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COADJOINT ORBITS, VORTICES, AND CLEBSCH VARIABLES FOR INCOMPRESSIBLE FLUIDS

Jerrold MARSDEN* and Alan WEINSTEIN*
Department of Mathematics, University of California, Berkeley, California 94720, USA

This paper is a study of incompressible fluids, especially their Clebsch variables and vortices, using symplectic geometry and the Lie–Poisson structure on the dual of a Lie algebra. Following ideas of Arnold and others it is shown that Euler's equations are Lie–Poisson equations associated to the group of volume-preserving diffeomorphisms. The dual of the Lie algebra is seen to be the space of vorticities, and Kelvin's circulation theorem is interpreted as preservation of coadjoint orbits. In this context, Clebsch variables can be understood as momentum maps. The motion of N point vortices is shown to be identifiable with the dynamics on a special coadjoint orbit, and the standard canonical variables for them are a special kind of Clebsch variables. Point vortices with cores, vortex patches, and vortex filaments can be understood in a similar way. This leads to an explanation of the geometry behind the Hald–Beale–Majda convergence theorems for vorticity algorithms. Symplectic structures on the coadjoint orbits of a vortex patch and filament are computed and shown to be closely related to those commonly used for the KdV and the Schrödinger equations respectively.

1. Introduction

The purpose of this paper is to use the methods of symmetry and reduction to study incompressible fluids, especially their Clebsch variables and vortices. The original techniques introduced in Marsden and Weinstein [41] were used in a study of the Hamiltonian structure of plasmas in Marsden and Weinstein [42]. Compressible flow, magnetohydrodynamics, and elasticity require the use of semidirect products, which will be the subject of another publication (Marsden, Ratiu and Weinstein [40]).

Symmetry and Hamiltonian systems are related in the following way to the topic "order in chaos" of this conference:

1) *Order*. Physical systems often exhibit "order" simultaneous with symmetries. For example, soliton-like behavior is frequently linked with symmetry and complete integrability.

2) *Chaos*. When the symmetry of a system is broken, Hamiltonian structures can be useful in detecting chaos by the method of Melnikov (see Holmes and Marsden [28–30]).

Hamiltonian systems written in "non-canonical" variables can be elegantly understood in terms of reduction of the standard canonical variables, such as the Lagrangian configuration map and its conjugate momentum in fluid mechanics. When this is done, one obtains a Poisson manifold which is a union of symplectic leaves. An orbit beginning on a leaf stays on it, so these leaves appear as "constraint manifolds". In fluid mechanics these are the "Lin constraint" manifolds and are exactly the coadjoint orbits for the configuration manifold, which is a group.

'Constraints' are of three types. First of all there are the type above, which correspond to symplectic leaves in a larger phase space and indicate that a reduction has taken place. Second, there are constraints which are imposed on a model for purposes of idealization or simplification, such as incompressibility, rigidity, or the passage from electrofluid dynamics to magnetohydrodynamics. We expect that this second kind can be understood in the context of the first kind using an enlarged Poisson manifold and the framework of Weinstein

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[59]. Finally, there are constraints like div $E = \rho$, which are, from the four-dimensional point of view, some of the Euler–Lagrange equations of the theory. The latter can also be understood in terms of zero sets of momentum maps and reduction, as in Marsden and Weinstein [42].

Clebsch, or canonical, variables for a system can be understood in terms of Poisson maps from symplectic to Poisson manifolds. For many systems these maps are, or are constructed from, momentum maps for symmetry groups. We note that Holm and Kuperschmidt [27] have taken the opposite approach, using Clebsch representations to derive the non-canonical Poisson structures.

In addition to the topics above, this paper contains discussions of point vortices, vortex patches, and vortex filaments. These objects form coadjoint orbits whose symplectic structures are related respectively to those for particles on $\mathbb{R}^1$, the KdV equation, and the Schrödinger equation. Canonical variables for these systems are particular instances of Clebsch variables.

Space does not permit the inclusion of extensive background material. Readers should consult our earlier papers and lectures listed in the bibliography, along with Arnold [6] (especially appendices 2 and 5) and Abraham and Marsden [1] (especially chapter 4).

2. Poisson manifolds, momentum maps, and reduction

A *Poisson manifold* is a manifold P together with a Lie algebra structure $\{\ ,\ \}$ on the space $C^\infty(P)$ of smooth real valued functions on P such that $\{f, g\}$ is a derivation in each argument.

If G is a Lie group and $\mathfrak{G}$ is its Lie algebra, the dual space $\mathfrak{G}^*$ carries a natural Poisson structure defined as follows. For $\mu \in \mathfrak{G}^*$ and $F: \mathfrak{G}^* \to \mathbb{R}$, define $\delta F/\delta\mu \in \mathfrak{G}$ by

$$\mathrm{D}F(\mu)\cdot v = \left\langle v, \frac{\delta F}{\delta \mu}\right\rangle,$$

where $\mathrm{D}F$ is the derivative of F, $v \in \mathfrak{G}^*$, and $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{G}^*$ and $\mathfrak{G}$. For $F, G \in C^\infty(\mathfrak{G}^*)$, define

$$\{F, G\}_-(\mu) = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle,$$

where $[\ ,\ ]$ is the standard (left) Lie bracket on $\mathfrak{G}$. The bracket $\{\ ,\ \}_-$ is the one induced on $\mathfrak{G}^*$ by identifying $C^\infty(\mathfrak{G}^*)$ with the *left* invariant functions on T^*G. We denote this structure by $\mathfrak{G}^*_-$. The corresponding bracket with the $+$ sign is associated to *right* invariant functions and is denoted $\mathfrak{G}^*_+$. In finite dimensions the formula for the bracket on $\mathfrak{G}^*_\pm$ in terms of a basis e_i and dual basis e^i, with $\mu = \Sigma\, \mu_i e^i$, is

$$\{F, G\}_\pm(\mu) = \pm \sum c^k_{ij} \frac{\partial F}{\partial \mu_i}\frac{\partial G}{\partial \mu_j}\mu_k,$$

where c^k_{ij} are the structure constants for the Lie algebra, defined by $[e_i, e_j] = \Sigma_k\, c^k_{ij} e_k$.

This formula for the bracket on $\mathfrak{G}^*_+$ is due to Lie [34], pp. 235 and 294. It was rediscovered by Berezin [9] and is closely related to results obtained by Arnold, Kirillov, Kostant, and Souriau around the same time.

If P is a Poisson manifold, the Hamiltonian system on P corresponding to a function $H: P \to \mathbb{R}$ is the vector field X_H on P such that real-valued functions on P evolve by $\dot F = \{F, H\}$. Since Poisson structures define maps of covectors to vectors, X_H is just the image of $\mathrm{d}H$.

Every Poisson manifold is a union of symplectic manifolds, its "symplectic leaves". Trajectories of X_H starting in a particular leaf necessarily stay there. Thus, these leaves may be viewed as constraint surfaces. For $\mathfrak{G}^*_+$ or $\mathfrak{G}^*_-$, the symplectic leaves are coadjoint orbits. More is known about the structure of Poisson manifolds, extending some of Lie's work [34] on function groups. Namely, Weinstein [59] shows that, at least on a linearized level, a Poisson manifold is near each point the product of a symplectic space and the dual of a Lie algebra. This helps to explain why Lie–Poisson brackets on $\mathfrak{G}^*$ are so fundamental.

Let P be a Poisson manifold and G a Lie group. Assume that G acts on P by a left (resp. right) action by Poisson maps, i.e. maps $\phi: P \to P$ such that $\{F \circ \phi, G \circ \phi\} = \{F, G\} \circ \phi$ for all $F, G \in C^\infty(P)$. By a *Hamiltonian map* for this action we mean a Lie algebra homomorphism (resp. anti-homomorphism) $\hat{J}: \mathfrak{G} \to C^\infty(P)$ such that $X_{\hat{J}(\xi)} = \xi_P$ for each $\xi \in \mathfrak{G}$, where ξ_P denotes the associated infinitesimal generator of the action. Define $J: P \to \mathfrak{G}^*$ by $\langle J(x), \xi \rangle = \hat{J}(\xi)(x)$. We call J the *momentum map*.

Proposition 2.1. Let $J_L: P \to \mathfrak{G}^*$ be a momentum map for a left action of G on P. Then

$J_L: P \to \mathfrak{G}^*_+$ is a Poisson map.

Likewise, if J_R is the momentum map for a right action, then

$J_R: P \to \mathfrak{G}^*_-$ is a Poisson map.

Proof. By definition of the Lie–Poisson bracket,

$$\{F, G\}_+(\mu) = \left\langle J(x), \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle = \hat{J}\left(\left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right)(x).$$

Since $\hat{J}$ is a Lie algebra homomorphism,

$$\hat{J}\left(\left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right)(x) = \left\{\hat{J}\left(\frac{\delta F}{\delta \mu}\right), \hat{J}\left(\frac{\delta F}{\delta \mu}\right)\right\}(x).$$

The proof will be complete if we can show that

$$\mathrm{d}\left(\hat{J}\left(\frac{\delta F}{\delta \mu}\right)\right) = \mathrm{d}(F \circ J),$$

where $\delta F/\delta \mu$ is regarded as a constant element of $\mathfrak{G}$ evaluated at $\mu = J(x)$. Indeed, we have

$$\mathrm{d}(F \circ J) \cdot v_x = \mathrm{d}F(\mu) \cdot \mathrm{d}J(x) \cdot v_x = \left\langle \mathrm{d}J(x) \cdot v_x, \frac{\delta F}{\delta \mu}\right\rangle$$

for $v_x \in TxP$. Also,

$$\mathrm{d}\left(\hat{J}\left(\frac{\delta F}{\delta \mu}\right)\right) \cdot v_x = \mathrm{d}\left(\left\langle J(x), \frac{\delta F}{\delta \mu}\right\rangle\right) \cdot v_x = \left\langle \mathrm{d}J(x) \cdot v_x, \frac{\delta F}{\delta \mu}\right\rangle,$$

since $\delta F/\delta \mu$ is regarded as a constant element of $\mathfrak{G}$. ■

Note that Ad*-equivariant momentum maps in the usual sense for actions on symplectic manifolds are momentum maps in the present sense. In what follows, most of the momentum maps we consider are standard ones from symplectic geometry (see Abraham and Marsden [1], sec. 4.2).

A consequence of the formula $\mathrm{d}(\hat{J}(\delta F/\delta \mu)) = \mathrm{d}(F \circ J)$ proved above is the following fact about collective Hamiltonians. (cf. Marle [35] and Guillemin and Sternberg [23]):

Corollary 2.2. Let $F \in C^\infty(\mathfrak{G}^*)$. Then

$$X_{F \circ J}(x) = \left(\frac{\delta F}{\delta \mu}\right)_P (x),$$

where $\delta F/\delta \mu$ is evaluated at $\mu = J(x)$. (This holds for momentum maps associated with either left or right actions).

The distinction between left and right in 2.1 may be clarified by the following remarks. Let G be a Lie group and T^*G be its cotangent bundle. Consider the map $\lambda: T^*G \to \mathfrak{G}^*$ given by

$$\langle \lambda(\alpha_g), \xi \rangle = \langle \alpha_g, TL_g \cdot \xi \rangle,$$

i.e. left translate covectors to the identity; $\mathfrak{G}^* = (T_eG)^*$. This map is a Poisson map of T^*G to $\mathfrak{G}^*_-$ because it is the momentum map associated with *right* translations of G on T^*G. (Abraham and Marsden [1], p. 302). Thus, T^*G/G (quotient by the *left* action) yields $\mathfrak{G}^*_-$ with the Lie–Poisson structure (including the minus sign).

There is a converse to 2.1. Namely, if $J: P \to \mathfrak{G}^*$ is a Poisson map, then J is a momentum map for an action (of the simply connected covering group) of G. See Fong and Meyer [22] for the symplectic case. Thus, when the range space is the dual of a Lie algebra one loses no generality in the search for Poisson maps by looking among momentum maps.

If P is a Poisson manifold and G acts on P by Poisson maps, then P/G is a Poisson manifold. (Assume for present purposes that P/G has no singularities.) Indeed, we may identify functions on P/G with G-invariant functions on P, so that the bracket on P is inherited by P/G. We call P/G the *reduced Poisson manifold*. For example, T^*G reduced by the left action of G is just $\mathfrak{G}^*_-$.

Example 2.4. (The rigid body). Here we take $G = \mathrm{SO}(3)$ so that $\mathfrak{G}$, its Lie algebra, is identifiable with $\mathbb{R}^3$ and the Lie bracket with the cross product. A point $m \in \mathfrak{G}^*_-$ represents the angular momentum in "body coordinates". (See Abraham and Marsden [1] or Arnold [6] for the explanation of this terminology.) The Hamiltonian H is the kinetic energy of the body, a positive definite quadratic function of m. By choosing an appropriate orthonormal basis of $\mathbb{R}^3$ (and corresponding orthonormal dual basis of $\mathbb{R}^{3*}$) we can assume that H is diagonal:

$$H(m) = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right),$$

where I_1, I_2, I_3 are positive constants, the moments of inertia. Let us work out the Lie–Poisson equations $\dot{F} = \{F, H\}_-$ in this case. Clearly $\delta F/\delta m$ is just the vector in $\mathbb{R}^3$ with components $(\partial F/\partial m_1, \partial F/\partial m_2, \partial F/\partial m_3)$. Thus

$$\{F, H\}_-(m) = -\left\langle m, \frac{\delta F}{\delta m} \times \frac{\delta H}{\delta m}\right\rangle,$$

the triple product. If we choose $F(m) = m_1$, the equation $\dot{F} = \{F, H\}_-$ reads

$$\dot{m}_1 = -\begin{vmatrix} m_1 & m_2 & m_3 \\ 1 & 0 & 0 \\ \dfrac{m_1}{I_1} & \dfrac{m_2}{I_2} & \dfrac{m_3}{I_3} \end{vmatrix} = \frac{I_2 - I_3}{I_2 I_3} m_2 m_3.$$

The equations for $\dot{m}_2$ and $\dot{m}_3$ are obtained by cyclic permutation. These are the famous Euler equations for a force-free rigid body. It is trivial to check that $(\mathrm{d}/\mathrm{d}t)(m_1^2 + m_2^2 + m_3^2) = 0$; i.e. $\|m\|^2$ is constant in time. The spheres $\|m\| =$ constant are exactly the coadjoint orbits for SO(3). Thus $\mathfrak{so}(3)^*$ is the union of these symplectic manifolds (plus the origin). Their preservation by the Euler equations corresponds to the conservation of angular momentum.

The *heavy* top requires the semi-direct product $E(3) = SO(3) \times \mathbb{R}^3$; see Vinogradov and Kupershmidt [54] p. 236, Guillemin and Sternberg [23], and Marsden, Ratiu, and Weinstein [40].

The reduction of Poisson manifolds is related to reduction of symplectic manifolds with symmetry. Let J be a momentum map for the G-action on P, and assume that P is symplectic. Suppose that $\mathcal{O} \subset \mathfrak{G}^*$ is a coadjoint orbit in $\mathfrak{G}^*$ and that J is transversal to $\mathcal{O}$. Then $J^{-1}(\mathcal{O})/G$ (assuming it is without singularities) is the *reduced symplectic manifold* [41].

Proposition 2.3. The symplectic leaves of P/G are $J^{-1}(\mathcal{O})/G$.

This follows readily from the definitions and the fact that $T_x(J^{-1}(\mathcal{O}))$ splits into $T_x(G \cdot x) + \ker \mathrm{d}J(x)$, whose summands are symplectic orthogonal complements of each other.

3. Symplectic variables and gauge groups

We begin with a Poisson manifold P as the basic space of physical variables for a theory. Suppose that $H: P \to \mathbb{R}$ is a given energy function, so that the trajectories of X_H describe the dynamics of the system.

Definition 3.1. By *symplectic variables* (*or "Clebsch variables"*) we mean a symplectic manifold R and a Poisson map $\psi: R \to P$.

Recall that if $P = \mathfrak{G}^*$, then ψ will be a momentum map. If coordinates in R are found which bring the symplectic form into canonical form, they are called *canonical variables*. In many examples R is a cotangent bundle. One can always in principle use Darboux' theorem to find the canonical variables.

If we let $H_\psi = H \circ \psi$, then X_{H_ψ} projects to X_H, and so integral curves for H_ψ project to those for X_H. Thus, by introducing possibly redundant information, one can write the equations in the new R variables in symplectic Hamiltonian form, and using canonical variables, in canonical form.

Example 3.2. For the rigid body, $P = \mathfrak{G}^*_-$ where $G = \mathrm{SO}(3)$, as in example 2.4. Now SO(3) has the same Lie algebra as SU(2), which acts symplectically on $\mathbb{C}^2$. The induced momentum map $\psi : \mathbb{C}^2 \to P$ defines Cayley–Klein parameters as special symplectic variables. A related construction for general Lie groups can be found in Weinstein [59].

Definition 3.3. Let $\psi : R \to P$ be symplectic variables for a Poisson manifold P. The associated *gauge transformations* are the symplectic diffeomorphisms $\phi : R \to R$ such that $\psi \circ \phi = \psi$.

Let K be a Lie group of gauge transformations such that K acts on R from the left (resp. right) and is transitive on each fiber $\psi^{-1}(p)$ and has a momentum map $J_K : R \to \mathfrak{K}^*$ where $\mathfrak{K}$ is the Lie algebra of K. By 2.1, J_K is a Poisson map from R to $\mathfrak{G}^*_+$ (resp. $\mathfrak{G}^*_-$). We call K a *gauge group*; J_K is the corresponding conserved quantity.

Example 3.4. In electromagnetism, $R = T^*\mathfrak{A}$ where $\mathfrak{A}$ is the space of vector potentials A, and P is the Poisson space of E's and B's with $\operatorname{div} B = 0$ and with bracket

$$\{F, G\} = \int_{\mathbb{R}^3} \left(\frac{\delta F}{\delta E} \cdot \operatorname{curl} \frac{\delta G}{\delta B} - \frac{\delta G}{\delta E} \cdot \operatorname{curl} \frac{\delta F}{\delta B} \right) \mathrm{d}x$$

(see Born and Infeld [10] and Marsden and Weinstein [42, §4].) Here we choose K to be the standard group of gauge transformations $A \mapsto A + \nabla \phi$. The symplectic leaves in P are the sets where $\operatorname{div} E = \rho$ is given. (These leaves are obtained from R by reduction, as shown in the preceding reference. It is also explained there how to couple electromagnetism with other continuum theories such as fluids and plasmas.) Here the momentum map is given by $J_K(A, Y) = -\operatorname{div} Y$ where $Y = -E$ is the variable conjugate to A, and $\psi(A, Y) = (-Y, \operatorname{curl} A)$.

Example 3.5. A gauge group for Cayley–Klein parameters is U(1), and $\mathbb{C}^2$ reduced by U(1) is $\mathfrak{so}(2)^* \approx \mathfrak{so}(3)^*$.

The reduction result mentioned in the preceding examples holds in general.

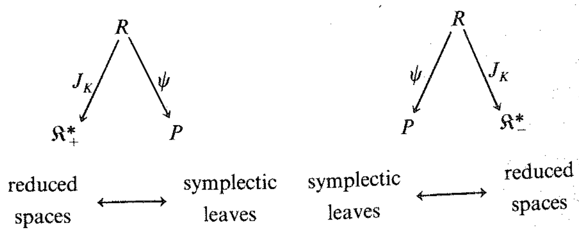
Proposition 3.6. Reduced spaces for $J_K : R \to \mathfrak{K}^*$ are symplectic leaves in P.

Proof. Let $\mathcal{O} \subset \mathfrak{K}^*$ be a coadjoint orbit. Let $\psi_{\mathcal{O}} : J_K^{-1}(\mathcal{O}) \to P$ be ψ restricted to $J_K^{-1}(\mathcal{O})$. By K-equivariance, $\psi_{\mathcal{O}}$ induces a Poisson map:

$$\Psi_{\mathcal{O}} : J_K^{-1}(\mathcal{O})/K \to P.$$

Since K acts transitively on fibers, $\Psi_{\mathcal{O}}$ is one-to-one. Since $J^{-1}(\mathcal{O})/K$ is symplectic, it embeds via $\Psi_{\mathcal{O}}$ as a symplectic leaf. ■

The following diagrams summarize the situation:

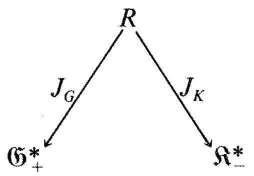


A left action of K A right action of K

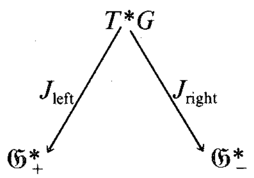
Notice that if $H : P \to \mathbb{R}$ is our given Hamiltonian and $H_\psi = H \circ \psi$, then J_K is a conserved quantity for

H_ψ, since H_ψ is invariant under gauge transformations. If J_K is constant on ψ-fibers (for example this holds if K is abelian) then it induces a map $\bar{J}_K: P \to \mathfrak{K}^*$ which gives constants of the motion for H. However these give no new information on conserved quantities in view of the preceding proposition and conservation of symplectic leaves in P. (For example, the conservation laws of Levich [33] are of this type and are explained in section 5 below).

In some cases, such as incompressible fluid mechanics, P itself is a Lie–Poisson space $\mathfrak{G}^*_+$ and ψ is a momentum mapping associated to a left action. Then the above diagram becomes



The situation is now symmetric, and G is also a gauge group for R regarded as symplectic variables for the Lie–Poisson space $\mathfrak{G}^*_+$. We say that we have a *dual pair* (see Weinstein [59]). A simple example of a dual pair is obtained by considering the left and right actions of a group G on T^*G:



For applications of this idea to semi-direct products, see Marsden, Ratiu and Weinstein [40].

4. Ideal fluid flow as a Lie–Poisson system

The configuration space for ideal (incompressible, homogeneous) fluid flow on a region Ω (a region in $\mathbb{R}^n$ or a compact Riemannian manifold with boundary) is $\mathscr{D}_{\text{vol}}$, the group of volume-preserving diffeomorphisms of Ω to itself. The phase space is $T^*\mathscr{D}_{\text{vol}}$ (where dual spaces are understood in the sense of L^2 pairings). As we shall detail below, the kinetic energy of a fluid is *right* invariant on $T^*\mathscr{D}_{\text{vol}}$ and so induces, by reduction, a Hamiltonian system on the Poisson manifold $\mathscr{X}^*_{\text{vol}}$ where $\mathscr{X}_{\text{vol}}$, the space of divergence-free fector fields on Ω parallel to $\partial\Omega$, is the Lie algebra of $\mathscr{D}_{\text{vol}}$, and the + Lie–Poisson structure (+ because of *right* invariance) is used. The Lie bracket on $\mathscr{X}_{\text{vol}}$ (the left Lie algebra of $\mathscr{D}_{\text{vol}}$) is the negative of the standard commutator bracket of vector fields*.

The picture above, but using $\mathscr{X}_{\text{vol}}$ and $T\mathscr{D}_{\text{vol}}$ rather than the dual spaces was known to Arnold [4]; its functional analytic details were established by Ebin and Marsden [21]. The use of $\mathscr{X}^*_{\text{vol}}$ in this problem has also been emphasized by Morrison [48]. (For compressible flow, see Morrison and Greene [49] and Marsden, Ratiu and Weinstein [40].)

We arrive naturally at the space $\mathscr{X}^*_{\text{vol}}$ starting only with these two assumptions:

1) the phase space is $T^*\mathscr{D}_{\text{vol}}$;

2) the Hamiltonian is right invariant.

Assumption 1 is taken for granted: the deformation and its conjugate momentum describe the state of a fluid (or generally a continuum) in material coordinates. This has been accepted as basic since the time of Euler and Lagrange. Assumption 2 is simply a fact, following from the change of variables theorem (see below or one of the aforementioned references for details).

Now $\mathscr{X}^*_{\text{vol}}$ consists of the linear functionals on $\mathscr{X}_{\text{vol}}$. These we identify with the one-forms α modulo exact one-forms. Note that exact one-forms are L^2-orthogonal to the divergence-free vector fields:

$$\int_\Omega \mathrm{d}f \cdot v \, \mathrm{d}x = 0$$

if $\operatorname{div} v = 0$, where $\mathrm{d}x$ is the volume-form on Ω.

One can represent an element $[\alpha]$ of $\mathscr{X}^*_{\text{vol}}$ by the

* Arnold [6] uses the left Lie algebra of vector fields so has conventions on brackets of vector fields that is opposite from the standard convention.

two-form $\mathrm{d}\alpha$ and the integrals $\Gamma_1, \ldots, \Gamma_l$ of α over a basis $\gamma_1, \ldots, \gamma_l$ of the first homology of Ω; i.e. elements ω of $\mathscr{X}^*_{\mathrm{vol}}$ consist of the *vorticity* $\mathrm{d}\alpha$ and circulations around non-contractible loops: $\omega = (\mathrm{d}\alpha, \Gamma_1, \ldots, \Gamma_l)$. For simplicity of exposition we shall assume that Ω is simply connected so we may ignore the Γ_i's. As in the abstract theory, for $F: \mathscr{X}^*_{\mathrm{vol}} \to \mathbb{R}$, we define $\delta F/\delta\omega \in \mathscr{X}_{\mathrm{vol}}$ by

$$\mathrm{D}F(\omega) \cdot [\sigma] = \int_\Omega \left\langle \frac{\delta F}{\delta \omega}, \sigma \right\rangle \mathrm{d}x\,,$$

where $\langle\ ,\ \rangle$ denotes vector–covector pairing and σ stands for any representative of its class $[\sigma]$. The + Lie–Poisson bracket we use is

$$\{F, G\}(\omega) = \int_\Omega \left\langle \omega, \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}\right]\right\rangle \mathrm{d}x\,.$$

This bracket was explicitly written down in Kuznetsov and Mikhailov [32], Morrison [48] and Olver [64]. It was implicitly known to Arnold [5]. Its connection with Lie–Poisson structures and reduction is due to the present authors.

Now the Hamiltonian function for incompressible flow is the kinetic energy. This is defined on $T\mathscr{D}_{\mathrm{vol}}$ by

$$H(V_\eta) = \int_\Omega \tfrac{1}{2}\langle V_\eta, V_\eta \rangle\, \mathrm{d}x\,, \qquad (\mathrm{H}_{\mathrm{material}})$$

where V_η, a vector field over $\eta \in \mathscr{D}_{\mathrm{vol}}$, is the material velocity of the fluid (i.e. $V_\eta(X)$ is the velocity at $\eta(X)$ of the particle with material point $X \in \Omega$). By the change of variables theorem, H is right invariant on $T\mathscr{D}_{\mathrm{vol}}$ (see Ebin and Marsden [21] for details). In terms of the Eulerian velocity variable,

$$H(v) = \tfrac{1}{2} \int \|v\|^2\, \mathrm{d}x\,. \qquad (\mathrm{H}_{\mathrm{Eulerian}})$$

Also, H induces a right-invariant function on $T^*\mathscr{D}_{\mathrm{vol}}$ given at the identity, i.e. on $\mathscr{X}^*_{\mathrm{vol}}$ by

$$H(\omega) = \tfrac{1}{2} \int \langle \Delta^{-1}\omega, \omega \rangle\, \mathrm{d}x\,, \qquad (\mathrm{H}_{\mathrm{vorticity}})$$

where $\langle\ ,\ \rangle$ is the metric pairing of two-forms and Δ is the Laplace–DeRham operator, $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$. (At the identity, ω is identified with $\mathrm{d}v^\flat$, where $v \in \mathscr{X}_{\mathrm{vol}}$, $v^\flat$ meaning the corresponding one-form); note that $\int \langle \Delta^{-1}\omega, \omega \rangle\, \mathrm{d}x = \int \langle \Delta^{-1}\mathrm{d}v^\flat, \mathrm{d}v^\flat \rangle\, \mathrm{d}x = \int \langle \delta\Delta^{-1}\mathrm{d}v^\flat, v^\flat \rangle, v^\flat \rangle\, \mathrm{d}x = \int \langle v^\flat, v^\flat \rangle\, \mathrm{d}x$ since $\delta v^\flat = 0$ (i.e. $\mathrm{div}\, v = 0$).

There are three equivalent ways of representing the following Euler equations for perfect incompressible flow:

$$\left.\begin{aligned} \frac{\partial v}{\partial t} + v \cdot \nabla v &= -\nabla p, \\ \mathrm{div}\, v &= 0, \\ v \text{ parallel to } \partial\Omega&, \end{aligned}\right\} \qquad (\mathrm{E})$$

in which the density is $\rho = 1$*.

First, one views (E) directly on $\mathscr{X}_{\mathrm{vol}}$ via reduction from the corresponding right-invariant system on $T\mathscr{D}_{\mathrm{vol}}$. This is the perspective developed by Arnold [4], and expositions are available in several places (Arnold [6], Marsden [36], and Abraham and Marsden [1]).

Second, one views (E) directly on $T\mathscr{D}_{\mathrm{vol}}$ or $T^*\mathscr{D}_{\mathrm{vol}}$ as an evolution equation for the configuration $\eta \in \mathscr{D}_{\mathrm{vol}}$ of the fluid and its conjugate velocity or momentum. As Arnold showed, *the equations (E) are equivalent to the geodesic equations on* $T\mathscr{D}_{\mathrm{vol}}$ (or $T^*\mathscr{D}_{\mathrm{vol}}$) for the right-invariant (weak) Riemannian metric on $T\mathscr{D}_{\mathrm{vol}}$ whose value at the identity is the L^2 inner product

$$\langle v, w \rangle = \int_\Omega \langle v(x), w(x) \rangle\, \mathrm{d}x\,.$$

It was shown in Ebin and Marsden [21] that the spray of this metric (that is the vector field on a suitable Sobolev completion of $T\mathscr{D}_{\mathrm{vol}}$ whose inte-

* For ideal but, inhomogeneous flow on $T\mathscr{D}_{\mathrm{vol}}$ together with the existence theory, see Marsden [37]. The system is not right invariant, but rather yields a Lie–Poisson system on the semi-direct product $\mathscr{X}_{\mathrm{vol}} \times$ Functions; see Marsden, Ratiu and Weinstein [40] and Henyey [26].

gral curves give solutions on $T\mathscr{D}_{\mathrm{vol}}$) is C^∞, so the Picard method for ordinary differential equations suffices to prove existence and uniqueness of solutions for short time.

Third, we can view (E) in terms of the vorticity. To do, let us temporarily use traditional fluid mechanics notation and write $\omega = \nabla \times v$, rather than $\omega = \mathrm{d}v^\flat$. Taking the curl of (E) gives the *vorticity form* of the equations:

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} - \omega \cdot \nabla v = 0, \tag{V}$$

where $\mathrm{D}\omega/\mathrm{D}t = \partial\omega/\partial t + v \cdot \nabla\omega$ is the material derivative (see, for instance, Chorin and Marsden [18, p. 32]). Here v is determined by ω through the equations

$$\omega = \nabla \times v,$$

$$\operatorname{div} v = 0, \quad v \parallel \partial\Omega.$$

If we regard ω as a two-form again, then

$$v^\flat = \delta\psi \quad \text{and} \quad \psi = \Delta^{-1}\omega,$$

where ψ is a 2-form, the 'stream function', δ is the codifferential, and $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ is the Laplace–DeRham operator. (The solution for v in terms of ω is to be supplemented by specified circulations if Ω contains non-contractible loops.)

Theorem 4.1. The vorticity equations (V) are equivalent to the Lie–Poisson equations $\dot F = \{F, H\}$ on $\mathscr{X}^*_{\mathrm{vol}}$ where H is given by ($H_{\mathrm{vorticity}}$).

This theorem follows directly from the general facts about reduction already mentioned, but we shall verify it by hand. If we let

$$\frac{\delta H}{\delta\omega} = v \in \mathscr{X}_{\mathrm{vol}},$$

then by definition

$$DH(\omega)\cdot[\sigma] = \int \langle v, \sigma\rangle \,\mathrm{d}x\,,$$

where σ is a 1-form on Ω and $[\sigma]$ is its equivalence class in $\mathscr{X}^*_{\mathrm{vol}}$, identified with the 2-form $\mathrm{d}\sigma$. Thus,

$$DH(\omega)\cdot[\sigma] = \int \langle \Delta^{-1}\omega, \mathrm{d}\sigma\rangle\,\mathrm{d}x = \int \langle v, \sigma\rangle\,\mathrm{d}x\,.$$

Since δ and d are adjoints, we get $v^\flat = \delta\Delta^{-1}\omega$. In other words, $\delta H/\delta\omega$ is nothing other than the corresponding velocity field of the vorticity. Thus,

$$\{F, H\} = \int \left\langle \omega, \left[\frac{\delta F}{\delta\omega}, \frac{\delta H}{\delta\omega}\right]\right\rangle \mathrm{d}x = \int \left\langle \omega, \left[\frac{\delta F}{\delta\omega}, v\right]\right\rangle \mathrm{d}x\,.$$

Now the Lie algebra bracket $[\delta F/\delta\omega, v]$ is the negative of the usual Lie bracket and so is given by $\mathrm{L}_v\,\delta F/\delta\omega$, where L_v is Lie differentiation. Integrating by parts, we get

$$\{F, H\} = -\int \left\langle \mathrm{L}_v\omega, \frac{\delta F}{\delta\omega}\right\rangle \mathrm{d}x\,,$$

where, as above, the pairing between $\mathscr{X}^*_{\mathrm{vol}}$ and $\mathscr{X}_{\mathrm{vol}}$ is understood in terms of representatives of the classes of two forms. Thus, by definition of $\delta F/\delta\omega$,

$$\{F, H\} = -DF(\omega)\cdot \mathrm{L}_v\omega.$$

Therefore, the equations $\dot F = \{F, H\}$ are, by the chain rule, equivalent to the *Lie form of the vorticity equation*

$$\frac{\partial\omega}{\partial t} + \mathrm{L}_v\omega = 0, \tag{LV}$$

where L_v is Lie differentiation of two-forms. Equation (LV) just says that ω is Lie transported by the flow. Now (LV) is equivalent to the form (V) using some simple vector identities†. This completes the verification of theorem 4.1. (Notice that the use of differential forms enables us to replace $\Omega \subset \mathbb{R}^3$ (or

† $\mathrm{L}_v\omega = \mathrm{d}\mathrm{i}_v\omega + \mathrm{i}_v\,\mathrm{d}\omega = \mathrm{d}\mathrm{i}_v\omega$. If $\hat\omega^\flat = (*\omega)$, then $(\mathrm{i}_v\omega) = (\hat\omega \times v)^\flat$ so $\mathrm{d}\mathrm{i}_v\omega = \operatorname{curl}(\hat\omega \times v) = -\hat\omega\cdot\nabla v + v\cdot\nabla\hat\omega$.

$\mathbb{R}^2$) by any Riemannian manifold and reveals the geometric interpretation of the vorticity equation.) ■

The equation (LV) enables us to check directly some other general facts about Lie–Poisson equations. Let us verify that *solution curves to (LV) remain on coadjoint orbits* in $\mathscr{X}^*_{\text{vol}}$.

The (right) coadjoint action of $\mathscr{D}_{\text{vol}}$ on $\mathscr{X}^*_{\text{vol}}$ is readily checked to be the pull-back action $(\eta, \omega) \mapsto \eta^*\omega$. Thus, the coadjoint orbit through $\omega \in \mathscr{X}_{\text{vol}}$ is

$$\mathcal{O}_\omega = \{\eta^*\omega \mid \eta \in \mathscr{D}_{\text{vol}}\},$$

but the solution to (LV) for given initial condition $\omega(0)$ is simply

$$\omega(t) = \eta(t)^*\omega(0),$$

where $\eta(t)$ is the flow of $v(t)$. Hence it is clear that the vorticity stays on $\mathcal{O}_{\omega(0)}$. This transport of vorticity by the flow is nothing other than Kelvin's circulation theorem. Thus, *the preservation of coadjoint orbits, right invariance on $T^*\mathscr{D}_{\text{vol}}$, and Kelvin's circulation theorem are all equivalent.*

We know from the general theory that $\mathcal{O}_\omega$ is a symplectic manifold. Let us compute its symplectic structure. Tangent vectors to $\mathcal{O}_\omega$ at ω are given by elements $\mathrm{L}_u\omega$, where $u \in \mathscr{X}_{\text{vol}}$.

Theorem 4.2. *The symplectic structure* Ω_ω on $T_\omega\mathcal{O}_\omega$ is given by

$$\Omega_\omega(\mathrm{L}_{u_1}\omega, \mathrm{L}_{u_2}\omega) = \int \omega(u_1, u_2)\,\mathrm{d}x\,.$$

Proof. We use the general Kirillov–Kostant–Souriau formula $\Omega_\mu(\xi_{\mathfrak{G}^*}(\mu), \eta_{\mathfrak{G}^*}(\mu)) = \langle \mu, [\xi, \eta]\rangle$ for the symplectic structure on coadjoint orbits (see Abraham and Marsden [1], p. 303. Here there is a + sign since we are dealing with a *right* invariant system). In our case this reads

$$\Omega_\omega(\mathrm{L}_{u_1}\omega, \mathrm{L}_{u_2}\omega) = \langle \omega, -[u_1, u_2]\rangle.$$

(Recall that our convention is to always use the left Lie bracket. For $\mathscr{D}$ or $\mathscr{D}_{\text{vol}}$, this is the *negative* of the usual Lie bracket; see Abraham and Marsden [1, Ex. 4.1G]). Let $\omega = \mathrm{d}\alpha$. Then the preceding equation gives

$$\Omega_\omega(\mathrm{L}_{u_1}\omega, \mathrm{L}_{u_2}\omega) = -\int \alpha \cdot [u_1, u_2]\,\mathrm{d}x\,,$$

according to our definition of the pairing. Now write $[u_1, u_2] = L_{u_1}u_2$ and integrate by parts to get

$$\Omega_\omega(\mathrm{L}_{u_1}\omega, \mathrm{L}_{u_2}\omega) = \int (\mathrm{L}_{u_1}\alpha)u_2\,\mathrm{d}x$$
$$= \int (\mathrm{i}_{u_1}\,\mathrm{d}\alpha + \mathrm{d}\mathrm{i}_{u_1}\alpha)\cdot u_2\,\mathrm{d}x\,.$$

The second term vanishes since $\operatorname{div} u_2 = 0$. Thus we get

$$\Omega_\omega(\mathrm{L}_{u_1}\omega, \mathrm{L}_{u_2}\omega) = \int (\mathrm{i}_{u_1}\omega)\cdot u_2\,\mathrm{d}x = \int \omega(u_1, u_2)\,\mathrm{d}x$$

as claimed. ■

5. Clebsch variables for ideal flow

According to our general scheme in section 3, symplectic variables for ideal flow are provided by momentum maps $J: R \to \mathscr{X}^*_{\text{vol}}$. To find such variables it suffices to seek symplectic manifolds on which $\mathscr{D}_{\text{vol}}$ acts and compute their momentum maps. Since $\mathscr{X}^*_{\text{vol}}$ carries the + Lie–Poisson structure, we should seek a *left* action. The classical Clebsch variables, and more, can be readily found by such an approach.

Consider the action of $\mathscr{D}_{\text{vol}}$ on the space $\mathscr{F}$ of real valued functions on Ω by $(\eta, \lambda) \to \eta_*\lambda = \lambda \circ \eta^{-1}$. This induces in the usual way a symplectic left action on $T^*\mathscr{F} = \mathscr{F} \times \mathscr{F}^*$. Identify $\mathscr{F}^*$ with $\mathscr{F}$ via the L^2 pairing using the given volume element.

Proposition 5.1. The momentum map of the above action is given by $J: (\lambda, \mu) \mapsto \omega = \mathrm{d}\lambda \wedge \mathrm{d}\mu$.

Proof. Let $u \in \mathscr{X}_{\text{vol}}$. The corresponding infinitesimal generator on $\mathscr{F}$ is $-\mathrm{L}_u\lambda$. Thus, using the formula

$$\langle \xi, J(\alpha_q)\rangle = \langle \alpha_q, \xi_Q(q)\rangle$$

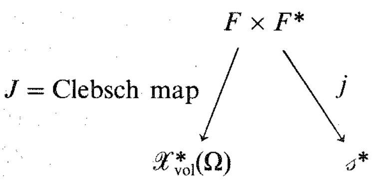
for the momentum map of a lifted action (Abraham and Marsden [1, p. 283]) we get

$$\begin{aligned}\langle u, J(\lambda, u)\rangle &= \int \mu \cdot (-\mathrm{L}_u\lambda)\,\mathrm{d}x \\ &= \int (\mathrm{L}_u\mu)\lambda\,\mathrm{d}x \\ &= \int \lambda\,\mathrm{d}\mu \cdot u\,\mathrm{d}x\,.\end{aligned}$$

Thus the one form representing $J(\lambda, \mu)$ is $\lambda\,\mathrm{d}\mu$. The corresponding two-form is $\mathrm{d}(\lambda\,\mathrm{d}\mu) = \mathrm{d}\lambda \wedge \mathrm{d}\mu$. Thus $J(\lambda, \mu) = \mathrm{d}\lambda \wedge \mathrm{d}\mu$. ■

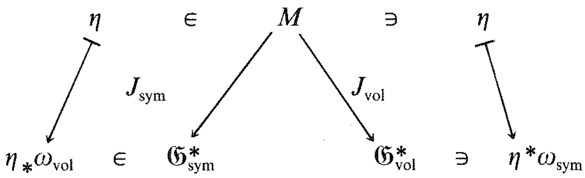
It follows directly that if the Euler equations are expressed in terms of λ and μ they will be in canonical Hamiltonian form, a result already known to Clebsch [19]. We call the canonical variables λ, μ *Clebsch variables.*

The gauge group of Clebsch variables consists of all canonical transformations $\mathscr{S}$ of $\mathbb{R}^2$. Indeed, these are the transformations of the (λ, μ) variables that leave ω invariant. Now we get a dual pair as one can readily check:



The momentum map j for the action of $\mathscr{S}$ on $\mathscr{F} \times \mathscr{F}^*$ is given by $(\lambda, \mu) \mapsto (\lambda \times \mu)_*\,\mathrm{d}\mu$ where we identify $\mathfrak{s}^*$, the dual of the Lie algebra $\mathfrak{s}$ of $\mathscr{S}$, with densities on $\mathbb{R}^2$, as in Marsden and Weinstein [42].

The example above is a member of a family of dual pairs. If $(P_{\text{sym}}, \omega_{\text{sym}})$ is a symplectic manifold and $(P_{\text{vol}}, \omega_{\text{vol}})$ is a volume manifold, i.e. a manifold carrying a volume element ω_{vol}, let $\mathscr{M}$ be the space of maps from P_{vol} to P_{sym}. Then the groups G_{vol} and G_{sym} of volume-preserving and symplectic diffeomorphisms of P_{vol} and P_{sym} respectively act on $\mathscr{M}$ by compositions. Their momentum maps are $J_{\text{vol}}(\eta) = \eta^*\omega_{\text{sym}} \in \mathfrak{G}^*_{\text{vol}}$ regarded as two-forms on P_{vol} as before and $J_{\text{sym}}(\eta) = \eta_*\omega_{\text{vol}} \in \mathfrak{G}^*_{\text{sym}}$, regarded as densities on P_{sym}. Then



is a dual pair.

Notice that in the case of fluids j is a constant of the motion for the induced Hamiltonian system on $\mathscr{F} \times \mathscr{F}^*$ since $\mathscr{S}$ is, by construction, an invariance group. For example, if we compose j with evaluation against an element ϕ of $\mathfrak{s}$, we get the constant of the motion

$$\int_{\mathbb{R}^2} \phi(\lambda, \mu)[(\lambda \times \mu)_*\,\mathrm{d}x] = \int_{\Omega} \phi(\lambda, \mu)\,\mathrm{d}x\,.$$

These invariants were found by Levich [33]. From the general theory, we know that these give no more information than the invariance of coadjoint orbits. In particular, even though one can find infinitely many such invariants in involution, they can never suffice to prove complete integrability. One might call these invariants "kinematic", since their conservation depends only on the Poisson structure and not on the specific Hamiltonian which generates the dynamics. Similar remarks apply to Ebin [63].

Note that if $\omega = \mathrm{d}\lambda \wedge \mathrm{d}\mu$, then the velocity field $v^\flat$ is such that $v^\flat - \lambda\,\mathrm{d}\mu$ is closed; if it is exact, we can write

$$v^\flat = \mathrm{d}\alpha + \lambda\,\mathrm{d}\mu,$$

which is the classical Clebsch representation for the velocity field. This expression is interpreted in terms of canonical variables in Marsden, Ratiu and Weinstein [40].

Notice next that if $\omega = \mathrm{d}\lambda \wedge \mathrm{d}\mu$, and we are in $\mathbb{R}^3$, say, then the *helicity* is given by

$$\mathscr{H} \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} v^\flat \wedge \omega = \int_{\mathbb{R}^3} v^\flat \wedge \mathrm{d}(\lambda \wedge \mathrm{d}\mu)$$

$$= -\int_{\mathbb{R}^3} \mathrm{d}v^\flat \wedge \lambda\, \mathrm{d}\mu \quad \text{(integration by parts)}$$

$$= -\int_{\mathbb{R}^3} (\mathrm{d}\lambda \wedge \mathrm{d}\mu) \wedge \lambda\, \mathrm{d}\mu = 0.$$

However, in a flow with linked vortex lines, the helicity does not vanish (see Moffatt [46] for a discussion), so *the Clebsch representation leaves out many interesting coadjoint orbits.*

Remark. Invariance of the helicity follows at once from preservation of coadjoint orbits: $\partial\omega/\partial t + \mathrm{L}_v\omega = 0$ (cf. Moffatt [46], p. 41).

It has likewise been noted by Bretherton [12] that the variational principle of Seliger and Whitham [52], which is closely related to the Clebsch representation, does not permit knotted vortex lines.

Helicity can be reincorporated if, in the general (P_{sym}, P_{vol}) dual pair, one replaces $\mathbb{R}^2$ by the two-sphere S^2 and imposes certain boundary conditions on $\mathbb{R}^3$ to allow compactification to S^3. The helicity is then a multiple of the topological Hopf invariant; see Kuznetsov and Mikhailov [32].

6. Two-dimensional flow

The "vorticity-bracket"

$$\{F, G\}(\omega) = \int_\Omega \left\langle \omega, \left[\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta\omega}\right]\right\rangle \mathrm{d}\mu$$

introduced in section 4 has a particularly simple formulation in two dimensions. A volume (i.e. area) element in two dimensions in also a symplectic structure, so that each divergence-free vector field v may be thought of as a Hamiltonian vector field X_ψ; the Hamiltonian function ψ coincides with the *stream function* for v discussed in section 4. Assuming that the manifold Ω on which the fluid flows is connected, the function ψ is determined by v up to a constant, so we may identify the Lie algebra $\mathscr{X}_{\text{vol}}$ with $C^\infty(\Omega)$/constants. The dual space $\mathscr{X}^*_{\text{vol}}$ is then identified with generalized functions ω on Ω with $\int_\Omega \omega\, \mathrm{d}\mu = 0$, and the reader may check that this correspondence is consistent with the previous identification of $\mathscr{X}^*_{\text{vol}}$ with the vorticities.

Now the Lie algebra bracket $[v_1, v_2]$ in $\mathscr{X}^*_{\text{vol}}$ is the negative of the Lie bracket of vector fields $[v_1, v_2] = -\mathrm{L}_{v_1}v_2 = -\mathrm{L}_{X_{\psi_1}}X_{\psi_2} = X_{\{\psi_1,\psi_2\}}$ (Abraham and Marsden [1], p. 194). Thus the Lie algebra bracket corresponds to the Poisson bracket $\{\psi_1, \psi_2\}$ of stream functions, (with the standard sign conventions), so we may write the vorticity bracket in the form

$$\{F, G\}(\omega) = \int_\Omega \omega \left\{\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta\omega}\right\} \mathrm{d}\mu,$$

where ω, $\delta F/\delta\omega$, and $\delta G/\delta\omega$ are all thought of as functions on Ω, and the bracket inside the integral is the "ordinary" two-dimensional Poisson bracket* on the symplectic manifold Ω.

In the remainder of this section, we will exhibit two different kinds of Clebsch variables for the vorticity bracket variables on the individual coadjoint orbits.

The first Clebsch representation, introduced by Morrison [47, 48] is to write ω as a bracket $\omega = \{\mu, \lambda\}$. Then if we consider the pair (λ, μ) to

* As Morrison [48] has also noted, this the same as the Poisson bracket for the Poisson–Vlasov equation in one space dimension, as well as for the guiding center plasma beam equation. It is shown in Weinstein [60] that the same bracket can also be used for geostrophic fluid flow and plasma drift waves.

be canonical variables, the mapping $(\lambda, \mu) \to \omega$ turns out to be a Poisson mapping. This construction is a special case of one proposed in a general Lie group context by Kazhdan, Kostant, and Sternberg [31], who apply it to show the integrability of the Calogero dynamical system. If $\mathfrak{G}$ is any Lie algebra, then G acts on $\mathfrak{G}$ by the adjoint representation, and this action lifts to a symplectic action on $T^*\mathfrak{G} \approx \mathfrak{G} \times \mathfrak{G}^*$. The momentum map is $(\xi, \mu) \mapsto \mathrm{ad}_\xi^* \mu$. If $\mathfrak{G}$ carries an Ad-invariant metric, we may identify $\mathfrak{G}^*$ with $\mathfrak{G}$, and the momentum map of the lifted adjoint action is simply $(\xi, \eta) \mapsto [\eta, \xi]$. (The ad-invariant metric for the vorticity algebra is the L^2 inner product.)

We note that the Lagrangian description in $T^*\mathscr{D}_{\mathrm{vol}}$ is itself a "Clebsch representation". Since $T^*\mathscr{D}_{\mathrm{vol}} \approx \mathscr{D}_{\mathrm{vol}} \times \mathscr{X}^*_{\mathrm{vol}}$, and elements of $\mathscr{D}_{\mathrm{vol}}$ can be parametrized by their generating functions, this representation also involves two functions of two variables. The advantage of Morrison's representation $(\lambda, \mu) \mapsto \{\mu, \lambda\}$ is that canonical variables are readily available, and no generating functions are necessary.

The second Clebsch representation is more complicated to describe, but more "efficient" in that ω is represented by one function of two variables and one function of *one* variable. It will be motivated by a general construction for a Lie group G with a bi-invariant metric, following ideas from group theory (see Weinstein [59].) We seek a symplectic invariant submanifold S of $T^*G \approx TG \approx G \times \mathfrak{G}$. Such a submanifold must necessarily be of the form $S = G \times K \subset G \times \mathfrak{G}$. If K is an open subset in a subspace K_1 of $\mathfrak{G}$, the condition that S be symplectic is that for each $\theta \in K$, the form on $K_1^\perp$ ($\perp$ relative to the bi-invariant metric) given by $(\xi, \eta) \mapsto \langle \theta, [\xi, \eta] \rangle$ be nondegenerate. It is natural to take K_1 as a maximal commuting subalgebra of $\mathscr{X}_{\mathrm{vol}}$, with K the regular elements in K_1. The space $K_1^\perp$ is then the sum of the "root spaces"; i.e. the two-dimensional invariant subspaces for the adjoint action of K on $\mathfrak{G}$.

For *two*-dimensional flow we let $G = \mathscr{D}_{\mathrm{vol}}$; the L^2 pairing is the bi-invariant metric. At this point we can drop our motivating ideas from group theory and concentrate on specific calculations in the algebra $\mathscr{X}_{\mathrm{vol}} = C^\infty(\Omega)$. For simplicity, we shall take Ω to be the (x, y) plane and K_1 to consist of the functions which depend upon y alone. (The case where Ω is the 2-sphere and K_1 consists of functions of the latitude is also an instructive example to work out.)

The orthogonal complement $K_1^\perp$ consists of those ω for which $\int \omega(x, y)\, dx$ is identically zero in y and can therefore be equally well expressed as those ω for which the horizontal Fourier transform $\hat{\omega}(k_x, y) = \int e^{-ixk_x} \omega(x, y)\, dx$ vanishes along the line $x = 0$. In the same representation, ω belongs to K_1 if and only if $\hat{\omega}(k_x, y)$ is the form $\delta(k_x) f(y)$.

Now if ω_1 and ω_2 belong to $K_1^\perp$ and $f(y)$ belongs to K_1, we have

$$\begin{aligned} \langle f, \{\omega_1, \omega_2\} \rangle &= -\langle \omega_1, \{f, \omega_2\} \rangle \\ &= \int \omega_1(x, y) f'(y) \frac{\partial \omega_2}{\partial x}(x, y)\, dx\, dy \\ &= \int \hat{\omega}_1(k_x, y) \hat{\omega}_2(k_x, y)\, i k_x f'(y)\, dx\, dy. \end{aligned}$$

We can expect this to be nondegenerate when $f'(y)$ is nowhere vanishing, i.e. when the flow for which f is the stream function has no stationary points. These, then, are the "regular elements" and define the set K.

Now we can define our Clebsch representation to be in the space $\mathscr{D}_{\mathrm{vol}} \times K = \{(\gamma, f) \mid \gamma$ is an area-preserving diffeomorphism of $\mathbb{R}^2$ and f is a function of y such that $f'(y) > 0\}$. The Clebsch representation is $\omega = f \circ \gamma$; its image consists of those vorticity functions ω such that whenever $a < b$, the set $\omega_{a,b} = \{(x, y) \mid a \leqslant \omega(x, y) \leqslant b\}$ is a band of infinite area stretching to infinity in both directions. The Poisson structure on $\mathscr{D}_{\mathrm{vol}} \times K$ can be calculated in terms of that on $T^*\mathscr{D}_{\mathrm{vol}}$; although this structure is symplectic, we do not know of any explicit construction of canonical variables on it. (Implicit constructions are guaranteed by the version of Darboux's theorem in Marsden [38], lec. 1.)

7. Point vortices

We now consider point vortices in two dimensions with $\Omega = \mathbb{R}^2$. To do so, we need to allow delta functions as possibilities for the vorticity. Given N vortices in the xy plane with positions (x_i, y_i) and circulations Γ_i, the associated vorticity is

$$\omega = \sum_{i=1}^{N} \Gamma_i \delta_{(x_i, y_i)}\, \mathrm{d}x \wedge \mathrm{d}y.$$

For fixed $\Gamma_1, \ldots, \Gamma_N$, the set of all such vorticities forms a coadjoint orbit in $\mathscr{X}^*_{\mathrm{vol}}$. The coadjoint action is just the action that moves the points (x_i, y_i) by the diffeomorphism $\eta \in \mathscr{D}_{\mathrm{vol}}$. If $u_1, u_2 \in \mathscr{X}_{\mathrm{vol}}$ are divergence-free vector fields on $\mathbb{R}^2$, the symplectic structure on the coadjoint orbit of ω at ω is given from Theorem 4.2 by

$$\begin{aligned}\Omega_\omega(\mathrm{L}_{u_1}\omega, \mathrm{L}_{u_2}\omega) &= \int \omega(u_1, u_2)\, \mathrm{d}\mu \\ &= \sum_{i=1}^{N} \Gamma_i (\mathrm{d}x \wedge \mathrm{d}y)(u_1(x_i, y_i), u_2(x_i, y_i)).\end{aligned}$$

This may be identified with the symplectic structure*

$$\Omega_{\Gamma_1, \ldots, \Gamma_N} = \sum \Gamma_i\, \mathrm{d}x_i \wedge \mathrm{d}y_i$$

on $\mathbb{R}^{2N}$ applied to the pairs of vectors $(u_1(x_i, y_i), u_2(x_i, y_i))$. In fact, if we consider $\mathbb{R}^{2N}$ with the symplectic structure above, then $\mathscr{D}_{\mathrm{vol}}$ acts on it symplectically and its momentum map is precisely

$$V : (x_i, y_i) \mapsto \omega = \sum_{i=1}^{N} \Gamma_i \delta_{(x_i, y_i)}\, \mathrm{d}x \wedge \mathrm{d}y.$$

Notice that this is a diffeomorphism onto the coadjoint orbit.

Thus, *the variables* (x_i, y_i) *on* $\mathbb{R}^{2N}$ *are canonical (or Clebsch) variables for the motion of* N *vortices.* This Poisson map V clearly has only a trivial gauge group if the Γ_i's are distinct, otherwise the gauge group is a finite group of permutations.

Our construction is a special case of the example involving $\mathscr{M}$ = maps of P_{vol} to P_{sym} considered in section 5. Let P_{vol} be the set $\{1, \ldots, N\}$ with the ith point having mass Γ_i, and consider $P_{\mathrm{sym}} = \mathbb{R}^2$, with $\omega_{\mathrm{sym}} = \mathrm{d}x \wedge \mathrm{d}y$. Thus $\mathscr{M}$ is $\mathbb{R}^{2N}$ with symplectic structure $\Sigma\, \Gamma_i\, \mathrm{d}x_i \wedge \mathrm{d}y_i$. Now $\mathscr{D}_{\mathrm{sym}}$ acts on $\mathscr{M}$ on the left with momentum map $\eta \mapsto \eta_* \omega_{\mathrm{vol}}$. This is just the map V above.

We next consider the representation of the (kinetic energy) Hamiltonian for perfect fluids written in the above canonical variables. From $H(\omega) = \frac{1}{2}\int \langle \Delta^{-1}\omega, \omega \rangle\, \mathrm{d}\mu$ (section 4) and the fact that $\Delta^{-1}(\delta_{(x_i, y_i)}\, \mathrm{d}x \wedge \mathrm{d}y) = -(1/2\pi) \log\|(x - x_i, y - y_i)\|\, \mathrm{d}x \wedge \mathrm{d}y$, the Green's function† for Δ, we get

$$\begin{aligned}&H((x_1, y_1), \ldots, (x_N, y_N)) \\ &\quad = -\sum_{i,j} \frac{1}{4\pi} \Gamma_i \Gamma_j \log\|(x_i - x_j, y_i - y_j)\| \,.\end{aligned}$$

Of course the term with $i = j$ is infinite, corresponding to the infinite self-energy of a point vortex. If this term is removed we get the standard Hamiltonian picture of vortex motion (cf. Chorin and Marsden [18], p. 85). To summarize: *the usual Hamiltonian description of N vortices is just the restriction of the standard Euler equation Lie–Poisson Hamiltonian description to a particular coadjoint orbit, with the (infinite) self-energy terms ignored.* We did this in $\mathbb{R}^2$, but the description also works for bounded domains or curved surfaces (cf. Hally [24]). The necessary renormalization in the description of point vortices is related to difficulties in taking the limit $N \to \infty$ which are briefly discussed in the next section.

We conclude with some remarks on integrability and chaotic motions of vortices. For the Euler equations on $\mathbb{R}^2$, the Euclidean group E(2) acts on $\mathscr{D}_{\mathrm{vol}}$ by composition on the left, commutes with right composition, and leaves the Hamiltonian invariant. The momentum map for the E(2) action

* This symplectic structure for point vortices is well known. Its derivation from the Poisson structure for smooth vorticities was also found by Morrison [47].

† Recall that Δ is the Laplace–DeRham operator; $\Delta(f\, \mathrm{d}x \wedge \mathrm{d}y) = (-\nabla^2 f)\, \mathrm{d}x \wedge \mathrm{d}y$.

is the total linear and angular momentum. The momentum map is right invariant, so it induces a momentum map on $\mathscr{X}^*_{\mathrm{vol}}$; cf. Marsden and Weinstein [41], p. 127, Thm. 2. The momentum map on vortices is determined by the map of e(2) to $C^\infty(\mathscr{X}^*_{\mathrm{vol}})$ given by

$$\frac{\partial}{\partial x} \mapsto \int y\omega(x, y)\,\mathrm{d}x\,\mathrm{d}y = J_x,$$

$$\frac{\partial}{\partial y} \mapsto \int -x\omega(x, y)\,\mathrm{d}x\,\mathrm{d}y = J_y,$$

$$\frac{\partial}{\partial \theta} \mapsto \int -\tfrac{1}{2}(x^2 + y^2)\omega(x, y)\,\mathrm{d}x\,\mathrm{d}y = J_\theta,$$

which satisfy the commutation relations

$$\{J_x, J_y\} = \Omega \overset{\mathrm{def}}{=} \int \omega(x, y)\,\mathrm{d}x\,\mathrm{d}y,$$

$$\{J_x, J_\theta\} = J_y,$$

$$\{J_y, J_\theta\} = -J_x.$$

The vorticities which actually arise from momentum densities which vanish at infinity are those for which $\Omega = 0$; on this space the momentum map is Ad* equivariant.

On the full space of vorticities (i.e. all densities), Ω is a Casimir function, and the corresponding group which acts is the extension of E(2) (called the oscillator group), whose algebra is generated by $(J_x, J_y, J_\theta, \Omega)$ satisfying the commutation relations above. These quantities comprise an Ad*-equivariant momentum map for the oscillator group.

On point vortices,

$$\Omega = \sum_{i=1}^{N} \Gamma_i,\quad J_x = \sum_{i=1}^{N} \Gamma_i x_i,\quad J_y = -\sum_{i=1}^{N} \Gamma_i y_i$$

and

$$J_\theta = -\tfrac{1}{2}\sum_{i=1}^{N} \Gamma_i(x_i^2 + y_i^2).$$

For $N = 3$ one can check that the motion is (completely) integrable in the sense that the (non-abeilan) reduced phase spaces are points. However one can also see that the dynamics of 3 point vortices is (completely) integrable by exhibiting 3 independent integrals in involution such as H, J, and $J_x^2 + J_y^2$. (This is a special case of the replacement of non-abelian by abelian complete integrability, as discussed by Mischenko and Fomenko [45].)

The general point of view presented here has the advantage that it extends to related, and perhaps more realistic, situations. For example, the motion of three vortices with cores (defined in section 8) is also completely integrable.

The motion of four vortices is generally believed to be chaotic. There are many papers, (see the references in Aref [2]) giving numerical evidence for this belief. Using a perturbation argument of Melnikov, Ziglin [62] outlines a proof of non-integrability. The proof involves the introduction of a fourth vortex ("the restricted four vortex problem") moving in the field of three vortices in stable triangular relative equilibrium (for the E(2) symmetry); cf. Palmore [51]. However, Ziglin's proof has some gaps involving exponentially small terms, as noted in Holmes and Marsden [28]. As shown in Synge [53], Aref and Pomphrey [3], and Aref [2], there is a different configuration of three vortices having a homoclinic orbit joining two saddle points in its reduced phase space. (The saddle points are configurations of three identical vortices on a line.) The methods of Holmes and Marsden [30] can now be applied—the result is almost surely that a nearby four vortex model (with Γ_4 small) will be chaotic in the sense of having Smale horseshoes in its dynamics. These facts suggest, but certainly do not prove, that two-dimensional Euler flow is not completely integrable.

8. Vortex cores

In this section we outline a modification of point vortices intended to model vortices with cores. This model has the advantage that it involves no renormalization, and its solutions converge as $N \to \infty$ to solutions of the Euler equations. Vortex cores have been successfully used in numerical computations by Chorin [14]; the analytical convergence proof is due to Beale and Majda [8] with important earlier work due to Hald. Our purpose is to indicate how this model fits into the Lie–Poisson picture and how, using it, one can gain geometric insight into "why" the scheme converges. We hope that such insight may inspire similar results in (geometrically related) plasma problems, which also have a Lie–Poisson description.

The motion of N vortex cores still forms a finite-dimensional Hamiltonian system. The idea is to cut off the logarithmic singularity in the Hamiltonian of the point vortex model. This cutoff is gradually removed as the number N of blobs gets large.

Let ψ be a function on $\mathbb{R}^2$ with integral 1 and let $\psi_\delta(z) = \delta^{-2}\psi(z/\delta)$, for $z \in \mathbb{R}^2$ and $\delta > 0$. Let

$$G_\delta(z) = \int G(z - z')\psi_\delta(z')\,\mathrm{d}z',$$

where $G(z) = (-1/2\pi)\log\|z\|$ is the Green's function for Δ. Consider the modified Lie–Poisson Hamiltonian system on $\mathscr{X}^*_{\mathrm{vol}}$ obtained by replacing Δ by Δ_δ, the operator whose kernel function is G_δ. Thus we consider

$$H_\delta : \mathscr{X}^*_{\mathrm{vol}} \to \mathbb{R},$$

$$H_\delta(\omega) = \tfrac{1}{2}\int \langle \Delta_\delta^{-1}\omega, \omega\rangle\,\mathrm{d}\mu$$

(See equation ($\mathrm{H}_{\mathrm{vorticity}}$) in section 4).

We consider the following three systems:

1) H_δ on the coadjoint orbit with N points: $\omega = \Sigma\,\Gamma_i \delta_{(x_i, y_i)}\,\mathrm{d}x \wedge \mathrm{d}y$;

2) H_δ on smooth vorticities; and

3) H on smooth vorticities.

Beale and Majda show that solutions of system 1, with initial conditions obtained by discretizing a smooth vorticity field, converge as $\delta \to 0$ and $N \to \infty$, with δ and N linked in a certain way, to the solutions of system 3 with the given initial condition. (It is a classical theorem of Wolibner [61] that there are smooth global solutions of the Euler equations in $\mathbb{R}^2$). This convergence includes particle paths as well as velocity fields. In other words, the convergence holds in material coordinates; i.e. on $T\mathscr{D}_{\mathrm{vol}}$ or $T^*\mathscr{D}_{\mathrm{vol}}$. How δ and N are linked depends on the norms used and on ψ. See Beale and Majda [8] for details.

We introduce the intermediate system 2 in order to make the following series of remarks:

a) With the cut off Hamiltonian H_δ, the vortex cores form a Hamiltonian system on $\mathbb{R}^{2n}$ and coincide with the cores defined by Beale and Majda.

b) For δ fixed, and N large (depending on δ) solutions of system 1 approximate solutions of system 2. This result is similar to that of Braun and Hepp [11]. (With a singular interaction it is easy to see directly that one cannot get convergence, by considering clouds of vortices passing through each other with close encounters).

c) For δ small, trajectories of H_δ with smooth initial conditions converge to those of H with the same conditions as $\delta \to 0$. In fact, it follows as in Ebin and Marsden [21] that H_δ and H generate smooth vector fields on $T^*\mathscr{D}_{\mathrm{vol}}$ (completed in suitable Sobolev topologies) that are C^1 close for δ small. Thus the convergence in this step merely results from elementary facts about trajectories of smooth vector fields on Banach manifolds.

d) Step c) does not involve N; δ and N are linked in step b).

The remarks above are not meant to replace the analytic estimates needed for the actual convergence proof. Rather, they are intended to show the overall structure of the method and to give a rather different argument for why it works.

9. Vortex patches

A vortex patch is a vorticity distribution in the plane which is the characteristic function of a region with smooth boundary (times $\mathrm{d}x \wedge \mathrm{d}y$ to regard it as a two form). Early work on stationary and steadily rotating patches was done by Kirchoff and Kelvin. We refer to Aref [2] and Burbea [13] for recent work and references. The Euclidean group and the breaking of its symmetry are an important part of this analysis. Numerical investigations of vortex patches have been made by Deem and Zabusky [20].

Here we are interested in the dynamics of vortex patches as a Hamiltonian system. We shall first show that the Lie–Poisson structure reduces to a symplectic structure used in the KdV equation. The set of vortex patches supported on a set of fixed topological type and area forms a coadjoint orbit in $\mathscr{X}^*_{\text{vol}}$, since any two such patches are related by an area-preserving map. (This is proved as in Moser [50]; see Ebin and Marsden [21] p. 126.) For a vortex patch ω, let $\mathcal{O}_\omega$ be its coadjoint orbit. Let $v \in \mathscr{X}_{\text{vol}}$, with stream function ψ. Using our earlier notation,

$$v^\flat = \delta(\psi \, \mathrm{d}x \wedge \mathrm{d}y),$$

$$\psi \, \mathrm{d}x \wedge \mathrm{d}y = \Delta^{-1}\omega,$$

$$\omega = \mathrm{d}v^\flat.$$

A tangent vector to $\mathcal{O}_\omega$ is represented by $\mathrm{L}_v\omega$. Now we use theorem 4.2 to compute the symplectic structure:

Proposition 9.1. The symplectic structure Ω_ω on $T_\omega\mathcal{O}_\omega$, where ω is a vortex path associated with the set $M \subset \mathbb{R}^2$, is given by

$$\Omega_\omega(\mathrm{L}_{v_1}\omega, \mathrm{L}_{v_2}\omega) = \int_{\partial M} \psi_1 \, \mathrm{d}\psi_2,$$

where ψ_1 and ψ_2 are the stream functions for v_1 and v_2, and ∂M is the boundary of M.

Proof. By theorem 4.2,

$$\Omega_\omega(\mathrm{L}_{v_1}\omega, \mathrm{L}_{v_2}\omega) = \int_M \omega(v_1, v_2) \, \mathrm{d}\mu.$$

But on M, ω is the standard symplectic structure on the plane, so $\omega(v_1, v_2) = \{\psi_1, \psi_2\}$, the Poisson bracket.

Now $\{\psi_1, \psi_2\} \, \mathrm{d}\mu = \mathrm{d}(\psi_1 \, \mathrm{d}\psi_2)$, a simple identity, so the result follows by Stokes' theorem. ■

If M is diffeomorphic to a disc then ω is determined by the boundary loop ∂M. We observe that the symplectic structure in 9.1 is the same as that for the loop space in $\mathbb{R}^2$. (See Weinstein [56].)

In plasma physics, the analog of the vortex patch is called the water bag model†.

Let us briefly mention the evolution of the shape of the patch. Consider a patch near the unit disc whose shape is described in polar coordinates by

$$\tfrac{1}{2}(r^2 - 1) = \phi(\theta),$$

where $\int_0^{2\pi} \phi(\theta) \, \mathrm{d}\theta = 0$ (so that the area is always π). In this representation the Poisson bracket corresponding to the symplectic structure in proposition 9.1 is

$$\{F, G\} = \int_0^{2\pi} \frac{\delta F}{\delta\phi} \frac{\mathrm{d}}{\mathrm{d}\theta} \frac{\delta G}{\delta\phi} \, \mathrm{d}\theta \, .$$

Using this bracket and truncating the Hamiltonian $\tfrac{1}{2}\langle \Delta^{-1}\omega, \omega\rangle$ to third order in ϕ, one finds that the evolution equation for ϕ truncated at second order is

$$\phi_t = c_1\phi_x + c_2\mathscr{H}\phi + c_3\phi\phi_x,$$

where c_1, c_2 and c_3 are constants and $\mathscr{H}$ is the Hilbert transform on the circle (convolution with $\tfrac{1}{2}\Sigma_{k \neq 0} (\operatorname{sgn} k) \, \mathrm{e}^{ik\theta}$). It appears that the dispersion operator $\mathscr{H}$ is too weak to support travelling solitary waves without shocks; cf. the computations of Deem and Zabusky [20] and compare with the Benjamin–Ono equation, when the term $\mathscr{H}\phi$ is replaced by $\mathscr{H}\phi_{xx}$ (see Meiss and Pereira [43]).

We suspect that a Hamiltonian treatment of free boundary problems and surface waves for the

† For the Poisson–Vlasov equation, the analogue of 9.1 is

$$\Omega_f(\{\psi_1, f\}, \{\psi_2, f\}) = \int_{\partial M} \psi_2(\mathrm{i}_{X_{\psi_1}}\mu) \, .$$

Euler equations themselves (see Miles [44]) will enable one to see how the Hamiltonian description of the KdV equation fits in with that for the Euler equations as described in this paper. What is missing is a suitable framework for taking limits of Hamiltonian systems. We hope that the ideas in Weinstein [59] will be of help in this direction.

10. Vortex filaments

The volution of a vortex filament in space, in the "self-induction approximation" is given by a motion at each point of the filament in the direction of the binormal, with velocity equal to the curvature (cf. Batchelor [7]). Hasimoto [25] showed that this motion is equivalent to the (completely integrable) nonlinear Schrödinger equation for the quantity

$$(\text{curvature})\, e^{i \int \text{torsion}},$$

which determines the filament up to a rigid motion. It is our purpose in this section to describe the Hamiltonian structure of the self-induction equation as *deduced* from that for the Euler equation; it remains to be seen how Hasimoto's transformation should be interpreted in our framework.

We consider our Lie–Poisson manifold $\mathscr{X}^*_{\text{vol}}$ on $\mathbb{R}^3$ and look at the set of vorticity distributions of the following form. Let C be a curve in $\mathbb{R}^3$ extending to infinity in both directions. (One can study closed loops in a similar way.) Let δ_C be the delta-function given by integration along C with respect to arc length. Let ω_C be the 2-form along C defined by $i_T\,dx \wedge dy \wedge dz$, where T is the unit tangent vector to C. Then if Γ is any constant, $\Gamma\omega_C\delta_C$ is the vorticity corresponding to C with strength Γ. (The constancy of Γ is equivalent to the vorticity being a *closed* generalized 2-form.)

The set of all such vorticities with a fixed Γ forms a coadjoint orbit $\mathcal{O}_\Gamma$ in $\mathscr{X}^*_{\text{vol}}$ and so it carries a symplectic structure Ω_Γ. The tangent space to $\mathcal{O}_\Gamma$ at a curve C consists of all vector fields normal to C, and for two such fields v and ω one finds that $\Omega_\Gamma(v, \omega) = \Gamma \int_C (T \times v)\cdot \omega \, ds$ using theorem 4.2. If we think of the space perpendicular to T as a copy of the complex numbers, then the operator $T\times$ corresponds to multiplication by $\sqrt{-1}$, and Ω_Γ is then equivalent to the symplectic structure relative to which the Schrödinger equation (linear or nonlinear) is Hamiltonian. (See Abraham and Marsden [1], p. 461.)

The Hamiltonian for the self-induction equations now turns out to be simply the arc-length functional on the curves C. Of course, this is only formal, since the curves C all have infinite arc length, and so one must renormalize the Hamiltonian somehow, such as by considering filaments asymptotic at infinity to a reference curve and then taking the difference of the two arc-length integrands. For closed loops the symplectic structure Ω_Γ is given by the same formula as above. We notice that in the coadjoint orbit corresponding to Γ, the loops C can have arbitrary lengths; i.e. *the stretching of vortex filaments is allowed*, although it does not occur for the arc length Hamiltonian flow.

Our approach and our discussion in section 8 make it clear how to introduce *vortex filaments with cores* (following ideas of Chorin [15], the cutoff is uniform along the filament and is not curvature dependent). Again one gets a Hamiltonian system, and the symplectic structure is still that of the Schrödinger equation. Vortex filaments (actually segments) with cores are useful in numerical work (see, for instance Chorin [15]) and there is a convergence theorem for them due to Beale Majda [8]. The convergence is valid for time periods for which smooth solutions for the Euler equations exist, just as in related algorithms for the three-dimensional Euler equations (cf. Ebin and Marsden [21] and Chorin et al. [17]). The geometry behind this convergence, which helps to explain why it works, is similar to that for point vortices with cores, discussed in section 8.

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