Lectures in Applied Mathematics
Volume 23, 1986

NONLINEAR STABILITY OF THE KELVIN-STUART CAT'S EYES FLOW

Darryl D. Holm, Jerrold E. Marsden[1] and Tudor Ratiu[2]

ABSTRACT. Conditions which ensure the nonlinear stability of the Kelvin-Stuart cat's eyes solution for two dimensional ideal flow are given. The solution is periodic in the x direction and is bounded by two streamlines, which contain the separatrix, in the y-direction. The stability conditions are given explicitly in terms of the solution parameters and the domain size. The method is based on a technique originally developed by Arnold [1969].

1. EQUATIONS OF MOTION AND CONSERVED QUANTITIES. The Euler equations for an ideal, homogeneous incompressible fluid in a domain D in the plane $\mathbb{R}^2$ are:

$$\left.\begin{aligned} &\frac{\partial v}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\underline{\nabla}p \\ &\text{div } \underline{v} = 0 \\ &\underline{v}\cdot\underline{\hat{n}} = 0 \end{aligned}\right\} \tag{1.1}$$

where $\underline{v} = (v_1, v_2)$ is the velocity field, p is the pressure, and $\underline{\hat{n}}$ is the outward unit normal of the boundary ∂D.

1980 Mathematics Subject Classification 76E30, 58F10.
[1]Partially supported by DOE contract DE-AT03-82ER12097.
[2]Partially supported by an NSF postdoctoral fellowship.

Let $\omega = \hat{\underline{z}} \cdot \text{curl}\, \underline{v} = v_{2,x} - v_{1,y}$ be the scalar vorticity and ψ the stream function, i.e. $\underline{v} = \text{curl}(\psi\, \hat{\underline{z}}) = (\psi_{,y}, -\psi_{,x})$, where $\hat{\underline{z}}$ is the upward unit vector, orthogonal to the xy plane. The existence of ψ is proved in the following manner. Since $\underline{v}$ is tangent to each component $(\partial D)_i$ of ∂D, $i = 0, 1, \ldots, g$, the integral of $v_1\, dy - v_2\, dx$ around each $(\partial D)_i$ is zero. Since $\text{div}\, \underline{v} = 0$, we conclude that its integral around any closed loop is zero. Thus, by elementary calculus, $v_1 dy - v_2\, dx = d\psi$ for some ψ, i.e. $v_1 = \psi_{,y}$ and $v_2 = -\psi_{,x}$. Since $\underline{v}$ is tangent to $(\partial D)_i$, ψ is constant on each $(\partial D)_i$, $i = 0, \ldots, g$, so adding a suitable constant to ψ, we can assume it is zero on $(\partial D)_0$. Since

$$\underline{v} \cdot d\underline{\ell} = \text{curl}(\psi \hat{\underline{z}}) \cdot d\underline{\ell} = (\hat{\underline{z}} \times d\underline{\ell}) \cdot \underline{\nabla}\psi$$
$$= -\underline{\nabla}\psi \cdot \hat{\underline{n}}\, ds = -\frac{\partial\psi}{\partial \underline{n}}\, ds,$$

where $d\underline{\ell}$ and ds are the vectorial and scalar infinitesimal arc elements, we see that the circulations around $(\partial D)_i$, $i = 0, \ldots, g$ have the expressions

$$\Gamma_i := \int_{(\partial D)_i} \underline{v} \cdot d\underline{\ell} = -\int_{(\partial D)_i} \frac{\partial\psi}{\partial \underline{n}}\, ds.$$

In a bounded domain D, given the scalar vorticity ω, the stream function ψ is uniquely determined by the elliptic problem

$$\left.\begin{aligned} &-\nabla^2\psi = \omega \\ &\psi|(\partial D)_0 = 0 \\ &\psi_i := \psi|(\partial D)_i = \text{constant}, \quad i = 1, \ldots, g \\ &\Gamma_i := -\int_{(\partial D)_i} \frac{\partial\psi}{\partial n}\, ds = \text{constant}. \end{aligned}\right\} \tag{1.2}$$

Applying the operator $\hat{\underline{z}}\cdot$curl to the momentum conservation equation in (1.1) written in the form

$$\partial \underline{v}/\partial t = \underline{v} \times \omega\hat{\underline{z}} - \underline{\nabla}(\tfrac{1}{2}\,|\underline{v}|^2 + p),$$

yields the vorticity equation

$$\partial\omega/\partial t = \{\psi,\omega\}, \tag{1.3}$$

where $\{\psi,\omega\} = \psi_{,x}\,\omega_{,y} - \psi_{,y}\,\omega_{,x}$ is the usual Poisson bracket in $\mathbb{R}^2$.

Fix the vectors $\underline{\psi} = (\psi_1, \ldots, \psi_y)$ and $\underline{\Gamma} = (\Gamma_0, \ldots, \Gamma_g)$, and consider the following space of vorticities (with appropriate smoothness properties):

$$F_{\underline{\psi},\underline{\Gamma}} = \{\omega : D \to \mathbb{R} \mid \text{there exists a function } \psi : D \to \mathbb{R} \text{ satisfying (1.2) with the constants } \psi_1, \ldots, \psi_g, \Gamma_0, \ldots, \Gamma_g\}.$$

For $\omega \in F_{\underline{\psi},\underline{\Gamma}}$, we will write $\psi = -(\nabla^2)^{-1}\omega$ for the unique solution of (1.2). On this space, the total energy takes the form

$$\begin{aligned} H(\omega) &= \frac{1}{2}\int_D |\underline{v}|^2\, dx\, dy = \frac{1}{2}\int_D |\underline{\nabla}\psi|^2\, dx\, dy \\ &= \frac{1}{2}\int_D \psi\, \omega dx\, dy + \frac{1}{2}\int_{\partial D} \psi\, \underline{\nabla}\psi\cdot\hat{\underline{n}}\, ds \\ &= \frac{1}{2}\int_D \omega(-\nabla^2)^{-1}\omega\, dx\, dy - \frac{1}{2}\sum_{i=1}^{g} \psi_i\Gamma_i . \end{aligned} \tag{1.4}$$

In addition to this conserved energy, from (1.3), the identity

$$\int_D f\{g,h\}\,dx\,dy = \int \{f,g\}h\,dx\,dy - \int_{\partial D} fh\underline{\nabla}g\cdot d\underline{\ell}\ ,$$

and the fact that ω and ψ satisfy (1.2), it follows that the functionals

$$C_\Phi(\omega) = \int_D \Phi(\omega)\,dx\,dy \qquad (1.5)$$

are also conserved, for any $\Phi:\mathbb{R} \to \mathbb{R}$

2. VARIATIONAL PRINCIPLE FOR THE CAT'S EYES. Stationary solutions ω_e, ψ_e of (1.3) are characterized by having $\underline{\nabla}\psi_e$ and $\underline{\nabla}\omega_e$ parallel. A sufficient condition for this to hold is the functional relationship

$$\psi_e = \Psi(\omega_e). \qquad (2.1)$$

The stationary solution treated in this paper is the Kelvin [1880]-Stuart [1967] cat's eyes solution given by

$$\psi_e(x,y) = \log[a\cosh y + \sqrt{a^2-1}\cos x], \qquad (2.2)$$

in the domain $0 \leq x \leq 2\pi$, $-\infty < y < \infty$, where a is a real parameter satisfying $a \geq 1$ ($a = 1$ gives a shear flow). The streamlines ψ = constant have the form shown in Figure 1. The vorticity is given by

$$\omega_e(x,y) = -\nabla^2\psi_e(x,y) = -e^{-2\psi_e} \qquad (2.3)$$

so that Ψ in (2.1) is given by

$$\Psi(\lambda) = -\frac{1}{2}\log(-\lambda),\ \lambda < 0. \qquad (2.4)$$

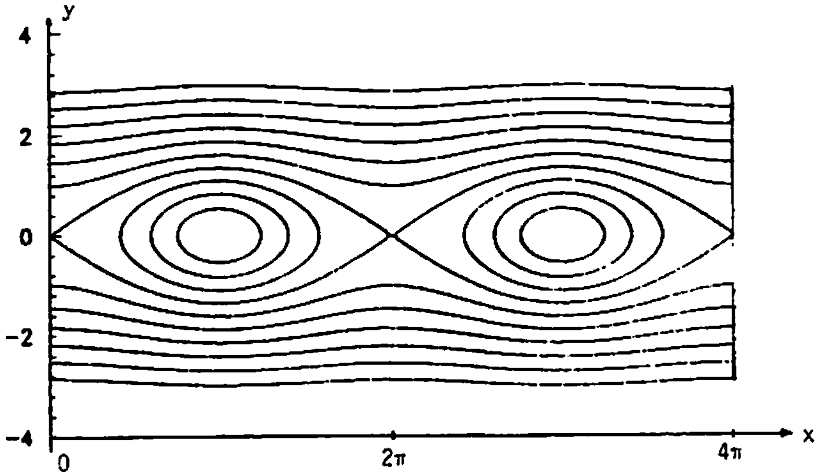

Figure 1. Computer plot of the cat's eyes streamlines for the stream function $\psi(x,y)$ in (2.2) with a = 1.175.

The components of the velocity are

$$
\begin{aligned}
v_1 &= \psi_{,y} = \frac{a \sinh y}{a \cosh y + \sqrt{a^2-1} \cos x}, \\
v_2 &= -\psi_{,x} = \frac{\sqrt{a^2-1} \sin x}{a \cosh y + \sqrt{a^2-1} \cos x}
\end{aligned}
\tag{2.5}
$$

so that $v_2 \to 0$ as $y \to \pm\infty$, whereas $v_1 \to \pm 1$ as $y \to \pm\infty$, i.e. in the limit, the velocity is a shear flow in each half-plane in the opposite direction. We shall consider in this paper only domains bounded by a pair of streamlines below the upper separatrix and above the lower separatrix, i.e. we shall require that the finite domain D be given by $0 \le x \le 2\pi$ and y bounded by the streamlines $\psi = \pm \log[ac + a + \sqrt{a^2-1}]$ for some $c > 0$. The reason for this restriction is that the infinite domain allows arbitrary wave numbers, which prevent the estimates below from being carried out.

In the domain D, we shall seek the stationary solution ω_e as a critical point of the conserved functional

$$H_\phi(\omega) = \int_D [\tfrac{1}{2}\,\omega(-\nabla^2)^{-1}\omega + \Phi(\omega)]\,dx\,dy - \frac{1}{2}\sum_{i=1}^{g} \psi_i \Gamma_i . \tag{2.6}$$

Integrating twice by parts and using the fact that $\delta\psi|(\partial D)_i = 0$, $\int_{(\partial D)_i} \partial(\delta\psi)/\partial \underline{n}\,ds = 0$, $i = 0, \ldots, g$, we get

$$DH_\phi(\omega_e)\cdot\delta\omega = \int_D (\psi_e + \Phi'(\omega_e)\delta\omega\,dx\,dy$$

$$= \int_D (\Psi(\omega_e) + \Phi'(\omega_e))\delta\omega\,dx\,dy.$$

By (2.4), the function Φ equals (up to a constant)

$$\begin{aligned}\Phi(x) &= -\int_0^\lambda \Psi(s)\,ds \\ &= \frac{1}{2}\int_0^\lambda \log(-s)\,ds \\ &= \frac{1}{2}\,\lambda(\log(-\lambda) - 1).\end{aligned} \tag{2.7}$$

This function has the graph shown in Figure 2. The function is concave since

$$\Phi''(\lambda) = \frac{1}{2\lambda} < 0 \quad \text{for} \quad \lambda < 0.$$

Bounding the domain in the y direction will keep ω away from the bad point $\lambda = 0$ in Figure 2, where Φ'' is unbounded below.

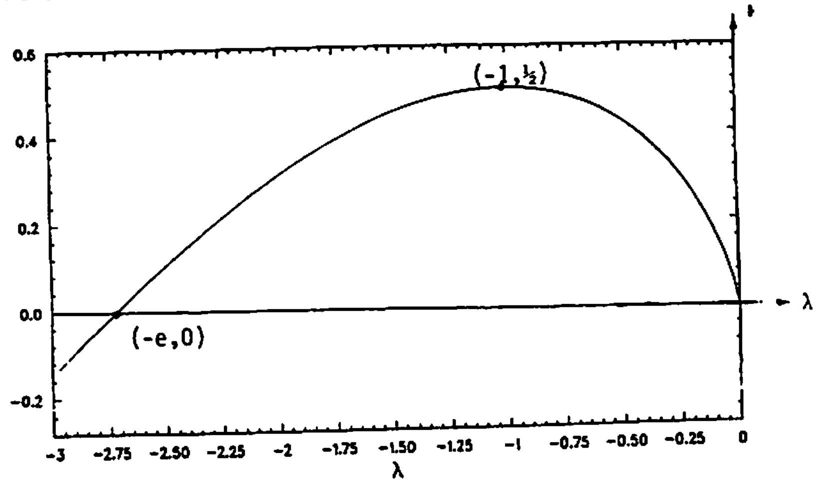

Figure 2. The Casimir function $\Phi(\lambda)$ in equation (2.7) is convex downward.

3. STABILITY ESTIMATES. To study the stability of the Cat's Eyes solution in a finite domain, consider a finite perturbation $\Delta\omega$. The quantity

$$\hat{H}_\Phi(\Delta\omega) := H_\Phi(\omega_e + \Delta\omega) - H_\Phi(\omega_e) - DH_\Phi(\omega_e)\cdot\Delta\omega$$

$$= \int_D [\tfrac{1}{2}\,\Delta\omega(-\nabla^2)^{-1}\Delta\omega + \Phi(\omega_e + \Delta\omega) - \Phi(\omega_e) - \Phi'(\omega_e)\Delta\omega]\,dx\,dy \qquad (3.1)$$

is conserved since $DH_\Phi(\omega_e) = 0$ for Φ given by (2.7)

To establish nonlinear stability, we shall bound the conserved quantity (3.1) above and below, in a way that implies bounds on the L^2 norm of the vorticity perturbation for all time. To get this stability estimate, we modify Φ to a function $\tilde{\Phi}$, in such a manner that Φ'' is bounded above and below and $DH_{\tilde{\Phi}}(\omega_e) = 0$. For these bounds on $\tilde{\Phi}''$, we first compute $\min \omega_e(x,y)$ and $\max \omega_e(x,y)$, where $D = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 2\pi,\ |y| \le \cosh^{-1}\left[c + 1 + \frac{\sqrt{a^2-1}}{a}(1 - \cos x)\right],\ c > 0\}$ is the domain bounded in the y-direction, by the two streamlines $\psi_e(x,y) = \pm\log(ac + a + \sqrt{a^2-1})$ and over a 2π-period in x. Since

$$\omega_e(x,y) = -e^{-2\psi_e(x,y)} = -[a \cosh y + \sqrt{a^2-1} \cos x]^{-2} \quad (3.2)$$

the critical points of the stream function and, thus, of the vorticity are at $x = 0, \pi, 2\pi$ and $y = 0$. The critical values of vorticity are

$$-[a + \sqrt{a^2-1}]^{-2}, \quad \text{for} \quad x = 0,\ 2\pi$$

and

$$-[a - \sqrt{a^2-1}]^{-2}, \quad \text{for} \quad x = \pi.$$

The value of $\omega_e(x,y)$ on the y-boundary is

$$-[ac + a + \sqrt{a^2-1}]^{-2}$$

and on the vertical boundaries $x = 0, 2\pi$ is

$$-[a + \sqrt{a^2-1}]^{-2} \le -[a \cosh y + \sqrt{a^2-1}]^{-2} \le -[ac + a + \sqrt{a^2-1}]^{-2}.$$

Consequently,

$$\min_D \omega_e(x,y) = -[a - \sqrt{a^2-1}]^{-2}$$
$$\max_D \omega_e(x,y) = -[ac + a + \sqrt{a^2-1}]^{-2}. \quad (3.3)$$

Thus, on the interval $[\min_D \omega_e(x,y), \max_D \omega_e(x,y)]$ the function Φ has its second derivative bounded by

$$\frac{1}{2\max_D \omega_e(x,y)} \leq \Phi''(\omega_e) = \frac{1}{2\omega_e} \leq \frac{1}{2\min_D \omega_e(x,y)} < 0.$$

Now define the following C^2-function:

$$\tilde{\Phi}(\lambda) = \begin{cases} -\frac{1}{2}(a-\sqrt{a^2-1})^2\lambda^2 + (\sqrt{a^2-1} - a - 1)\lambda + \alpha, \\ \qquad \text{for } \lambda \leq -(a-\sqrt{a^2-1})^{-2} \\ \Phi(\lambda) = \frac{1}{2}\lambda(\log(-\lambda) - 1), \\ \qquad \text{for } -(a-\sqrt{a^2-1})^{-2} \leq \lambda \leq -(ac+a+\sqrt{a^2-1})^{-2} \\ -\frac{1}{2}(ac+a+\sqrt{a^2-1})^2\lambda^2 - (ac+a+\sqrt{a^2-1} - 1)\lambda + \beta, \\ \qquad \text{for } \lambda \geq -(ac+a+\sqrt{a^2-1})^{-2} \end{cases}$$

where
$$\alpha = (a-\sqrt{a^2-1})^{-2}(\log(a-\sqrt{a^2-1}) + 2) + \frac{1}{2}(a-\sqrt{a^2-1})^{-2} + (\sqrt{a^2-1}) - a - 1)(a-\sqrt{a^2-1})^{-2},$$

and
$$\beta = (ac+a+\sqrt{a^2-1})^{-2}(\log(ac+a+\sqrt{a^2-1}) + 2) + \frac{1}{2}(ac+a+\sqrt{a^2-1})^{-2} - (ac+a+\sqrt{a^2-1})(ac+a+\sqrt{a^2-1} - 1)^{-2}.$$

Since $\tilde{\Phi}$ and Φ coincide on the interval $[\min_D \omega_e(x,y),$

$\max_D \omega_e(x,y)]$, it follows that $DH_{\tilde{\Phi}}(\omega_e) = 0$. But unlike Φ, $\tilde{\Phi}$ has its second derivative bounded on the entire axis, namely

$$\frac{1}{2}(a - \sqrt{a^2-1})^2 \leq -\tilde{\Phi}''(\lambda) \leq \frac{1}{2}(ac + a + \sqrt{a^2-1})^2 \tag{3.4}$$

for all $-\infty < \lambda < +\infty$. Consequently, the function $-\tilde{\Phi}$ is convex, i.e.,

$$\frac{1}{4}(a - \sqrt{a^2-1})^2(\Delta\omega)^2 \leq -\tilde{\Phi}(\omega_e + \Delta\omega) + \tilde{\Phi}(\omega_e) + \tilde{\Phi}'(\omega_e)\Delta\omega$$
$$\leq \frac{1}{4}(ac + a + \sqrt{a^2-1})^2(\Delta\omega)^2 . \tag{3.5}$$

Considering the negative of (3.1) with Φ replaced by $\tilde{\Phi}$ we get from (3.5) the estimates

$$-\hat{H}_{\tilde{\Phi}}(\Delta\omega) \leq \frac{1}{2}\int_D [\frac{1}{2}(ac + a + \sqrt{a^2-1})^2(\Delta\omega)^2 + \Delta\omega(\nabla^2)^{-1}\Delta\omega]\, dx\, dy$$

$$-\hat{H}_{\tilde{\Phi}}(\Delta\omega) \geq \frac{1}{2}\int_D [\frac{1}{2}(a - \sqrt{a^2-1})^2(\Delta\omega)^2 + \Delta\omega(\nabla^2)^{-1}\Delta\omega]\, dx\, dy.$$

Let $\Delta\omega_0$ denote the value of the perturbation $\Delta\omega$ at $t = 0$. Then by conservation of $-\hat{H}_{\tilde{\Phi}}$ we get

$$\int_D [\frac{1}{2}(a - \sqrt{a^2-1})^2(\Delta\omega)^2 + \Delta\omega\,(\nabla^2)^{-1}\Delta\omega]dx\, dy \leq -2\,\hat{H}_{\tilde{\Phi}}(\Delta\omega)$$

$$= -2\hat{H}_{\tilde{\Phi}}(\Delta\omega_0) \leq \int_D [\frac{1}{2}(ac + a + \sqrt{a^2-1})(\Delta\omega_0)^2$$
$$+ (\Delta\omega_0)(\nabla^2)^{-1}(\Delta\omega_0)^2\,]\, dx\, dy$$

$$\leq \int_D [\frac{1}{2}(ac + a + \sqrt{a^2-1})^2(\Delta\omega_0)^2\, dx\, dy,$$

since $(\nabla^2)^{-1}$ is negative. Thus we have the a priori estimate

$$\frac{1}{2}(a-\sqrt{a^2-1})^2\|\Delta\omega\|^2_{L^2}+\int_D \Delta\omega(\nabla^2)^{-1}\Delta\omega\,dx\,dy \tag{3.6}$$
$$\leq [\tfrac{1}{2}(ac+a+\sqrt{a^2-1})^2]\|\Delta\omega_0\|^2_{L^2}.$$

To prove nonlinear stability, we still need an estimate in terms of the L^2-norm of $\Delta\omega$ for the second (negative) integral on the left hand side of (3.6). This will be done by using the following Poincaré type inequality.

<u>LEMMA</u>. *Let* k^2_{min} *be the minimal eigenvalue of* $-\nabla^2$ *in the space* $F_{\psi,\underline{\Gamma}}$ *on the domain* D. *Then*

$$\int_D \Delta\omega(\nabla^2)^{-1}\Delta\omega\,dx\,dy \geq -k^{-2}_{min}\|\Delta\omega\|^2_{L^2}.$$

<u>PROOF</u>. Let k_i^2 be the eigenvalues of $-\nabla^2$, $i = 0, 1, \ldots,$ with $k_0^2 = k^2_{min}$ and let ϕ_i be an L^2 orthonormal basis of eigenfunctions, i.e.

$$-\nabla^2\phi_i = k_i^2\phi_i, \int_D \phi_i\phi_j\,dx\,dy = \delta_{ij}.$$

Then $-k_i^{-2}$ are the eigenvalues of $(\nabla^2)^{-1}$, i.e.

$$(\nabla^2)^{-1}\phi_i = -k_i^{-2}\phi_i,\ i = 0, 1, \ldots,$$

setting $\Delta\omega = \sum_{i=0}^{\infty} c_i\phi_i$, we have

$$\int_D \Delta\omega(\nabla^2)^{-1}\Delta\omega \ dx\,dy = \sum_{i,j}^{\infty} c_i c_j \int_D \phi_i(\nabla^2)^{-1}\phi_i \ dx\,dy$$

$$= -\sum_{i,j} c_i c_j k_j^{-2} \int_D \phi_i\phi_j \ dx\,dy$$

$$= -\sum_{i=0}^{\infty} k_j^{-2} c_j^2 \int_D \phi_j^2 \ dx\,dy$$

$$\geq -k_{min}^{-2} \sum_{i=0}^{\infty} c_j^2 \int_D \phi_j^2 \ dx\,dy$$

$$= -k_{min}^{-2} \|\Delta\omega\|_{L^2}^2$$

since $k_j^{-2} \leq k_{min}^{-2}$ for all $j = 0, 1, 2, \ldots$ ∎

This lemma, and (3.6) yield the estimate

$$[\tfrac{1}{2}(a - \sqrt{a^2-1})^2 - k_{min}^{-2}]\|\Delta\omega\|_{L^2}^2$$

$$\leq [\tfrac{1}{2}(ac + a + \sqrt{a^2-1})^2]\|\Delta\omega_0\|_{L^2}^2 \tag{3.7}$$

The final requirement for nonlinear stability is to ensure the positivity of the coefficient in the left hand side of (3.7).

According to the characterization of the minimal eigenvalue of the Laplacian on bounded domains by the Rayleigh-Ritz quotient, we see that this minimal eigenvalue is a decreasing function of the size of the domain. Thus, we shall replace k_{min}^2 with the first eigenvalue of the Laplacian on the rectangle $0 \leq x \leq 2\pi$,

$$|y| \leq \ell := \cosh^{-1}\left(c + 1 + \frac{2\sqrt{a^2-1}}{a}\right), \tag{3.8}$$

i.e. the height of the rectangle is the distance between the highest points of the streamlines $\psi = \pm \log(ac + a + \sqrt{a^2-1})$. The minimal eigenvalue of $-\nabla^2$ on the space of functions vanishing on the boundary and having zero circulations on each component of the boundary belongs to the eigenfunction $\cos x \sin \frac{\pi y}{\ell}$ and is $1 + \frac{\pi^2}{\ell^2}$. Thus, for (3.7) to provide a meaningful estimate, we need to satisfy the inequality

$$(a - \sqrt{a^2-1})^2 > 2/\left(1 + \frac{\pi^2}{\ell^2}\right) = \frac{2\ell^2}{\pi^2 + \ell^2} \tag{3.9}$$

Solutions of this inequality exist, since, for example, the pair $a = 1$, $c = 1$ satisfies it, but there is clearly an implicit trade-off between a and ℓ in (3.9), by virtue of (3.8).

We study inequality (3.9) for $c = 0$, i.e. we take the y-boundary of the domain to be the separatrix. The inequality becomes

$$(a - \sqrt{a^2-1})^2 > \frac{2\left[\cosh^{-1}\left(1 + \frac{2\sqrt{a^2-1}}{a}\right)\right]^2}{\pi^2 + \left[\cosh^{-1}\left(1 + \frac{2\sqrt{a^2-1}}{a}\right)\right]^2} \quad . \tag{3.10}$$

Numerically one verifies that this inequality holds for $1 \leq a \leq 1.175 \ldots$; see Figure 3.

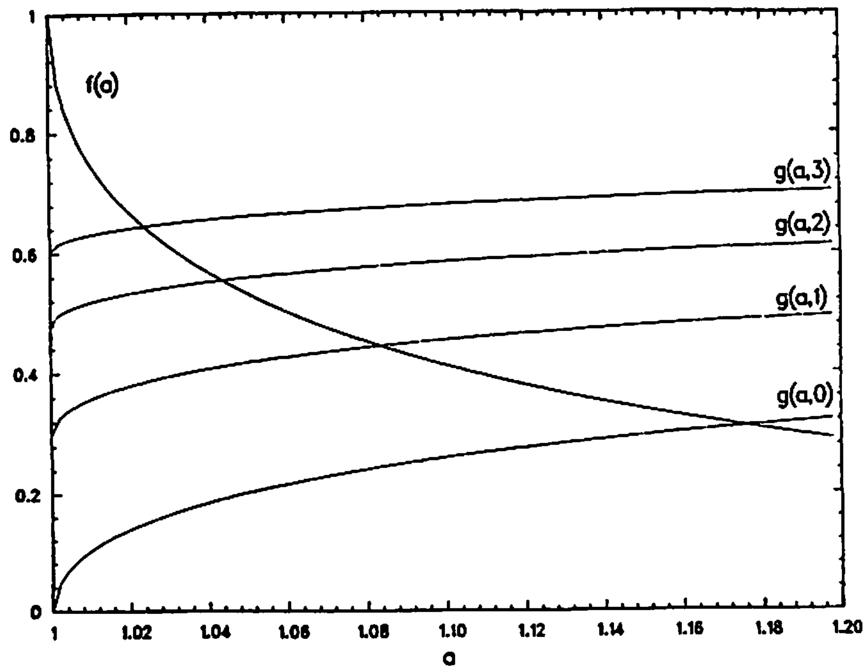

Figure 3. Graphs of $f(a) = (a - \sqrt{a^2 - 1})^2$ and $g(a,c) = 2\ell^2/(\pi^2 + \ell^2)$, where $\ell(a,c) = \cosh^{-1}[1 + c + 2a^{-1}\sqrt{a^2 - 1}]$ for a between 1 and 1.2 and c = 0,1,2,3. The inequality (3.9) is satisfied, so that the cat's eyes flow is stable, for values of (a,c) such that $f(a) > g(a,c)$.

We summarize our results in the following.

THEOREM. *The Kelvin-Stuart cat's eyes solution (2.2) of the Euler equation (1.1) is nonlinearly stable in the* L^2 *norm on vorticities for perturbations of the initial vorticity which preserve the flow rate* (ψ = *constant on the boundaries) and the circulations, in a region bounded by the streamlines* $\psi_e = \pm \log[ac + a + \sqrt{a^2-1}]$, *provided* a *and* c *satisfy (3.8) and (3.9). For* $1 \leq a \leq 1.175, \ldots,$ *this region contains a separatrix in the cat's eyes flow.*

Note that in the special case $a = 1$, the cat's eyes solution reduces to the $v_1 = \tanh(y)$ shear flow, which is stable according to the present analysis, provided the domain is limited in the y direction by $|y| \leq \cosh^{-1}(1 + c)$ where (using 3.9),

$$c < \cosh \pi^2 - 1 = 9,665.8 \ldots \qquad (3.11)$$

4. FURTHER REMARKS

(i) Variants of the basic flow can be treated by the same method. For example, consider

$$\psi_e = \log[\, a \cosh y + \sqrt{a^2-1} \; \cos x]$$

as before, but on $[0,4\pi]$ rather than $[0,2\pi]$; i.e. include two "eyes" rather than one. The same analysis shows that (3.9) is replaced by

$$(a - \sqrt{a^2-1})^2 > \frac{2}{\frac{1}{4} + \frac{\pi^2}{\ell^2}} \qquad (4.1)$$

This restricts the stability region somewhat, but by considering $a = 1$, it holds for $c < \cosh\left(\frac{4\pi^2}{7}\right) - 1 = 139.7 \ldots$ (the analogue of (3.11)), and so (4.1) holds for a nontrivial range of $a > 1$ and $c > 0$ and again we get stability on a region containing the cat's eye separatrix. (These results are consistent with known linearized and nonlinear results; cf. Stuart [1971]).

(ii) Although the computations are more complex, in principle, the method applies to the sinh-Poisson solutions of Ting, Chen and Lee [1984].

(iii) Cat's eye solutions provide interesting equilibria for a plasma confining Grad-Shafranov solution of reduced magnetohydrodynamics (where the current and magnetic potential replace the vorticity and stream function). The present method applies directly to give a nonlinear stability result in that case; see Hazeltine, Holm, Marsden and Morrison [1984]. The known coalescence instability for magnetic islands in that case is avoided by having sufficient transverse constriction for the Poincaré inequality to ensure positivity in (3.7).

ACKNOWLEDGEMENTS. We thank John Gibbon for suggesting this problem and Jerry Kazdan for useful conversations about the Poincaré inequality. We also thank George Nickel for making the figures.

REFERENCES

1. V. Arnold, [1969]. On an a priori estimate in the theory of hydrodynamical stability, Transl. Am. Math. Soc. 79, 267-269.

2. R.D. Hazeltine, D.D. Holm, J.E. Marsden and P.J. Morrison [1984]. Generalized Poisson Brackets and nonlinear Liapunov stability -- Application to reduced MHD. Proc. Lausanne Plasma Phys. Conf., June 1984.

3. D.Holm, J. Marsden, T. Ratiu and A. Weinstein [1984]. Nonlinear stability of fluid and plasma equilibria, Physics Reports (to appear).

4. Lord Kelvin [1880]. On a disturbing infinity in Lord Rayleigh's solution for waves in a plane vortex stratum, Nature 23, 45-46.

5. J.T. Stuart [1971]. Stability problems in fluids, Lectures in Applied Math. AMS 13, 139-155.

6. J.T. Stuart [1967]. On finite amplitude oscillations in laminar mixing layers, J. Fluid Mech. 29, 417-440.

7. A.C. Ting, H.H. Chen and Y.C. Lee [1984]. Exact vortex solutions of two dimensional guiding center plasma (preprint).

CENTER FOR NONLINEAR STUDIES
LOS ALAMOS NATIONAL LABORATORY
MS B 258
LOS ALAMOS, N.M. 87545

DEPARTEMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIF. 94720