Stress and Riemannian Metrics in Nonlinear Elasticity

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§1. Introduction

In Doyle and Ericksen [1956, p. 77] it is observed that the Cauchy stress tensor σ can be derived by varying the internal energy e with respect to the Riemannian metric on space: $\sigma^{ab} = 2\rho \partial e/\partial g_{ab}$. Their formula has gone virtually unnoticed in the elasticity literature. In this lecture we shall explain some of the reasons why this formula is, in fact, of fundamental significance. Some additional reasons for its importance follow. First of all, it allows for a rational derivation of the Duhamel-Neumann hypothesis on a decomposition of the rate of deformation tensor (see Sokolnikoff [1956, p. 359]), which is useful in the identification problem for constitutive functions. This derivation, due to Hughes, Marsden and Pister, is described in Marsden and Hughes [1983, p. 204-207]. Second, it is used in extending the Noll-Green-Naghdi-Rivlin balance of energy principle (using invariance under rigid body motions) to a covariant theory which allows arbitrary mappings. This is described in Section 2.4 of Marsden and Hughes [1983] and is closely related to the discussion herein. Finally, in classical relativistic field theory, it has been standard since the pioneering work of Belinfante [1939] and Rosenfeld [1940] to regard the stress-energy-momentum tensor as the derivative of the Lagrangian density with respect to the spacetime (Lorentz) metric; see for example, Hawking and Ellis [1973, Sect. 3.3] and Misner, Thorne and Wheeler [1973, Sect. 21.3]. This modern point of view has largely replaced the construction of "canonical

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stress-energy-momentum tensors". Thus, for the Lagrangian formulation of elasticity (relativistic or not) the Doyle-Ericksen formulation plays the same role as the Belinfante-Rosenfeld formula and brings it into line with developments in other areas of classical field theory.

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§2. Some Basic Notation

Let $\mathcal{B}$ and $\mathcal{S}$ be oriented smooth n-manifolds (usually n = 3). We call $\mathcal{B}$ the reference configuration and $\mathcal{S}$ the ambient space. Let $\mathcal{C}$ denote the set of smooth embeddings of $\mathcal{B}$ into $\mathcal{S}$, so $\phi \in \mathcal{C}$, $\phi:\mathcal{B} \to \mathcal{S}$ represents a possible configuration of an elastic body. (In many situations one needs to put Sobolev or Hölder differentiability conditions on elements of $\mathcal{C}$; such conditions will not interfere with, nor play a significant role in our discussions, so we work with C^∞ objects for simplicity.)

A motion is a curve $\phi(t) \in \mathcal{C}$. For $X \in \mathcal{B}$, we write $x = \phi(X,t) = \phi(t)(X)$. The material velocity of a motion is the curve of vector functions over $\phi(t)$ defined by

$$V(t)(X) \equiv V(X,t) = \frac{\partial}{\partial t}\,\phi(X,t);$$

thus $V(t)(X) \in T_x\mathcal{S}$, the tangent space to $\mathcal{S}$ at x. The spatial velocity is $v(t) = V(t)\circ\phi(t)^{-1}$, a vector field on the image $\phi(t)(\mathcal{B}) \subset \mathcal{S}$.

Let $\mathcal{B}$ and $\mathcal{S}$ carry Riemannian metrics G and g, respectively, and associated volume elements dV and dv. Using the standard pull-back notation from analysis on manifolds (see Abraham, Marsden and Ratiu [1983]), we let $C = \phi^*g$, the Cauchy-Green deformation tensor. In coordinates $\{X^A\}$ on $\mathcal{B}$ and $\{x^a\}$ on $\mathcal{S}$, if $F^a{}_A$ denote the components of the derivative $F = T\phi$, then $C_{AB} = g_{ab}F^a{}_AF^b{}_B$ (for the relationship with the formula $C = F^TF$, see Marsden and Hughes [1983, Sect. 1.3]).

Let us recall the usual approach to the Cauchy equations of elasticity (see Truesdell and Noll [1965]). Let W be a materially frame indifferent stored energy function; that is, W is a function of X $\in \mathcal{B}$ and the point values of C. We write W(X,C) and let ρ_{Ref} denote the mass density in the reference configuration. Let

$$P = \rho_{Ref} \frac{\partial W}{\partial F}$$

denote the <u>first</u> <u>Piola-Kirchhoff</u> <u>stress</u> <u>tensor</u> and A(t) be the material acceleration of a motion $\phi(t)$ (the acceleration is defined using the Levi-Civita connection of g). Cauchy's equations are:

$$\rho_{Ref} A = \mathrm{DIV}\ P + B$$

where B is an external body force. In the spatial picture these equations read

$$\rho a = \mathrm{div}\ \sigma + b$$

where $\rho_{Ref} = J\rho$ (J is the Jacobian of ϕ), $J\sigma = PF^T$, so σ is the Cauchy stress, and b is an external spatial body force. These equations are usually derived by postulating the integral form of balance of momentum and sufficient smoothness. (There are also boundary conditions to be imposed, but we shall not explicate them here.)

Although one can do it, it is not trivial or especially natural to pass directly from the integral balance of momentum assumption to the weak form of Cauchy's equations; see Antman and Osborne [1979]. Another difficulty with balance of momentum is that it does not make sense on manifolds; in particular, this approach is not useful for studying continuum mechanics in general relativity.

Let us now outline some different approaches based on covariant energy principles which overcome the preceding objections.

§3. Energy Principles in the Material Picture

Let us first recall how one introduces stress by an energy principle. Let W be a given materially frame indifferent stored energy function. Let us regard W as a function of ϕ, g and G. Assume that

1. W is spatially covariant; that is, for any diffeomorphism ξ of S,

$$W(\phi, g, G) = W(\xi \circ \phi, \xi_* g, G)$$

and

2. W is local; that is, if (ϕ_1, g_1, G_1) and (ϕ_2, g_2, G_2) agree in a neighborhood of X (with g evaluated at $x = \phi(X)$), then $W(\phi_1, g_1, G_1)(X) = W(\phi_2, g_2, G_2)(X)$.

As in the usual Coleman-Noll [1959] argument these assumptions together with energy balance laws below enable one to deduce that W depends only on the point values of C. For simplicity let us assume this at the outset.

Notice that we do not necessarily assume that W is materially covariant; that is, $W(\phi \circ \eta, g, \eta^* G) = W(\phi, g, G) \circ \eta$ for diffeomorphisms η of B. Indeed, this holds if and only if the material is isotropic.

The energy function is

$$\mathcal{H}(U, \phi, g, G) = \int_U \rho_{Ref}\left(\frac{1}{2}\,\|V(t)\|^2 + W(\phi, g, G)\right)dV$$

where $U \subset B$ is a compact region with smooth boundary.

Now assume that there is a traction field, namely for each ϕ, g, G we have a map $T(\phi, g, G): TB \to T^* S$ covering ϕ such that the following holds:

Cauchy's Axiom of Power. Admissible motions satisfy the following condition:

$$\frac{d}{dt}\mathcal{H}(U,\phi,g,G) = \int_{\partial U} \langle T(\phi,g,G)(N),V(t)\rangle dA + \int_U B\cdot V(t)dV$$

where N is the outward unit normal to ∂U.

Cauchy's theorem states that if this holds for all U, then necessarily T is linear in N and so defines a two tensor P.

A. The Hamiltonian Systems Approach

In this approach we first prove that $P = \rho_{\rm Ref}\partial W/\partial F$ by assuming that there are enough motions so that V can be varied arbitrarily at any fixed ϕ; for example one often assumes "any" motion is possible by choosing B appropriately. Then the divergence theorem shows that indeed, $P = \rho_{\rm Ref}\partial W/\partial F$. With this equation in hand one can then assume that motions come from a Hamiltonian system (adapted to take care of external forces) on $T\mathcal{C}$ or $T^*\mathcal{C}$ with energy function $\mathcal{H}(B,\phi,g,G)$. These yield directly the weak form of the Cauchy equations (cf. Marsden [1981, Lectures 1 and 2]).

B. Covariance Approach

In the covariant approach one assumes that an admissible motion $\phi(t)$ satisfies Cauchy's axiom of power and that for any curve $\xi(t)$ of diffeomorphisms of S, the new motion $\phi'(t) = \xi(t)\circ\phi(t)$ also satisfies balance of energy in which g is replaced by $\xi_* g$ and the velocities, forces and accelerations are transformed according to the standard dictates of the Cartan theory of classical spacetimes (see Marsden and Hughes [1983, Sect. 2.4]). In the material picture this approach directly yields a version of the weak form of the equations and a formula for the "rotated stress":

$$\Sigma = 2\rho\partial W/\partial G, \text{ i.e., } \Sigma^{AB} = 2\rho\frac{\partial W}{\partial G_{AB}},$$

which is a material version of the Doyle-Ericksen formula. See Simo and Marsden [1984] for details.

Remark. The symmetry of the stress is built into the assumption of material frame indifference. Indeed, if $S = F^{-1}P$ is the second Piola-Kirchhoff stress, then one finds from the chain rule that

$$S = 2\rho_{Ref}\frac{\partial W}{\partial C}$$

which is symmetric; see the appendix. This formula for the stress is related to the covariance approach in the convected picture; again see Simo and Marsden [1984].

§4. Energy Principles in the Spatial Picture

Spatially, the energy is also regarded as a function of ϕ, g and G by

$$e(\phi,g,G) = W(\phi,g,G)\circ\phi^{-1}$$

Balance of energy for a moving region $\phi(t)(U) = U(t)$ where $U \subset \mathcal{B}$, now takes the form

$$\frac{d}{dt}\int_{U(t)} \rho(e + \frac{1}{2}\,\|v\|^2)dv = \int_{\partial U(t)} t\cdot v\ da + \int_{U(t)} b\cdot v\ dv$$

The spatial form of Cauchy's axiom of power states that admissible motions satisfy the preceeding equation. As before, this implies that t, the Cauchy traction vector, is linear in n, the unit outward normal to $\partial U(t)$, so defines a two tensor σ (depending on ϕ,g and G), the Cauchy stress tensor.

To complete the spatial description two routes are possible.

A. The Hamiltonian Systems Approach

If one assumes that any motion is possible with suitable forces, then as before, one gets an equation for the stress. Using the spatial picture, this equation is

$$\sigma = 2\rho\frac{\partial e}{\partial g},\ \text{i.e.,}\ \sigma^{ab} = 2\rho\frac{\partial e}{\partial g_{ab}}$$

In this approach one then assumes that the variables (m, ρ, C), where $m = \rho v$ is the momentum density, form a Hamiltonian system in a sense involving Lie-Poisson brackets for Lie groups analogous to the way the Euler equations for a rigid body are Hamiltonian when written in terms of its three angular velocities or momenta. See Holm and Kuperschmidt [1983] and Marsden, Ratiu and Weinstein [1984] for details. As before, this directly yields the weak form the spatial equations.

B. The Covariant Approach

In this approach we assume that a motion satisfies balance of energy and that balance of energy is still valid under any superposed curve of diffeomorphisms $\xi(t)$, where as above, the superposed motion uses the metric $\xi_* g$ and the other quantities are transformed by the dictates of classical mechanics. This assumption directly yields the weak form of the evolution equations, conservation of mass and the Doyle-Ericksen formula $\sigma = 2\rho \partial e/\partial g$. The proof of this is similar to that of Theorem 4.13 in Marsden and Hughes [1983].

§5. Concluding Remarks

1. Since all of the approaches sketched are equivalent, the four basic formulas for the stress

$$P = \rho_{\text{Ref}} \frac{\partial W}{\partial F}, \quad \sigma = 2\rho \frac{\partial e}{\partial g},$$

$$S = 2\rho_{\text{Ref}} \frac{\partial W}{\partial C}, \text{ and } \Sigma = 2\rho \frac{\partial W}{\partial G}$$

must be equivalent as well. Indeed, their equivalence is easily checked directly using the chain rule and the relation $C_{AB} = g_{ab} F^a{}_A F^b{}_B$. (This chain rule argument is how Doyle and Ericksen [1956] present the formula.) This is detailed in the appendix for the first three formulas. (The fourth requires special interpretations and is omitted only for brevity.)

2. Some special peculiarities with the Hamiltonian formalism arise when one is considering electromagnetic fields coupled to

elasticity (the Cauchy-Maxwell equations). The sense in which the equations written in the variables (m, ρ, C) and (E, B) are Hamiltonian is especially interesting. For the corresponding structures for charged fluids and plasmas, see Marsden and Weinstein [1982] and Spencer [1982].

3. A deep understanding of the Hamiltonian formalism for incompressible fluids enabled Arnold [1966a,b] to prove the nonlinear stability of plane flows studied by Rayleigh in a situation where one would otherwise expect the usual difficulties with potential wells (Knops and Wilkes [1973], Marsden and Hughes [193, Sect. 6.6] and Ball and Marsden [1984]). It is hoped that a similar understanding in elasticity will shed light on the energy criterion.

4. Finally, we note that the Doyle-Ericksen formula is the spatial part of the stress-energy-momentum tensor that naturally arises when one couples elasticity to the gravitational field in Einstein's theory. The material picture is derived from this by choosing a reference body and slicing of spacetime relative to which the motion may be represented. Thus, in this sense, the Doyle-Ericksen formula may be regarded as the most basic of the four equivalent formulas

$$P = \rho_{\text{Ref}}\frac{\partial W}{\partial F},\ S = 2\rho_{\text{Ref}}\frac{\partial W}{\partial C},\ \sigma = 2\rho\frac{\partial e}{\partial g},\ \text{and}\ \Sigma = 2\rho\frac{\partial W}{\partial G}$$

Appendix

A Direct Verification of the Equivalence of the Stress Formulas

The components of the Green deformation tensor are defined by

$$C_{AB} = F^a{}_A F^b{}_B g_{ab} \tag{1}$$

where $F^a{}_A$ and g_{ab} represent the deformation gradient and spatial metric tensor, respectively. (The summation convention is assumed to hold except with respect to the arguments of functions.) By virtue of

(1), we may think of the C_{AB}'s as functions of the $F^a{}_A$'s and g_{ab}'s; viz.

$$(2) \qquad C_{AB} = \hat{C}_{AB}(F^a{}_A;\ g_{ab}) = \hat{C}_{AB}(F^1{}_1, F^1{}_2, \ldots;\ g_{11}, g_{12}, \ldots).$$

The physical interpretation of (2) goes as follows: The "strains" (i.e., C_{AB}'s) are functions of the gradients of the motion (i.e., $F^a{}_A$'s), and the length scales and angle measures of the ambient space, as manifested by the g_{ab}'s.

We will need to use the partial derivatives of $\hat{C}_{AB}$; namely

$$(3) \qquad \partial\hat{C}_{AB}/\partial F^b{}_C = \delta_{AC}F^a{}_B g_{ba} + \delta_{BC}F^a{}_A g_{ab}$$

$$(4) \qquad \partial\hat{C}_{AB}/\partial g_{ab} = F^a{}_A F^b{}_B$$

Let $W = W(C_{AB})$ be a given stored energy function and let us start by assuming that, say,

$$(5) \qquad S^{AB} = 2\rho_{Ref}\frac{\partial W}{\partial C_{AB}}$$

where S^{AB} and ρ_{Ref} represent the (symmetric) second Piola-Kirchhoff stress tensor and density in the reference configuration, respectively. A related potential, $\hat{W}$, may be defined by using (2);

$$(6) \qquad \hat{W}(F^a{}_A;\ g_{ab}) = W(\hat{C}_{AB}(F^a{}_A;\ g_{ab}))$$

Substituting (3) and (4) into (5) yields

$$(7) \qquad \rho_{Ref}\frac{\partial\hat{W}}{\partial F^a{}_A} = g_{ab}F^b{}_B S^{BA} = P_a{}^A$$

and

(8) $$2\rho\frac{\partial\hat{W}}{\partial g_{ab}} = J^{-1}F^{a}{}_{A}S^{AB}F^{b}{}_{B} = \sigma^{ab}$$

where $\rho J = \rho_{Ref}$; ρ is the density in the current configuration; J is the determinant of the deformation gradient; and $P_a{}^A$ and σ^{ab} are the (unsymmetric) first Piola-Kirchhoff and Cauchy stress tensors, respectively.

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