qquad (5.54)$$

for $\varepsilon = \pm 1$. This is clearly six-dimensional, coordinatized by $(J_{11}, J_{12}, J_{13}, J_{22}, J_{33}, u_1)$. Therefore, the Kovalevski top in space coordinates has as phase space one of the generic leaves of our Lie-Poisson manifold and the trick that works in the Lagrange top case, fails here. We recall, however, that the Kovalevski top _is_ completely integrable, the second integral on a generic leaf T^*S^2 in the convective picture being given by

$$\left|(m_2 + im_1)^2 - 4Mg\ell I_3 i(\gamma_2 + i\gamma_1)\right|^2 \quad . \tag{5.55}$$

Kovalevski has shown that the three cases we mentioned above are the only completely integrable cases of the heavy top equations admitting polynomial integrals of motion. Ziglin [1981], [1983] has extended this result by showing that these are the only completely integrable cases admitting meromorphic integrals.

The nature of the Kovalevski integral (5.55) remains to this day one of the outstanding problems in the theory of completely integrable systems.

6. IDEAL COMPRESSIBLE ADIABATIC FLUIDS

The results of Sections 3 and 4 are now applied to the equations of motion of an ideal, compressible, adiabatic fluid in a fixed region $D \subset \mathbb{R}^3$ with smooth boundary ∂D. Starting with this example, all conventions are the same as in the main body of the text in Section 3, namely group representations are by action on the right.

6.1 The Material Phase Space

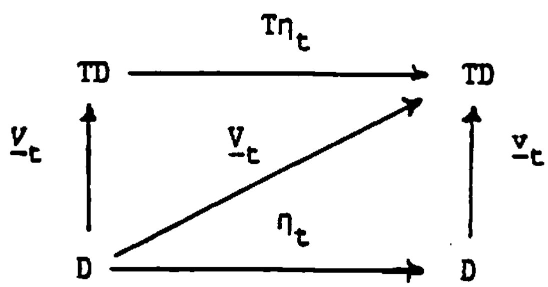
Let D be a compact region in $\mathbb{R}^3$ with smooth boundary ∂D. In accordance with Section 4, let C be the space of configurations of the fluid, i.e., the space of orientation preserving smooth embeddings of D into $\mathbb{R}^3$. Since we assume the boundaries are fixed, the images of these embeddings are all submanifolds of D. Let us make the simplifying physical assumption that cavitation and infinite density are excluded. Thus, $C \subset \mathrm{Diff}(D)$, the group of diffeomorphisms of D.

As in Section 4, we shall denote material points by capital letters $\underline{X}$ and spatial points by lower case letters $\underline{x}$. Given the mass density $\rho_0(\underline{X})$ and specific entropy $\sigma_0(\underline{X})$ of the fluid in the reference configuration, both functions of $\underline{X}$, denoting by $J_\eta(\underline{X})$ the Jacobian determinant $d^3\underline{x}/d^3\underline{X}$ of the motion η_t at $\underline{X}$, we shall see in this section that the mass density $\rho_t = \rho(\cdot,t)$ and specific entropy $\sigma_t = \sigma(\cdot,t)$ satisfy [cf. Holm [1986] Eq. (A.1)]

$$\rho(\underline{x},t)J_{\eta_t}(\underline{X}) = \rho_0(\underline{X}) \quad \text{and} \quad \sigma(\underline{x},t) = \sigma_0(\underline{X}) \quad .$$

Consequently, the Eulerian mass density ρ and entropy σ are completely determined by the motion, given initial conditions ρ_0 and σ_0, respectively. Hence, the configuration space of compressible fluid flow with a given mass and entropy density in the reference configuration is the group of diffeomorphisms Diff(D) of D. Consequently, the phase space is the cotangent bundle $T^*(\text{Diff}(D))$.

For later reference, we summarize here the relationship between the Lagrangian ($\underline{V}_t$), Eulerian ($\underline{v}_t$) and convective ($\underline{\mathcal{V}}_t$) velocities for a fluid motion in a domain with fixed boundary in the following diagram:



The vertical arrows in this diagram are vector fields, whereas $\underline{V}_t$ is a vector field over η_t, i.e.,

$$\underline{V}_t(\underline{X}) \in T\eta_t(\underline{X})D \quad .$$

Next we turn to the study of the configuration space $\mathcal{C}$ = Diff(D)).

6.2 The Lie Group Diff(D) and Its Lie Algebra $\mathcal{X}(D)$

There are two ways in which Diff(D) can be made into a Lie group. The most obvious one is to consider only C^∞ diffeomorphisms. It turns out that in this way Diff(D) becomes a Fréchet manifold, i.e., its model space is a locally convex, Hausdorff, complete vector space. Composition of diffeomorphisms and taking the inverse are smooth operations, so Diff(D) becomes a Fréchet Lie group (see, e.g., Ebin and Marsden [1970]). The main drawback of this approach is that in Fréchet spaces special hypotheses are needed for inverse function theorems to hold; the same is true of existence and uniqueness theorems for integral curves of differential equations.

The second approach is to use diffeomorphisms of Sobolev or Hölder class. It turns out that if the Sobolev class $W^{s,p}$ or Hölder class $C^{k+\alpha}$ is high enough so that such diffeomorphisms are at least C^1, then they form a C^∞ Banach manifold and one has the usual existence and uniqueness theorems for solutions of differential equations. Unfortunately only right translation is smooth, whereas left translation and taking inverses are only continuous. Thus $W^{s,p}$ Diff(D) (or $C^{k+\alpha}$-Diff(D)) is now a topological group, which is a Banach manifold on which right translation is smooth. One may now make Diff(D) into a "Lie" group by taking the inverse limit as the differentiability class goes to ∞ (Ebin and Marsden [1970], Omori [1974]).

We next determine the tangent space $T_\eta(\mathrm{Diff}(D))$ of Diff(D) at η. Let $t \to \eta_t$ be a smooth curve with $\eta_0 = \eta$. Then $(d\eta_t/dt)|_{t=0}$ is, by definition, a tangent vector at η to Diff(D). If $\underline{X} \in D$, then $t \to \eta_t(\underline{X})$ is a smooth curve in D through $\eta(\underline{X})$ and thus

$$\left.\frac{d\eta_t(\underline{X})}{dt}\right|_{t=0} \in T_{\eta(\underline{X})}D \quad ,$$

where $T_{\eta(\underline{X})}D$ is the tangent space to D at $\eta(\underline{X})$. Consequently we have a map $\underline{X} \in D \to (d\eta_t(\underline{X})/dt)|_{t=0} \in T_{\eta(\underline{X})}D$, i.e., $(d\eta_t/dt)|_{t=0}$ is a vector field over η. Thus,

$$T_\eta(\mathrm{Diff}(D)) = (\underline{V}_\eta : D \to TD \,|\, \underline{V}_\eta(\underline{X}) \in T_{\eta(\underline{X})}D) \quad . \tag{6.1}$$

In coordinates, if $\underline{x} = \eta(\underline{X})$, then $\underline{V}_\eta(\underline{X}) = V^i(\underline{X})(\partial/\partial x^i)$.

In particular, if e denotes the identity map of D, then $T_e(\mathrm{Diff}(D)) = \mathcal{X}(D)$, the Lie algebra of vector fields on D. One computes that the Lie algebra bracket of $\mathcal{X}(D)$ is minus the usual Lie bracket of vector fields, i.e., $[U,V]^i = V^j(\partial U^i/\partial X^j) - U^j(\partial V^i/\partial X^j)$. Thus, <u>the Lie algebra of</u> Diff(D) <u>may be identified with</u> $\mathcal{X}(D)$, with the negative of the usual Lie algebra structure.

To determine the dual of $\mathcal{X}(D)$ and the cotangent bundle of Diff(D), we take a geometric point of view. Instead of considering the functional analytic dual of all linear continuous functionals on $\mathcal{X}(D)$, we will be content to find another vector space $\mathcal{X}(D)^*$ and a weakly nondegenerate pairing

$$\langle \; , \; \rangle : \mathcal{X}(D)^* \times \mathcal{X}(D) \to \mathbb{R} \quad ;$$

this means that $\langle \; , \; \rangle$ is a bilinear mapping such that if $(\underline{M},\underline{V}) = 0$ for all $\underline{V} \in \mathcal{X}(D)$, then $\underline{M} = 0$. Clearly $\mathcal{X}(D)^*$ is a subspace of the functional analytic dual. With this definition, it is easy to see that $\mathcal{X}(D)^*$ <u>consists of all</u> <u>one-form densities on</u> D, i.e.,

$$\mathcal{X}(D)^* = \Lambda^1(D) \times |\Lambda^3(D)| \quad . \tag{6.2}$$

The notation in (6.2) is the standard one: $\Lambda^i(D)$ denotes the set of all exterior i-forms on D and $|\Lambda^3(D)|$ denotes the densities on D. Thus, a one-form density is of the form $\underline{a}d^3\underline{X}$ with $\underline{a}$ a one-form on D, so locally it is $(a_i(\underline{X})dX^i)d^3\underline{X}$. The pairing $\langle\ ,\ \rangle$ between $\mathcal{X}(D)^*$ and $\mathcal{X}(D)$ is

$$\langle \underline{a}d^3\underline{X}, \underline{V}\rangle = \int_D \underline{a}(\underline{V})\cdot\underline{X}\ d^3\underline{X}$$

or in local coordinates,

$$\int_D a_i(\underline{X})V^i(\underline{X})d^3\underline{X} \quad .$$

Finally, in view of (6.2), $T^*(\mathrm{Diff}(D))$consists of all one-form densities over η, i.e.,

$$T^*_\eta(\mathrm{Diff}(D)) = (\underline{a}_\eta : D \to T^*D \otimes |\Lambda^3(D)| \ | \underline{a}_\eta(\underline{X}) \in T^*_{\eta(\underline{X})}D \otimes |\Lambda^3_{\underline{X}}(D)|) \quad . \qquad (6.3)$$

This means that $\underline{a}_\eta = \underline{\xi}_\eta d^3\underline{X}$, where $\underline{\xi}_\eta$ is a one-form over η on D, i.e., $\xi_\eta(X) \in T^*_\eta(\underline{X})^D$. Locally, $\underline{a}_\eta = (\xi_i(\underline{X})dx^i)d^3X$, where $(x^i) = \underline{X} = \eta(\underline{X})$ and $\xi_\eta(\underline{X}) = \underline{\xi}_i(\underline{X})dx^i$. We shall denote the action of one-forms $\underline{\xi}$ over η on vector fields $\underline{V}_\eta$ over η by $\xi_\eta(\underline{V}_\eta)$; the result is a function of $\underline{X}$ which locally equals $\xi_i V^i$. The pairing $\langle\ ,\ \rangle$ between $T^*_\eta(\mathrm{Diff}(D))$ and $T_\eta(\mathrm{Diff}(D))$ is given by

$$\langle \underline{a}_\eta, \underline{V}_\eta\rangle = \int_D \underline{\xi}_\eta \underline{V}_\eta(\underline{X})d^3\underline{X} \quad , \quad \text{where} \quad \underline{a}_\eta = \underline{\xi}^3_\eta d\ \underline{X} \quad ;$$

locally this has the expression $\int_D \xi_i(\underline{X})\underline{V}^i(\underline{X})d^3\underline{X}$.

Left and right translations are defined by the composition of maps,

$$L_\eta : \mathrm{Diff}(D) \to \mathrm{Diff}(D) \quad , \quad L_\eta(\phi) = \eta\circ\phi \quad ,$$

$$R_\eta : \mathrm{Diff}(D) \to \mathrm{Diff}(D) \quad , \quad R_\eta(\phi) = \phi\circ\eta \quad ,$$

for η and $\phi \in \text{Diff}(D)$. Both are diffeomorphisms of the Lie group Diff(D). It is easy to see that their derivatives have the following expressions:

$$T_\phi L_\eta : T_\phi(\text{Diff}(D)) \to T_{\eta\circ\phi}(\text{Diff}(D)) \quad ; \quad T_\phi L_\eta(\underline{V}_\phi) = T\circ\underline{V}_\phi \tag{6.4}$$

and

$$T_\phi R_\eta : T_\phi(\text{Diff}(D)) \to T_{\phi\circ\eta}(\text{Diff}(D)) \quad ; \quad T_\phi R_\eta(\underline{V}_\phi) = T_{\phi}\circ\eta \quad , \tag{6.5}$$

for $\underline{V}_\phi \in T_\phi(\text{Diff}(D))$. The physical interpretation of these formulas is the following. Think of ϕ as a relabelling or rearrangement of the particles in D and of η as a fluid motion. Then (6.6) says that the material derivative of the motion η followed by the relabelling ϕ equals $T\eta\circ\underline{V}_\phi$. In local coordinates, if $\phi(\underline{X}) = \underline{Y}$ and $\eta(\underline{Y}) = \underline{y}$, then $\underline{V}_\phi(\underline{X}) = V^i(\underline{X})(\partial/\partial Y^i)$ and

$$(T\eta\circ\underline{V}_\phi)^i(\underline{X}) = \frac{\partial x^i}{\partial y^j}(\underline{Y})V^j(\underline{X})\frac{\partial}{\partial y^i} \quad . \tag{6.6}$$

On the other hand, (6.5) says that the material derivative of the relabelling ϕ followed by the motion η equals $\underline{V}_\phi\circ\eta$. In local coordinates, if

$$\eta(\underline{X}) = \underline{x} \quad \text{and} \quad \phi(\underline{x}) = y \quad , \quad \text{then} \quad \underline{V}_\phi(\underline{X}) = V^i(\underline{X})(\partial/\partial y^i) \quad ,$$

and

$$(\underline{V}_\phi\circ\eta)^i(\underline{X}) = (V^i\circ\eta)(\underline{X})(\partial/\partial y^i) \quad . \tag{6.7}$$

Simply put, left translation by η transforms $V_\phi(\underline{X})$, a vector at $\phi(\underline{X})$ to a vector at $\eta(\phi(\underline{X}))$, whereas right translation merely changes the argument from $\underline{X}$ to $\eta(\underline{X})$.

By (6.7), the derivative of right translation is again right translation, so R_η is C^∞. However, if η and ϕ are diffeomorphisms of a given finite Sobolev class, $T\eta$ loses one derivative. (This is basically the reason why left translation is only continuous in $W^{s,p}$-Diff(D). In C^∞-Diff(D) however with differentiability suitably interpreted, left translation is C^∞.)

As an application, note that the material velocity $\underline{V}_t$ is the right translate of the spatial velocity $\underline{v}_t$ and the left translate of the convective velocity $\underline{\mathcal{V}}_t$.

If $\underline{V} \in \mathcal{X}(D)$, a diffeomorphism $\eta \in$ Diff(D) acts on $\underline{V}$ by the adjoint action, the analogue of conjugation for matrices. The definition combined with (6.4) and (6.5) gives

$$\mathrm{Ad}_\eta \underline{V} := T_e(L_\eta \circ R_{\eta^{-1}})\underline{V} = T_{\eta^{-1}} L_\eta (T_e R_{\eta^{-1}}(\underline{V}))$$

$$= T\eta \circ \gamma \circ \eta^{-1} = \eta_* \underline{V} \quad ,$$

i.e., the adjoint action of η on $\underline{V}$ is the push-forward of vector fields:

$$\mathrm{Ad}_\eta \underline{V} = \eta_* \underline{V} \quad . \tag{6.8}$$

For example, [cf. Eq. (4.7)]

$$\underline{v}_t' = \mathrm{Ad}_{\eta_t} \underline{\mathcal{V}}_t \quad ,$$

which is similar to the formula which relates angular velocities $\hat{\underline{\omega}}_B$ and $\hat{\underline{\omega}}_S$ in the previous section. Finally, we compute the coadjoint action $Ad^*_{\eta^{-1}}\underline{a}$ of η on $\underline{a} \in \mathcal{X}(D)^*$. By the change of variables formula, we have

$$(Ad^*_{\eta^{-1}}\, \underline{a}\cdot\underline{V}) := (\underline{a}, Ad_{\eta^{-1}}\, \underline{V}) = \int_B \underline{a}\cdot\eta^*\underline{V} = \int_B \eta_*\underline{a}\cdot\underline{V} \quad ;$$

here $\underline{a}\cdot\underline{V}$ in the integrand signifies the pairing between one-form densities and vector fields so that $\underline{a}\cdot\underline{V}$ is a density on D. Thus

$$Ad^*_{\eta^{-1}}\underline{a} = \eta_*\underline{a} \quad ; \tag{6.9}$$

$\eta_*\underline{a}$ is the push-forward of the one-form density $\underline{a}$; the push-forward operates separately on the one-form and the density.

6.3 Equations of Motion

We review the derivation of the equations of motion in Eulerian coordinates from the principles of conservation of mass, entropy, and momentum. Conservation of energy will follow by imposing the adiabatic equation of state.

(a) The principle of conservation of mass stipulates that mass can be neither created or destroyed, i.e.,

$$\int_{\eta_t(W)} \rho_t(\underline{x})d^3\underline{x} = \int_W \rho_0(\underline{X})d^3\underline{X} \quad ,$$

for all compact W D with nonempty interior having smooth boundary. Changing variables, this becomes

$$\eta_t^*(\rho_t(\underline{x})d^3\underline{x}) = \rho_0(\underline{X})d^3\underline{X} \quad \text{or} \quad (\eta_t^*\rho_t)J_{\eta_t} = \rho_0 \quad , \tag{6.10}$$

where $J_{\eta_t} = |d\underline{x}/d\underline{X}|$ is the Jacobian determinant of η_t, and η_t^* is pull-back of forms or functions, as the case may be. Using the relation between Lie derivatives and flows show that (6.10) is equivalent to the continuity equation

$$\frac{\partial \rho_t}{\partial t} + \mathrm{div}(\rho_t \underline{v}) = 0 \quad . \tag{6.11}$$

(b) By the <u>principle of conservation of entropy</u> (no exchange of heat across flow lines), the heat content of the fluid cannot be altered, i.e.,

$$\int_{\eta_t(W)} \sigma_t(\underline{x})\rho_t(\underline{x})d^3\underline{x} = \int_W \sigma_0(\underline{X})\rho_0(\underline{X})d^3\underline{X} \quad ,$$

for all compact W with nonempty interior having smooth boundary. By a change of variables this becomes

$$\eta_t^*(\sigma_t(\underline{X})\rho_t(\underline{x})d^3\underline{x}) = \sigma_0(\underline{X})\rho_0(\underline{X})d^3\underline{X}$$

and by (6.10) one finds

$$\eta_t^*(\sigma_t(\underline{X})) = \sigma_0(\underline{X}) \quad , \quad \text{or} \quad \frac{\partial \sigma_t}{\partial t} + \underline{v}\cdot\underline{\nabla}\sigma_t = 0 \quad . \tag{6.12}$$

(c) <u>Balance of momentum</u> is Newton's second law: the rate of change of momentum of a portion of the fluid equals the total force applied to it. Since we assume that no external forces are present, the only forces acting on the fluid are forces of normal stress. The assumption of an <u>ideal fluid</u> means that the force of stress per unit area exerted across a surface element at $\underline{x}$, with outward unit normal $\hat{\underline{n}}$ at time t, is $-p(x,t)\hat{\underline{n}}$ for some function $p(\underline{x},t)$, called the pressure. With this hypothesis, the balance of momentum becomes Euler's equation of motion,

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\frac{1}{\rho}\underline{\nabla}p \quad , \tag{6.13}$$

with the boundary condition : $\underline{v}$ tangent to ∂D (no friction exists between fluid and boundary for an ideal fluid). The initial condition is of the form $\underline{v}(\underline{x},0) = \underline{v}_0(\underline{x})$ in D for a given vector field $\underline{v}_0$ defined on D.

The proof of conservation of energy is standard. The kinetic energy of the fluid is

$$\frac{1}{2}\int_D \rho(\underline{x})\ |\underline{v}(\underline{x})|^2\ d^3\underline{x} \quad .$$

The internal energy of the fluid is

$$\int_D \rho(\underline{x})e(\rho(\underline{x}),\sigma(\underline{x}))d^3\underline{x} \quad ,$$

with the equation of state $p(\underline{x}) = \rho(\underline{x})^2(\partial e/\partial\rho)(\underline{x})$ satisfying $\partial e/\partial\rho > 0$. (Also, the square of the sound speed $\partial p/\partial\rho = c_s^2$ should be positive.) In the next computation the following two vector identities are needed, where $\underline{\omega}$ = curl $\underline{v}$ is the vorticity:

$$(\underline{v}\cdot\underline{\nabla})\underline{v} = \underline{\nabla}(|\underline{v}|^2/2) + \underline{\omega}\times\underline{v} \quad ,$$

and

$$\underline{\nabla}(e + \rho\partial e/\partial\sigma) = \rho^{-1}\,\underline{\nabla}p + (\partial e/\partial\sigma)\underline{\nabla}\sigma \quad .$$

We have by (6.11), (6.12), and (6.13)

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\,|\underline{v}|^2 + \rho e(\rho,\sigma)\right] = -\,\mathrm{div}\left[\rho\underline{v}\left(\frac{1}{2}\,|\underline{v}|^2 + e(\rho,\sigma) + \rho\,\frac{\partial e}{\partial \rho}\right)\right]$$

$$-\,\rho\underline{v}\cdot\left[(\underline{v}\cdot\underline{\nabla})\underline{v} + \frac{1}{\rho}\,\underline{\nabla}p + \frac{\partial e}{\partial \sigma}\,\underline{\nabla}\sigma\right]$$

$$= -\,\mathrm{div}\left[\rho\underline{v}\left(\frac{1}{2}\,|\underline{v}|^2 + e(\rho,\sigma) + \rho\,\frac{\partial e}{\partial \rho}\right)\right] \quad .$$

Consequently, the total energy

$$H(\underline{v},\rho,\sigma) = \int_D \frac{1}{2}\,\rho(\underline{x})\left[|\underline{v}|^2 + e(\rho(\underline{x}),\ \sigma(\underline{x}))\right]\,d^3\underline{x} \quad , \tag{6.14}$$

which represents the Hamiltonian of the system, is conserved.

The physical problem to be solved now consists of the continuity equation (6.11), entropy convection (6.12), and Euler's equations (6.13), i.e.,

$$\begin{aligned}
&\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\,\frac{1}{\rho}\,\underline{\nabla}p \quad ,\\
&\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\underline{v}) = 0 \quad ,\\
&\frac{\partial \sigma}{\partial t} + \underline{v}\cdot\underline{\nabla}\sigma = 0 \quad ,
\end{aligned} \tag{6.15}$$

where

$$p = \rho^2\partial e/\partial\rho \quad , \quad e \text{ the internal energy density} \quad , \tag{6.16}$$

with boundary condition

$$v(\underline{x},t) \in T_{\underline{x}}\partial D \text{ if } \underline{x} \in \partial D \tag{6.17}$$

and initial conditions

$$v(\underline{x},0) = \underline{v}_0(\underline{x}) \quad , \quad \rho(\underline{x},0) = \rho_0(\underline{x}) \quad , \quad \sigma(\underline{x},0) = \sigma_0(\underline{x}) \quad , \tag{6.18}$$

where $\underline{v}_0$ is a given vector field on D, and ρ_0 and σ_0 are the material mass density and specific entropy of the fluid, respectively.

We also mention that the condition $\partial p/\partial\rho > 0$ is needed in the proof of local existence and uniqueness of the system (6.15) - (6.18) (see Majda [1985]).

6.4 The Hamiltonian in the Lagrangian and Convective Representations

The system (6.15) - (6.18) just described does not have $T^*(\text{Diff}(D))$ as its phase space. To describe the Hamiltonian dynamics on $T^*(\text{Diff}(D))$, the total energy (6.14) must be expressed on $T^*(\text{Diff}(D))$, i.e., in the material representation.

We start with the potential energy. Perform the change of variables $\underline{x} = \eta_t(X)$ in the potential energy and use (6.10) and (6.12) to get

$$\int_D \rho_t(\underline{x})e(\rho_t(\underline{x}),\sigma_t(\underline{x}))d^3\underline{x} = \int_D \rho_0(\underline{X})e(\rho_0(\underline{X})J_{\eta_t}^{-1}(\underline{X})),\sigma_0(\underline{X}))d^3\underline{X} \quad . \tag{6.19}$$

The right-hand side is a function of η_t and, hence, defined on Diff(D), so that by lifting we get a function on $T^*(\text{Diff}(D))$.

To express the kinetic energy on the cotangent bundle of Diff(D), we first need its expression in terms of the material velocity. This is accomplished by performing the same change of variables $\underline{x} = \eta_t(\underline{X})$. We have by (6.10) and $\underline{v}_t \circ \eta_t = \underline{V}_t$,

$$\frac{1}{2}\int_D \rho_t(\underline{x})\ |\underline{v}_t(x)|^2\ d^3\underline{x} = \frac{1}{2}\int_D \rho_0(\underline{X})\ |\underline{V}_t(\underline{X})|^2\ d^3\underline{X} \quad . \tag{6.20}$$

But

$$\underline{V}_t \in T_{\eta_t}(\mathrm{Diff}(D)) \quad ,$$

so that (6.20) is the expression of the kinetic energy on the tangent bundle. Define

$$\langle\langle \underline{V}_\eta, \underline{W}_\eta \rangle\rangle = \int_D \rho_0(\underline{X}) \underline{V}_\eta(\underline{X}) \cdot \underline{W}_\eta(\underline{X}) d^3\underline{X} \quad , \tag{6.21}$$

for $\underline{V}_\eta, \underline{W}_\eta \in T_\eta(\mathrm{Diff}(D))$, where the dot in the integrand means the metric on D (in our case, just the usual dot-product in $\mathbb{R}^3$). It is easily seen that (6.21) defines a weak Riemannian metric on Diff(D) whose kinetic energy is (6.20).

In finite dimensions, a metric on a manifold induces a bundle metric on the cotangent bundle, as we have seen in the case of the heavy top in Section 5. In infinite dimensions, as in the present case, this bundle metric is not guaranteed by general theory, so in examples it must be constructed explicitly. Let $\underline{\alpha}_\eta$ and $\underline{\beta}_\eta \in T^*_\eta(\mathrm{Diff}(D))$, i.e.,

$$\underline{\alpha} = \xi_\eta d^3\underline{X} \quad , \quad \underline{\beta}_\eta = \zeta_\eta d^3\underline{X} \quad ,$$

where ξ_η and ζ_η are one-forms over η. Consequently, $\underline{\alpha}_\eta/(\rho_0\, d^3\underline{X}) = \xi_\eta/\rho_0)$ and $\underline{\beta}_\eta/(\rho_0 d^3\underline{X}) = \zeta_\eta/\rho_0$ are one-forms over η, so evaluated at $\underline{X}$ they are elements of $T^*_{\eta(\underline{X})}D$. Now D is a finite dimensional Riemannian manifold (with the Euclidean metric in our case). Thus, to every one-form at $\eta(\underline{X})$ the metric associates a unique vector at $\eta(\underline{X})$. Explicitly, if $\underline{u}_{\underline{x}} \in T_{\underline{x}}D$, the one-form $\underline{u}^b \in T^*D$ is defined by $\underline{u}^b_{\underline{x}}(\underline{w}_{\underline{x}}) = \underline{u}_{\underline{x}}\cdot\underline{w}_{\underline{x}}$ for all $\underline{w} \in T_{\underline{x}}D$. In this way, the <u>index lowering action</u> $b : D \rightarrow T^*D$ is a bundle isomorphism. The inverse of b is denoted by $\# : T^*D \rightarrow TD$ and is called the <u>index raising action</u>. In coordinates, if $g = (g_{ij})$ is the metric and (g^{ij}) is the inverse matrix of (g_{ij}), we have for $\underline{u} = u^i(\partial/\partial x^i)$, $\underline{\alpha} = \alpha_i dx^i$,

$$\underline{u}^b = g_{ij}u^j dx^i \quad , \quad \underline{\alpha}^\# = g^{ij}\alpha_j(\partial/\partial x^i) \quad .$$

Now define the bundle metric on $T^*(\mathrm{Diff}(D))$ by

$$(\underline{\alpha}_\eta,\underline{\beta}_\eta) = \int_D \rho_0(\underline{X})\underline{V}_\eta(\underline{X})\cdot\underline{W}_\eta(\underline{X})d^3\underline{X} \quad , \tag{6.22}$$

for $\underline{V}_\eta = (\underline{\alpha}_\eta/\rho_0 d^3\underline{X})^\#$ and $\underline{W}_\eta = (\underline{\beta}_\eta/\rho_0 d^3X)^\# \in T_\eta(\mathrm{Diff}(D))$. Denote by $||\cdot||$ the bundle norm defined by the metric (6.22) and let

$$\underline{M}_\eta = \rho_0\underline{V}^b_\eta d^3\underline{X} \in T^*_\eta(\mathrm{Diff}(D)) \tag{6.23}$$

be the <u>material momentum density</u> of the fluid. With this notation, (6.20) becomes $||\underline{M}_\eta||^2/2$ and so by (6.19) the expression for the Hamiltonian on $T^*(\mathrm{Diff}(D))$ is

$$H(\underline{M}_\eta) = \frac{1}{2}\,||\underline{M}_\eta||^2 + \int_D \rho_0(\underline{X})e(\rho_0(\underline{X})J^{-1}_\eta(\underline{X}),\sigma_0(\underline{X}))d^3\underline{X} \quad .$$

Define the function $E(\eta, \rho_0, \sigma_0)$ by

$$E(\eta, \rho_0, \sigma_0) = e(\rho_0 J_\eta^{-1}, \sigma_0) \quad ,$$

and call it the Lagrangian internal energy density. Then the above formula for the Hamiltonian becomes

$$H(M_\eta) = \frac{1}{2} ||M_\eta||^2 + \int_D \rho_0(\underline{X})\; E\,(\eta, \rho_0, \sigma_0)(\underline{X}) d^3\underline{X} \quad . \tag{6.24}$$

Let us analyze carefully the parameters on which the Hamiltonian (6.24) depends. The mass density ρ_0 and specific entropy σ_0 are readily apparent. But, in addition, (6.24) depends on the bundle metric (6.22), which in turn is uniquely determined by ρ_0 and the metric g on D (the dot product in our case). Therefore, H in (6.24) depends in addition on the parameters $(\rho_0, \sigma_0, g) \in \mathcal{F}(D)^* \times \mathcal{F}(D) \times S_2(D)$, where $\mathcal{F}(D)$ denotes the vector space of smooth functions on D, $S_2(D)$ the covariant symmetric two-tensors on D, and $\mathcal{F}(D)^*$ is the space of smooth densities $\lambda(\underline{X})\, d^3\underline{X}$ in weak nondegenerate pairing with $\mathcal{F}(D)$ via the L^2-product on D.

Next, let us investigate the expression for the Hamiltonian in convective coordinates. We start with the potential energy in (6.24). The only expression involving material quantities is J_η. To transform it into a convective quantity, denote the metric on D in the reference configuration by G (it is the dot product in our case), and its volume element by $\mu(G)$, i.e., $d^3\underline{X} = \mu(G)$. Then, recalling that J_η is defined by $\eta^*(\mu(g)) = J_\eta \mu(G)$, where $\mu(g) = d^3\underline{x}$, and denoting

$$C = \eta^* g \quad , \tag{6.25}$$

the Cauchy-Green tensor on D, we find

$$\mu(C) = J_\eta \mu(G) \quad , \tag{6.26}$$

which we symbolically solve by writing

$$J_\eta = \mu(C)/\mu(G) = \sqrt{\frac{\det C_{AB}(\underline{X})}{\det G_{AB}(\underline{X})}} \quad . \tag{6.27}$$

Note that in this way J_η becomes a function of the volume elements of C and G, i.e., it is expressed in the convective picture. (In the case of elasticity, as opposed to fluids, the potential energy depends on the metrics, not just their volume elements.) Define the convective internal energy density $E(\rho_0, C, \sigma_0)$ by

$$E(\rho_0, C, \sigma_0)(\underline{X}) = e(\rho_0(X)\mu(G)/\mu(C), \sigma_0(\underline{X})) \quad ,$$

so that the potential energy has the following expression in the convective picture,

$$\int \rho_0(X) E(\rho_0, C, \sigma_0)(X)\, d^3\underline{X} \quad . \tag{6.28}$$

To express the kinetic energy in the convective picture, we need first to express it in terms of the convective velocity $\underline{V}$. We have by (6.20), the expression

$$\underline{V}_t = T\eta_t \circ \underline{V}_t \quad ,$$

and (6.25).

$$\int_D \rho_0(X) g(\eta_t(X))(\underline{V}_t(\underline{X}),\underline{V}_t(\underline{X}))\ d^3\underline{X}$$

$$= \int_D \rho_0(\underline{X}) g(\eta_t(\underline{X}))(T_{\underline{X}}\eta_t(\underline{V}_t(X),T_{\underline{X}}\eta_t(\underline{V}_t(\underline{X})))d^3\underline{X}$$

$$= \int_D \rho_0(\underline{X}) C(\underline{X})(\underline{V}_t(\underline{X}),\underline{V}_t(\underline{X}))d^3\underline{X}$$

$$= \int_D \frac{1}{\rho_0(X)}\ C(\underline{X})(M_t(\underline{X})^{\#},M_t(\underline{X})^{\#})d^3\underline{X}$$

where

$$M_t = \rho_0 \underline{V}_t^{b}\mu(G) \quad ,$$

the index raising ($\#$) and index lowering (b) actions being taken with respect to C. But the latter expression represents twice the kinetic energy of the metric on $\mathcal{X}(D)^*$ induced by C and ρ_0, i.e.,

$$(\alpha d^3\underline{X},\quad \beta d^3\underline{X}) = \int_D \frac{1}{\rho_0(X)}\ C(X)(\alpha(X)^{\#},\beta(X)^{\#})d^3\underline{X} \qquad (6.29)$$

Therefore, the total energy in the convective picture has the expression (see (6.28), (6.29))

$$H = \frac{1}{2}\ (M,M) + \int_D \rho_0(\underline{X}) E(\rho_0,\ C,\ \sigma_0)(\underline{X})d^3\underline{X} \quad . \qquad (6.30)$$

Proceeding as in (6.29) with $\underline{X}$ replaced by $\underline{x}$, ρ_0 by ρ, and C by g, we get another metric < , > on $\mathcal{X}(D)^*$ such that the kinetic energy,

$$\frac{1}{2}\int \rho|\underline{v}|^2 d^3\underline{x} \quad ,$$

coincides with the kinetic energy of the metric < , >. Summarizing, we have the following three expressions of the Hamiltonian:

$$H(\underline{M}_\eta, \rho_0, \sigma_0, g) = \frac{1}{2}\int_D \rho_0(\underline{X})g(\underline{V}_\eta(X), \underline{V}_\eta(X))\mu(G)(X)$$

$$+ \int_D \rho_0(\underline{X})E(\eta, \rho_0, \sigma_0)(\underline{X})\mu(G)(\underline{X}) \quad \text{(material)} \qquad (6.31)$$

where $\underline{V}_\eta = (\underline{M}_\eta/\rho_0\mu(G))^\#$, the index raising action being relative to g;

$$H(\underline{M}, \rho, \sigma, g) = \frac{1}{2}\int_D \rho(\underline{x})g(\underline{v}(x),\underline{v}(x))\mu(g)(\underline{x})$$

$$+ \int_D \rho(\underline{x})e(\rho(\underline{x}),\sigma(\underline{x}))\mu(g)(\underline{x}) \quad , \quad \text{(spatial)} \qquad (6.32)$$

where $\underline{v} = (\underline{M}/\rho\ \mu(g))^\#$ the index raising action being also relative to g;

$$H(\underline{\mathcal{M}}, \rho_0, \sigma_0, C) = \frac{1}{2}\int_D \rho_0(\underline{X})C(\underline{\mathcal{V}}(X),\mathcal{V}(\underline{X}))\mu(G)(\underline{X})$$

$$+ \int_D \rho_0(\underline{X})\mathcal{E}(\rho_0, C, \sigma_0)(\underline{X})\mu(G)(\underline{X}) \quad , \text{(convective)} (6.33)$$

where $\underline{\mathcal{V}} = (\underline{\mathcal{M}}/\rho_0\mu(G))^\#$, the index raising action being relative to C. In indices, (6.33) reads

$$H(M,\rho_0,\sigma_0,C) = \frac{1}{2}\int_D \rho_0 M_A M_B C^{AB} d^3\underline{X}$$

$$+ \int_D \rho_0 e\left(\rho_0\sqrt{\frac{\det C_{AB}}{\det G_{AB}}}\ ,\ \sigma_0\right)d^3\underline{X} \quad , \qquad (6.33')$$

where $\underline{M} = M_A dX^A d^3\underline{X}$ and $\mu(G) = d^3\underline{X} = \sqrt{\det G_{AB}}\,dX^1 \wedge dX^2 \wedge dX^3$. The relationships between the various internal energy densities are

$$E(\eta,\rho_0,\sigma_0) = e(\rho_0 J_\eta^{-1},\sigma_0)$$

$$E(\rho_0,C,\sigma_0) = e(\rho_0\mu(G)/\mu(C),\sigma_0) \quad .$$

6.5 The Ideal Compressible Adiabatic Fluid Equations in the Eulerian Picture

To apply Theorem 3.3(i), we investigate the invariance properties of H in (6.31) under right translations. We have for any $\phi \in \mathrm{Diff}(D)$ and $U_{\eta\circ\phi} \in T_{\eta\circ\phi}(\mathrm{Diff}(D))$,

$$M_\eta = \underline{V}_\eta^b \otimes \rho_0\mu(G) \in T^*(\mathrm{Diff}(D)) \quad ,$$

$$\langle T^*_{\eta\circ\phi}R_{\phi^{-1}}(M_\eta),\ U_{\eta\circ\phi}\rangle = \langle M_\eta, T_{\eta\circ\phi}R_{\phi^{-1}}(U_{\eta\circ\phi})\rangle$$

$$= \langle M_\eta,\ U_{\eta\circ\phi}\circ\phi^{-1}\rangle$$

$$= \langle \underline{V}_\eta^b \times \rho_0(\underline{X})d^3\underline{X},\ U_{\eta\circ\phi}\circ\phi^{-1}\rangle$$

$$= \int_D g(\eta(\underline{X}))(\underline{V}_\eta(\underline{X}),\ \mathcal{U}_{\eta\circ\phi}(\phi^{-1}(\underline{X}))\rho_0(\underline{X})d^3\underline{X}$$

$$= \int_D g((\eta\circ\phi)(Y))(\underline{V}_\eta(\phi(\underline{Y})),\ \underline{\mathcal{U}}_{\eta\circ\phi}(\underline{Y}))\ \rho_0(\phi(\underline{Y}))J_\phi(\underline{Y})d^3\underline{Y}$$

$$= \langle(\underline{V}^b_\eta\circ\phi) \otimes (\rho_0\circ\phi)J_\phi\mu(G),\ \underline{\mathcal{U}}_{\eta\circ\phi}\rangle \quad ,$$

i.e.,

$$T^*_\eta\ {}_\phi R_{\phi^{-1}}(\underline{V}^b_\eta \otimes \rho_0\mu(G)) = (\underline{V}^b_\eta\circ\phi) \otimes (\rho_0\circ\phi)J_\phi\mu(G) \quad . \tag{6.34}$$

Therefore, by (6.31) and a change of variables $X = \phi(\underline{Y})$, we get

$$H(T^*_{\eta\circ\phi}R_{\phi^{-1}}(\underline{V}^b_\eta \otimes \rho_0\mu(G)),\ (\rho_0\circ\phi)J_\phi,\ \sigma_0\circ\rho,\ g)$$

$$= \frac{1}{2}\int_D \rho_0(\phi(\underline{Y}))J_\phi(\underline{Y})g((\eta\circ\phi)(\underline{Y}))(V_\eta(\phi(\underline{Y})),\ \underline{V}_\eta(\phi(\underline{Y}))d^3\underline{Y}$$

$$+ \int_D \rho_0(\phi(\underline{Y}))J_\phi(\underline{Y})e(\rho_0(\phi(\underline{Y}))J_\phi(\underline{Y})J_{\eta\circ\phi}(\underline{Y})^{-1},\ \sigma_0(\phi(\underline{Y})))d^3\underline{Y}$$

$$= \frac{1}{2}\int_D \rho_0(\underline{X})g(\eta(\underline{X}))(\underline{V}_\eta(\underline{X}),\ \underline{V}_\eta(\underline{X}))d^3\underline{X}$$

$$+ \int_D \rho_0(\underline{X})e(\rho_0(\underline{X})J_\eta(\underline{X})^{-1},\ \sigma_0(\underline{X}))d^3\underline{X}$$

$$= H(\underline{V}^b_\eta \otimes \rho_0\mu(G),\ \rho_0,\ \sigma_0,\ g) \quad .$$

Consequently, by Theorem 3.3, H induces Lie-Poisson equations on $[(\mathcal{X}(D) \circledS (\mathcal{F}(D) \times \mathcal{F}(D)^*) \times S^2(D)]^*_+$. The Poisson bracket is hence given by (3.11). Denoting by $\underline{M}(\underline{x}) = \rho(\underline{x})v(\underline{x})$, this expression becomes

$$\{F,H\}(\underline{M},\rho,\sigma,g) = \int_D \underline{M}\cdot[(\frac{\delta H}{\delta \underline{M}}\cdot\underline{\nabla})\frac{\delta F}{\delta \underline{M}} - (\frac{\delta F}{\delta \underline{M}}\cdot\underline{\nabla})\frac{\delta H}{\delta \underline{M}}]d^3\underline{x}$$

$$+ \int_D \rho[\frac{\delta H}{\delta \underline{M}}\cdot(\underline{\nabla}\frac{\delta F}{\delta\rho}) - \frac{\delta F}{\delta \underline{M}}\cdot(\underline{\nabla}\frac{\delta H}{\delta\rho})]d^3\underline{x} \tag{6.35}$$

$$+ \int_D \sigma \operatorname{div}(\frac{\delta F}{\delta\sigma}\frac{\delta H}{\delta \underline{M}} - \frac{\delta H}{\delta\sigma}\frac{\delta F}{\delta \underline{M}})d^3\underline{x} \quad ,$$

where all dot products are taken in the metric g. The Hamiltonian of the fluid motion is easily checked in this case to be given by H in spatial coordinates (6.32) (see Theorem 3.3(i) and (6.34)) and the equations of motion are then computed via (6.35) to be (6.15) plus $\dot{g} = 0$.)

Finally, denoting by $\underline{\omega} = \operatorname{curl} \underline{v}$ the Eulerian vorticity, the scalar function

$$\Omega = \underline{\omega}\cdot\underline{\nabla}(\sigma/\rho) = \operatorname{div}[(\sigma\underline{\omega})/\rho] \tag{6.36}$$

is called the Eulerian potential vorticity. Using (6.35) it is easy to check that

$$C(\underline{M},\rho,\sigma,g) = \int_D \rho(\underline{x})\Phi(\sigma(x),\Omega(\underline{x}))d^3\underline{x} \tag{6.37}$$

for any smooth function $\Phi : \mathbb{R} \to \mathbb{R}$ is a Casimir functional. That is, $\{C,F\}$ vanishes for any $F(\underline{M},\rho,\sigma,g)$. In addition, any functional depending only on g is also a Casimir functional for the bracket (6.35).

6.6 The Ideal Compressible Adiabatic Fluid Equations in the Convective Picture

We shall now apply Theorem 3.3(ii) to get the equations of motion in the convective picture. We start by computing the left action of Diff(D) on $T^*(\mathrm{Diff}(D))$. If $\psi \in \mathrm{Diff}(D)$, let W_η^{b*} denote the index lowering action with respect to $\psi_* g$ applied to $W_\eta \in T_\eta(\mathrm{Diff}(D))$. Then for any $U_{\psi\circ\eta} \in T_{\psi\circ\eta}(\mathrm{Diff}(D))$, $V_\eta \in T_\eta(\mathrm{Diff}(D))$, we find

$$\langle T^*_{\psi\circ\eta} L_{\psi^{-1}} (\underline{V}_\eta^b \otimes \rho_0 \mu(G)), U_{\psi\circ\eta} \rangle$$

$$= \langle \underline{V}_\eta^b \otimes \rho_0 \mu(G), T_{\psi\circ\eta} L_{\psi^{-1}} (U_{\psi\circ\eta}) \rangle$$

$$= \langle \underline{V}_\eta^b \otimes \rho_0 \mu(G), T\psi^{-1} \circ (U_{\psi\circ\eta}) \rangle$$

$$= \int_D \rho_0(\underline{X}) g(\eta(\underline{X}))(\underline{V}_\eta(\underline{X}), T\psi^{-1}(U_{\psi\circ\eta}(\underline{X}))) d^3\underline{X}$$

$$= \int_D \rho_0(\underline{X}) (\psi_* g)(\psi(\eta(\underline{X})))((T\psi \circ \underline{V}_\eta)(\underline{X}), \underline{U}_{\psi\circ\eta}(\underline{X})) d^3\underline{X}$$

$$= \langle (T\psi \circ V_\eta)^{b^*} \otimes \rho_0 \mu(G), U_{\psi\circ\eta} \rangle$$

Note that

$$T^*_{\psi\circ\eta} L_{\psi^{-1}} (\underline{V}_\eta^b \otimes \rho_0 \mu(G)) = (T\psi \circ \underline{V}_\eta)^{b^*} \otimes \rho_0 \mu(G) \quad . \tag{6.38}$$

Now we check the condition in Theorem 3.3(ii). Denote by $J_\lambda^{\psi_* g}$ the Jacobian of λ relative to $\mu(G)$ and $\mu(\psi_* g)$. Then $J_{\psi\circ\eta}^{\psi_* g} = J_\eta$, so that by (6.31), (6.38) we have

$$H(T^*_{\psi\circ\eta} L_{\psi^{-1}}(\underline{V}^b_\eta \otimes \rho_0 \mu(G)),\ \rho_0,\ \sigma_0,\ \psi_* g)$$

$$= \frac{1}{2} \int_D \rho_0(\underline{X})(\psi_* g)(\psi(\eta(\underline{X})))((T\psi \circ \underline{V}_\eta)(\underline{X}), (T\psi \circ \underline{V}_\eta)(\underline{X}))d^3\underline{X}$$

$$+ \int_D \rho_0(\underline{X}) e(\rho_0(X) J^{\psi_* g}_{\psi\circ\eta}(\underline{X})^{-1}, \sigma_0(\underline{X}))d^3\underline{X}$$

$$= \frac{1}{2} \int_D \rho_0(\underline{X}) g(\eta(\underline{X}))(\underline{V}_\eta(\underline{X}), \underline{V}_\eta(\underline{X}))d^3\underline{X}$$

$$+ \int_D \rho_0(\underline{X}) e(\rho_0(\underline{X}) J_\eta(\underline{X})^{-1}, \sigma_0(\underline{X}))d^3\underline{X}$$

$$= H(\underline{V}^b_\eta \otimes \rho_0 \mu(G), \rho_0,\ \sigma_0,\ \psi_* g)$$

Therefore, by Theorem 3.3(ii), H induces Lie-Poisson equations on $[(\mathcal{X}(D) \circledS (S^2(D) \otimes \Lambda^3(D))) \times \mathcal{F}(D) \times \mathcal{F}(D)^*]^*_-$. The action of Diff(D) on the vector space of contravariant symmetric two-tensor densities $S^2(D) \otimes \Lambda^3(D)$ is given by pull-back, i.e.,

$$(T \otimes \mu(G)) \cdot \eta = \eta^*(T \otimes \mu(G)) \quad . \tag{6.39}$$

Therefore, the action of $\mathcal{X}(D)$ on $S^2(D) \times \Lambda^3(D)$ is by Lie-derivative, i.e.,

$$(T \otimes \mu(G)) \cdot \underline{V} = L_{\underline{V}}(T \otimes \mu(G)) \quad . \tag{6.40}$$

The dynamic variables are:

- on $\mathcal{X}(D)^*$ = one-form densities on D, $\underline{M}$ = convective momentum density of the fluid;
- on $(S^2(D) \otimes \Lambda^3(D))^* = S_2(D)$ = symmetric covariant two-tensors on D, paired with $S^2(D) \otimes \Lambda^3(D)$ by $(S, T \otimes \mu(G)) \in S_2(D) \otimes (S^2(D) \otimes \Lambda^3(D)) = \int_D S{:}T\mu(G) \in \mathbb{R}$ where $S{:}T$ denotes contraction on both indices of S and T, C = the Cauchy-Green tensor, defined by $C = \eta^* g$ for $\eta \in \mathrm{Diff}(D)$ a motion of the fluid;
- on $\mathcal{F}(D)^*$ = densities on D, $\rho_0\mu(G)$ = material mass density of the fluid; and
- on $\mathcal{F}(D)$ = functions on D, σ_0 = the material entropy function of the fluid.

If $F,H : \mathcal{X}(D)^* \times S_2(D) \times \mathcal{F}(D)^* \times \mathcal{F}(D) \to \mathbb{R}$ are functions, their Poisson bracket is given by (3.11), which in this case takes the form

$$\{F,H\}(\underline{M},\rho_0,\sigma_0,C) = \int_D \underline{M} \cdot \left[\frac{\delta F}{\delta \underline{M}}, \frac{\delta H}{\delta \underline{M}}\right]$$

$$+ \int_D C : \left(L_{\frac{\delta F}{\delta \underline{M}}} \frac{\delta H}{\delta C} - L_{\frac{\delta H}{\delta \underline{M}}} \frac{\delta F}{\delta C}\right) \quad , \tag{6.41}$$

where the dot in the first integral denotes the contraction of a one-form density with a vector field and the colon in the second integral denotes contraction of a covariant symmetric two-tensor with a contravariant symmetric two-tensor density.

Using (6.33) and the formula for the derivative of $\mu(C)$, i.e.,

$$D\mu(C) \cdot \delta C = \frac{1}{2}\, \mu(C)\, \mathrm{trace}_C\, \delta C \quad , \tag{6.42}$$

where trace_C denote the trace function with respect to C, we get

$$\frac{\delta H}{\delta \underline{M}} = \underline{V} \quad , \quad \frac{\delta H}{\delta C} = -\frac{1}{2}\, \underline{V} \otimes \underline{V} \otimes \rho_0 \mu(G) + \frac{1}{2}\, T \otimes \mu(C) \quad ,$$

where $T \in S^2(D)$ is the stress-tensor defined by

$$T = 2\rho_0 \frac{\delta E}{\delta C} \frac{\mu(G)}{\mu(C)} \quad . \tag{6.43}$$

Next, we carry out the computation of the equations of motion. If F is an arbitrary function of $(\underline{M}, \rho_0, \sigma_0, C)$, we have by the equations of motion $\dot{F} = \{F,H\}$, (6.41), and (6.42),

$$\int_D \frac{\delta F}{\delta \underline{M}} \cdot \dot{\underline{M}} + \int_D \frac{\delta F}{\delta \rho_0} (\rho_0 \mu(G))^{\cdot} + \int_D \frac{\delta F}{\delta \sigma_0} \dot{\sigma}_0\, \mu(G) + \int_D \frac{\delta F}{\delta C} : \dot{C}$$

$$= - \int_D \underline{M} \cdot L_{\underline{V}} \frac{\delta F}{\delta \underline{M}} +$$

$$\int_D C : [L_{\frac{\delta F}{\delta \underline{M}}} (-\frac{1}{2}\, \underline{V} \otimes \underline{V} \otimes \rho_0 \mu(G)) + \frac{1}{2}\, T \otimes \mu(C)) - L_{\underline{V}} \frac{\delta F}{\delta C}] \quad . \tag{6.44}$$

Comparing the coefficients of

$$\frac{\delta F}{\delta \rho_0} \quad \text{and} \quad \frac{\delta F}{\delta \sigma_0}$$

on both sides of (6.44) leads, respectively, to

$$\dot{\rho}_0 = 0 \quad \text{and} \quad \dot{\sigma}_0 = 0 \quad . \tag{6.45}$$

The coefficient of $\frac{\delta F}{\delta C}$ on the right-hand side of (6.44) is isolated via integration by parts. Namely, we write

$$L_{\underline{V}}(C : \frac{\delta F}{\delta C}) = L_{\underline{V}}C : \frac{\delta F}{\delta C} + C : L_{\underline{V}} \frac{\delta F}{\delta C} \quad ,$$

so, by Gauss' theorem,

$$- \int_D C : L_{\underline{V}} \frac{\delta F}{\delta C} = \int_D L_{\underline{V}}C : \frac{\delta F}{\delta C} - \int_D L_{\underline{V}}(C : \frac{\delta F}{\delta C})$$

$$= \int_D L_{\underline{V}}C : \frac{\delta F}{\delta C} - \int_D \mathrm{div}_\nu(\underline{V})\nu$$

$$= \int_D L_{\underline{V}}C : \frac{\delta F}{\delta C} - \int_{\partial D} \underline{V} \cdot \hat{\underline{n}}\, \nu_{\partial D} \quad , \tag{6.46}$$

where $\nu = C:\frac{\delta F}{\delta C}$, $\hat{\underline{n}}$ is the outward unit normal to ∂D, $\nu_{\partial D}$ is the induced volume form on ∂D, and $\mathrm{div}_\nu(\underline{V})$ is the divergence of V with respect to the volume form ν. The last integral is zero since $\underline{V}$ is tangent to ∂D at points of ∂D. Therefore, identifying the coefficient of $\frac{\delta F}{\delta C}$ in (6.44) yields

$$\dot{C} = L_{\underline{V}}C \quad , \tag{6.47}$$

i.e. C is dragged along by the flow of $\underline{V}$ during the motion; this result is predicted by Theorem 3.3 and the above computation is merely a confirmation of a general fact. Finally, we must isolate the coefficient of $\frac{\delta F}{\delta \underline{M}}$ in the right-hand side of (6.44). By integration by parts as in (6.46), we have

$$- \int_D \underline{M} \cdot L_{\underline{V}} \frac{\delta F}{\delta \underline{M}} = \int_D (L_{\underline{V}}\underline{M}) \cdot \frac{\delta F}{\delta \underline{M}} \quad . \tag{6.48}$$

Using the derivation properties of the Lie derivative and the relation between $\underline{M}$ and $\underline{V}$, namely,

$$\underline{V}^{b^*} \otimes \rho_0\mu(G) = \underline{M} \quad , \tag{6.49}$$

where b^* denotes the index lowering operation defined by C, leads to the following series of identities.

$$- \frac{1}{2} \int_D C : L_{\frac{\delta F}{\delta \underline{M}}} (\underline{V} \otimes \underline{V} \otimes \rho_0\mu(G))$$

$$= - \int_D C : (L_{\frac{\delta F}{\delta \underline{M}}} \underline{V}) \otimes \underline{V} \otimes \rho_0\mu(G) - \frac{1}{2} \int_D C(\underline{V},\underline{V}) \; L_{\frac{\delta F}{\delta \underline{M}}} (\rho_0\mu(G))$$

$$= - \int_D [\frac{\delta F}{\delta \underline{M}} , \underline{V}] \cdot \underline{M} - \frac{1}{2} \int_D \{L_{\frac{\delta F}{\delta \underline{M}}} [C(\underline{V},\underline{V})\rho_0\mu(G)] - L_{\frac{\delta F}{\delta \underline{M}}} (C(\underline{V},\underline{V})) \; \rho_0\mu(G)\}$$

$$= \int_D [- (L_{\underline{V}}\underline{M}) \cdot \frac{\delta F}{\delta \underline{M}} + \frac{1}{2} d||\underline{V}||^2_C \cdot \frac{\delta F}{\delta \underline{M}} \rho_0\mu(G)]$$

$$= \int_D [- L_{\underline{V}}\underline{M} + d(\frac{1}{2} \; ||\underline{V}||^2_C) \otimes \rho_0\mu(G)] \cdot \frac{\delta F}{\delta \underline{M}} \quad . \tag{6.49}$$

Finally, the last term involving $\frac{\delta F}{\delta \underline{M}}$ in (6.44) is

$$\frac{1}{2} \int_D C : L_{\frac{\delta F}{\delta \underline{M}}} [T \otimes \mu(C)]$$

$$= \int_D \mathrm{div}_C \; T \cdot \frac{\delta F}{\delta \underline{M}} \mu(C) \tag{6.50}$$

where $\mathrm{div}_C T$ denotes the divergence of T with respect to the metric C. {In coordinates, if ; denotes covariant differentiation with respect to C, then one has the identities

$$\int_D C : L_{\underline{V}}[T \otimes \mu(C)]$$

$$= - \int_D L_{\underline{V}} C : T \,\mu(C)$$

$$= - \int_D (V_{i;j} + V_{j;i}) T^{ij} \mu(C)$$

$$= 2 \int_D V^i T^j_{i;j} \mu(C)$$

by the divergence theorem and symmetry of T. Thus, $(\mathrm{div}_C T)$ is the one form with components $T^j_{i;j}$, where index raising and lowering and covariant differentiation is taken with respect to C. Thus, adding (6.48), (6.49) and (6.50) yields

$$\frac{\partial \underline{M}}{\partial t} + L_{\underline{V}} \underline{M} = d(\tfrac{1}{2} \, ||V||_C^2) \rho_0 \mu(G) + \mathrm{div}_C T \tag{6.51}$$

which is the desired equation of motion for $\underline{M}$.

7. CONCLUSIONS

In this paper we have laid the foundation for systematic passages among the Hamiltonian formalisms for the material, inverse material, spatial and convective pictures. The examples presented include the motion of incompressible fluids with a free boundary and surface tension (in §1), the heavy top (§5) and ideal compressible flow (§6). In other publications, these ideas will be used for other systems such as nonlinear elasticity (see Holm and Kupershmidt [1983], Marsden, Ratiu and Weinstein [1984], Simo and Marsden [1984] and Krishnaprasad, Marsden and Simo [1986]). The new feature here is the inclusion of the inverse material and convective pictures to complete the overall scenario of standard representations of continuum systems. The inverse material and convective pictures are useful in a number of situations such as for general relativistic fluids (Holm [1985]) and in the study of stability for ideal fluid equilibria in three dimensions. See Holm [1986] in this volume for the treatment of Lyapunov stability of three-dimensional ideal fluid equilibria using the inverse material representation.

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