Symmetry and Bifurcation in Three-Dimensional Elasticity

Part III: Stressed Reference Configurations

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§ 1. Introduction

Consider a hyperelastic material, with a region $\mathscr{B}$ in $\mathbb{R}^3$ as its reference configuration. Suppose that the reference configuration is in equilibrium under some initial system of forces (not necessarily zero).
The general problem: Describe the equilibrium solutions of this elastic material subject to a system of forces close to the initial one. Specifically, we want
(a) *to count the number of solutions,*
(b) *to determine their stabilities.*

(For general references on elasticity theory, see GRIOLI [1962], GURTIN [1972], MARSDEN & HUGHES [1983], TRUESDELL & NOLL [1965] and WANG & TRUESDELL [1973].)

In Parts I and II of this series we examined a special case, namely the traction problem near a natural state. In this case the system of forces was a dead load, consisting of a body force and a traction field applied to the material, and the reference configuration was stress-free (so that the initial system of forces was zero). A variational approach proved to be successful in dealing with this traction problem (see CHILLINGWORTH, MARSDEN & WAN [1982] and [1983], hereinafter referred to as [I] and [II]).

In this paper we shall extend this method to the general problem stated above. To be more precise, we shall investigate:
(1) The traction problem with a general reference configuration, so that the initial load need not be zero (as in BHARATHA & LEVINSON [1978] and CAPRIZ & PODIO GUIDUGLI [1979]).
(2) The pressure problem. Here we replace the traction boundary condition by a pressure boundary condition (*cf.* TRUESDELL & NOLL [1965]).

One often analyzes such problems in a setting in which there is symmetry for both the material and the reference configuration. (*E.g.*, the material may be isotropic and the reference configuration may be a ball.) Here we shall also examine:

(3) The effect of such symmetry assumptions on the solutions of the problems.

These topics are each obtained by dropping various hypotheses on the traction problem studied in [I] and [II].

Now we recall the main ideas in [I] and [II]. What we have dealt with is a bifurcation problem for a potential function in the presence of a symmetry group, which acts on both the space of deformations (state space) and the load space (parameter space). The "trivial" solutions $SO(3)$ for zero load, consist in a group orbit through $I_{\mathscr{B}}$, the identity map on $\mathscr{B}$. One seeks solutions nearby for small applied forces. Tubular neighborhoods provide a convenient way to parametrize the neighborhood of the trivial solutions. By the stability condition and Korn's inequality, one concludes that the potential function with zero load has a nondegenerate minimum transversal to the trivial solutions. Accordingly, by a Liapunov-Schmidt procedure, a reduced potential function on $SO(3)$ is obtained. A further Liapunov-Schmidt reduction can be made provided the *type* of the applied load is known. (The notion of "type" is defined in [I].) Consequently, one has a bifurcation problem on the group orbit.

For the pressure problem (2), we shall show that these ideas are appropriate and that similar results are valid. The same ideas apply equally well for the traction problem with a general reference configuration, provided a certain condition (S), which generalizes the stability condition, is fulfilled. In particular, the use of this condition leads to a classification of initial loads by the associated isotropy group, and it agrees with the classification given by BHARATHA & LEVINSON [1978] and CAPRIZ & PODIO GUIDUGLI [1979]. The condition (S) is satisfied in some cases, but fails to be in certain other interesting cases corresponding to buckling in related rod and shell models. These cases often result from assumptions of symmetry in items (1) and which fall under item (3). It is necessary to extend somewhat the ideas involved in order to handle the cases in which bifurcations also occur within the slice for the group action. As we shall see, there is a rather general theory based on this idea. For results about bifurcations of zeros of G-equivariant map near a fixed point rather than an orbit, consult SATTINGER [1979] and GOLUBITSKY & SCHAEFFER [1979].

This paper is arranged as follows: in Section 2, we analyze the traction problem with a general reference configuration under condition (S). To make the presentation self-contained, many details are given. After Sections 1 and 2, it becomes apparent how to make an abstract bifurcation theory concerning potential functions invariant under bisymmetry actions. This is the content of Section 3. An abstract version of the Signorini perturbation scheme is given in Section 4. As further applications, we present the pressure problem in Section 5 and also the traction problem under symmetry hypotheses in the final section.

The traction problem for the incompressible material using methods motivated by this paper are treated by WAN [1983] (*cf.* BALL & SCHAEFFER [1982], MARSDEN & HUGHES [1983], TRUESDELL & NOLL [1965] and WANG & TRUESDELL [1973]).

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§ 2. The traction problem with a general reference configuration

A. Statement of the Problem

Let $\mathscr{B} \subset \mathbb{R}^3$ be a *reference configuration* for an elastic body with $0 \in \mathscr{B}$, where $\mathscr{B}$ is an open bounded set with smooth boundary $\partial\mathscr{B}$ in $\mathbb{R}^3$. Denote by $\mathscr{C}$ *the space of deformations*, which consists of all orientation-preserving embeddings $\phi: \overline{\mathscr{B}} \to \mathbb{R}^3$, $\phi(0) = 0$, of Sobolev class $W^{s,2}$, $s > \frac{5}{2}$.

Assume that the elastic body possesses a *smooth stored energy function* $W = W(X, C)$, where $X \in \overline{\mathscr{B}}$, $C = F^T F$, and F is a nonsingular orientation-preserving linear transformation from $\mathbb{R}^3$ ($\approx T_X\mathscr{B}$) into $\mathbb{R}^3$. For any deformation ϕ in $\mathscr{C}$, $F(X) = D\phi(X)$ denotes the corresponding deformation gradient, and $C(X) = F^T(X)\,F(X)$ is the Cauchy-Green tensor. Thus the first *Piola-Kirchhoff stress tensor* is given by $P(X) = \frac{\partial W}{\partial F}(X, F(X)) \in L(T_X\mathscr{B}, \mathbb{R}^3)$, *i.e.*, $P_{ij}(X) = \frac{\partial W}{\partial F_{ij}}(X, F(X))$. We drop the variable X, when there is no danger of confusion, and identify $T_X\overline{\mathscr{B}}$ with $\mathbb{R}^3$.

For a given body force $B: \mathscr{B} \to \mathbb{R}^3$ of class $W^{s-2,2}$, and a surface traction $\tau: \partial\mathscr{B} \to \mathbb{R}^3$ of class $W^{(s-\frac{3}{2}),2}$, the equations for equilibrium solutions ϕ in $\mathscr{C}$ are:

$$
\text{(E)} \qquad \begin{cases} \operatorname{DIV} P(X, F(X)) + B(X) = 0, & \text{for } X \in \mathscr{B}, \\ P(X, F(X))\, N(X) = \tau(X), & \text{for } X \in \partial\mathscr{B}. \end{cases}
$$

where $N(X)$ stands for the outward unit normal to $\partial\mathscr{B}$ at $X \in \partial\mathscr{B}$ and DIV P is the divergence of $P(X, F(X))$ with respect to X $\left(\textit{i.e.}, (\operatorname{DIV} P)_i = \sum_{j=1}^{3} P_{ij,j} = \sum_{j=1}^{3} \frac{\partial P_{ij}}{\partial X_j}\right)$.

Using the divergence theorem, we easily verify that if the equation (E) can be solved for a given load (B, τ), then the total force on $\mathscr{B}$ must be zero:

$$
\int_{\mathscr{B}} B\, dV + \int_{\partial\mathscr{B}} \tau\, dA = 0.
$$

Hence, dV and dA are the respective volume and area elements on $\mathscr{B}$ and $\partial\mathscr{B}$.

Denote by $\mathscr{L}$ the space of all such pairs $l = (B, \tau)$ of forces, and call $\mathscr{L}$ *the space of* (dead) *loads*.

Let $\Phi: \mathscr{C} \to \mathscr{L}$ be the map defined by $\Phi(\phi) = (-\mathrm{DIV}\, P(\phi), P(\phi)\, N)$. Hence the equation (E) may be written $\Phi(\phi) = l$. Clearly, the proper orthogonal group $SO(3)$ acts on $\mathscr{C}$ and $\mathscr{L}$ by composition, and Φ is SO(3)-equivalent. Set $\Phi(I_{\mathscr{B}}) = l_{\mathscr{B}}$, where $I_{\mathscr{B}} \in \mathscr{C}$ stands for the identity map on $\mathscr{B}$, and let $G \subset \mathrm{SO}(3)$ be the isotropy group of $l_{\mathscr{B}}$ (*i.e.*, $Q \in G$ if and only if $Q l_{\mathscr{B}} = l_{\mathscr{B}}$). $l_{\mathscr{B}}$ is the *initial load* of our problem. By the SO(3)-equivariance of Φ, the configurations $G I_{\mathscr{B}}$ are solutions of the equation $\Phi(\phi) = l_{\mathscr{B}}$. Under suitable hypotheses, the only solutions of $\Phi(\phi) = l_{\mathscr{B}}$ near $G I_{\mathscr{B}} \subset \mathscr{C}$ are those in $G I_{\mathscr{B}}$.

Our basic problem can now be described as follows:

(T) *Describe the set $E(\lambda l)$ of all equilibrium solutions of $\Phi(\phi) = l_{\mathscr{B}} + \lambda l$ near $G I_{\mathscr{B}}\,(\subset \mathscr{C})$ for various loads $\lambda l \in \mathscr{L}$, with λ small and positive and with l near some fixed load $l_0 \in \mathscr{L}$. Specifically, we want*

(a) *to count the number of solutions in $E(\lambda l)$, and*

(b) *to determine their stabilities.*

In [I] the reference configuration $\mathscr{B}$ was taken to be *stress-free*; *i.e.*, $P(I_{\mathscr{B}}) = 0$. Thus $l_{\mathscr{B}} = 0$, $G = \mathrm{SO}(3)$ and we considered solutions near $G I_{\mathscr{B}} = \mathrm{SO}(3)$. Here we drop this hypothesis and study our traction problem with a stressed reference configuration.

Given $l \in \mathscr{L}$, the astatic load $k(l)$ is $\int_{\mathscr{B}} B \otimes X\, dV + \int_{\partial\mathscr{B}} \tau \otimes X\, dA$. The load l is said to be *equilibrated* if $k(l)$ is symmetric. Set $\mathscr{U} = \{u : \overline{\mathscr{B}} \to \mathbb{R}^3 \mid u$ is of class $W^{s,2}$, and $u(0) = 0\}$. A pairing between $\mathscr{L}$ and $\mathscr{U}$ is given by (the virtual work)

$$\langle l, u\rangle = \int_{\mathscr{B}} \langle B(X), u(X)\rangle\, dV + \int_{\partial\mathscr{B}} \langle \tau(X), u(X)\rangle\, dA, \text{ where } l = (B, \tau).$$

2.1 Proposition. *The initial load $l_{\mathscr{B}} = \Phi(I_{\mathscr{B}})$ must be equilibrated.*

Proof. For $K \in$ skew (the space of 3×3 skew symmetric matrices) the divergence theorem implies that

$$\mathrm{trace}\,(K^T k(l_{\mathscr{B}})) = \langle \Phi(I_{\mathscr{B}}), KX\rangle = \int_{\mathscr{B}} \mathrm{trace}\,(P^T K)\, dV.$$

Since $P = \dfrac{\partial W}{\partial F} = \dfrac{\partial W}{\partial C}\dfrac{\partial C}{\partial F} = 2\dfrac{\partial W}{\partial C}$ at $I_{\mathscr{B}}$ is symmetric, $\int_{\mathscr{B}} \mathrm{trace}\,(P^T K)\, dV = 0$ for all $K \in$ skew. Thus $\mathrm{trace}\,(K^T k(l_{\mathscr{B}})) = 0$ for all $K \in$ skew, so that $k(l_{\mathscr{B}})$ is symmetric. ∎

The section is concluded by introducing an extension of the stability condition on the elasticity tensor for the stress-free case (see [I]).

Denote by M_3 the inner product space of all linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^3$, with

$$\langle A, B\rangle = \mathrm{trace}\,(A^T B) = \sum_{i,j=1}^{3} A_{ij} B_{ij}.$$

Let $A(X, F(X)) = \frac{\partial P}{\partial F}(X, F(X)) : M_3 \to M_3$ be the *elasticity tensor*. Since $A = \frac{\partial^2 W}{\partial F^2}$, A is symmetric on M_3. Set $A(X) = A(X, I)$ and let $\mathfrak{g} = \{K \in \text{skew} \mid e^{Kt} \in G, \text{ for all } t\}$ be the Lie algebra of G. Write $M_3 = \mathfrak{g} \oplus \mathfrak{g}^\perp$ as the sum of $\mathfrak{g}$ and its orthogonal complement $\mathfrak{g}^\perp$ in M_3. Thus each $m \in M_3$ has the decomposition $m = m_{\mathfrak{g}} + m_{\mathfrak{g}^\perp} \in \mathfrak{g} \oplus \mathfrak{g}^\perp$. Now the desired stability condition can be stated as follows:

(S) there exists a constant $c > 0$, such that $\langle A(X)\, m, m \rangle \geqq c \, \|m_{\mathfrak{g}^\perp}\|^2$ for all $X \in \overline{\mathscr{B}}$, and for all $m \in M_3$.

As was mentioned in the Introduction, this condition will be studied in Section 2. Cases where it is not satisfied are studied in Sections 3 and 4.

B. Classification of the initial loads, $l_{\mathscr{B}}$

The initial loads $l_{\mathscr{B}}$ will be classified by symmetry via the associated isotropy group G or its Lie algebra $\mathfrak{g}$.

Recall that $l \in \mathscr{L}$ is called a *parallel system* of loads if $l(X) = g(X)\, a$ for some function $g : \overline{\mathscr{B}} \to \mathbb{R}$, and vector $a \in \mathbb{R}^3$.

2.2 Proposition. (a) $l_{\mathscr{B}} = 0$ *if and only if* $G = SO(3)$.

(b) $l_{\mathscr{B}}$ *is a non-zero parallel system if and only if* $G = S^1$, *a circle group, and*

(c) $l_{\mathscr{B}}$ *is a non-zero, non-parallel system if and only if* $G = \{0\}$.

The proof is elementary and is left to the reader.

The next proposition partially justifies our claim that our load classification is the appropriate one. It relates the symmetry of the loads to the kernel of the linearized problem. In particular, it can be interpreted as: symmetry implies bifurcation.

The space $\mathscr{C}$ of deformations is an open set of the Hilbert space $\mathscr{U}$. The derivative $D\Phi(I_{\mathscr{B}})$ of Φ at $I_{\mathscr{B}}$ will be considered as a linear map L from $\mathscr{U}$ into $\mathscr{L}$. Thus $L(u) = D\Phi(I_{\mathscr{B}})\,(u) = (-\text{DIV}\, A(\nabla u), A(\nabla u)\, N)$ represents the linearized elastostatic equations at $I_{\mathscr{B}}$.

2.3 Proposition. *An element* $K \in \text{skew}$ *satisfies* $L(KX) = 0$, *if and only if* $K \in \mathfrak{g}$, *the Lie algebra of* G.

Proof. By $SO(3)$-equivariance, $\Phi(e^{Kt} X) = e^{Kt}\, l_{\mathscr{B}}$ for each $K \in \text{skew}$. Hence $L(KX) = K l_{\mathscr{B}}$ for all $K \in \text{skew}$. One completes the proof by observing that $K l_{\mathscr{B}} = 0$ if and only if $e^{Kt} l_{\mathscr{B}} = l_{\mathscr{B}}$, *i.e.*, if and only if $e^{Kt} \in G$, for all t. ∎

Set $\mathscr{K} = \{KX \mid K \in \text{skew}, L(KX) = 0\}$. Geometrically, the above proposition says that $\mathscr{K}$ is the tangent space to $G I_{\mathscr{B}}$ at $I_{\mathscr{B}}$.

Combining the previous two propositions, we get:

2.4 Corollary (BHARATHA & LEVINSON [1978]).
(a) $\dim \mathscr{K} = 3$, 1 *or* 0.
(b) $\dim \mathscr{K} = 3$ *if and only if* $l_{\mathscr{B}} = 0$.
(c) $\dim \mathscr{K} = 1$ *if and only if* $l_{\mathscr{B}}$ *is a non-zero, parallel system.*
(d) $\dim \mathscr{K} = 0$ *if and only if* $l_{\mathscr{B}}$ *is a non-zero non-parallel system.*

We shall examine the traction problem by using the following condition for the linearized problem with kernel $\mathscr{K}$:

(F) $\ker L = \mathscr{K}$, *and* $\operatorname{Im} L = \mathscr{K}^{\perp}$, *where* $\mathscr{K}^{\perp} = \{l \in \mathscr{L} \mid \langle l, KX\rangle = 0$, *for all* $KX \in \mathscr{K}\}$,

which is the conclusion of the theorem of the Fredholm alternative (see Proposition 2.5 below).

Condition (F) agrees with that in BHARATHA & LEVINSON [1978], for hyperelastic materials. Condition (F) suffices to count the number of equilibrium solutions; for stability results, however, one needs to assume more, such as condition (S) introduced above.

For $L(u) = (-\mathrm{DIV}\, \mathbf{A}(\nabla u), \mathbf{A}(\nabla u)\, N)$, condition (F) is a condition on the elasticity tensor $\mathbf{A}(X)$. Indeed, we have

2.5 Proposition. *If the elasticity tensor* $\mathbf{A}(X)$ *satisfies condition* (S), *then it also satisfies condition* (F).

Proof. (a) As in FICHERA [1972], one shows the Fredholm alternative holds by establishing a Gårding inequality for the bilinear form $B(u, u) = \langle \mathbf{A}(\nabla u), \nabla u\rangle_V = \int_B \langle \mathbf{A}(X)\,(\nabla u(X)), \nabla u(X)\rangle\, dV$ over $W^{1,2}$, *i.e.*, $\langle A(\nabla u), \nabla u\rangle_V \geqq \gamma_0 \|u\|_1^2 - \lambda_0 \|u\|_0^2$, for some constants $\gamma_0 > 0$, $\lambda_0 \geqq 0$. By condition (S) $\langle \mathbf{A}(\nabla u), \nabla u\rangle_V \geqq c\,\|\tfrac{1}{2}(\nabla u + \nabla u^T)\|_0^2$. The desired inequality, then follows from Korn's second inequality (FICHERA [1972]):

$$\|(\nabla u + \nabla u^T)\|_0^2 + \|u\|_0^2 \geqq c_2\, \|u\|_1^2, \text{ for some constant } c_2 > 0.$$

(b) Since $\mathscr{K} \subset \ker L$ by proposition 2.3, it remains to show that $\mathscr{K} \supset \ker L$. Let $L(u) = 0$ and $u \in \mathscr{U}$. Using the divergence theorem and condition (S), we get $0 = \langle L(u), u\rangle = \langle \mathbf{A}(\nabla u), \nabla u\rangle_V \geqq c\,\|\tfrac{1}{2}(\nabla u + \nabla u^T)\|_0^2$. Thus $\|(\nabla u + \nabla u^T)\|_0^2 = 0$, and so $\nabla u + \nabla u^T = 0$ on $\overline{\mathscr{B}}$. Therefore, $u = KX$ for some $K \in$ skew. ∎

Another method for establishing the Fredholm alternative is to use directly strong ellipticity and the elliptic estimates. For this approach, see MARSDEN & HUGHES [1983, Ch. 6]. It is this latter method which will be useful when condition (S) is dropped.

From now on, we shall examine our traction problem with a general reference configuration under the stability condition (S). The initial loads are classified into three categories according to the cases (a), (b) and (c) in Proposition 2.2. In case (a), we have a load-free reference configuration and so the analysis is basically the same as that for a stress-free reference configuration studied in [I] and [II]. Hence we omit it. In case (c), $L = D\Phi(I_{\mathscr{B}})$ is an isomorphism (*cf.* Proposition 2.5) and so by the inverse mapping theorem, no bifurcation will occur.

Therefore we need only investigate case (b) in detail, where the initial load $l_{\mathscr{B}}$ is a parallel system. Note that condition (S) implies that $\langle \mathrm{A}(X)\, K, K\rangle > 0$ for all $K \in \text{skew} \setminus \{0\}$ in $\mathfrak{g}^{\perp}$, which, in turn, implies that the parallel system must be nontrivial with the eigenvalues of $k(l_{\mathscr{B}})$ given by 0, 0, c, where $c > 0$. Indeed, one can readily establish that

$$\langle \mathrm{A}(X)\, K, K\rangle = \langle L(KX), KX\rangle = \langle Kl_{\mathscr{B}}, KX\rangle = \langle -K^2, k(l_{\mathscr{B}})\rangle = \frac{1}{2}\,\|K\|^2\, c$$

by using the divergence theorem and the relation $L(KX) = Kl_{\mathscr{B}}$, from which our remark follows. It turns out that the same ideas used in [II] can be used here, as will be shown in the next subsection.

C. A non-zero parallel initial load system

Let $l_{\mathscr{B}} = \Phi(I_{\mathscr{B}})$ be an initial load parallel to some non-zero vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$. Thus the isotropy group $G = \{Q \in SO(3) \mid Ql_{\mathscr{B}} = l_{\mathscr{B}}\} = \{e^{\hat{a}t}\} = S^1$, and $\mathscr{K} = \{(\hat{a}t)\, X\}$, where

$$\hat{a} = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}.$$

To systematize the presentation, we divide this subsection into parts (I)–(IV).

(I) *The subspace* $\mathscr{U}_Z$. Write

$$\mathscr{L} = \operatorname{Im} L \oplus Z, \tag{1}$$

the L^2 orthogonal decomposition.

2.6 Definition. Let $\mathscr{U}_Z = \{u \in \mathscr{U} \mid \langle Z, u\rangle = 0\}$.

2.7 Lemma. (i) $\mathscr{U} = \mathscr{K} \oplus \mathscr{U}_Z$. (2)

(ii) *there is a neighborhood* U *of* 0 *in* $\mathscr{U}_Z$ *such that the map* $\varrho : S^1 \times (I + U) \to \mathscr{C}$ *defined by* $\varrho(Q, I + u) = Q^{-1}(I + u)$ *is a tubular neighborhood of* $S^1 I_{\mathscr{B}}$ *in* $\mathscr{C}$.

Proof. (i) Given $u \in \mathscr{U}$, consider the linear map $z \to \langle z, u\rangle$ on Z. By condition (F) on L, $\langle z, KX\rangle$ is a non-degenerate pairing between Z and $\mathscr{K}$. Thus there is a unique $KX \in \mathscr{K}$ such that $\langle z, u\rangle = \langle z, KX\rangle$. Hence $u = KX + (u - KX)$ is the required decomposition (2).

(ii) Part (ii) follows from (i) and the fact that $\mathscr{K} = T_{I_{\mathscr{B}}} GI_{\mathscr{B}}$, by Proposition 2.3. ■

2.8 Lemma. $Z = \mathscr{U}_Z^{\perp} = \{l \mid l \perp \mathscr{U}_Z\}$.

Proof. Clearly, $Z \subset \mathscr{U}_Z^{\perp}$. On the other hand, let $l \perp \mathscr{U}_Z$. By the non-degeneracy of the pairing between Z and $\mathscr{K}$, there is an element $z \in Z$ such that $\langle l, u\rangle = \langle z, u\rangle$ for all $u \in \mathscr{K}$. Therefore $\langle l - z, u\rangle = 0$ for all $u \in \mathscr{K} + \mathscr{U}_Z = \mathscr{U}$ and so $l = z \in Z$. ■

(II) *A potential function on* S^1. Recall that ϕ is an equilibrium solution with load $l_{\mathscr{B}} + \lambda l$ if and only if ϕ is a critical point of the function $V(\phi) = \mathscr{V}(\phi) - \langle l_{\mathscr{B}} + \lambda l, \phi\rangle$ on $\mathscr{C}$, where $\mathscr{V}(\phi) = \int_{\mathscr{B}} W(\phi)\, dV$. The solution ϕ is called *stable* if and only if ϕ is a local minimum of V. Lemma 2.7 implies that to consider equilibrium solutions near $S^1 I_{\mathscr{B}}$ one needs to examine the critical point (Q, ϕ), $\phi \in I + U$, of the function $V_\varrho(Q, \phi) = V \circ \varrho(Q, \phi) = V(Q^{-1}\phi) = S(\phi) - \langle Q(l_{\mathscr{B}} + \lambda l), \phi\rangle$.

2.9 Proposition. *Let* $(Q, \phi) \in S^1 \times (I + U)$. *Then* (Q, ϕ) *is a critical point of* V_ϱ *if and only if*

(i)
$$\Phi(\phi) - l_{\mathscr{B}} \equiv \lambda Q l_{\mathscr{B}} \,(\mathrm{mod}\ Z) \tag{3}$$

and

(ii)
$$\langle \lambda W Q l, \phi\rangle = 0 \text{ for all } W \in \mathfrak{g}. \tag{4}$$

Proof.
$$\begin{aligned} D_\phi V_\varrho(u) &= \int_{\mathscr{B}} \left\langle \frac{\partial W}{\partial F}, Du \right\rangle dV - \langle Q(l_{\mathscr{B}} + \lambda l), u\rangle \\ &= \langle \Phi(\phi), u\rangle - \langle Q(l_{\mathscr{B}} + \lambda l), u\rangle \\ &= \langle \Phi(\phi) - l_{\mathscr{B}} - \lambda Q l, u\rangle. \end{aligned}$$

Thus $D_\phi V_\varrho(u) = 0$ for all $u \in \mathscr{U}_Z$, if and only if (i) holds by Lemma 2.8. Since $D_Q V_\varrho(WQ) = -\langle WQ(l_{\mathscr{B}} + \lambda l), \phi\rangle$, it follows that $D_Q V_\varrho(WQ) = 0$ for all $W \in \mathfrak{g}$ if and only if (ii) holds. ■

Now we perform a Liapunov-Schmidt procedure on the function V_ϱ. By condition (F), $D\Phi(I_{\mathscr{B}})$ is an isomorphism from $\mathscr{U}_Z$ onto $\mathscr{L}$ mod Z. Thus by the inverse mapping theorem, there exists a unique solution $\phi = \phi_Q(\lambda l) \in I + U$ of equation (3) for $\lambda \geqq 0$ small and l near l_0. For simplicity in notation, we drop λl if there is no danger of confusion.

2.10 Definition. *Define* $f: S^1 \to \mathbb{R}$, *by* $f(Q) = V_\varrho(Q, \phi_Q)$.

Thus

2.11 Lemma. (Q, ϕ_Q) *is a critical point of* $V_\varrho(Q, \phi)$ *if and only if* Q *is a critical point of* f *on* S^1.

2.12 Lemma. *Suppose that condition* (S) *on* $\mathrm{A}(X)$ *holds. Then* $\langle \mathrm{A}(\nabla u), \nabla u\rangle_V \geqq c\, \|Du\|^2$ *on* $\mathscr{U}_Z$, *for some constant* $c > 0$.

Proof (*cf.* FICHERA [1972, p. 384]). If the lemma were not true, there would exist a sequence $v^{(n)} \in \mathscr{U}_Z + \mathbb{R}$, such that $\langle \mathbf{A}(\nabla v^{(n)}), \nabla v^{(n)} \rangle_V \to 0$, $\|Dv^{(n)}\|^2 = 1$ with $\int v^{(n)}\, dV = 0$. Therefore, by Rellich's compactness theorem, we may assume that $v^{(n)}$ converges to v in the space L^2. Notice that $\|\tilde{\nabla} v^{(n)}\|_0 = \|\frac{1}{2}[\nabla v^{(n)} + (\nabla v^{(n)})^T]\| \to 0$ as $n \to \infty$. By Korn's second inequality

$$\|v^{(n+p)} - v^{(n)}\|_1^2 \leqq c_2^{-1}\left(\int \|\tilde{\nabla} v^{(n+p)} - \tilde{\nabla} v^{(n)}\|_0^2 + \|v^{(n+p)} - v^{(n)}\|_0^2\right).$$

Thus $v^{(n)}$ converges to v in $W^{1,2}$, and $\langle \mathbf{A}(\nabla v), \nabla v \rangle_V = 0$, $\|Dv\|^2 = 1$. By condition (S), it follows from $\langle \mathbf{A}(\nabla v), \nabla v \rangle_V = 0$ that $(\nabla v) + (\nabla v)^T = 0$ and $v = a + KX$, with $KX \in \mathscr{K}$. Since $v \in \mathscr{U}_Z + \mathbb{R}$, it follows that $KX \in \mathscr{U}_Z$ and so K must be zero by Lemma 2.7. This contradicts the fact that $\|Dv\|^2 = 1$. ■

2.13 Lemma. *V_ϱ has a local minimum at (Q, ϕ_Q) if and only if f has a local minimum at Q.*

Proof. It suffices to show that to each (Q, ϕ_Q) there corresponds a neighborhood $\mathscr{N}_1 \times \mathscr{N}_2$ of (Q, ϕ_Q) such that $V_\varrho|_{\tilde{Q} \times \mathscr{N}_2}$ has a strict minimum at $\phi_{\tilde{Q}}$ for any $\tilde{Q} \in \mathscr{N}_1$. Let $\tilde{F} = D\phi_{\tilde{Q}}$. Then

$$W(\tilde{F} + \tilde{H}) = W(\tilde{F}) + \left\langle \frac{\partial W}{\partial F}(\tilde{F}), \tilde{H} \right\rangle + \frac{1}{2}\frac{\partial^2 W}{\partial F^2}(\tilde{F})(\tilde{H})^2 + O(|\tilde{H}|^3).$$

Thus

$$\begin{aligned} V_\varrho|_{\tilde{Q} \times \mathscr{N}_2}(\phi_{\tilde{Q}} + u) - V_\varrho|_{\tilde{Q} \times \mathscr{N}_2}(\phi_{\tilde{Q}}) &= \left[\int \left\langle \frac{\partial W}{\partial F}, Du \right\rangle dV - \langle \tilde{Q}(l_{\mathscr{B}} + \lambda l), u \rangle\right] \\ &\quad + \int \left[\frac{1}{2}\frac{\partial^2 W}{\partial F^2}(Du)^2 + O(|Du|^2)\right] dV \\ &= \int \left[\frac{1}{2}\frac{\partial^2 W}{\partial F^2}(Du)^2 + O(|Du|)^3\right] dV \\ &\geqq c\,\|Du\|^2 - k\,\|Du\|^3 > 0 \end{aligned}$$

for small $\eta_1, \eta_2, \lambda, \|l - l_0\|$, by the definition of critical points, continuity arguments and Lemma 2.12. ■

Summarizing, we have proved the following result:

2.14 Theorem. *Suppose that $l_{\mathscr{B}}$ is a non-zero parallel load and l_0 is an arbitrary load. For λ small and positive and for l near l_0, the set of solutions of the elastostatic equations $\Phi(\phi) = l_{\mathscr{B}} + \lambda l$ in a neighborhood of $S^1 I_{\mathscr{B}}$ in $\mathscr{C}$ is in 1-1 correspondence with the critical points of f. Furthermore, the stable solutions correspond to the local minima of f on S^1.*

2.15 Corollary. *For λ small and positive and for l near l_0, there exist at least two solutions, one of which must be stable.*

If $l = l_0$ is parallel to $l_{\mathscr{B}}$, then the function f can be made S^1-invariant, and thus must be constant on S^1. Hence we obtain:

2.16 Corallary. *Suppose that $l = l_0$ is parallel to $l_{\mathscr{B}}$, and that $\lambda \geqq 0$ is small. Then there exists exactly a circle of solutions $S^1\phi^*$ near $S^1 I_{\mathscr{B}}$.*

The condition "$l = l_0$ is parallel to $l_{\mathscr{B}}$" is very degenerate. Next we examine our problem in the non-degenerate case.

(III) *The second-order potential on S^1.* Expand the solution of equation (3) as

$$\phi_Q = I_{\mathscr{B}} + \lambda u_Q(l) + O(\lambda^2). \tag{5}$$

2.17 Lemma. $L(u_Q(l)) = (Ql)_{\mathrm{Im}L}$, *where* $u_Q(l) \in \mathscr{U}_Z$, *and* $Ql = (Ql)_{\mathrm{Im}L} \oplus (Ql)_Z$, *according to the decomposition* (1).

Proof. $\Phi(I_{\mathscr{B}} + \lambda u_Q(l) + O(\lambda^2)) - l_{\mathscr{B}} \equiv \lambda Ql\,(\mathrm{mod}\,Z)$. Thus $L(u_Q(l)) = (Ql)_{\mathrm{Im}L}$. ■

2.18 Proposition. *For any* $Q \in S^1$,

$$\begin{aligned} f(Q) &= \mathscr{V}(I_{\mathscr{B}}) - \langle l_{\mathscr{B}}, Q^T I_{\mathscr{B}}\rangle - \lambda\langle l, Q^T I_{\mathscr{B}}\rangle \\ &\quad - \frac{\lambda^2}{2}\langle \mathrm{A}(\nabla u_Q^0), \nabla u_Q^0\rangle + O(\lambda^2\,\|l - l_0\|) + O(\lambda^3), \end{aligned} \tag{6}$$

where $L(u_Q^0) = (Ql_0)_{\mathrm{Im}L}$.

Proof. Write $\phi_Q = I_{\mathscr{B}} + \lambda u_Q(l) + O(\lambda^2)$. Thus

$$\begin{aligned} f(Q) &= \int W(\phi_Q) - \langle Ql_{\mathscr{B}} + \lambda Ql, \phi_Q\rangle \\ &= \mathscr{V}(I_{\mathscr{B}}) + \int \frac{\partial W}{\partial F}(I_{\mathscr{B}})\,(\phi_Q - I_{\mathscr{B}})\,dV \\ &\quad + \frac{1}{2}\int \frac{\partial^2 W}{\partial F^2}(I_{\mathscr{B}})\,(\phi_Q - I_{\mathscr{B}})^2\,dV + O(\lambda^3) - \langle Ql_{\mathscr{B}} + \lambda Ql, \phi_Q\rangle \\ &= \mathscr{V}(I_{\mathscr{B}}) + \langle l_{\mathscr{B}}, \phi_Q - I_{\mathscr{B}}\rangle \\ &\quad + \frac{\lambda^2}{2}\langle \mathrm{A}(\nabla u_Q,(l)), \nabla u_Q(l)\rangle + O(\lambda^3) - \langle Ql_{\mathscr{B}} + \lambda Ql, \phi_Q\rangle \\ &= \mathscr{V}(I_{\mathscr{B}}) - \langle l_{\mathscr{B}} Q^T\rangle - \lambda\langle l, Q^T\rangle - \langle \lambda Ql, \phi_Q - I_{\mathscr{B}}\rangle \\ &\quad + \frac{\lambda^2}{2}\langle \mathrm{A}(\nabla u_Q(l)), u_Q(l)\rangle + O(\lambda^3) \\ &= \mathscr{V}(I_{\mathscr{B}}) - \langle l_{\mathscr{B}}, Q^T\rangle - \lambda\langle l, Q^T\rangle \\ &\quad - \frac{\lambda^2}{2}\langle \mathrm{A}(\nabla u_Q^0), \nabla u_Q^0\rangle + O(\lambda^2\,\|l - l_0\|) + O(\lambda^3). \end{aligned}$$

At this point we need to use the following facts which are readily verified with the divergence theorem:

(a) $\int \left\langle \frac{\partial W}{\partial F}(I_{\mathscr{B}}), \nabla v \right\rangle dV = \langle l_{\mathscr{B}}, v \rangle,$

(b) $\langle \mathrm{A}(\nabla u), \nabla v \rangle = \langle L(\nabla u), v \rangle,$

so that

$$\langle Ql, u_Q(l) \rangle = \langle \mathrm{A}(\nabla u_Q(l)), u_Q(l) \rangle$$

and

$$\langle \mathrm{A}(\nabla u_Q(l_0)), \nabla u_Q(l_0) \rangle = \langle \mathrm{A}(\nabla u_Q^0), \nabla u_Q^0 \rangle. \quad \blacksquare$$

Since $\mathscr{V}(I_{\mathscr{B}})$ and $\langle l_{\mathscr{B}}, Q^T \rangle = \mathrm{tr}\,(k(l_{\mathscr{B}}))$ are constant, first order considerations concerning the critical points of f lead to the following classification of l_0.

2.19 Definition. *The load l_0 is said to be of*

type (α) *if* $\langle l_0, Q^T I_{\mathscr{B}} \rangle \neq$ *constant on* S^1,

type (β) *if* $\langle l_0, Q^T I_{\mathscr{B}} \rangle =$ *constant on* S^1.

If l_0 is a load of type (α), then clearly $\langle l_0, Q^T \rangle$ is a Morse function, with exactly two critical points on S^1.

Remarks. (a) This classification is directly related to the classification of loads l_0 into 5 types in the load-free or stress-free case (see [I]).

(b) Suppose that $\langle l_0, Q^T I_{\mathscr{B}} \rangle$ has a critical point on S^1 at $Q = I$. Then $\langle l_0, Q^T I_{\mathscr{B}} \rangle$ has a non-degenerate critical point on S^1 at $Q = I$ if and only if $(A - \mathrm{Tr}\, A)\, a \cdot a \neq 0$ for $A = k(l_0)$ (where l_0 is parallel to a). Thus the condition (α) on the load l_0 renders global the condition on l_0 in Theorem 4 of BHARATHA & LEVINSON [1978].

(IV) *Bifurcation analysis*: *the non-degenerate cases.* From equation (6) one sees that the bifurcation analysis of loads of types (α) and (β) is similar to that for loads of type 0 and type 1 for a stress-free reference configuration. Only those parts of the proofs of the results that are different need be given.

2.20 Theorem. *Let condition* (S) *hold. Suppose l_0 is of type* (α). *Then for $\lambda > 0$ and l near l_0, there exist exactly two solutions, one stable and one unstable.*

Proof. For $\lambda > 0$, set $f^* = \dfrac{f - S(I_{\mathscr{B}}) + \langle l_{\mathscr{B}}, Q^T I_{\mathscr{B}} \rangle}{\lambda}$. The theorem follows from the facts that $f^* = -\langle l, Q^T I_{\mathscr{B}} \rangle + O(\lambda)$ and $\langle l, Q^T I_{\mathscr{B}} \rangle$ is a Morse function on S^1. $\blacksquare$

Next we consider a load l_0 of type (β). Without loss of generality, we can assume that l_β is parallel to (0, 0, 1). Thus

$$S^1 = \left\{ \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x^2 + y^2 = 1 \right\}.$$

Define constants b_i and a_i by

$$\begin{aligned} &\langle l, Q^T I_{\mathscr{B}} \rangle + \frac{\lambda}{2} \langle \mathrm{A}(\nabla u_Q^0), \nabla u_Q^0 \rangle \\ &= (b_0 + b_1 x + b_2 y) + \frac{\lambda}{2} (a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6), \end{aligned} \tag{7}$$

where

$$Q = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x^2 + y^2 = 1.$$

By rotation of l_0 by some $Q \in S^1$ if necessary, we can take $a_2 = 0$. Fix α_1, α_3 with $\alpha_1 \neq \alpha_3$, and set

$$\Delta(\alpha_4, \alpha_5) = [2(\alpha_1 - \alpha_3)^2 - \alpha_4^2 - \alpha_5^2]^3 - 108 \alpha_4^2 \alpha_5^2 (\alpha_1 - \alpha_3)^2. \tag{8}$$

Since $\Delta = 0$ if and only if $[2(\alpha_1 - \alpha_3)]^{\frac{2}{3}} = \alpha_4^{\frac{2}{3}} + \alpha_5^{\frac{2}{3}}$, $\Delta = 0$ defines an astroid in the (α_4, α_5)-plane (just as in [I]).

We are now ready to state our main results on the number of solutions near loads of type (β). Let $l = l(c)$ depend smoothly on a parameter c in $\mathbb{R}^m$, with $l(0) = l_0$.

2.21 Theorem. *Suppose the load l_0 is of type (β) with $a_2 = 0$, $a_1 \neq a_3$. Then there exists a (smooth) function $\tilde{\Delta}(\lambda, c)$ with $\tilde{\Delta}(\lambda, 0) = \Delta(a_4, a_5) + O(\lambda)$ defined for (λ, c) sufficiently small and for $\lambda > 0$, such that our traction problem has*

(i) *two solutions for the load $\lambda l(c)$, one of which is stable, if $\tilde{\Delta}(\lambda, c) < 0$.*

(ii) *four solutions for the load $\lambda l(c)$, two of which are stable, if $\tilde{\Delta}(\lambda, c) > 0$.*

The proof of this theorem is basically the same as that for Theorem 8.4 in [I].

The following theorem describes the "generic" structure of the bifurcation set $\Sigma = \{(\lambda, c) \mid \lambda > 0, \tilde{\Delta} = 0\}$. Denote by $\tilde{k}: \mathbb{R}^m \to (\alpha_4, \alpha_5)$-space, the affine map defined by $\tilde{c} \to \left(\frac{b_1 A(\tilde{c})}{2} + a_4, \frac{b_2 A(\tilde{c})}{2} + a_5 \right)$, where $A(\tilde{c})$ is the linear part of $l(\tilde{c})$.

2.22 Theorem. *Suppose that the affine map k is of maximal rank and is transversal to the astroid $\Delta = 0$. Then the bifurcation set Σ is*

(0) *empty for $m = 0$,*

(1) *as in Figures 1 or 2 for $m = 1$,*

(2) *as in Figure* 3 *for* $m = 2$,
(3) *a cylinder-like set with height* $\mathbb{R}^{m-2}$ *and base as in Figure* 3 *for* $m > 2$.

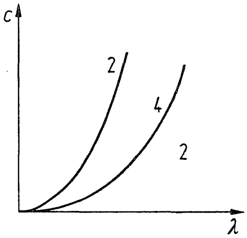

Fig. 1

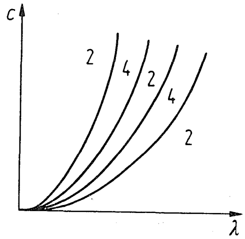

Fig. 2

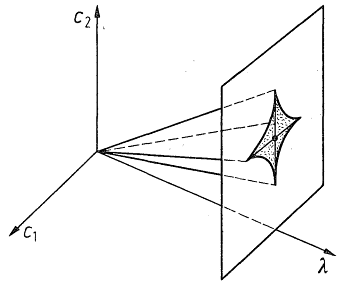

Fig. 3

2.23 An example. Expand the first and second Piola-Kirchhoff stress tensors around $I_{\mathscr{B}}$ as follows:

$$P = T_0 + \mathbf{A}(H) + O(|H|^2),$$

$$S = \tilde{T}_0 + \mathbf{C}(H) + O(|H|^2).$$

From the standard relation $S = F^{-1}P$, or from

$$(I - H + \ldots)(T_0 + \mathbf{A}(H) + \ldots) = \tilde{T}_0 + \mathbf{C}(H) + \ldots,$$

one gets

$$T_0 = \tilde{T}_0,$$

$$\mathbf{A}(H) = \mathbf{C}(H) + HT_0.$$

Conversely, given any $T_0(X) \in \mathrm{sym}$, and a symmetric linear map $\mathbf{C}(X)\colon \mathrm{sym} \to \mathrm{sym}$, define

$$W(\mathbf{C}) = \langle T_0, D\rangle + \frac{1}{2}\langle D, \mathbf{C}(D)\rangle, \text{ where } D = \frac{1}{2}(\mathbf{C} - 1).$$

Straightforward computations show that

$$P = \frac{\partial W}{\partial F} = FT_0 + F\mathbf{C}(D),$$

$$\mathbf{A}(H) = \frac{\partial P}{\partial F}(H) = \mathbf{C}(H) + HT.$$

Thus there exists a hyperelastic material with prescribed T_0 and $\mathbf{C}$, provided $T_0 \in \mathrm{sym}$, and $\mathbf{C}$ is symmetric on sym.

2.24 Example. Consider a homogeneous material, occupying a unit volume in $\mathbb{R}^3$, with $\tau_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\mathbf{C}(e) = e - \frac{1}{2}\operatorname{diag}(e)$, $e \in \mathrm{sym}$. Let us study the traction problem with loads λl_0, where $l_0 = (0, \tau_0)$, and $\tau_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} N$.

Since $l_{\mathscr{B}} = \begin{pmatrix} -\operatorname{Div} T_0 \\ \\ T_0 N \end{pmatrix} = g \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, for some $g : \overline{\mathscr{B}} \to \mathbb{R}$, the initial load $l_{\mathscr{B}}$ is parallel to (0, 0, 1), and $S^1 = \left\{ \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x^2 + y^2 = 1 \right\}$. We claim that

(i) condition (S) is satisfied,
(ii) l_0 is of type (β),

and

(iii) $\int \langle \mathbf{A}(\nabla u_Q^0), \nabla u_Q^0 \rangle \, dV = 4x^2 + 2y^2$.

Therefore Theorem 2.21 can be applied, and our traction problem has four solutions for λl_0, with λ small and positive, exactly two of which are stable.

Proof. (i) Since $\mathbf{A} \begin{pmatrix} 0 & h_3 & 0 \\ -h_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$ where $h_3 \in \mathbb{R}$, it suffices to show that

$$\langle \mathbf{A}(e + H), e + H \rangle > 0 \quad \text{for} \quad (e, H) \neq 0,$$

where

$$H = \begin{pmatrix} 0 & 0 & -h_2 \\ 0 & 0 & h_1 \\ h_2 & -h_1 & 0 \end{pmatrix}, \quad \text{and} \quad h_1, h_2 \in \mathbb{R}.$$

Direct computations give

$$\begin{aligned} \langle \mathbf{A}(e + H), e + H \rangle &= \left\langle e - \frac{1}{2}\operatorname{diag} e + H^+, e + H \right\rangle \\ &= \|e\|^2 - \frac{1}{2}(e_{11}^2 + e_{22}^2 + e_{33}^2) + \langle H^+, e \rangle + \langle H^+, H \rangle \\ &\geqq \frac{1}{2}\|e\|^2 + \langle H^+, e \rangle + 2\,\|H^+\|^2 > 0, \end{aligned}$$

where

$$H^{+} = \begin{pmatrix} 0 & 0 & -h_2 \\ 0 & 0 & h_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(ii) $\langle l_0, Q^T l_{\mathscr{B}} \rangle = \int 0 \, dV = 0$, for all $Q \in S^1$.

(iii) Since $u_Q^0(X) = \mathbf{C}^{-1}\left[Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] X$, where $\mathbf{C}^{-1}(e) = e + \operatorname{diag} e$,

we have

$$\langle \mathsf{A}(\nabla u_Q^0), \nabla u_Q^0 \rangle = \left\langle \begin{pmatrix} 2x & y & 0 \\ y & -2x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} x & y & 0 \\ y & -x & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$$

$$= 4x^2 + 2y^2. \quad \blacksquare$$

§ 3. Bifurcation of potential functions under bisymmetry

The use of two Liapunov-Schmidt reductions and the explicit expressions for the associated reduced function seem to be the reasons for our success in treating the problems in § 2. To pave the way for further applications, we shall present these methods in a general and abstract form.

A. The abstract setting, and a first reduction

Let G be a compact Lie group acting linearly on two Banach spaces $\mathscr{U}$ and $\mathscr{A}$ (called the state space and parameter space respectively, and let $\mathscr{C}$ be an G-invariant submanifold in $\mathscr{U}$. Consider a function $\mathscr{V}: \mathscr{C} \times \mathscr{A} \to \mathbb{R}$ invariant under G, so that $V(x, a) = V(gx, ga)$ for all $g \in G$, $x \in \mathscr{C}$ and $a \in \mathscr{A}$. We are interested in the bifurcation of critical points of $V(\cdot, a)$ (*i.e.*, points where the x-derivative of V vanishes), for x near the orbit Ge and for a near the point a_0. We assume that $V(\cdot, a_0)$ restricted to $\mathscr{C}$ has a critical point at a given point $e \in \mathscr{C}$ and that $ga_0 = a_0$ for all $g \in G$. Notice that the group G acts on both the state space $\mathscr{C}$ and the parameter space $\mathscr{A}$.

To fix the idea, we take $\mathscr{C}$ to be an open set in $\mathscr{U}$, and, as before, we examine the bifurcation problem for $V(\cdot, a)$ with $a = a_0 + \lambda a_1$, for λ small and positive and for a_1 near some fixed a_1^* in $\mathscr{A}$. Let G also act on a Banach space $\mathscr{L}$ with a G-equivariant non-degenerate pairing $\langle \, , \rangle$ between $\mathscr{L}$ and $\mathscr{U}$. We postulate that $V_a = V(\cdot, a)$ has a gradient $\nabla V_a = \Psi_a$ with values in $\mathscr{L}$ with respect to this pairing, *i.e.*, $DV_a(x)(u) = \langle \Psi_a(x), u \rangle$ for all $u \in \mathscr{U}$. One can readily see that $\Psi: \mathscr{C} \times \mathscr{A} \to \mathscr{L}$, $\Psi(x, a) = \Psi_a(x)$, is G-equivariant. Denote by $\Gamma = \{g \in G \mid ge = e\}$ the isotropy group of e. Clearly, $\Psi_{a_0}: \mathscr{C} \to \mathscr{L}$ is Γ-equivariant. Unless stated otherwise, partial derivatives will be evaluated at (e, a_0).

Indeed, for the traction problem studied in [I] and [II], we take $\mathscr{U} = \{\phi : \mathscr{B} \to \mathbb{R}^3, \phi(0) = 0\}$ the space of infinitesimal deformations, $\mathscr{C} = \{\phi \in \mathscr{U} \mid \det D\phi > 0\}$ the space of deformations, and $\mathscr{A} = \mathscr{L} = \{(b, \tau)\, b : \mathscr{B} \to \mathbb{R}^3, \tau : \partial\mathscr{B} \to \mathbb{R}^3, \int_{\mathscr{B}} b\, dV + \int_{\partial\mathscr{B}} \tau\, dA = 0\}$, the space of loads. $\langle l, \phi\rangle = \int_{\mathscr{B}} \langle b, \phi\rangle\, dV + \int_{\partial\mathscr{B}} \langle \tau, \phi\rangle\, dA$, for $(b, \tau) = l \in \mathscr{L}$ and $\phi \in \mathscr{U}$. Thus the parameters take their values in the "dual" space $\mathscr{L}$ of $\mathscr{U}$, $G = SO(3)$ acts on $\mathscr{U}$, $\mathscr{A}$, and $\mathscr{L}$ by composition, V is given by $V(\phi) = \int W(D\phi)\, dV - \langle l, \phi\rangle$ where W is a stored energy function, $(e, a_0) = (I_{\mathscr{B}}, 0)$, and $\Gamma = \{\text{identity}\}$. We also have $\Psi_l(\phi) = \Phi(\phi) - l$ with $a = l$, where $\Phi(\phi) = (-\mathrm{DIV}\, P(\phi), P(\phi)\, N)$ is as in the previous section.

Now we assume that the Fredholm alternative holds for the Γ-equivariant symmetric operator $L = D\Psi_{a_0} : \mathscr{U} \to \mathscr{L}$. Thus $\langle Lu, v\rangle = \langle Lv, u\rangle$. We assume the condition

(F) $\dim \ker L < \infty$, *and* $\operatorname{Im} L = (\ker L)^{\perp}$, *with*

$$(\ker L)^{\perp} = \{l \in \mathscr{L} \mid \langle l, k\rangle = 0, \textit{ for all } k \in \ker L\}.$$

Under this condition, we obtain a Γ-invariant decomposition,

$$\mathscr{L} = \operatorname{Im} L \oplus Z, \quad \text{with } \dim Z = \dim(\ker L).$$

Let $\mathscr{U}_Z = \{u \in \mathscr{U} \mid \langle l, u\rangle = 0$ for all $l \in Z\}$. We also get a Γ-invariant decomposition

$$\mathscr{U} = \ker L \oplus \mathscr{U}_Z.$$

By G-invariance of Ψ_{a_0}, $\ker L \supset T_eGe$, the tangent space of Ge at e. Take any Γ-invariant complement V of T_eGe in $\ker L$, and set $W = V \oplus \mathscr{U}_Z$, a Γ-invariant subspace transversal to T_eGe. Thus, for some Γ-invariant open neighborhood $W_\varepsilon = V_\varepsilon \times U_\varepsilon$ of a in W, the map $\varrho : G \times \{e + W_\varepsilon\} \to \mathscr{U}$ defined by $\varrho(g, \phi) = g\phi$ induces a G-equivariant diffeomorphism of $G \times_\Gamma \{e + W_\varepsilon\}$ onto $G(e + W_\varepsilon)$, a tubular neighborhood of Ge in $\mathscr{U}$. Here $G \times_\Gamma \{e + W_\varepsilon\}$ denotes the quotient manifold of $G \times \{e + W_\varepsilon\}$ under the free action of the compact group $\Gamma : \gamma(g, \phi) = (g\gamma^{-1}, \gamma\phi)$. Observe that $G(e + V_\varepsilon)$ is a submanifold, and $Ge \subset G(e + V_\varepsilon) \subset G(e + W_\varepsilon)$. Clearly,

$$V_a \circ \varrho(g, v + \phi) = V(g(v + \phi), a) = V(v + \phi, g^{-1}a).$$

By varying ϕ, v, and g in $e + U_\varepsilon$, V and G separately, we obtain

3.1 Proposition. *Let* $(g, v, \phi) \in G \times V_\varepsilon \times (e + U_\varepsilon)$. *Then* (g, v, ϕ) *is a critical point of* $V_a \circ \varrho$ *if and only if*

(a) $\Psi_{g^{-1}a}(v + \phi) \in Z$,

(b) $\Psi_{g^{-1}a}(v + \phi), w\rangle = 0$, *for all* $w \in V$,

and

(c) $V(v + \phi, g^{-1}a)$ *has a critical point on* G *at* g.

Since $L = D\Psi_{a_0} : \mathscr{U}_Z \to \operatorname{Im} L$ is an isomorphism, we can apply the implicit function theorem to Ψ modulo $\operatorname{Im} L$. Thus for some Γ-invariant neighborhood $W_2' = V_\varepsilon' \times U_\varepsilon'$ of 0 in W_ε and a δ-neighborhood of a_0, there exists a unique $\phi_a(g, v) \in e + U_\varepsilon'$, defined on $G \times V_\varepsilon'$, $\|a - a_0\| < \delta$, such that $\Psi_{g^{-1}a}(v + \phi_a(g, v)) \in Z$. Indeed, $\phi_a(g\gamma^{-1}, \gamma v) = \phi_a(g, v)$ for $\gamma \in \Gamma$.

3.2 Lemma. $\phi_a(g, v) = e + \lambda u_{a_1}(g) + O(|\lambda|^2 + |\lambda|\,|v| + |v|^2)$, *where* $u_{a_1}(g) \in \mathscr{U}_Z$ *satisfies the equation*

$$L(u) + \frac{\partial\Psi}{\partial a}(g^{-1}a_1) \in Z \quad \left(\text{here } \frac{\partial\Psi}{\partial a} = \frac{\partial\Psi}{\partial a}(a, e_0)\right). \tag{9}$$

Proof. Write $\phi_a(g, v) = e + \lambda u + O(v) + O(|\lambda|^2 + |\lambda|\,|v| + |v|^2)$. Therefore,

$$\Psi_{a_0 + \lambda g^{-1}a_1}(e + \lambda u + O(v) + \ldots) \in Z.$$

Gathering terms of first order in λ, v gives

$$\lambda \frac{\partial\Psi}{\partial a}(g^{-1}a_1) + L(\lambda u + O(v)) \in Z.$$

Consequently, $L(u) + \dfrac{\partial\Psi}{\partial a}(g^{-1}a_1) \in Z$. ∎

Applying the Liapunov-Schmidt procedure to $V_a \circ \varrho$, we have

$$\begin{aligned} f_a(g, v) &= V_a \circ \varrho(g, v + \phi_a(g, v)) \\ &= V(v + \phi_a(g, v), a_0 + \lambda g^{-1}a_1) \\ &= V(v + \phi_{a_0}(e, v), a_0) + \lambda \left(\frac{\partial V}{\partial a}(g^{-1}a_1)\right) + O(v) + O(\lambda^2). \end{aligned} \tag{10}$$

Here, $\dfrac{\partial V}{\partial a} = \dfrac{\partial V}{\partial a}(e, a_0)$ and each term is invariant under the action of $\Gamma : \gamma \cdot g = g\gamma^{-1}, \gamma \cdot v = \gamma v$. Thus $(f_a \mid \Gamma)(g(e + v)) = f_a(g, v)$ is well-defined on $G(e + V')$. Therefore we can make a reduction of V_a to a function $f_a \mid \Gamma$ on the submanifold $G(e + V_\varepsilon')$.

To obtain stability results, one needs the following condition.

(Σ) *There exists a real number* $c > 0$, *a norm* $\|\;\|_1$ *on* $\mathscr{U}$ *satisfying* $\|\cdot\|_1 \leqq \|\cdot\|$, *and such that* $\dfrac{\partial^2 V}{\partial x^2}(x, a)(u^2) \geqq c\|u\|_1^2$ *for all* $u \in \mathscr{U}_Z$ *and* $(x, a) \in (e + W) \times \mathscr{A}$ *sufficiently close to* (e, a_0).

Stable solutions are defined to be strict local minima of V_a. Fix a small neighborhood $\tilde{W}_\varepsilon = \tilde{V}_\varepsilon \times \tilde{U}_\varepsilon$ of a in W_ε', and small $\tilde{\delta}$ with $0 < \tilde{\delta} < \delta$, such that condition ($\Sigma$) holds for $x \in e + \tilde{W}_\varepsilon$, and $\|a - a_0\| < \tilde{\delta}$. Let $(\bar{g}, \bar{v}, \bar{\phi}) \in G \times \tilde{V}_\varepsilon \times \tilde{W}_\varepsilon$ be a critical point of $V_a \circ \varrho$ with $\|a - a_0\| < \tilde{\delta}$. By the splitting lemma (Tromba

[1976] and GOLUBITSKY & MARSDEN [1983]), there exists a local diffeomorphism $\phi_a : (g, v, \phi) \to (g, v, u)$ near $(\bar{g}, \bar{v}, \bar{\phi})$ sending $(g, v, \phi_a(gv))$ to $(g, v, 0)$ such that

$$V_a \circ \varrho(g, v, \phi) = V(v + \phi, g^{-1}a)$$

$$= \frac{1}{2} \frac{\partial^2 V}{\partial x^2} (v + \phi_a(g, v), a)(u^2) + f_a(g, v)$$

for (g, v, ϕ) near $(\bar{g}, \bar{v}, \bar{\phi})$. Hence, by condition (Σ), V_a has a (strict) local minimum at $\bar{g}(\bar{v} + \tilde{\phi})$ if and only if $f_a \mid \Gamma$ has a (strict) local minimum at $\bar{g}(e + \bar{v})$ in the submanifold $G(e + \tilde{V}_\varepsilon)$. It is natural to define index $V_a =$ index $f_a \mid \Gamma$. For a non-degenerate $f_a \mid \Gamma$, the fact that index $V_a =$ index $f_a \mid \Gamma = 0$ simply means stability.

B. A second reduction

To obtain additional results, we make more hypotheses and consider three cases. As we shall see, we have a bifurcation problem on the slice in the first case (A) considered below, a bifurcation problem on the group orbit in case (B), and a combination of both in case (C).

Case (A). $Ga_1 = a_1$ and $a = a_0 + \lambda a_1$.

Thus $\phi_a(g, v) = \phi_a(e, v)$ and $f_a(g, v) = f_a(e, v)$ for all $g \in G$. Hence (g, v) is a critical point of f_a if and only if v is a critical point of the Γ-invariant function $f_a(e, \cdot)$ on V. It may be easier to understand this fact by the following reasoning. Since $V(gx, a) = V(x, a)$, for all $g \in G$, it follows that if x is a critical point of V_a, then so is the whole orbit Gx. Thus, one need only look for solutions in the cross-section $e + W_\varepsilon$. Since $V(gx, a)$ is a constant along the orbit Gx, these solutions are critical points of the function V_a restricted to $e + W_\varepsilon$. Our problem is thereby reduced to a more familiar one: study the family of Γ-invariant functions $\tilde{V}_a(w) = V_a(e + w)$, on W_ε, with a varying in a neighborhood of a_0. By the Liapunov-Schmidt reduction, we find that the reduced potential is precisely the function $f_a(e, v)$. To study such a bifurcation problem for $f_{a_0}(e, v)$, we may consider its Γ-codimension, and examine its universal Γ-unfolding. We may also consider an imperfect bifurcation for such Γ-invariant functions. For more details, consult POÉNARU [1976], WASSERMAN [1974] and GOLUBITSKY & SCHAEFFER [1979], *etc.*

Case (B). $\ker L = T_e Ge$ (*i.e.*, $V = \{0\}$).

Here critical points of V_a are in 1–1 correspondence with those of $f_a \mid \Gamma$ on $Ge \approx G_a \Gamma$ (the isomorphism being $g\phi_a(g) \leftrightarrow ge$).

The following is an immediate consequence of the Liapunov-Schmidt reduction discussed above and of critical point theory.

3.3 Theorem. *Suppose condition* (F) *on* L *is fulfilled with* $\ker L = T_eGe$. *Then, for* $\|a - a_0\|$ *small, there exists at least* $\operatorname{cat}(G/\Gamma)$ *solutions near* Ge.

Here, $\operatorname{cat}(G/\Gamma)$ denotes the category of the space G/Γ in the sense of Liusternik and Šnirel′man.

To get more information on the solution set, we must obtain an expression for $f_a(g)$ up to second order in λ^2, which is invariant under the action of $\Gamma : \gamma \cdot g = g\gamma^{-1}$ for $\gamma \in \Gamma$. Equation (9) yields

$$\frac{\partial^2 V}{\partial x^2}(u, u_{a_1}(g)) + \frac{\partial^2 V}{\partial x\,\partial a}(u, g^{-1}a_1) = \left\langle L(u_{a_1}(g)) + \frac{\partial \Psi}{\partial a}(g^{-1}a_1), u \right\rangle = 0,$$

$$\text{for all } u \in \mathscr{U}_Z.$$

In particular,

$$-\frac{\partial^2 v}{\partial x\,\partial a}(u_{a_1}(g), g^{-1}a_1) = \frac{\partial^2 V}{\partial x^2}(u_{a_1}(g), u_{a_1}(g)) = \langle L(u_{a_1}(g)), u_{a_1}(g)\rangle. \tag{11}$$

Therfore, expanding V at (e, a_0) into Γ-invariant terms, we obtain

$$\begin{aligned} f_a(g) &= V(\phi_a(g), a_0 + \lambda g^{-1}g_1) = V(e + (\phi_a(g) - e), a_0 + \lambda g^{-1}a_1) \\ &= V(e, a_0) + \frac{\partial V}{\partial x}(\phi_a(g) - e) + \frac{\partial V}{\partial a}(\lambda g^{-1}a_1) \\ &\quad + \frac{1}{2}\left\{\frac{\partial^2 V}{\partial x^2} + 2\frac{\partial^2 V}{\partial x\,\partial a} + \frac{\partial^2 V}{\partial a^2}\right\} + O(\lambda^3). \end{aligned}$$

Using equation (11), we obtain the Γ-invariant expression,

$$\begin{aligned} f_a(g) = {}& V(e, a_0) + \lambda\frac{\partial V}{\partial a}(g^{-1}a_1) - \frac{\lambda^2}{2}\langle L(u(g)), u(g)\rangle \\ & + \frac{\lambda^2}{2}\frac{\partial^2 V}{\partial a^2}(g^{-1}a_1^{*2}) + O(|\lambda|\,\|a_1 - a_1^*\|) + O(|\lambda|^3). \end{aligned} \tag{12}$$

Here we abbreviate $u_{a_1^*}(g)$ as $u(g)$. First order considerations lead one to consider $\dfrac{\partial V}{\partial a}(g^{-1}a_1^*) : G \to \mathbb{R}$, a linear functional on the orbit of a group representation. For the special cases of $G = SO(n)$, $U(n)$, *etc.*, consult FRAENKEL [1965] and RAMANUJAM [1969].

Now we further assume the following condition on the Γ-invariant function $\dfrac{\partial V}{\partial a}(g^{-1}a_1^*)$, which will allow a second Liapunov-Schmidt reduction:

($\mathfrak{B}$) *The function* $\dfrac{\partial V}{\partial a}(g^{-1}a_1^*) : G \to \mathbb{R}$ *is non-degenerate in the sense of Bott with a critical manifold* S_1.

Set $f_a^*(g) = (f_a(g) - V(e, a_0))/\lambda$, for $\lambda > 0$. Fix any Γ-invariant normal bundle of S_1 in G, and let $k(g) \in G$ be the critical point of f_a^* restricted to the fibre through $g \in S_1$. Since

$$\frac{\partial V}{\partial a}(k^{-1}(g)\, a_1) = \frac{\partial V}{\partial a}(g^{-1}a_1) + O(\lambda \,\|a_1 - a_1^*\|) + O(|\lambda|^2),$$

we get

$$\tilde{f}_a(g) = f_a^*(k(g)) = \frac{\partial V}{\partial a}(g^{-1}a_1) - \frac{\lambda}{2}\langle L(u(g)), u(g)\rangle$$

$$+ \frac{\lambda}{2}\frac{\partial^2 V}{\partial a^2}(g^{-1}a_1^*)^2 + O(\lambda \,\|a_1 - a_1^*\|) + O(\lambda^2) \tag{13}$$

on S_1. Again, each term in the above equation is invariant under Γ, where $\gamma \cdot g = g\gamma^{-1}$ for $\gamma \in \Gamma$. Clearly,

$$\text{Index } V_a \text{ at } k(g)\, \Phi_a(k(g)) = \text{Index}\frac{\partial V}{\partial a}(g^{-1}a_1^*) \text{ at } g + \text{Index}\,\tilde{f}_a(g) \text{ at } g. \tag{14}$$

Let us examine the situation in which $a_1 = a_1(\lambda)$ depends on λ. Thus, $a = a_0 + \lambda a_1(\lambda) = a_0 + \lambda a_1^* + \lambda^2 a_2 + O(\lambda^3)$. Denote by $\theta(g)$ the Γ-invariant function on S_1, $2\frac{\partial V}{\partial a}(g^{-1}a_2) - \langle L(u(g)), u(g)\rangle + \frac{\partial^2 V}{\partial a^2}(g^{-1}a_1^{*2})$. Any Γ-invariant function ξ on S_1 induces a function ξ/Γ on S_1e via $(\xi/\Gamma)(ge) = \xi(g)$.

3.4 Theorem. *Let conditions* (F), *and* ($\mathfrak{B}$) *hold for* V_a, *and let* $a = a_0 + \lambda a_1^* + \lambda^2 a_2 + O(\lambda^3)$. *Suppose* θ/Γ *is a Morse function on* S_1e *with critical points* $g_1e, \ldots, g_ne$. *Then, for* $\lambda > 0$ *small, the solution set* $\{x \mid \Psi_a(x) = 0\}$ *near* Ge *is given by* $\{x_1(\lambda), \ldots, x_n(\lambda)\}$, *where* $x_i(\lambda) \to g_ie$ *as* $\lambda \to 0$ *for* $i = 1, \ldots, n$.

Proof. Since $\frac{\partial V}{\partial a}(g^{-1}a_1^*)$ is locally constant on S_1, one obtains the Γ-invariant expression, $\frac{\partial V}{\partial a}(g^{-1}a_1) = \text{const. (local)} + \lambda\frac{\partial V}{\partial a}(g^{-1}a_2) + O(\lambda^2)$. Hence $f_a/\Gamma = \text{const. (local)} + \frac{1}{2}\,\theta/\Gamma + O(\lambda)$. Since Morse functions are stable under small perturbations, the results follows. ■

Often the variations $a = a_0 + \lambda a_1$ preserve certain symmetry of a_0, *i.e.*, $Ha = a$ for some subgroup H of G. When this happens V_a and θ/Γ are H-invariant functions, so their critical sets consists of critical H-orbits. An H-invariant function is said to be an H-Morse function if to each critical H-orbit, the Hessian is non-degenerate on the transversal cross-section.

A simple generalization of Theorem 3.4 is as follows.

3.5 Theorem. *Let conditions* (F) *and* ($\mathfrak{B}$) *hold for* V_a *and let* $a = a_0 + \lambda a^* + \lambda^2 a_2 + O(\lambda^2)$. *Suppose* $Ha = a$ *for a subgroup* $H \subset G$. *Suppose* θ/Γ *is an* H-*Morse function on* S_1e *with critical orbits* $H(g_1e), \ldots, H(g_ne)$. *Then for* $\lambda > 0$

small, the solution set $\{x \mid \Psi_a(x)=0\}$ *near* Ge *is given by orbits of the form* $\{Hx_1(\lambda), \ldots, Hx_n(\lambda)\}$, *where* $x_i(\lambda) \to g_i e$ *for* $i=1, \ldots, n$.

Proof. It suffices to observe that one can write $\tilde{f}_a/\Gamma = \text{const. (local)} + \frac{1}{2}\,\theta/\Gamma + O(\lambda)$, an H-invariant expression, by choosing a $(H\times\Gamma)$-invariant normal bundle of S_1 in G in reducing f_a^* to $\tilde{f}_a$. ∎

Case (C). *The* Γ*-invariant function* $f_{a_0}(e, v) = V(v + \phi_{a_0}(e, w, a_0))$ *has a strict local minimum in* V *at* $v=0$ *with finite codimension* k.

3.6 Lemma (a). *If there are solutions arbitrarily close to* $\bar{g}e \in Ge$ *for* λ *and* $a_1 - a_1^*$ *small, then* $\bar{g}$ *is a critical point for* $\dfrac{\partial V}{\partial a}(g^{-1}a_1^*)$ *on* G.

(b) $\dfrac{\partial V}{\partial a}(g^{-1}a_1) = \dfrac{\partial V}{\partial a}(ge, a_0)(a_1)$ *and* $\dfrac{\partial^2 V}{\partial a^2}(g^{-1}a_1)^2 = \dfrac{\partial^2 V}{\partial a^2}(ge, a_0)(a_1)^2$, *for any* $a_1 \in \mathscr{A}$.

Proof. (a) By hypotheses, there exist $\lambda_n \searrow 0$, $a_n^* \to a_1$, $g_n \to \bar{g}$, and $v_n \in V \to 0$, as $n\to\infty$, such that $f_{a_n^*}(g, v)$ has a critical point at (g_n, v_n). From equation (10), it follows that the function $\left\{\dfrac{\partial V}{\partial a}(g^{-1}a_n^*) + O(v_n) + O(\lambda_n)\right\}$ has a critical point at $g=g_n$. Since $g_n^{-1}a_n^* \to (\bar{g})^{-1}a_1^*$, $\dfrac{\partial V}{\partial a}(g^{-1}a^*)$ must have a critical point at $\bar{g}$.

(b) Expanding the equation $V(ge, a_0 + \lambda a_1) = V(e, a_0 + \lambda g^{-1}a_1)$ in λ, one obtains $\dfrac{\partial V}{\partial a}(ge, a_0)(a_1) = \dfrac{\partial V}{\partial a}(g^{-1}a_1)$, $\dfrac{\partial^2 V}{\partial a^2}(ge, a_0)(a_1^2) = \dfrac{\partial^2 V}{\partial a^2}(g^{-1}a_1)^2$ *etc.* ∎

From this lemma, it is natural to perform another Liapunov-Schmidt reduction on $f_a(g, v)$, by assuming that

(M) $\dfrac{\partial V}{\partial a}(ge, a_0)(a_1^*)$ *is a Morse function on* Ge.

Let $\bar{\sigma}$ be a critical point of $\dfrac{\partial V}{\partial a}(g^{-1}a_1^*)$ on G. Recall that $f_a(g, v) = f_{a_0}(e, v) + \lambda H_a(g, v)$ and $H_a = \dfrac{\partial V}{\partial a}(g^{-1}a_1) + O(v) + O(\lambda)$, a Γ-invariant function. Let Σ be a cross-section of $\Gamma\cdot\bar{\sigma}$ through $\bar{\sigma}$ in G. Thus $(\sigma, v) \in \Sigma\times V_\varepsilon \to \sigma(e+v) \in \mathscr{C}$ provides a local diffeomorphism. By condition (M), we can uniquely solve the equation

$$\frac{\partial f_a}{\partial\sigma}(\sigma, v) = \lambda\frac{\partial H_a}{\partial\sigma}(\sigma, v) = 0 \quad \text{for} \quad \sigma = \sigma(v, a) \in \Sigma \quad \text{with} \quad \bar{\sigma} = \sigma(0, a_0).$$

Consequently

$$f_a(\sigma(v, a), v) = f_{a_0}(e, v) + \lambda H_a(\sigma(v, a), v). \tag{15}$$

3.7 Theorem. *Suppose conditions* (F) *and* (M) *are satisfied. Then for* λ *small and positive and for* $\|a_1 - a_1^*\|$ *small, there exist at most* $(k+1)\,h$ *solutions, where* $h =$ *the number of critical points of* $\frac{\partial V}{\partial a}(ge, a_0)\,(a_1^*)$ *on* Ge.

Proof. Since the codimension of $f_{a_0}(e, v)$ is k, the function $f_a(\sigma(v,a), v)$, which is a small perturbations of $f_{a_0}(e, v)$, possesses at most $(k+1)$ solutions. The theorem now follows from Lemma 3.6. ■

If $Ha = a$ for some subgroup H in G, then V_a, f_a, and $\frac{\partial V}{\partial a}(ge, a_0)\,(a_1^*)$ are H-invariant functions. Thus it is natural to consider the following condition:

($\mathrm{M_H}$) $\frac{\partial V}{\partial a}(ge, a_0)\,(a_1^*)$ *is an* H*-Morse function on* Ge.

Let $\bar{\sigma}$ be a critical point of $\frac{\partial V}{\partial a}(g^{-1}a_1^*)$ on G. Choose a cross-section Σ of $H\bar{\sigma}\Gamma^{-1}$ through $\bar{\sigma}$ in G. Thus $(\sigma, v) \in \Sigma \times V_\varepsilon \to \sigma(e+v) \in \mathscr{C}$ provides a cross-section for H-orbits. By condition ($\mathrm{M_H}$), $\frac{\partial V}{\partial a}(\sigma e, a_0)\,(a_1^*) = \frac{\partial V}{\partial a}(\sigma^{-1}a_1^*)$ has a non-degenerate critical point at $\sigma = \bar{\sigma}$ on Σ. Hence one can solve $\frac{\partial f_a}{\partial \sigma}(\sigma, v) = \lambda \frac{\partial H_a}{\partial \sigma}(\sigma, v) = 0$ for $\sigma = \sigma(v, a) \in \Sigma$ with $\bar{\sigma} = \sigma(0, a_0)$. The reduced function is

$$f_a(\sigma(v, a), v) = f_{a_0}(e, v) + \lambda H_a(\sigma(v, a), v). \tag{16}$$

3.8 Theorem. *Suppose that conditions* (F) *and* ($\mathrm{M_H}$) *are satisfied with* $Ha = a$. *Then for* λ *small and positive and for* $\|a_1 - a_1^*\|$ *small, there exist at most* $(k+1)\,h$ *critical* H*-orbits, where* h *is the number of critical* H*-orbits of* $\frac{\partial V}{\partial a}(ge, a_0)\,(a_1^*)$ *in* Ge.

The proof of this result is similar to that of the preceding theorem and is thus omitted.

For the sake of completeness, let us describe how our earlier work on the traction problem fits into this general framework.

3.9 *Example* (*cf.* [I] and [II]). Denote by $\mathscr{B}$ a bounded open set with smooth boundary $\partial\mathscr{B}$ in $\mathbb{R}^3$. Let $\mathscr{U} = \{\phi : \mathscr{B} \to \mathbb{R}^3, \phi(0) = 0\}$, $\mathscr{C} = \{\phi \in \mathscr{U} \mid \det D\phi > 0\}$, the space of deformations, and $\mathscr{A} = \mathscr{L} = \{(b, \tau) \mid b : \mathscr{B} \to \mathbb{R}^3,\ \tau : \partial\mathscr{B} \to \mathbb{R}^3,\ \int_{\mathscr{B}} b\, dv + \int_{\partial\mathscr{B}} \tau\, dA = 0\}$ the space of loads. Let $\langle l, \phi\rangle = \int_{\mathscr{B}} \langle b, \phi\rangle\, dV + \int_{\partial\mathscr{B}} \langle \tau, \phi\rangle\, dA$, for $(b, \tau) = l \in \mathscr{L}$ and $\phi \in \mathscr{U}$. Then $G = SO(3)$ acts on $\mathscr{U}$, $\mathscr{A}$, and $\mathscr{L}$ by composition. Set $V(\phi) = \int W(D\phi)\, dV - \langle l, \phi\rangle$, where W is a stored energy function, and choose $(e, a_0) = (I_{\mathscr{B}}, 0)$. Thus $\Gamma = \{\text{identity}\}$ and $\Psi_l(\phi) = \Phi(\phi) - l$. Assume that the classical stability hypotheses on W holds at $I_{\mathscr{B}}$. Then the condition (F) with $V = \{0\}$, and the conditions (Σ) and ($\mathfrak{B}$) introduced in our case (B) are satisfied. Condition (F) follows from classical linear stability, condition (Σ) from Korn's inequality, and condition ($\mathfrak{B}$) by direct computations. Indeed, the loads a_1^* may be classified according to the type of critical manifolds S_1 obtained in condition ($\mathfrak{B}$). Consequently, Theorem 3.4 can be applied. For parallel loads, $H = S^1$, Theorem 3.5 becomes relevant.

3.10 *Example.* Let $\mathscr{U}$, $\mathscr{C}$, $\mathscr{A}$, $\mathscr{L}$, $\langle l, \phi\rangle$, and V have the same meaning as in the preceding Example 3.9. Set $(e, a_0) = (I_{\mathscr{B}}, l_{\mathscr{B}})$. Thus we consider a traction problem with initial load $l_{\mathscr{B}}$, which need not be zero. Let G be the isotropy group of $l_{\mathscr{B}}$ in $SO(3)$, and let it act on $\mathscr{U}$, $\mathscr{L}$ and $\mathscr{A}$ by composition as before. The verification of condition ($\mathfrak{B}$) offers no problem; it is basically the same as in Example 3.9. In § 2A, we introduced a condition (S) on W, and under this condition, condition (F) with $V = \{0\}$ and condition (Σ) become valid. (See Proposition 2.5 and Lemma 2.12.) Therefore the results in case (B) can again be applied.

3.11 *Remark.* When comparing the results obtained by the general method with what we got before, we need only replace g by Q^T in appropriate places. This follows from the different ways of parametrizing the neighborhood of "trivial" solutions, namely, by the parametrizations $(Q, \varphi) \mapsto Q^{-1}\varphi$ in § 2 and $(Q, \varphi) \mapsto Q\varphi$ in § 3.

§ 4. Signorini Series

In this section we present an abstract version of Signorini's scheme for finding a power series solution $x = e + \lambda u_1 + \lambda^2 u_2 + \dots$ of the equation $\Psi(x, a) = 0$ with a given in the form $a = a_0 + \lambda a_1 + \lambda^2 a_2 + \dots$. Here we employ the abstract context described in the previous section. When we specialize our results to the traction problems as in Examples 3.9, 3.10, we recover the usual Signorini scheme, as in Truesdell & Noll [1965], Marsden & Wan [1983] (for zero initial load) and Bharatha & Levinson [1978] (for a general initial load).

Let $x = e + \lambda u_1 + \lambda^2 u_2 + \dots$ be a series solution. We wish to determine the u_i in a systematic way by solving linear problems. We have

$$\Psi(e + \lambda u_1 + \lambda^2 u_2 + \dots, a_0 + \lambda a_1 + \lambda^2 a_2 + \dots) = 0$$

or

$$L(\lambda u_1 + \lambda^2 u_2 + \dots) + \frac{\partial \Psi}{\partial a}(\lambda a_1 + \lambda^2 a_2 + \dots) + \dots = 0.$$

Comparing orders in λ, we get

$$\text{order } \lambda: \quad L(u_1) + \frac{\partial \Psi}{\partial a}(a_1) = 0,$$

$$\vdots$$

$$\text{order } \lambda^n: \quad L(u_n) + \frac{\partial \Psi}{\partial a}(a_n) + \mathscr{H}(u_1, \dots, u_{n-1}) = 0 \qquad (L_n)$$

where $\mathscr{H}$ is a polynomial in $u_1, \dots, u_{n-1}$ of degree n.

Since $V(gx, a) = V(x, g^{-1}a)$, by varying g near the identity in G, we obtain $\frac{\partial V}{\partial a}(x, a)(wa) = 0$ where $w \in \mathfrak{g}$, the Lie algebra of G. Therefore

$$\frac{\partial V}{\partial a}(e + \lambda u_1 + \lambda^2 u_2 + \dots, a_0 + \lambda a_1 + \lambda^2 a_2 + \dots)(\lambda w a_1 + \lambda^2 w a_2 + \dots) = 0$$

(since $wa_0 = 0$). Thus

$$\frac{\partial V}{\partial a}(\lambda w a_1 + \lambda^2 w a_2 + \dots) + \frac{\partial^2 V}{\partial x\, \partial a}(\lambda u_1 + \lambda^2 u_2 + \dots, \lambda w a_1 + \lambda^2 w a_2 + \dots)$$
$$+ \frac{\partial^2 V}{\partial a^2}(\lambda a_1 + \lambda^2 a_2 + \dots, \lambda w a_1 + \lambda^2 w a_2 + \dots)$$
$$+ \dots = 0.$$

Comparing orders in λ, we obtain the *compatibility conditions*:

order λ: $\frac{\partial V}{\partial a}(w, a_1) = 0$, for all $w \in \mathfrak{g}$

$\vdots$

order λ^n: $\frac{\partial^2 V}{\partial x\, \partial a}(u_{n-1}, wa_1) + \mathscr{K}(u_1, \dots, u_{n-2}) + \frac{\partial V}{\partial a}(wa_n) = 0$, for all $w \in \mathfrak{g}$, (C_n),

where $\mathscr{K}(u_1, \dots, u_{n-2})$ denotes a polynomial in $u_1, \dots, u_{n-2}$ of degree n. The condition $\frac{\partial V}{\partial a}(wa_1) = 0$ simply means that the function $\frac{\partial V}{\partial a}(g^{-1}a_1)\left(= \frac{\partial V}{\partial a}(ge, a_0)(a_1)\right)$ on Ge has a critical point at e.

The following result gives a first-order sufficient condition for the existence of such a series solution.

4.1 Theorem. *Suppose condition* (F) *is fulfilled with* $\ker L = T_e Ge$ *(case* (B)). *Let* $a = a_0 + \lambda a_1 + \lambda^2 a_2 + \dots$ *be given with the function* $\frac{\partial V}{\partial a}(g^{-1}a_1)$ *having a non-degenerate critical point on Ge at e. Then there exists a unique solution* $x(\lambda)$ *of the equation* $\Psi(x(\lambda), a(\lambda)) = 0$ *with* $x(\lambda) \to e$ *as* $\lambda \to 0$.

Proof. From equation (10) or (12), $f_a = V(e, a_0) + \lambda \frac{\partial V}{\partial a}(g^{-1}(a_1 + \lambda a_2 + \dots)) + O(\lambda^2) = V(e, a_0) + \lambda \frac{\partial V}{\partial a}(g^{-1}a_1) + O(\lambda^2)$. By the non-degeneracy condition on $\frac{\partial V}{\partial a}$, the critical points of f_a, V_a depend on λ smoothly so the result follows. ∎

Let us now examine the Hessian of the function $\frac{\partial V}{\partial a}(g^{-1}a_1)$ on Ge at e. For simplicity, we take G to be a subgroup of some ring M_n of $n \times n$ matrices with the actions on $\mathcal{U}, \mathcal{A}$ induced from actions of M_n on $\mathcal{U}, \mathcal{A}$. Thus for $k, w \in \mathfrak{g}$, $kw - wk \in \mathfrak{g}$ and $\frac{\partial V}{\partial a}(kwa_1) = \frac{\partial V}{\partial a}(wka_1)$. Consequently, the Hessian $\mathcal{H}(ke, we)$ of $\frac{\partial V}{\partial a}(g^{-1}a_1)$ is $\frac{\partial V}{\partial a}(kwa_1)$.

4.2 Lemma. $\frac{\partial^2 V}{\partial x\, \partial a}(ke, wa_1) = -\frac{\partial V}{\partial a}(kwa_1)$ *for* $k, w \in \mathfrak{g}$.

Proof. Write $h = 1 + tk + \frac{t^2}{2}k^2 + \dots$ so $h^{-1} = 1 - tk + \frac{t^2}{2}k^2 + \dots$. Then

$$V(he, a_0) = V(e, h^{-1}a_0) = V(he, a_0) = V\left(\left(1 + tk + \frac{t^2k^2}{2} + \dots\right) e, a_0\right)$$

$$= V(e, a_0) + \frac{\partial V}{\partial x}\left(\left(tk + \frac{t^2k^2}{2} + \dots\right) e\right) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(tke)^2 + \dots.$$

Thus $\frac{\partial^2 V}{\partial x^2}(ke)^2 = 0$. Now write $V(he, a_0 + twa_1) = V(e, h^{-1}(a_0 + twa_1)) = V(e, a_0 + th^{-1}wa_1)$. The left-hand side equals

$$V(e, a_0) + \frac{\partial V}{\partial x}\left(tk + \frac{t^2k^2}{2} + \dots\right) + \frac{\partial V}{\partial a}(twa_1)$$

$$+ \frac{1}{2}\left\{t^2\frac{\partial^2 V}{\partial x^2}(ke)^2 + 2t^2\frac{\partial^2 V}{\partial x\,\partial a}(ke, wa_1) + t^2\frac{\partial^2 V}{\partial a^2}(wa_1)^2\right\} + O(t^3),$$

while the right-hand side equals $V(e, a_0 + twa_1 - t^2kwa_1 + \dots) = V(e, a_0) + \frac{\partial V}{\partial a}(twa_1 - t^2kwa_1) + \frac{1}{2}\frac{\partial^2 V}{\partial a^2}(twa_1)^2 + O(t^3)$. Since $\frac{\partial V}{\partial x}\left(tk + \frac{t^2k^2}{2} + \dots\right) = 0$, $\frac{\partial^2 V}{\partial x^2}(ke)^2 = 0$, the terms of order t^2 give $\frac{\partial^2 V}{\partial x\,\partial a}(ke, wa_1) = -\frac{\partial V}{\partial a}(kwa_1)$. ■

4.3 Theorem. (*Abstract Signorini Scheme*). *Let the hypothesis be the same as in Theorem* 4.1. *Suppose* $u_1, \dots, u_{n-1}$ *are determined by* $(L_1), (C_1), \dots, (L_{n-1}), (C_{n-1})$. *Then* u_n *is determined by* (L_n), (C_n).

Proof. Let $x = e + \lambda u_1 + \lambda^2 u_2 + \dots$ be the solution obtained in Theorem 4.1. Thus $u_1, \dots, u_m$ satisfy the equations $(L_1), (C_1), \dots, (L_m), (C_m)$. It suffices to show for any m, that if $u_1^*, \dots, u_m^*$ satisfy the equations $(L_1), (C_1), \dots, (L_m), (C_m)$, then $u_1^* = u_1, \dots, u_m^* = u_m$. To see this, suppose to the contrary that there exists q, $1 \leqq q \leqq m$, such that $u_i^* = u_i$ for $i < q$ and $u_q \neq u_q^*$. The condition (L_q) implies that $u_q - u_q^* = ke$, for some $k \in \mathfrak{g}$. Condition (C_q) gives $\frac{\partial^2 V}{\partial x\,\partial a}(u_q, wa_1) = \frac{\partial^2 V}{\partial x\,\partial a}(u_q^*, wa_1)$ for all $w \in \mathfrak{g}$. Therefore $\frac{\partial^2 V}{\partial x\,\partial a}(ke, wa_1) = 0$ for all $w \in \mathfrak{g}$.

By the non-degeneracy hypothesis on $\frac{\partial V}{\partial a}(g^{-1}a_1)$ and Lemma 4.2, we obtain $k = 0$, which is impossible. ∎

When we specialize the theorem above to the traction problem for a stress-free reference configuration, as in Example 3.9, the non-degeneracy condition on $\frac{\partial V}{\partial a}(g^{-1}a_1) = \langle g, l_1 \rangle$ simply says that the load $l'(0) = l_1$ has no axis of equilibrium. When we specialize the theorem above to the traction problem with a general reference configuration, as in Example 3.10, the non-degeneracy condition on $\langle g, l_1 \rangle$ holds if and only if $(A_1 - \operatorname{tr} A_1)\, e_1 \cdot e_1 \neq 0$, where $A_1 = k(l_1)$ and the load $l_{\mathscr{B}}$ is parallel to e_1. This condition is the same as in BHARATHA & LEVINSON [1978].

Our development here leads naturally to the study of linearization stability; *i.e.* whether or not the solutions predicted by the linearized theory are the linear terms in a convergent series expansion for solutions of the the nonlinear theory. As in MARSDEN & WAN [1983], one can show that there are such series for the nonlinear theory provided u_1 satisfies the compatibility conditions of second order, namely (C_1), (L_1), (C_2).

§ 5. The Pressure Problem

Here we examine a third variant of our basic problem studied in [I] and [II]. Instead of a (dead) traction field along $\partial\mathscr{B}$, we consider a constant pressure boundary condition along $\phi(\partial\mathscr{B})$. Thus this boundary condition depends on the configuration or the current position of the deformed body. This problem will be treated as an application of the theory developed in § 3.

For simplicity, assume that the reference configuration is stress-free and that the classical stability condition holds for the classical elasticity tensor **c**. These are the same hypotheses as those in [I] and [II].) The equations for equilibrium solutions are:

$$\begin{cases} -\operatorname{Div} P = \lambda b & \text{in } \mathscr{B}, \\ \sigma n = -\lambda p n & \text{on } \phi(\partial\mathscr{B}) \end{cases} \tag{17}$$

where $P = J\sigma F^{-T}$ is the first Piola-Kirchhoff stress tensor, σ is the Cauchy stress tensor, b is a dead body force, p is a constant pressure, and n is the outward unit normal along $\phi(\partial\mathscr{B})$.

5.1 Lemma. *$\sigma n = -\lambda p n$ on $\phi(\partial\mathscr{B})$ if $PN = -\lambda p J F^{-T} N$ on $\partial\mathscr{B}$.*

Proof. It suffices to show

$$\sigma n\, da = -\lambda p n\, da \quad \text{if and only if} \quad \sigma J F^{-T} N\, dA = -\lambda p J F^{-T} N\, dA,$$

which follows from the Piola identity: $JF^{-T} N\, dA = n\, da$. ∎

Therefore the equations for our pressure problem become:

$$\begin{cases} -\operatorname{Div} P = \lambda b & \text{in } \mathscr{B}, \\ PN = -\lambda p J F^{-T} N & \text{on } \partial\mathscr{B}. \end{cases} \tag{18}$$

Equation (18) implies that $\int \lambda b \, dV + \int -\lambda p J F^{-T} N \, dA = 0$. Since $\int JF^{-T}N \, dA = \int n \, da = 0$, one must have $\int b \, dV = 0$. In what follows, we assume that $\int b \, dV = 0$ and we consider b near a given b^* with $\int b^* \otimes X \, dV \in$ sym.

Set $V(\phi) = \mathscr{V}(\phi) - \langle \lambda b, \phi \rangle_V + \lambda p \int_{\mathscr{B}} \det F \, dV$, where $\mathscr{V}(\phi) = \int_{\mathscr{B}} W(F) \, dV$ and $F = D\phi$.

5.2 Proposition. *A deformation ϕ is a solution of* (18) *with given λb and λp if and only if ϕ is a critical point of V.*

To see this, we first establish the following.

5.3 Lemma. $D(\int J \, dV)(u) = \int \langle JF^{-T}N, u \rangle \, dA$ *for all* $u \in \mathscr{U}$. (19)

Proof. Since $DJ(H) = J \operatorname{tr}(F^{-T}H)$ for any $H \in M_3$,

$$\begin{aligned} D\left(\int J \, dV\right)(u) &= \int J \operatorname{tr}(F^{-T} Du) \, dV = \int \langle JF^{-T}, Du \rangle \, dV \\ &= -\int \langle \operatorname{Div} JF^{-T}, u \rangle \, dV + \int \operatorname{Div}(JF^{-1}u) \, dV. \end{aligned}$$

Since $\operatorname{Div}(JF^{-T}) = 0$ and $\int \operatorname{Div}(JF^{-1}u) \, dV = \int \langle JF^{-T}N, u \rangle \, dA$, we obtain $D(\int J \, dV)(u) = \int \langle JF^{-T}N, u \rangle \, dA$. ■

Using the divergence theorem, we obtain

$$D\mathscr{V}(u) = \int -\langle \operatorname{Div} P, u \rangle \, dV + \int \langle PN, u \rangle \, dA. \tag{20}$$

Proposition 5.2 now follows easily from equations (19) and (20).

Now we want to put the pressure problem into the general framework of § 3. Denote the fixed pressure by p^*.

Set $\mathscr{C}$ = the space of deformations as before $\subset \mathscr{U}$:

$$\mathscr{A} = \{(b, p) \mid \textstyle\int b \, dV = 0\},$$

$$\mathscr{L} = \{(b, \tau) \mid \textstyle\int b \, dV + \int \tau \, dA = 0\}, \text{ the space of loads,}$$

$$\langle l, u \rangle = \int \langle b, u \rangle \, dV + \int \langle \tau, u \rangle \, dA,$$

$$e = I_{\mathscr{B}}, \; a_0 = 0,$$

$$a_1^* = (b^*, p^*),$$

$$a_1 = (b, p^*),$$

$$V_a(\phi) = \int W(F) \, dV - \langle b, \phi \rangle_V + \int_{\mathscr{B}} \det F \, dV \text{ with } a = (b, p),$$

$$\Psi_a(\phi) = \begin{pmatrix} -\operatorname{Div} P - b \\ PN + pJF^{-T}N \end{pmatrix}. \tag{21}$$

By equations (19) and (20),

$$DV_a(\phi)(u) = \langle \Psi_a(\phi), u \rangle.$$

Let $G = SO(3)$ act on $\mathscr{C}$, $\mathscr{A}$, and $\mathscr{L}$ by compositions $(Q(b, p) = (Q \circ b, p))$. Clearly,

$$\langle Q \cdot l, Q \cdot u \rangle = \langle l, u \rangle,\ V(Q \cdot \phi, Q \cdot a) = V(\phi, a),\ \text{and}\ \Gamma = 1.$$

Next we check, the appropriate conditions on V. At $a_0 = 0$, $\Psi_{a_0}(\phi) = \begin{pmatrix} -\operatorname{Div} P \\ PN \end{pmatrix}$ and $L(u) = \begin{pmatrix} -\operatorname{Div} \mathbf{c} \\ \mathbf{c}N \end{pmatrix}$. Thus condition (F) holds as before, with $\ker L = \{KX \mid K \in \text{skew.}\}$ Let us take $Z = \text{Skew} = \{(0, KN) \mid K \in \text{skew}\}$ and $\mathscr{U}_Z = \mathscr{U}_{\text{sym}} = \{u \in \mathscr{U} \mid \int u_{i,j}\, dV = \int u_{j,i}\, dV\}$. Condition (Σ) follows from Korn's inequality as before. From Theorem 3.3 we obtain

5.4 Theorem. *For small $\lambda > 0$ and b near b^*, the pressure problem has at least* 4 *solutions.*

Now notice that $\dfrac{\partial V}{\partial a}(g^{-1} a_1^*) = \dfrac{\partial V}{\partial a}(Q^{-1}(b^*, p^*)) = -\langle Q^{-1} b^*, I_{\mathscr{B}} \rangle_V + p^* \int_{\mathscr{B}} dV$ $= \text{const.} - \langle l^*, Q I_{\mathscr{B}} \rangle$ with $l^* = (b^*, 0) \in \mathscr{L}$. Thus condition $(\mathfrak{B})$ can be verified in exactly the same way as that for the traction problem considered in Example 3.9 of § 3. Indeed, one may classify b^* according to the critical manifold $S_1 = \{Q \in SO(3) \mid k(Q^T l^*) \in \text{sym}\}$ of $-\langle b^*, Q I_{\mathscr{B}} \rangle_V$ on $SO(3)$. There are five different types as classified in [I].

Next, observe that $\dfrac{\partial V}{\partial a}(g^{-1} a_2) = \dfrac{\partial V}{\partial a}(Q^{-1}(b_2, 0)) = -\langle Q^{-1} b_2, I_{\mathscr{B}} \rangle_V$ where $(b_2, 0) = a_2$ and $\dfrac{\partial^2 V}{\partial a^2} = 0$. Since $\dfrac{\partial \Psi}{\partial a}(g^{-1} a_1^*) = \begin{pmatrix} -Qb^* \\ p^* n \end{pmatrix}$, Lemma 3.2 implies that $u(Q)$ satisfies the equation $L(u) + \begin{pmatrix} -Qb^* \\ p^* N \end{pmatrix} \in \text{skew}$. Thus we obtain the following result from Theorem 3.4:

5.5 Theorem. *Let $b = b^* + \lambda b_2 + O(\lambda^2)$. Suppose that $\theta(Q) = -2\langle Q^{-1} b_2, I_{\mathscr{B}} \rangle_V - \langle L(u(Q)), u(Q) \rangle$ is a Morse function on S_1 having n critical points $Q_1, \ldots, Q_n$. Then, for λ small and positive, the solution set near $SO(3)$ is $\{x_1(\lambda), \ldots, x_n(\lambda)\}$ with $x_i(\lambda) \to Q_i$ as $\lambda \to 0$ for $i = 1, \ldots, n$.*

Exactly the same arguments as in [II] provide the same upper bound for n, a task we leave to the reader.

§ 6. Relaxation of Condition (S)

Let us study the traction problem in general form once more. It has been shown in [II], § 2 that for an isotropic, homogeneous hyperelastic material there are no bifurcations near a stress-free reference state. On the other hand, many interesting cases are concerned with solutions near a state that has non-zero initial loads and bifurcations do happen. Although the condition (S) introduced in § 2 A is fulfilled in certain situations, it fails in many other cases. In particular, the kernel of its linearized problem contains elements transversal to the (group orbit of) trivial solutions. Often in such a problem, the reference configuration possesses a non-trivial symmetry. In what follows, we show that such a traction problem can, in principle, be put into our general framework.

Consider a homogeneous, isotropic material with reference configuration a region $\mathscr{B}$ in $\mathbb{R}^3$. Just as in Examples 3.9 and 3.10 of Section 3, we let

$$\mathscr{C} = \text{the space of deformations} \subset \mathscr{U},$$

$$\mathscr{A} = \mathscr{L} = \{(b, \tau)\} = \text{the space of loads},$$

$$\langle\, ,\rangle = \text{the non-degenerate pairing between } \mathscr{L} \text{ and } \mathscr{U},$$

$$V(\phi) = \int W(F)\, dV - \langle l, \phi\rangle, \text{ the potential function}.$$

Thus $\Psi_a(\phi) = \begin{pmatrix} -\operatorname{Div} P - b \\ Pn - \tau \end{pmatrix}$ with $a = (b, \tau)$,

$$e = I_{\mathscr{B}},\ a_0 = l_{\mathscr{B}} \text{ the initial load}, \quad a_1^* = l_0, \quad \text{and} \quad a_1 = l.$$

Let G_s be a symmetry group of $\mathscr{B}$; *i.e.*, $Q_2(B) = B$ for $Q_2 \in G_s$. (In our treatment of the traction problem given in Examples 3.9 and 3.10 in Section 3, we took $G_s = \{e\}$). The group $SO(3) \times G_s$ acts on $\mathscr{U}$, $\mathscr{L}$, $\mathscr{A}$ by compositions: $(Q_1, Q_2) \cdot u = Q_1 \circ u \circ Q_2^{-1}$, $(Q_1, Q_2) \cdot l = Q_1 \circ l \circ Q_2^{-1}$. Define $G = \{(Q_1, Q_2) \in SO(3) \times G_s \mid (Q_1, Q_2) \cdot l_{\mathscr{B}} = l_{\mathscr{B}}\}$. Clearly, $\langle Q \cdot l, Q \cdot \phi\rangle = \langle l, \phi\rangle$ and $V(Q \cdot \phi, Q \cdot l) = V(\phi, l)$ for $Q \in G$. Thus with this choice of the group G, this problem fits into our general setting. Clearly, $\Gamma = \{(Q, Q) \in G\} \equiv \{Q \in G_s \mid Q \circ l_{\mathscr{B}} \circ Q^{-1} = l_{\mathscr{B}}\}$, and the Γ-action on $\mathscr{U}$, $\mathscr{L}$, $\mathscr{A}$ becomes the usual conjugate actions. One often encounters the situation $Q \circ l_{\mathscr{B}} \circ Q^{-1} = l_{\mathscr{B}}$ for $Q \in G_s$; thus $\Gamma \approx G_s$.

Usually the elasticity tensor A is assumed to be strongly elliptic. Thus the condition (F) or the Fredholm alternative holds for the linear operator $L(u) = \begin{pmatrix} -\operatorname{Div} a \\ aN \end{pmatrix} : \mathscr{U} \to \mathscr{L}$ (*cf.* MARSDEN & HUGHES [1983, Ch. 6]). The dimension of Ker L in the transversal direction (*i.e.*, dim V) depends not only on the material, but also on the geometry of the reference configuration. (For example for the von Kármán equation for the thin plate, the ratio between the lengths of the sides enters into the determination the kernel of the linearized problem (*cf.* GOLUBITSKY & SCHAEFFER [1979]).

If the applied force maintains the same symmetry as that of the initial loads, $G \cdot l = l$, then the studies in Case (A) of Section 3 become applicable. Indeed, one needs to solve a bifurcation problem under Γ-symmetry. If the applied force breaks the symmetry, then the studies in Case (C) of Section 3 become valid.

As an illustration, we examine the following.

6.1 Example. Consider a rectangular block $B = [-1, 1] \times [-l, l] \times [-h, h]$ of a homogeneous isotropic hyperelastic material that is in equilibrium with an initial load

$$l = \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} N\right).$$

We claim that

(1) $G_s = \{e, R_x, R_y, R_z\} = \{R_x\} \oplus \{R_y\} = Z_2 \oplus Z_2$ where R_x stands for the rotation by 180 degrees about the x-axis, *etc.*

(2) G is a semi-direct product of $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix} \middle| \, a^2 + b^2 = 1 \right\} \times \{e\} \approx S^1$ and $\{(e, e), (R_x, R_x), (R_y, R_y), (R_z, R_z\} \approx Z_2 \oplus Z_2$.

(3) $\Gamma = \{(e, e), (R_x, R_x), (R_y, R_y), (R_z, R_z)\} \approx Z_2 \oplus Z_2$,

$$Ge = \left\{ \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & a & -b \\ 0 & b & a \end{array}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, a^2 + b^2 = 1 \right\}.$$

(4) Let $(k_{ij}) = k(l_0)$, the astatic load. Then condition (M) is satisfied if and only if $k_{22} + k_{33} \neq 0$ or $k_{23} \neq k_{32}$.

(5) Let $l_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} N, a \neq 0$. Then $H \approx Z_2 \oplus Z_2$, and condition $(\mathrm{M_H})$ is fulfilled if and only if $k_{22} + k_{23} \neq 0$ or $k_{23} \neq k_{32}$.

Proof. These facts follow from straightforward computations. For (2), it suffices to show that $G = \{\eta S^1 \times \eta \mid \eta = e, R_x, R_y,$ or $R_z\}$, with $S^1 = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & a & -b \\ 0 & b & a \end{array}\right)$ and for (4), it suffices to observe that $\dfrac{\partial V}{\partial a}(g^{-1}a_1^*) = -\langle g^{-1}l_0, I\rangle = \langle l_0, gI\rangle = -\langle k(l_0), g\rangle$, for $g \in S^1$. ∎

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