Proceedings of the Royal Society of Edinburgh, 95A, 171-180, 1983

Linearization stability and Signorini Series for the traction problem in elastostatics

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(Ms received 28 September 1982. Revised MS received 10 February 1983)

Synopsis

This paper uses previous results of Chillingworth, Marsden and Wan on symmetry and bifurcation for the traction problem in three dimensional elastostatics to establish new results on the Signorini expansion. We show that the Signorini compatibility conditions are necessary and sufficient for linearization stability and analogies with results known for other field theories are pointed out. Under an explicit non-degeneracy condition, a new series expansion is given in which successive terms are inductively determined in pairs rather than singly. Our results include as special cases, classical results of Signorini, Tolotti and Stoppelli.

1. Introduction

This paper studies linearization stability of the equations of non-linear elastostatics and the closely related notion of the expansion of the solution in a series whose terms are found by solving a hierarchy of linear problems. The latter topic was developed extensively by Signorini starting in [12] and is thoroughly described in Grioli [10] and Truesdell and Noll [16]. Linearization stability originated in perturbation theory for the Einstein equations of general relativity by Fischer and Marsden [7, 8] but is a notion that is useful in the study of non-linear partial differential equations rather generally.

Chillingworth, Marsden and Wan [4, 5] studied the bifurcation of solutions of the traction problem for small loads, as the loads are varied. Some of the results obtained there will be used in an essential way here. In particular, those papers used the Liapunov-Schmidt procedure to study the bifurcations near a given load l; this process reduced the problem to studying the critical points of a reduced potential function on a manifold S_A, where A is the associated astatic load. The manifold S_A is either four points, two points and a circle, one point and $\mathbb{RP}^2$, two

* Research partially supported by NSF grant 81-07086, The Miller Institute and DOE contract DE-AT03-82ER12097.
† Research partially supported by NSF grant MCS 81-02463 and DOE contract DE-AT03-82ER12097.

disjoint circles or $\mathbb{RP}^3$, depending on the type (either 0, 1, 2, 3 or 4) of the load. Crucial in this bifurcation study is the *Betti form*, a function on S_A closely related to the symmetric bilinear form on linearized solutions that occurs in the Betti reciprocity theorem (see for instance, Marsden and Hughes [11]).

This paper contains three principal theorems. First of all, in Theorem 1, we relate the Signorini compatibility conditions to the critical points of the Betti form on S_A. This has, as a consequence, an extension of a theorem of Tolotti [15]. Secondly, in Theorem 2, we show that the Signorini compatibility conditions are necessary and sufficient for linearization stability. This is analogous to the theorem in general relativity which states that the Taub conditions are necessary and sufficient for linearization stability, a result of Fischer, Marsden and Moncrief [9] and Arms, Marsden and Moncrief [1, 2]. However, the technical details in the two theorems seem to have little if anything in common. Thirdly, in Theorem 3, we establish a generalization and modification of the classical Signorini–Stoppelli schemes for a power series solution which is valid even when the loads have axes of equilibrium. Our scheme is different from and is not restricted by the special series used by Stoppelli [14] in his analysis of the bifurcation of solutions for type 1 loads.

As in [4, 5], we assume the material is hyperelastic, materially frame indifferent, the reference configuration is stress free and the linearized elasticity tensor $\mathbf{c}$ at the identity is stable (and hence strongly elliptic).

2. Linearization stability and critical points of the Betti form

We begin with the definition of linearization stability.

DEFINITION. Suppose a pair $(\boldsymbol{u}_1, \boldsymbol{l}_1)$ (displacement, load) satisfies the equations of elastostatics linearized about the (stress free) reference state I (= Identity); i.e.

$$L\boldsymbol{u}_1 = \boldsymbol{l}_1.$$

Here, $L\boldsymbol{u}_1 = D\Phi(I)\cdot\boldsymbol{u}_1$ and $\Phi(\phi) = (-\mathrm{DIV}\,\boldsymbol{P}, \boldsymbol{P}\cdot\boldsymbol{N})$, where $\phi: \mathscr{B}\to\mathbb{R}^3$ is a configuration of the body $\mathscr{B}$, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress at ϕ and $\boldsymbol{N}$ is the unit outward normal on $\mathscr{B}$. Let us call the pair $(\boldsymbol{u}_1, \boldsymbol{l}_1)$ *linearization stable* (*or integrable*) if there exists a C^∞ curve $(\phi(\lambda), \boldsymbol{l}(\lambda))\in\mathscr{C}\times\mathscr{L}$ (configuration, loads) such that

(i) $\phi(0) = I$, $\boldsymbol{l}(0) = \boldsymbol{0}$,
(ii) $\phi'(0) - \boldsymbol{u}_1 \in \ker D\Phi(I)$, $\boldsymbol{l}'(0) = \boldsymbol{l}_1$,
and
(iii) $\Phi(\phi(\lambda)) = \boldsymbol{l}(\lambda)$.

Here, $(\phi(\lambda), \boldsymbol{l}(\lambda))$ should be defined in some interval; say $[0, \varepsilon)$, $\varepsilon > 0$.

For $\boldsymbol{l}_1 \in \mathscr{L}_e$ (the equilibrated loads), let us say that $\boldsymbol{l}_1$ is *linearization stable* when there is a curve $(\phi(\lambda)), \boldsymbol{l}(\lambda))\in\mathscr{C}\times\mathscr{L}$ satisfying (i) and (iii) above with $\boldsymbol{l}'(0) = \boldsymbol{l}_1$. Then, $D\Phi(I)\cdot\phi'(0) = \boldsymbol{l}_1$ is automatic and we can take $\boldsymbol{u}_1 = \phi'(0)$.

A classical result of Signorini and Stoppelli is the following:

PROPOSITION 1. *Suppose $\boldsymbol{l}_1 \in \mathscr{L}_e$ has no axis of equilibrium and $D\Phi(I)\boldsymbol{u}_1 = \boldsymbol{l}_1$. Then $(\boldsymbol{u}_1, \boldsymbol{l}_1)$ is linearization stable.*

Proof. Let $\boldsymbol{l}(\lambda) = \lambda\boldsymbol{l}_1$. Then there is a unique smooth curve $\phi(\lambda)$ through I such that $\Phi(\phi(\lambda)) = \boldsymbol{l}(\lambda)$ by [4, Theorem 5.1]. By differentiating at $\lambda = 0$, we obtain $D\Phi(I)\,.\,\phi'(0) = \boldsymbol{l}_1$, so $\phi'(0) - \boldsymbol{u}_1 \in \ker D\Phi(I)$. □

In this proposition and elsewhere in the paper, the spaces of loads and configurations are appropriate Sobolev spaces; see [4, 5] for details. We note, however, that elements of the space $\mathscr{C}$ of configuration are at least C^1.

The following Signorini compatibility conditions produce a potential obstruction to linearization stability. Let us write

$$\int_{\mathscr{B}} \boldsymbol{u}\times\boldsymbol{l} \quad \text{for} \quad \int_{\mathscr{B}} \boldsymbol{u}(X)\times\boldsymbol{B}(X)\,dV(X) + \int_{\partial\mathscr{B}} \boldsymbol{u}(X)\times\boldsymbol{\tau}(X)\,dA(X),$$

where $\boldsymbol{l} = (\boldsymbol{B}, \boldsymbol{\tau}) =$ (body force, surface traction).

PROPOSITION 2. *Suppose $\boldsymbol{l}_1 \in \mathscr{L}_e$ is linearization stable with $\boldsymbol{l}(\lambda)\in\mathscr{L}_e$. On letting $\boldsymbol{u}_1$ be as above, we have*

$$\int_{\mathscr{B}} \boldsymbol{l}_1\times\boldsymbol{u}_1 = 0. \tag{C}$$

Proof. We have identity $\int_{\mathscr{B}} \boldsymbol{l}(\lambda)\times\phi(\lambda) = \boldsymbol{0}$ from balance of moment of momentum. By differentiating twice in λ and setting $\lambda = 0$, we obtain

$$\int_{\mathscr{B}} \boldsymbol{l}''(0)\times I + 2\int_{\mathscr{B}} \boldsymbol{l}'(0)\times\phi'(0) = \boldsymbol{0}.$$

Since $\boldsymbol{l}(\lambda)\in\mathscr{L}_e$, we have $\boldsymbol{l}''(0)\in\mathscr{L}_e$, so the first integral is zero. Thus $\int_{\mathscr{B}} \boldsymbol{l}_1\times\boldsymbol{u}_1 = 0$. □

If we write

$$\boldsymbol{l}(\lambda) = \lambda\boldsymbol{l}_1 + \lambda^2\boldsymbol{l}_2 + \dots$$

and

$$\phi(\lambda) = I + \lambda\boldsymbol{u}_1 + \lambda^2\boldsymbol{u}_2 + \dots$$

and assume $\boldsymbol{l}(\lambda)\in\mathscr{L}_e$, then $\int \boldsymbol{l}(\lambda)\times\phi(\lambda) = 0$ gives a hierarchy of conditions:

order λ: $\int \boldsymbol{l}_1\times I = 0$ (i.e. $\boldsymbol{l}_1\in\mathscr{L}_e$),
order λ^2: $\int \boldsymbol{l}_1\times\boldsymbol{u}_1 = 0$ (using $\boldsymbol{l}_2\in\mathscr{L}_e$),
⋮
order λ^n: $\int \boldsymbol{l}_1\times\boldsymbol{u}_{n-1} + \int \boldsymbol{l}_2\times\boldsymbol{u}_{n-2} + \dots + \int \boldsymbol{l}_{n-1}\times\boldsymbol{u}_1 = 0.$

Stoppelli [13] proved that the curve $\phi(\lambda)$ needed for Proposition 1 can be obtained by the preceding Signorini scheme with $\boldsymbol{l}(\lambda) = \lambda\boldsymbol{l}_1$.

We shall now reformulate the compatibility condition (C) in geometric terms. Let $\boldsymbol{A} = \boldsymbol{k}(\boldsymbol{l})\in\mathrm{sym}$ (the symmetric 3×3 matrices) be the astatic load defined by

$$\boldsymbol{k}(\boldsymbol{l}) = \int_{\mathscr{B}} \boldsymbol{B}(X)\otimes X\,dV(X) + \int_{\partial\mathscr{B}} \boldsymbol{\tau}(X)\otimes X\,dA(X),$$

and let $\boldsymbol{u}_0(\boldsymbol{l})\in\mathscr{U}_{\mathrm{sym}}$, the L^2 orthogonal complement to Skew in $\mathscr{U} = T_I\mathscr{C}$, the space of linearized displacements, be defined by the linear problem $L\boldsymbol{u}_0(\boldsymbol{l}) = Q\boldsymbol{l}$

when $Q \in S_A$, i.e. when $Ql \in \mathscr{L}_e$. (By the linear theory, $L: \mathscr{U}_{sym} \to \mathscr{L}_e$ is an isomorphism; see Fichera [6] or Marsden and Hughes [11, Chapter 6].) Let $\mathfrak{B}(Q) = \langle c(\nabla u_Q(l)), \nabla u_Q(l) \rangle$, the *Betti form, defined on* S_A.

LEMMA 1. *$\mathfrak{B}$ restricted to S_A has a critical point at $Q \in S_A$, if and only if $\langle c(\nabla u_{WQ}(l)), \nabla u_Q(l) \rangle = 0$ for all $W \in$ skew (the 3×3 skew matrices) such that $WQk(l) \in$ sym.*

Proof. This follows from the definition of $\mathfrak{B}$. ☐

The following is readily verified.

LEMMA 2. *Let $A \in$ sym be fixed and let p: skew $\times$ skew $\to \mathbb{R}$ be defined by $p(K, W) = \langle KA, W \rangle$. Then p is a symmetric bilinear form with kernel $\{K \in$ skew $| KA \in$ sym$\}$.*

Note that $\langle KA, W \rangle$ is the Hessian of $-\langle l, Q^T I \rangle = -\langle Ql, I \rangle$ at $Q = I$, where

$$\langle l, \phi \rangle = \int_{\mathscr{B}} B(X) \cdot \phi(X)\, dV(X) + \int_{\partial \mathscr{B}} \tau(X) \cdot \phi(X)\, dA(X).$$

Here is our first main result.

THEOREM 1. Let $l_1 \in \mathscr{L}_e$. *Then there exists a $u_1 \in \mathscr{U}$ such that $L(u_1) = l_1$ and $\int l_1 \times u_1 = 0$ if and only if the Betti from (for l_1) restricted to S_{A_1} has a critical point at the identity I.*

Proof. First assume u_1 exists. Thus, $\int l_1 \times u_1 = 0$, so $\langle Wl_1, u_1 \rangle = 0$ for all $W \in$ skew. We can write $u_t(X) = u_1(X) + KX$ for some $K \in$ skew. Then

$$\langle c(\nabla u_W), \nabla u_t \rangle = \langle Wl_1, u_t \rangle$$
$$= \langle Wl_1, u_1 + KX \rangle = \langle Wl_1, KX \rangle = \langle k, k(Wl_1) \rangle = 0, \quad \text{when } k(Wl_1) \in \text{sym}.$$

Thus, $\mathfrak{B}$ has a critical point at I by Lemma 1. For the converse, we need to find $K \in$ skew, so that, with $u_1 = u_t - KX$, we have $\langle Wl_1, u_1 \rangle = 0$ for all $W \in$ skew. Now $\langle Wl_1, u_t \rangle = \langle c(\nabla u_W), u_t \rangle$ is a linear function of $W \in$ skew, vanishing for $Wk(l_1) \in$ sym (by hypotheses). Thus, by Lemma 2, there is a $K \in$ skew such that $\langle Wl_1, u_t \rangle = \langle WA, K \rangle$ for all $W \in$ skew. Therefore, $\langle Wl_1, u_1 \rangle = \langle Wl_1, u_t \rangle - \langle Wl_1, KX \rangle = \langle Wl_1, u_t \rangle - \langle WA, K \rangle = 0$ for all $W \in$ skew. ☐

The next corollary is an extension of results of Tolotti [15].

COROLLARY. *There exist at least 4 rotations Q in SO(3) such that the Signorini 0th and 1st order compatibility conditions hold for $Ql_1 = l^*$, i.e. $l^* \in \mathscr{L}_e$, and $\int l^* \times u^* = 0$ for some u^* satisfying $L(u^*) = l^*$.*

Proof. Let Q be a critical point of $\langle c(\nabla u_Q(l_1)), \nabla u_Q(l_1) \rangle$ on S_{A_1}. By Lemma 1, $\langle c(\nabla u_{WQ}(l_1)), \nabla u_Q(l_1) \rangle = 0$ when $WQk(l_1) \in$ sym. Since $u_{WQ}(l_1) = u_W(Ql_1) = u_W(l^*)$, and $u_Q(l_1) = u_t(Ql_1) = u_t(l^*)$, we get $\langle c(\nabla u_W(l^*)), \nabla u_t(l^*) \rangle = 0$ for all $WQ(l^*) \in$ sym. By Lemma 1 and Theorem 1 with $Q = I$ and $l = l^*$, we see that this critical point has the desired property. Since, by critical point theory, at least 4 such critical points can always be found, the corollary follows. ☐

3. Linearization stability and the compatibility conditions

Besides the case of no-axis of equilibrium, Stoppelli also showed that the Signorini scheme can be made to work for parallel loads. In fact this result, which we now recall, follows directly from [5, Theorem 4.7].

PROPOSITION 3 (Stoppelli). *Let $l(\lambda) = \lambda l_1$, where l_1 is a non-trivial load paralllel to the z-axis. Then there is a solution curve $\phi(\lambda)$, which can be obtained by Signorini's scheme supplemented by the condition $u_{1,2}(0) - u_{2,1}(0) = 2c(\lambda)$, where $c(\lambda) \to 0$ but is otherwise an arbitrary given function.*

We now recall some developments that combine the classical cases of Propositions 1 and 3, following Capriz and Podio-Guidugli [3].

DEFINITION. Let $l_1 \in \mathscr{L}_e$ and set $\phi(\lambda) = k(l_1, I + \lambda u)$ for $u \in \mathscr{U}$. The load l_1 is said to be *infinitesimally stable* when, for any $u \in \mathscr{U}$, there exists a smooth curve $Q(\lambda) \in$ SO(3), with $Q(0) = I$ such that $Q(\lambda)\phi(\lambda) \in$ sym.

This is motivated by the following. One seeks a solution in the form $\phi = Q^{-1}(I + \lambda u)$; where $Q = Q(\lambda)$ and $u = u(\lambda)$. Thus, $k(l_1, Q^{-1}(I + \lambda u)) \in$ sym or $Q(\lambda)k(l_1, I + \lambda u(\lambda)) \in$ sym. Requiring a solvability condition depending on the load only, leads to the notion of an infinitesimally stable load. The next result follows readily; see the aforementioned reference for details.

PROPOSITION 4. *A load $l_1 \in \mathscr{L}_e$ is infinitesimally stable if and only if*

l_1 has no axis of equilibrium

or, *l_1 is a non-trivial parallel load.*

Thus, the load l_1 is linearization stable if it is infinitesimally stable.

The next theorem generalizes the classical results by showing that the necessary condition (C) is also sufficient. No special non-degeneracy hypotheses are required.

THEOREM 2. *Let $l_1 \in \mathscr{L}_e$. If there is a $u_1 \in \mathscr{U}$ such that $\int l_1 \times u_1 = 0$, then l_1 is linearization stable.*

Proof. By Theorem 1, $\langle c(\nabla u_Q), \nabla u_Q \rangle$ has a critical point at $Q = I$. Choose l_2 so that $-2\langle l_2, Q^T I \rangle - \langle c(\nabla u_Q), \nabla u_Q \rangle$ restricted to S_{A_1} is non-degenerate at I. For example, we can choose $l_2 = (0, aN)$ for a large, $a > 0$. Let $l(\lambda) = \lambda(l_1 + \lambda l_2)$. Then the reduced bifurcation potential on S_A *is, by* [5, *formula* (30)],

$$\bar{f} = -\lambda \langle l_2, Q^T I \rangle - \frac{\lambda}{2} \langle c(\nabla u_Q), \nabla u_Q \rangle + O(\lambda^2).$$

Thus $\bar{f}$ has non-degenerate critical points which vary smoothly in λ and thus, from them, $\phi(\lambda)$ can be reconstructed by the Liapunov-Schmidt procedure (see [5, §2].) ☐

Notice that the second order term $\lambda^2 l_2$ is necessary to allow construction of $\phi(\lambda)$.

An example due to Signorini of a type 4 load l_1 for which no u_1 exists satisfying (C) is described in Capriz and Podio-Guidugli [3, §9].

Two specific cases in which the construction of a curve $\phi(\lambda)$ corresponding to $l(\lambda)=\lambda l_1$ is possible and which employ non-degeneracy hypotheses on the Betti form, are as follows.

COROLLARY. (a) *Let $l_1 \in \mathscr{L}_e$, and suppose $\langle c(\nabla u_Q), \nabla u_Q\rangle$ is non-degenerate along S_{A_1} at $Q=I$. There is a unique solution curve $\phi(\lambda)$ with $\phi(0)=I$ such that $\Phi(\phi(\lambda))=\lambda l_1$ (here one can choose $l_2=0$ by examination of the preceding proof).*

(b) *Let $l_1 \in \mathscr{L}_e$ be a "trivial" load (i.e. $A_1=0$) parallel to the z-axis. Suppose that $\langle c(\nabla u_Q), \nabla u_Q\rangle$ on SO(3) has a critical point at I which is non-degenerate transversal to $S^1=\{Q \mid Ql_1=l_1\}$. Then there is a solution curve $\phi(\lambda)$ with $\phi(0)=I$ and $\Phi(\phi(\lambda))=\lambda l_1$.*

The classical results Propositions 1 and 3 are also corollaries of Theorem 2.

4. Signorini expansions

Now we turn to the problem of finding a generalization of the Signorini scheme which will work in the generality of Theorem 2. We begin by setting up the perturbation series using slightly different notation.

We consider the problem of solving $\Phi(\phi(\lambda))=l(\lambda)$, where $l(\lambda)\in\mathscr{L}$ is a given curve and $l(0)=0$. (Note that $l(\lambda)$ is not assumed to lie in $\mathscr{L}_e$.) Write the Taylor expansion of Φ in u at I as $\Phi=\Phi_1+\Phi_2+\Phi_3+\dots$, (Thus $\Phi_1=L$, and Φ_2 is quadratic in u, etc.) and expand l as a series in λ by setting $l(\lambda)=\lambda l_1+\lambda^2 l_2+\lambda^3 l_3+\dots$. Write the unknown $\phi(\lambda)$ as $\phi=I+\lambda u_1+\lambda^2 u_2+\dots$. Hence,

$$\Phi(I+\lambda u_1+\lambda^2u_2+\dots)=L(\lambda u_1+\lambda^2 u_2+\dots)+\Phi_2(\lambda u_1+\lambda^2u_2+\dots,\lambda u_1+\lambda^2u_2+\dots)$$
$$=\lambda l_1+\lambda^2 l_2+\dots.$$

By comparing orders in λ, we get

$$\text{order } \lambda: \quad L(u_1)=l_1, \quad (\text{so } l_1\in\mathscr{L}_e),$$
$$\text{order } \lambda^2: \quad L(u_2)+\Phi_2(u_1,u_1)=L_2,$$
$$\vdots$$
$$\text{order } \lambda^n: \quad L(u_n)+2\Phi_2(u_1,u_{n-1})+\text{a polynomial in}$$
$$u_1,\dots,u_{n-2}=l_n, \quad (n\geqq 3). \qquad (\mathrm{L}_n)$$

Therefore, one hopes to determine $u_1,\dots,u_n$ inductively, with the help of the compatibility conditions:

$$\int l_1\times u_n+\dots+\int l_n\times u_1+\int l_{n+1}\times I=0. \qquad (\mathrm{C}_n)$$

Theorem 1 shows that *if $l_1=l'(0)\in\mathscr{L}_e$ and has no axis of equilbrium, then there is a unique solution $\phi(\lambda)$, and can be obtained by Signorini's scheme. Indeed, suppose $u_1,\dots,u_{n-1}$ are determined, then (L_n) and (C_n) define u_n*. The uniqueness and existence of the formal Signorini's scheme follows from a special case of the next two lemmas.

LEMMA 3. *Let $K\in$* skew, then $\int l_1\times KX=0$ *if and only if $Kk(l_1)\in$* sym *(i.e. $K\in T_IS_{A_1}$).*

LEMMA 4. *Suppose $u_1,\dots,u_{n-1}$ satisfy $(\mathrm{L}_1),(\mathrm{C}_1),\dots,(\mathrm{L}_{n-1})$. Then (C_{n-1}) is the solvability condition for u_n in (L_n).*

These simple facts are discussed and proved in [16].

Let us state the results obtained in Theorems 1 and 2 in a slightly different and more general form.

THEOREM 1'. *Let $l_1\in\mathscr{L}_e$. Then there exists a $u_1\in\mathscr{U}$ such that $L(u_1)=l_1$ and $\int l_1\times u_1+\int l_2\times I=0$ if and only if $2\langle l_2, Q^TI\rangle+\langle c(\nabla u_Q),\nabla u_Q\rangle$ restricted to S_{A_1} has a non-degenerate critical point at I.*

THEOREM 2'. *Consider the problem $\Phi(\phi(\lambda))=l(\lambda)$, with $l_1\in\mathscr{L}_e$ and $l(\lambda)\in\mathscr{L}$ given. Suppose that $2\langle l_2, Q^TI\rangle+\langle c(\nabla u_Q), u_Q\rangle$ restricted to S_{A_1} has a non-degenerate critical point at I. Then there is a unique $\phi(\lambda)$ such that $\Phi(\phi(\lambda))=l(\lambda)$, where $\phi(0)=I$.*

These theorems are proved in the same way as Theorems 1 and 2.

We claim that $\phi(\lambda)$ determined by Theorem 2' can be obtained by a modification of Signorini's scheme. The new scheme determines the solutions in pairs.

THEOREM 3. *Suppose $u_1,\dots,u_{n-2}(n\geqq 2)$, and u_{n-1}* mod KX, $k\in T_IS_{A_1}$ *are determined; then, equations (L_n) and (C_n) define u_{n-1} and u_n* mod KX, where $K\in T_IS_{A_1}$.

From Theorem 1', one can see readily that u_1 mod KX, with $K\in T_IS_{A_1}$, is determined by equations (L_1) and (C_1), provided that the non-degeneracy hypothesis in Theorem 2' is fulfilled. Thus, starting from u_1 mod KX with $K\in T_IS_{A_1}$, one can find $u_1,u_2,\dots$, inductively by Theorem 3.

Our proof of Theorem 3 consists of a brute force computation. Lemmas 5, 6, and 7 are collections (and extensions) of relevant facts we have already established, in [4, 5].

LEMMA 5. (a) *The function $2\langle l_2, Q^TX\rangle+\langle c(\nabla u_Q),\nabla u_Q\rangle$ has a critical point on S_{A_1} at I if and only if $0=\langle l_2, K^TX\rangle+\langle c(\nabla u_I),\nabla u_K\rangle$, for all $K\in T_IS_{A_1}$.*

(b) *The Hessian of this function is $\langle l_2, K^2X\rangle+\langle c(\nabla u_K),\nabla u_K\rangle+\langle c(\nabla u_I),\nabla u_{K^2}\rangle$, for $K\in T_IS_{A_1}$.*

Write the first Piola–Kirchhoff tensor in a perturbation series $P=P_1+P_2+P_3+\dots$, with $P_1=\mathrm{a}$ and write, as above,

$$\Phi=\Phi_1+\Phi_2+\Phi_3+\dots, \quad \text{with} \quad \Phi_1=L.$$

LEMMA 6.

(a) $\langle L(u), w\rangle=\langle \mathrm{a}(\nabla u),\nabla w\rangle.$

(b) $\langle \Phi_2(u^2), w\rangle=\langle P_2(\nabla u^2),\nabla w\rangle,$ *for all* $u, w\in\mathscr{U}$,

where $\Phi_2(u^2)=\Phi_2(u,u)$, etc.

Now, write the stored energy function as $W=W(D)$, where $D=\frac{1}{2}(F^TF-1)$, and $F=D\phi$ is the deformation gradient. Thus

$$P=\frac{\partial W}{\partial F}, S=\frac{\partial W}{\partial D} \quad \text{(the second Piola–Kirchhoff stress)},$$

and

$$P(F)=FS(D).$$

Computations show that:

LEMMA 7.

(a) $\dfrac{\partial P}{\partial F}(H)=\dfrac{\partial S}{\partial D}(H)$, (i.e. a = c).

(b) $\dfrac{\partial^2 P}{\partial F^2}(H,K)=H\dfrac{\partial S}{\partial D}(K)+K\dfrac{\partial S}{\partial D}(H)+\dfrac{\partial S}{\partial D}(K^{T}H)+\dfrac{\partial^2 S}{\partial D^2}(H,K).$

LEMMA 8.

(a) $\Phi_2(KX^2)=L(\tfrac{1}{2}K^{T}KX).$

(b) $2\Phi_2(u_1,KX)=Kl_1+L(K^{T}u_1),$ for $K\in T_IS_{A_1}$.

Proof.

(a)

$$\langle 2\Phi_2(KX,KX),w\rangle=\left\langle\frac{\partial S}{\partial D}(K^{T}K),\nabla w\right\rangle \quad \text{(by Lemmas 6(b) and 7(b))}$$

$$=\langle L(K^{T}KX),w\rangle \quad \text{(by Lemmas 6(a) and 7(a)).}$$

(b)

$$\langle 2\Phi_2(u_1,KX),w\rangle=\left\langle K\frac{\partial S}{\partial D}(H)+\frac{\partial S}{\partial D}(K^{T}H),\nabla w\right\rangle \quad \text{(by Lemmas 6(b) and 7(b))}$$

$$=\langle Kl_1+L(K^{T}u_1),w\rangle \quad \text{(by Lemma 6(a)).} \ \square$$

LEMMA 9. *$\langle Wl_1,K^{T}u_1\rangle-\langle Wl_2,KX\rangle$ is symmetric in W and K where $W,K\in T_IS_{A_1}$.*

Proof. $\int l_1\times u_1+\int l_2\times I=0$ means that $k(l_1,u_1)+k(l_2,I)\in\text{sym}$. Thus $\langle\tilde K^{T},k(l_1,u_1)\rangle+\langle\tilde K^{T},k(l_2,I)\rangle=0$ for all $\tilde K\in\text{skew}$, or $\langle\tilde Kl_1,u_1\rangle+\langle\tilde Kl_2,X\rangle=0$. Let $\tilde K=WK-KW$. Then one obtains $\langle WKl_1,u_1\rangle+\langle WKl_2,X\rangle=\langle KWl_1,u_1\rangle+\langle KWl_2,X\rangle$ or $\langle Kl_1,W^{T}u_1\rangle-\langle Kl_2,WX\rangle=\langle Wl_1,K^{T}u_1\rangle-\langle Wl_2,KX\rangle$. $\square$

Now, we are ready to prove Theorem 3.

(A) Let $n>2$. We need to show that there is a unique $K\in T_IS_{A_1}$, such that $u_{n-1}=u^{*}_{n-1}+KX$ (u^{*}_{n-1} is given by hypotheses) and a corresponding u_n, obtained by Lemma 4, which solve (L_n) and (C_n). For each $K\in T_IS_{A_1}$, from (L_n) for u_n and u^{*}_n given by Lemma 4,

$$L(u_n-u^{*}_n)+2\Phi_2(u_1,KX)=0.$$

By Lemma 8, $u_n-u^{*}_n+u_K+K^{T}u_1+\tilde KX=0$ for some $\tilde K\in\text{skew}$ (to be determined). On substituting in (C_n) one has

$$\int l_1\times(u^{*}_n-u_K-K^{T}u_1-\tilde KX)+\int l_2\times(u^{*}_{n-1}+KX)+\int l_3\times u_{n-2}+\ldots+\int l_{n+1}\times I=0$$

or

$$-k(l_1,u_K+K^{T}u_1+\tilde KX)+k(l_2,KX)+M\in\text{sym},\tag{1}$$

where

$$M=k(l_1,u^{*}_n)+k(l_2,u^{*}_{n-1})+k(l_3,u_{n-2})+\ldots+k(l_{n+1},I).$$

Since $\langle W,k(l_1,\tilde KX)\rangle=\langle -Wl_1,\tilde KX\rangle=\langle -WA_1,\tilde K\rangle=0$ for $W\in T_IS_{A_1}$,

$$\langle W,-k(l_1,u_K+K^{T}u_1)\rangle+\langle W,k(l_2,KX)\rangle+\langle W,M\rangle=0 \quad \text{for all} \quad W\in T_IS_{A_1}.\tag{2}$$

Now, $\langle W,-k(l_1,u_K+K^{T}u_1)\rangle+\langle W,k(l_2,KX)\rangle=\langle Wl_1,u_K\rangle+\langle Wl_1,K^{T}u_1\rangle-\langle Wl_2,KX\rangle$ is a non-degenerate form, by Lemmas 9 and 5(b). Thus, there is a unique $K\in T_IS_{A_1}$, such that equation (2) holds. Now, choose this K and consider the equation (1) for $\tilde K$, i.e.

$$k(l_1,\tilde KX)=-k(l_1,u_K+K^{T}u_1)+k(l_2,KX)+M \bmod \text{sym}$$

$$:=N \bmod \text{sym}.$$

From $\langle\tilde K,k(W^{T}l_1)\rangle=\langle k(l_1,\tilde KX),W\rangle$, the solvability condition for $\tilde K$ becomes $\langle W,N\rangle=0$, for $W\in T_IS_{A_1}$. Therefore, for the unique solution K determined by equation (2), one can obtain a $\tilde K$ so that equation (1) holds. In other words, for such a K (unique) and $\tilde K$ (modulo $T_IS_{A_1}$),

$$u_{n-1}=u^{*}_{n-1}+KX$$

$$u_n=u^{*}_n-u_K-K^{T}u_1-\tilde KX$$

have the desired properties.

(B) The proof for $n=2$ is basically the same as in (A), where one needs Lemma 8(a). $\square$

Acknowledgments

We thank Robin Knops and Clifford Truesdell for suggesting this area of research and David Chillingworth for valuable comments on the logical structure of the Signorini scheme.

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(*Issued* 20 *September* 1983)