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EXAMPLES FOR THE INFINITE DIMENSIONAL MORSE LEMMA*

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Abstract. Examples are presented which show how to use the Morse lemma in specific infinite dimensional examples and what can go wrong if various hypotheses are dropped. One of the examples shows that the version of the Morse lemma using singularity theory can hold, yet the hypotheses of the Morse–Palais and Morse–Tromba lemmas fail. Another example shows how to obtain a concrete normal form in infinite dimensions using the splitting lemma and hypotheses related to those in the Morse–Tromba lemma. An example of Dancer is given which shows that for the validity of the Morse lemma in Hilbert space, some hypotheses on the higher order terms must be made in addition to smoothness, if the quadratic term is only weakly nondegenerate. A general conjecture along these lines is made.

Introduction. In this paper we discuss several examples relevant to the Morse lemma and singularity theory in infinite dimensions.

We begin with some historical comments on the various methods that have been used to prove the Morse lemma. The *original method of Morse* uses induction on the dimension of the space and does not, as given, apply to infinite dimensions. See Milnor [1963] for this proof. The *Palais method* was introduced in Palais [1963]. It is a modification of the original method that works in Hilbert space under the hypothesis of strong nondegeneracy of the quadratic term.

The *Moser–Weinstein method* is a variant of the singularity theory method described in Golubitsky and Marsden [1983] (this issue, pp. 1037–1044). It was adapted to the Morse lemma by Palais [1969]. Rather than directly join the quadratic part f to $f+p$ by $f+tp$, as in the preceding paper, one joins df to $df+dp$ by $df+t\,dp$. Palais' [1969] theorem states the following: *if E is a Banach space, h: $E\to\mathbb{R}$ is C^3, $Dh(0)=0$, and $D^2h(0)$, regarded as a map of E to E^*, is an isomorphism, then there is a C^1 diffeomorphism ϕ defined on a neighborhood of* 0 *in E such that*

$$\phi(0)=0, \qquad D\phi(0)=I(=\textit{identity}),$$

and

$$h(\phi(x))=h(0)+\frac{1}{2}D^2h(0)\cdot(x,x).$$

In Hilbert space this result reduces to that in Palais [1963]. We call the condition on $D^2h(0)$ *strong nondegeneracy*. If the map of E to E^* associated to $D^2h(0)$ is injective, we say $D^2h(0)$ is *weakly nondegenerate*.

The *Morse–Tromba lemma* was introduced in Tromba [1976]. It is motivated by the fact that in many elliptic variational problems one does not have strong nondegeneracy of the quadratic term. Rather, this is changed to weak nondegeneracy at the expense of putting special hypotheses on the nonlinear terms. The necessity of weak nondegeneracy occurred already for Hamiltonian systems in Marsden [1968]. Tromba's original proof was an adaptation of Palais' [1963] proof. A proof of the Morse–Tromba lemma using the Moser–Weinstein method was given in Choquet-Bruhat, Fischer and Marsden [1979]. The Morse–Tromba lemma is Theorem B of Golubitsky and Marsden [1983]. The *singularity theory method*, described in that paper, yields a result strictly stronger than Tromba's. Examples 5 and 6 below illustrate this.

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For spaces admitting a duality map (such as Hilbert space or $W^{s,p}$ spaces with p even), the Morse–Tromba lemma is valid for C^2 functions with C^1 changes of coordinates (by Remark (e) following Theorem B of Golubitsky and Marsden [1983]). We do not know a C^2 counterexample if E is a general Banach space. We conjecture that there is not such an example.

The Morse–Tromba lemma suggests the question: can the Morse–Palais lemma be generalized *without* putting conditions on the higher order terms? We conjecture that the answer is no. More precisely,

CONJECTURE. *Let E be a Banach space and let B: $E\times E\to\mathbb{R}$ be a continuous symmetric bilinear map such that $x\mapsto B(x,\cdot)$ is not an isomorphism of E and E^*. Let $f(x)=\frac{1}{2}B(x,x)$. Then there is a C^3 map p: $E\to\mathbb{R}$ with $p(0)=0$, $Dp(0)=0$, and $D^2p(0)=0$ such that f and $f+p$ are not C^1 right equivalent.*

For E a Hilbert space, this conjecture has been verified by E. N. Dancer (private communication). His class of examples is presented below in Example 8.

In the examples that follow, the labels $(E1)$, $(E2)$, $(T1)$, $(T2)$, $(S1)$, $(S2)$, Theorem A and Theorem B refer to Golubitsky and Marsden [1983]. A couple of these examples are simple and well known but are included for completeness.

Example 1. This example shows that *nondegeneracy* of $D^2h(0)$ *in the sense of* $(T1)$ *is not sufficient for the validity of the Morse lemma.* Let $E=l_2$ and let h be the C^∞ function

$$h(x)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}x_n^2-\frac{1}{3}\sum_{n=1}^{\infty}x_n^3.$$

Let $\langle x,y\rangle=\sum_{n=1}^{\infty}(1/n)x_ny_n$. Then $(T1)$ holds with $T=I$. However $(T2)$ fails, since the only possibility would be

$$\nabla h(x)_n=x_n-nx_n^2,\qquad n=1,2,\cdots$$

which is not defined on open sets in l_2. Indeed, the Morse lemma fails for this function. The quadratic term has no zeros other than the origin, yet h vanishes on the sequence $(0,0,\cdots,3/2n,0,\cdots)$, which approaches 0 in l_2. If the cubic term is changed to $\frac{1}{3}\sum_{n=1}^{\infty}(1/n)x_n^3$ then the gradient exists and the Morse–Tromba lemma applies.

Example 2. This example shows that *Tromba's hypotheses* $(T1)$ *and* $(T2)$, *but not those of the Morse–Palais lemma, can be expected to hold for many elliptic variational problems.* If Ω is a bounded region in $\mathbb{R}^n$ with smooth boundary, $W^{s,p}(\Omega,\mathbb{R}^m)$ denotes the Sobolev space of maps u: $\Omega\to\mathbb{R}^m$ whose derivatives up to order s are in L^p (see Friedman [1969], for example). For $p=2$ we write $W^{s,2}=H^s$. If $m=1$ we write $W^{s,p}(\Omega,\mathbb{R})=W^{s,p}(\Omega)$.

Let us begin with the one-dimensional case.

(a) Let $E=H^1([a,b])$. We define the function g: $E\to\mathbb{R}$ by

$$g(u)=\int_a^b[u(x)]^2dx+\int_a^b[u(x)]^3dx=f(u)+p(u).$$

Composition properties of Sobolev spaces (Palais [1968]) show that g is C^∞. Considered as a linear map $E\to E^*$, the bilinear map $D^2g(0)$ is $u\mapsto(v\mapsto 2\int_a^b uv)$. This map is injective but not surjective. For example the delta function $\delta_x(v)=v(x)$ for $a<x<b$ is in E^* but not in the image of $D^2g(0)$: $E\to E^*$. Thus the hypotheses of the Morse–Palais lemma do not hold. (If $\int_a^b[u(x)]^2dx$ is replaced by $\int_a^b[u(x)]^2dx+\int_a^b[u'(x)]^2dx$, then

the hypotheses of the Morse–Palais lemma *do* hold; this quadratic functional is similar to the functionals used in the variational approach to geodesics.)

On the other hand, let $\langle \cdot , \cdot \rangle$ be the L^2 inner product on H^1. Then the gradient $\nabla g(u)$ relative to $\langle \cdot , \cdot \rangle$ is given by $\nabla g(u) = 2u + 3u^2$, which is C^∞. Moreover, $D\nabla g(0) = 2I$. Consequently, Tromba's Morse lemma applies, so g can be transformed to the functional $\int_a^b [u(x)]^2\,dx$.

In this example the transformation can be seen directly. Observe that $g(u)$ can be written as $g(u) = \int_a^b [u(x)(1+u(x))^{1/2}]^2\,dx$. Now if ϕ: $(c,d) \subset \mathbb{R} \to \mathbb{R}$ is C^∞ then $u \mapsto \phi \circ u$ is C^∞ on $\{u \in H^1 | c < u(x) < d$ for all $x \in [a,b]\}$. Hence the map $u \mapsto u(1+u)^{1/2}$ is C^∞ on $\{u \in H^1 | -1 < u(x) < \infty$ for all $x \in [a,b]\}$, has derivative the identity at $u=0$, and hence is a local diffeomorphism.

Tromba's proof of his Morse lemma applied to this example also yields the transformation $u \mapsto u(1+u)^{1/2}$. So does the proof of Theorem A. For if one solves $p = -df \cdot A$ by $A(u) = -u^2/2$ and $dp = df \circ R$ by $R(u) \cdot v = 3uv/2$, one obtains $A(u) = -u^2/2$ and for $A_t(u)$ we get the expression $-[1+3tu/2]^{-1}u^2/2$. Note that $A(u) = -\int_0^1 \tau R(\tau u) \cdot u\,d\tau$, in agreement with Remark (b) following Theorem A of Golubitsky and Marsden [1983]. Integrating this vector field leads to the inverse of the transformation $u \mapsto u(1+u)^{1/2}$.

(b) We now sketch a typical multiple integral variational problem in higher dimensions. (Proofs rely on standard elliptic theory and Sobolev estimates, which are omitted here.) Let E be $W_\partial^{s,p}(\Omega)$, the $W^{s,p}$ functions which are zero on $\partial\Omega$, and let $s > n/p + 1$. Consider h: $E \to \mathbb{R}$ defined by

$$h(u) = \int_\Omega W(Du)\,dx + \int_\Omega K(u)\,dx,$$

where W is a smooth function of $\mathbb{R}^n$ to $\mathbb{R}$, K is a smooth function of $\mathbb{R}$ to $\mathbb{R}$, and $Du(x)$ is identified with a column vector or a point in $\mathbb{R}^n$. Suppose that

$$W(0) = 0, \quad DW(0) = 0, \quad K(0) = DK(0) = D^2K(0) = 0$$

and

$$D^2W(0) \cdot (\xi, \eta) \geq c\|\xi\|\,\|\eta\| \quad \text{for all } \xi, \eta \in \mathbb{R}^n, \quad \text{where } c > 0.$$

Standard Sobolev inequalities (cf. Palais [1968, Thm. 9.10]) show that h is a smooth function. Let $\langle \cdot , \cdot \rangle$ on E be given by

$$\langle u, v \rangle = \int_\Omega Du \cdot Dv\,dx.$$

Then

$$Dh(u) \cdot v = \int_\Omega DW(Du) \cdot Dv\,dx + \int_\Omega DK(u) \cdot v\,dx$$

and

$$D^2h(0) \cdot (u, v) = \int_\Omega (Du)^T M(Dv)\,dx$$

where $D^2W(0) \cdot (\xi, \eta) = \xi^T M \eta$ for an $n \times n$ positive definite matrix M. Then $(T1)$ holds for $(Tu)(x) = \Delta^{-1}\operatorname{div}(MDv)$, using the classical fact that Δ: $W_\partial^{s,p}(\Omega) \to W^{s-2,p}(\Omega)$ is an isomorphism (Friedman [1969]). Also, T is an isomorphism on these spaces, for, as is readily checked, $\operatorname{div}(MDu)$ is elliptic with trivial kernel. ($T2$) holds with

$$\nabla h(u)=\Delta^{-1}\operatorname{div}[DW(Du)]-\Delta^{-1}DK(u).$$

For this example, again the hypotheses of the Morse–Tromba lemma hold, but those of the Morse–Palais lemma do not. Examples like this occur in minimal surfaces (see Tromba [1981]) and in elasticity (see Chillingworth, Marsden and Wan [1982] and Marsden and Hughes [1983]).

Example 3. (a) This example will show that *Theorem* A *is not limited to functions of the form quadratic + higher order* (*as the Morse–Tromba lemma is*). Let $E=H^1([a,b])$ and let

$$g(u)=\int_a^b[u(x)]^3dx+\int_a^b[u(x)]^4dx.$$

Let

$$f(u)=\int_a^b[u(x)]^3dx \quad \text{and} \quad p(u)=\int_a^b[u(x)]^4dx.$$

The equation $p(u)=-Df(u)\cdot A(u)$ can be solved by $A(u)=-u^2/3$, and $Dp(u)=Df(u)\circ R(u)$ is solved by $R(u)\cdot v=4uv/3$. Note that

$$A(u)=-\int_0^1\tau^2R(\tau u)\cdot u\,d\tau.$$

(See Remark (b) following Theorem A in Golubitsky and Marsden [1983].) Hence g can be transformed to f by a C^∞ transformation. This transformation, as in Example 2a, can be found directly by writing $g(u)=\int_a^b[u(x)(1+u(x))^{1/3}]^3\,dx$. Then $\phi(u)=u(1+u)^{1/3}$ is a suitable transformation.

An easy calculation shows that the diffeomorphism obtained by integrating the vector field $A_t(u)=-[1+4tu/3]^{-1}u^2/3$ is the inverse of $\phi(u)=u(1+u)^{1/3}$.

(b) Let $E=H^1([a,b])$ and let $g(u)=f(u)+p(u)$ where $f(u)=\int_a^b[u(x)]^3\,dx$ and $p(u)=\{\int_a^b[u(x)]^2\,dx\}^2$. Theorem A applies to g and shows that g can be transformed to f. Indeed, $p(u)=-Df(u)\cdot A(u)$ can be solved by $A(u)=-\frac{1}{3}\int_a^b[u(x)]^2\,dx$ and $Dp(u)=Df(u)\circ R(u)$ is solved by $R(u)\cdot v=\frac{4}{3}\int_a^b u(x)v(x)\,dx$ (both $A(u)$ and $R(u)\cdot v$ are constant functions). Note that $A(u)=\int_0^1\tau R(\tau u)\cdot u\,d\tau$. On the other hand there does not seem to be any explicit diffeomorphism that one could write down by inspection of g. The conjugating diffeomorphism is given by integrating

$$\frac{\partial\phi}{\partial t}(u,t)=\frac{-\frac{1}{3}\int_a^b[\phi(u,t)]^2dx}{1+\frac{4}{3}t\int_a^b\phi(u,t)\,dx}, \qquad \phi(u,0)=u$$

and setting $t=1$. It seems unlikely this could be solved explicitly in any simple fashion.

Example 4. We now give an example of *a function h which is* C^3, ($T1$) *holds*, ∇h *exists and is continuous but is not* C^1, *and yet the Morse lemma fails.*

Thus the hypothesis that ∇h be C^1 cannot be weakened to C^0 in the Morse–Tromba lemma.

Let $E=L^q([0,1])$ and let $\phi\colon \mathbb{R}\to\mathbb{R}$ be a C^∞ function such that $\phi'(\lambda)=1$, $-1\le\lambda\le 1$ and $\phi'(\lambda)=0$ if $|\lambda|\ge 2$. We assume ϕ is monotone increasing with $\phi=-M$ for $\lambda\le -2$

and $\phi=M$ for $\lambda\geq 2$. Let $h\colon E\to\mathbb{R}$ be given by

$$h(u)=\frac{1}{2}\int_0^1[u(x)]^2dx+\frac{1}{3}\int_0^1\phi([u(x)]^3)\,dx=f(u)+p(u).$$

For $q\geq 2$, f is clearly C^∞. Let $\langle\cdot,\cdot\rangle$ be the L^2 inner product on $L^q([0,1])$; then $(T1)$ holds with $T=I$. We claim that if q is an integer, then p is C^{q-1} but not C^q, and ∇p exists, is continuous, but is not C^1. Thus with $q\geq 4$ we get a C^3 function. Let us indicate the proof of these facts for $q=4$.

To prove that p is C^3, we let $\psi(\lambda)=\phi(\lambda^3)$. By Taylor's theorem,

$$\psi(\lambda)=\sum_{k=0}^{3}\psi^{(k)}(\lambda_0)\frac{(\lambda-\lambda_0)^k}{k!}+R(\lambda,\lambda_0)(\lambda-\lambda_0)^3$$

where $\lim_{\lambda\to\lambda_0}R(\lambda,\lambda_0)=0$ and, from the definition of ϕ, $\psi^{(k)}$ and R are bounded smooth functions. Thus, suppressing the argument x, for u and u_0 continuous functions of x we have the identity

$$3p(u)=\sum_{k=0}^{3}\int_0^1\psi^{(k)}(u_0)\frac{(u-u_0)^k}{k!}dx+\int_0^1R(u,u_0)(u-u_0)^3dx.$$

Since $\psi^{(k)}(\lambda_0)$ and $R(\lambda,\lambda_0)$ are bounded continuous, $\psi^{(k)}(u_0)$ (resp. $R(u,u_0)$) extends to a continuous mapping from L^4 (resp. $L^4\times L^4$) to L^4. Using this fact and the Schwarz inequality, it follows that $p(u)$ depends continuously on $u\in L^4$, and each integral above depends continuously on $(u,u_0)\in L^4\times L^4$. Thus the identity holds for all $(u,u_0)\in L^4\times L^4$. Since $\psi^{(k)}(u_0)$ $(k=0,1,2,3)$ is bounded, $(v_1,\cdots,v_k)\mapsto\int_0^1\psi^{(k)}(u_0)v_1\cdots v_k\,dx$ is a bounded multilinear functional on L^4. Using the Schwarz inequality and the Lebesgue dominated convergence theorem we see that the mapping that associates to $u_0\in L^4$ the k-multilinear functional $(v_1,\cdots,v_k)\mapsto\int_0^1\psi^{(k)}(u_0)v_1\cdots v_k\,dx$ is continuous from L^4 to the bounded multilinear functionals on L^4. Also, $\lim_{u\to u_0}R(u,u_0)=0$. It follows from the converse to Taylor's theorem (Abraham and Robbin [1967]) that p is C^3.

Now an easy check shows that $\nabla p(u)$ exists and is given by

$$\nabla p(u)=\frac{\psi'(u)}{3}.$$

If this were C^1, its derivative would be

$$u\mapsto\left(v\to\frac{\psi''(u)v}{3}\equiv P(u)\cdot v\right).$$

Choose a number a such that $\psi''(a)/3\neq 0$ and let

$$u_n=\begin{cases}a & \text{on } [0,1/n],\\ 0 & \text{elsewhere}\end{cases}$$

and

$$v_n=\begin{cases}\sqrt[4]{n} & \text{on } [0,1/n],\\ 0 & \text{elsewhere.}\end{cases}$$

Then one sees that $u_n\to 0$ in L^4, $\|v_n\|=1$ in L^4, but $P(u_n)\cdot v_n\not\to 0$ in L^4. Since $P(0)=0$, $\nabla p(u)$ is not C^1. (One sees in a similar way that p is not C^4.)

Finally, we note that h has a sequence of critical points u_n approaching the origin, namely

$$u_n=\begin{cases}-1 & \text{on } [0,1/n],\\ 0 & \text{on } (1/n,1].\end{cases}$$

Since this is not true for f, the Morse lemma cannot hold for h.

Example 5. We give an example to show that $(E1)$, $(E2)$ *and* $(T1)$ *can hold, without* $(T2)$ *holding. Thus, the Morse lemma is valid, Theorem* A *applies, but Tromba's Theorem* B *does not.*

Let $E=l_1$, the space of sequences x_n with $\Sigma_{n=1}^{\infty}|x_n|<\infty$. Let $h=f+p$ where

$$f(x)=\frac{1}{2}\sum_{n=1}^{\infty}x_n^2=\frac{1}{2}\langle x,x\rangle,$$

$\langle\cdot,\cdot\rangle$ being the usual l_2 inner product, and

$$p(x)=\left(\sum_{n=1}^{\infty}x_n\right)x_1^2+x_2^3+x_3^3+\cdots.$$

Since p is induced by a continuous trilinear map, h is C^∞. Also, $(T1)$ holds with $T=I$. $(E1)$ holds with

$$A(x)=-\left(\left(\sum_{n=1}^{\infty}x_n\right)x_1,x_2,x_3,\cdots\right)$$

and $(E2)$ holds with

$$R(y)\cdot u=\left(\left(\sum_{i=1}^{\infty}u_i\right)y_1+2\left(\sum_{i=1}^{\infty}y_i\right)u_1,3u_2y_2,3u_3y_3,\cdots\right)$$

as is easily checked. However $(T2)$ cannot hold using the l_2 inner product (or, by Remark (c), following Theorem B, any inner product such that $(T1)$ holds). If ∇h exists, so does ∇p (since $\nabla f(x)=x$). But ∇p would be

$$\nabla p(x)=\left(x_1^2+2x_1\left(\sum_{n=1}^{\infty}x_n\right),3x_2^2+x_1^2,3x_3^2+x_1^2,\cdots\right)$$

which is not in l_1. Note also that $R(y)$ does not have an everywhere defined l_2 adjoint; see Remark (b) following Theorem B.

Example 6. A variation on Examples 2 and 5 gives an example which is a prototype for problems in elasticity in which two bodies are in contact at a point. Like Example 5, this *example has* $(E1)$, $(E2)$ *and* $(T1)$ *holding, but <u>not</u>* $(T2)$.

Let $\Omega\subset\mathbb{R}^n$ be a region with smooth boundary and $0\in\Omega$; for instance, let Ω be the unit disk in the plane. Let $E=W_\partial^{s,p}$, $s>n/p+1$, the Sobolev space $W^{s,p}$ with Dirichlet boundary conditions, and let $h\colon E\to\mathbb{R}$ be given by

$$h(u)=\frac{1}{2}\int_\Omega\|Du\|^2dx+u(0)\int_\Omega u^2\,dx.$$

As above, h is C^∞. Let $\langle u,v\rangle$ on E be defined by $\langle u,v\rangle=\int_\Omega(Du\cdot Dv)\,dx$. Then $(T1)$ holds with $T=I$. Let

$$f(u)=\frac{1}{2}\int_\Omega\|Du\|^2dx \text{ and } p(u)=u(0)\int_\Omega u^2\,dx.$$

We show that p cannot have a gradient ∇p with respect to $\langle\cdot,\cdot\rangle$ (except in the case $s=1$, $n=1$), and thus $(T2)$ cannot hold (except in the case $s=1$, $n=1$, in which case $(T2)$ holds) for ∇p would have to satisfy

$$\int_\Omega Dv\cdot D(\nabla p(u))\,dx=\int_\Omega Dv\cdot Du\,dx+v(0)\int_\Omega u^2\,dx+2u(0)\int_\Omega uv\,dx.$$

This implies

$$\Delta(u-\nabla p(u))=\left(\int_\Omega u^2\right)\delta_0+2u(0)u \quad \text{(as distributions)}$$

where δ_0 is the Dirac delta function at the origin. But δ_0 is not in $W^{s-2,p}$ unless $s=1$ and $n=1$, in which case $\delta_0\in W^{-1,p}$. In this latter case $\nabla p(u)$ is given by the formula $u-(\int_\Omega u^2)\Delta^{-1}\delta_0-2u(0)\Delta^{-1}u$ (using the fact that Δ: $W_\partial^{1,p}\to W^{-1,p}$ is an isomorphism). Thus Tromba's hypotheses are satisfied only in the case $s=1$, $n=1$. In the case $s=1$, $n=1, p=2$, the Palais–Morse lemma hypotheses are also satisfied since $\langle\cdot,\cdot\rangle$ is the Hilbert space inner product for H_∂^1.

On the other hand, for *arbitrary* s and n (with $s>n/2+1$), $p=-df\cdot A$ is solved by $A(u)=u(0)\Delta^{-1}u$, and $dp=df\circ R$ is solved by $R(u)\cdot v=-2u(0)\Delta^{-1}v-v(0)\Delta^{-1}u$ (note that $A(u)=-\int_0^1\tau R(\tau u)\cdot u\,d\tau$), so Theorem A applies.

Example 7 below concerns the splitting lemma under hypotheses compatible with Tromba's Morse lemma. We shall use the splitting lemma from the previous paper for problems in which there is an additional parameter.

Example 7. As in Example 2, let $s>n/p+1$ and $E=W_\partial^{s,p}(\Omega)$, the $W^{s,p}$ space with Dirichlet boundary conditions. Assume that λ_0 is a simple eigenvalue of the Laplacian Δ on Ω and define h: $E\times\mathbb{R}\to\mathbb{R}$ by

$$h(u,\lambda)=\int_\Omega\left[\frac{1}{2}\|Du\|^2+\frac{1}{2}(\lambda_0+\lambda)u^2+G(u)\right]dx,$$

where $G(t)=t^3+$(higher order terms) is a C^∞ function from $\mathbb{R}$ to $\mathbb{R}$. We shall apply the splitting lemma to h and bring it to normal form. We find that

(a) $Dh(u,\lambda)\cdot(v,\mu)=\int_\Omega[Du\cdot Dv+(\lambda_0+\lambda)uv+\frac{1}{2}\mu u^2+G'(u)v]\,dx$,

(b) $D^2h(u,\lambda)\cdot((v,\mu),\ (w,\nu))=\int_\Omega\ [Dv\cdot Dw+(\lambda_0+\lambda)vw+G''(u)vw+\nu uv+\mu uw]\,dx$,

(c) $D^3h(u,\lambda)\cdot((v,\mu),(w,\nu),(y,\sigma))=\int_\Omega[G'''(u)vwy+\sigma vw+\nu yv+\mu yw]\,dx$.

Define $\langle v,w\rangle=\int_\Omega Dv\cdot Dw\,dx$. Then $D^2h(0,0)\cdot((v,\mu),\ (w,\nu))=\int_\Omega[Dv\cdot Dw+\lambda_0 vw]\,dx=\langle Tv,w\rangle$ where $Tv=(I-\lambda_0\Delta^{-1})v$. Since Δ is elliptic, T is Fredholm of index 0. The null space of T is $N(T)=\langle u_0\rangle$, where u_0 is an eigenfunction of Δ for the eigenvalue λ_0. The range of T is $R(T)=\langle u_0\rangle^\perp$, the space of vectors in E that are L^2-orthogonal to u_0. Let P be projection onto $\langle u_0\rangle^\perp$. Write $u\in E$ as $u=\alpha u_0+\tilde{u}$, $\tilde{u}\in\langle u_0\rangle^\perp$.

Let $\nabla h(u,\lambda)=(I-(\lambda_0+\lambda)\Delta^{-1})u-\Delta^{-1}G'(u)$. Then $Dh(u,\lambda)\cdot(v,0)=\langle\nabla h(u,\lambda),v\rangle$, and $P\nabla h$ is what was called $\nabla_y h$ in the splitting lemma. Solving $P\nabla h(u,\lambda)=0$ using the implicit function theorem gives a function $\tilde{u}(\alpha,\lambda)$ such that

$P\nabla h(\alpha u_0+\tilde{u}(\alpha,\lambda),\lambda)=0$. Now $\tilde{u}(0,0)=0$, and $D\tilde{u}(0,0)=0$ because the kernel of $P\circ D\nabla h(0,0)$ is $\langle u_0\rangle\times\mathbb{R}$. Clearly $\tilde{u}(0,\lambda)=0$ for all λ, since $\nabla h(0,\lambda)=0$ for all λ.

Let v denote a typical element of $\langle u_0\rangle^\perp$. Let $k(\alpha,v,\lambda)=h(\alpha u_0+\tilde{u}(\alpha,\lambda)+v,\lambda)$, so that $Dk(\alpha,0,\lambda)\cdot(0,w,0)=0$ for all $w\in\langle u_0\rangle^\perp$. There is then an (α,λ)-dependent change of coordinates $v=\eta_{(\alpha,\lambda)}(\bar{v})$ with $\eta_{(\alpha,\lambda)}(0)=0$ and $D\eta_{(\alpha,\lambda)}(0)=I$, such that

$$k\big(\alpha,\eta_{(\alpha,\lambda)}(\bar{v}),\lambda\big)=k(\alpha,0,\lambda)+\frac{1}{2}D^2k(\alpha,0,\lambda)(\bar{v},\bar{v}).$$

To find a normal form for k (and hence h) it remains to find a normal form for $g(\alpha,\lambda)\equiv k(\alpha,0,\lambda)=h(\alpha u_0+\tilde{u}(\alpha,\lambda),\lambda)$. Now,

(a) $Dg(\alpha,\lambda)\cdot(\beta,\mu)=Dh(\alpha u_0+\tilde{u}(\alpha,\lambda),\lambda)\cdot(\beta u_0+D\tilde{u}(\alpha,\lambda)\cdot(\beta,\mu),\mu)$,

(b) $$D^2g(\alpha,\lambda)\cdot(\beta,\mu)^2=D^2h(\alpha u_0+\tilde{u}(\alpha,\lambda),\lambda)\cdot(\beta u_0+D\tilde{u}(\alpha,\lambda)\cdot(\beta,\mu),\mu)^2 +Dh(\alpha u_0+\tilde{u}(\alpha,\lambda),\lambda)\cdot D^2\tilde{u}(\alpha,\lambda)\cdot(\beta,\mu)^2,$$

(c)

$$\begin{aligned}D^3g(\alpha,\lambda)\cdot(\beta,\mu)^3&=D^3h(\alpha u_0+\tilde{u}(\alpha,\lambda),\lambda)\cdot(\beta u_0+D\tilde{u}(\alpha,\lambda)\cdot(\beta,\mu),\mu)^3\\&\quad+3D^2h(\alpha u_0+\tilde{u}(\alpha,\lambda),\lambda)\\&\qquad\cdot\big[(\beta u_0+D\tilde{u}(\alpha,\lambda)\cdot(\beta,\mu),\mu),\ D^2\tilde{u}(\alpha,\lambda)\cdot(\beta,\mu)^2\big]\\&\quad+Dh(\alpha u_0+\tilde{u}(\alpha,\lambda),\lambda)\cdot D^3\tilde{u}(\alpha,\lambda)\cdot(\beta,\mu)^3.\end{aligned}$$

Therefore

(i) $g(0,0)=h(0,0)=0$,
(ii) $Dg(0,0)=0$ because $Dh(0,0)=0$,
(iii) $D^2g(0,0)=0$ because

$$D^2h(0,0)\cdot(\beta u_0,\mu)^2=\beta^2D^2h(0,0)\cdot(u_0,0)^2=\beta^2\langle Tu_0,u_0\rangle=\beta^2\langle 0,u_0\rangle=0,$$

(iv) $D^3g(0,0)\cdot(\beta,\mu)^3=\beta^3\int_\Omega G'''(0)u_0^3\,dx+3\mu\beta^2\int_\Omega u_0^2\,dx$,
using the formula for D^3h; the terms involving D^2h and Dh give 0.

Assume (by normalizing) that $\int_\Omega u_0^2\,dx=1$ and *assume* $\int_\Omega u_0^3\,dx\neq 0$. Since $G'''(0)\neq 0$, we have

$$g(\alpha,\lambda)=\frac{1}{3!}D^3g(0,0)\cdot(\alpha,\lambda)^3+\cdots=k\alpha^3+\frac{1}{2}\lambda\alpha^2+\cdots,\qquad k\neq 0.$$

Let us multiply α and λ by constants to put this in the form

$$g(\alpha,\lambda)=\alpha^3+3\lambda\alpha^2+\cdots.$$

The higher order terms are divisible by α^2, since $g(0,\lambda)=h(0,\lambda)=0$ (recall $\tilde{u}(0,\lambda)=0$), and $g_\alpha(0,\lambda)=0$ because $Dh(0,\lambda)=0$. Let us put g into normal form using the ideas in Wasserman [1975]. First note that $g(\alpha,\lambda)$ has the form

$$g(\alpha,\lambda)=\alpha^3z(\alpha)+3\alpha^2\lambda q(\alpha,\lambda)$$

where $z(0)=1$ and $q(0,0)=1$. By the universal unfolding theorem for cubic singularities, there are functions $\beta(\alpha,\lambda)$, $\sigma(\lambda)$ and $\tau(\lambda)$ such that

$$g(\alpha,\lambda)=h(\beta(\alpha,\lambda),\sigma(\lambda),\tau(\lambda))$$

where

$$h(\beta,\sigma,\tau)=\beta^3+\sigma\beta+\tau$$

and

$$\beta(0,0)=0,\quad \beta_\alpha(0,0)>0,\quad \sigma(0)=0,\quad \tau(0)=0.$$

Using the chain rule in some straightforward calculations we find that

$$\sigma'(0)=0 \quad\text{and}\quad \sigma''(0)<0.$$

Thus, there is a further change of coordinates $\mu=\mu(\lambda)$ with $\mu(0)=0$, $\mu'(0)>0$, such that

$$g(\alpha,\lambda)=[\beta(\alpha,\lambda)]^3-3[\mu(\lambda)]^2\beta(\alpha,\lambda)+\tau(\lambda).$$

Since $g(0,\lambda)=0$, $\{[\beta(0,\lambda)]^2-3[\mu(\lambda)]^2\}\beta(0,\lambda)+\tau(\lambda)=0$; and since $g_\alpha(0,\lambda)=0$, $3\{[\beta(0,\lambda)]^2-[\mu(\lambda)]^2\}\beta_\alpha(0,\lambda)=0$. Since $\beta_\alpha(0,\lambda)\neq 0$, $\beta(0,\lambda)=\varepsilon\mu(\lambda)$ where $\varepsilon=\pm1$. Hence $\tau(\lambda)=2\varepsilon[\mu(\lambda)]^3$, so $g=\beta^3-3\mu^2\beta+2\varepsilon\mu^3$. Letting $\gamma=\beta-\varepsilon\mu$, we get $g=\gamma^3+3\varepsilon\gamma^2\mu$. If we differentiate each side of this equation twice with respect to α and once with respect to λ, and set $(\alpha,\lambda)=(0,0)$, we find that $1=[\gamma_\alpha(0,0)]^2[\gamma_\lambda(0,0)+\varepsilon\mu'(0)]$. But $\gamma_\lambda(0,0)=\beta_\lambda(0,0)-\varepsilon\mu'(0)=0$ and $\mu'(0)>0$; therefore $\varepsilon=1$. Thus we obtain the normal form

$$g(\alpha,\lambda)=\gamma^3+3\mu\gamma^2$$

in the new coordinates (γ,μ). Hence there is a change of coordinates respecting the parameter such that the higher order terms can be eliminated.

Note that g is the potential function for a *transcritical bifurcation*: if we set $g_\alpha(\alpha,\lambda)=0$ we get

$$3\alpha^2+6\lambda\alpha+2\alpha(\cdots)=0.$$

The solution set is therefore the λ-axis and a curve tangent at $(0,0)$ to the line $\alpha+2\lambda=0$. The expression $g(\alpha,\lambda)=\gamma^3+3\mu\gamma^2$ puts the potential function for this bifurcation problem into normal form.

Normal forms for the equations $g_\alpha=0$ by coordinate changes respecting the parameter are found in Golubitsky and Schaeffer [1979]; see Marsden and Hughes [1983, Chap. 7] for simple proofs adequate for the present example. Golubitsky and Schaeffer point out that for many bifurcation problems, the equation $g_\alpha=0$ can be put into normal form by a coordinate change respecting the parameter, but the potential function g cannot.

Our approach to Example 7 should be compared with, for example, Chillingworth [1974] and Zeeman [1976], which consider a one-dimensional problem in which difficulties with the function spaces do not occur (i.e the energy norm is a complete Hilbert space norm) and for which the bifurcation parameter is not treated as distinguished. The example of Beeson and Tromba [1981] has the function-space complications of our example (i.e. the energy norm is not complete) but has additional complications due to a group action. However there is no distinguished bifurcation parameter.

Example 8 (E. N. Dancer). *This example proves the conjecture in the introduction for Hilbert spaces.* Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let B: $H\times H\to\mathbb{R}$ be a continuous symmetric bilinear map. There is a bounded self-adjoint operator L: $H\to H$ such that $B(x,y)=\langle Lx,y\rangle$. Suppose that L is *not* an isomorphism. Let $f(x)=\frac{1}{2}B(x,x)$. We shall find a continuous homogeneous cubic polynomial p: $H\to\mathbb{R}$ such that f and $f+p$ are not C^1 right equivalent in any neighborhood of the origin. Thus any generalization of the Morse–Palais lemma in Hilbert space *must* place restrictions on the perturbation p.

Let $\sigma(L)$ denote the spectrum of L. Since L is not an isomorphism, $0\in\sigma(L)$.

Case 1. $N(L)=\varnothing$. Then $\nabla f(x)=Lx\neq 0$ for $x\neq 0$. We shall find a continuous homogeneous cubic polynomial p: $H\to\mathbb{R}$ such that $\nabla(f+p)(x)=Lx+\nabla p(x)=0$ at points x arbitrarily close to 0. Then f and $f+p$ cannot be C^1 right equivalent.

There exist $w_n\in\sigma(L)$, $w_n\neq 0$, such that $w_n\to 0$. For each n let I_n be a closed interval centered at w_n such that the I_n are disjoint and radius $(I_n)<|w_n|/2$. Let P_n be the orthogonal projection corresponding to I_n that is given by the spectral theorem, and let $H_n=P_nH$. The subspaces H_n are mutually orthogonal subspaces of H, invariant under L, and $L|H_n$ has spectrum lying in I_n.

Choose $e_n\in H_n$ such that $\|e_n\|=1$. Then $\|Le_n-w_ne_n\|<|w_n|/2$. Let $z_n=Le_n-w_ne_n$. Decompose z_n as $z_n=\sigma_ne_n+\tau_ny_n$ where $\langle e_n,y_n\rangle=0$ and $\|y_n\|=1$. If z_n is a multiple of e_n, set $y_n=0$, $\tau_n=0$. Then $y_n\in H_n$ (by invariance of H_n under L) and $|\sigma_n|$, $|\tau_n|<|w_n|/2$. We conclude that $Le_n=\mu_ne_n+\tau_ny_n$ where $\mu_n=w_n+\sigma_n$. Thus $|w_n|/2<|\mu_n|<3|w_n|/2$.

Define p_n on span $\{e_n,y_n\}$ by $p_n(\alpha e_n+\beta y_n)=\alpha^3+(3\tau_n/\mu_n)\alpha^2\beta$. Notice that $|3\tau_n/\mu_n|<(3|w_n|/2)/(|w_n|/2)=3$. We find that $L(\gamma_ne_n)+\nabla p_n(\gamma_ne_n)=0$ provided

$$3\gamma_n^2+\gamma_n\mu_n=0$$

and

$$\frac{3\tau_n}{\mu_n}\gamma_n^2+\gamma_n\tau_n=0,$$

i.e., provided $\gamma_n=-\mu_n/3$.

Finally, define p on H by $p=\Sigma p_n$. Since all the e_n's and y_n's are mutually orthogonal, p is a continuous cubic polynomial.

(*Proof.* $p(x)=T(x,x,x)$ where $T(u,v,w)$ is the symmetric trilinear map defined by

$$T(u,v,w)=\sum\langle u,e_n\rangle\langle v,e_n\rangle\langle w,e_n\rangle+\sum\frac{\tau_n}{\mu_n}\langle u,y_n\rangle\langle v,e_n\rangle\langle w,e_n\rangle$$
$$+\sum\frac{\tau_n}{\mu_n}\langle u,e_n\rangle\langle v,y_n\rangle\langle w,e_n\rangle+\sum\frac{\tau_n}{\mu_n}\langle u,e_n\rangle\langle v,e_n\rangle\langle w,y_n\rangle.$$

Each of these four sums is a *bounded* trilinear map. For example, the second sum is estimated as follows:

$$\left|\sum\frac{\tau_n}{\mu_n}\langle u,y_n\rangle\langle v,e_n\rangle\langle w,e_n\rangle\right|<\sum|\langle u,y_n\rangle\langle v,e_n\rangle\langle w,e_n\rangle|$$
$$\leq\left[\sum\langle u,y_n\rangle^2\right]^{1/2}\left[\sum\langle v,e_n\rangle^2\langle w,e_n\rangle^2\right]^{1/2}$$
$$\leq\left[\sum\langle u,y_n\rangle^2\right]^{1/2}\cdot\left[\sum\langle v,e_n\rangle^2\cdot\sum\langle w,e_n\rangle^2\right]^{1/2}\leq\|u\|\|v\|\|w\|.\ )$$

We have $L(\gamma_ne_n)+\nabla p(\gamma_ne_n)=0$ where $\gamma_n\to 0$.

Case 2. $N(L)\neq\varnothing$. Let $\{e_n\}$ be an orthonormal basis for $N(L)$. Let $p(x)=\Sigma\langle x,e_n\rangle^3$. Then $\nabla f(x)=0$ for all $x\in N(L)$, but $\nabla(f+p)(x)\neq 0$ if $x\neq 0$. Thus f and $f+p$ are not C^1 right equivalent.

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