BIFURCATION AND LINEARIZATION STABILITY IN THE TRACTION PROBLEM

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In this lecture I will provide some insight into the recent work of Chillingworth, Marsden and Wan [1982] on the traction problem in elasto-statics. This work extends earlier results of Signorini and Stoppelli on the same problem. After briefly describing the principal features of the approach, I shall explain how they are related to linearization stability and to other bifurcation problems in mechanics by means of a series of remarks.

Let $\mathcal{B} \subset \mathbb{R}^3$ be the reference configuration and $\phi:\mathcal{B} \longrightarrow \mathbb{R}^3$ a configuration. Let $F = D\phi$ be the associated deformation gradient and $W(F)$ a materially frame indifferent stored energy function; that is, W depends on F only through the point values of the Cauchy-Green tensor $C = F^TF$, where F^T is the transpose of F. Let $\ell = (B,\tau)$ denote a (dead) load, where $B:\mathcal{B} \longrightarrow \mathbb{R}^3$ is the body force and $\tau:\partial\mathcal{B} \longrightarrow \mathbb{R}^3$ is the surface traction. If $P = \frac{\partial W}{\partial F}$ denotes the first Piola-Kirchhoff stress, the elasto-static equations are

$$\begin{aligned} \mathrm{DIV}\, P + B &= 0 \quad \text{in} \quad \mathcal{B} \\ P\cdot N &= \tau \quad \text{on} \quad \partial\mathcal{B}, \end{aligned} \qquad \text{(E)}$$

where N is the unit outward normal to $\partial\mathcal{B}$ and P is evaluated on the (unknown) configuration ϕ. Letting

$$\langle \ell,\phi \rangle = \int_{\mathcal{B}} B\cdot\phi \, dV + \int_{\partial\mathcal{B}} \tau\cdot\phi \, dA,$$

we recall that under the assumption of sufficient regularity, ϕ

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is a solution of (E) if and only if ϕ is a critical point of the function

$$V_\ell(\phi) = \int_{\mathcal{B}} W(F)\, dV - \langle \ell, \phi \rangle$$

If (ϕ, ℓ) solves (E) then the divergence theorem shows that the total force is zero: $\int \ell = \int_{\mathcal{B}} B\, dV + \int_{\partial\mathcal{B}} \tau dA = 0$. By using a suitable rotation we can also assume that ℓ is equilibrated relative to the reference configuration:

$$\int \ell \times X = \int_{\mathcal{B}} B(X) \times X\, dV(X) + \int_{\partial\mathcal{B}} \tau(X) \times X\, dA(X) = 0.$$

Let $SO(3) = \{Q | Q$ is a linear map of $\mathbb{R}^3$ to $\mathbb{R}^3$, Q^TQ = Identity = Id and $\det Q = 1\}$ denote the rotation group in $\mathbb{R}^3$. Observe that from material frame indifference,

$$V_{Q\ell}(Q\phi) = V_\ell(\phi) \quad \text{for} \quad Q \in SO(3)$$

We make three assumptions:

(H1) *the reference configuration is stress-free; that is,* $P = 0$ *when* $\phi = \mathrm{Id}$

(H2) *the reference configuration is strongly elliptic*

and in fact, for purposes of stability calculations, assume the stronger condition

(H3) *the classical elasticity tensor*

$$c = 4 \left.\frac{\partial^2 W}{\partial C \partial C}\right|_{C = \mathrm{Id}}$$

is stable:

$$c(e,e) \geq \eta \|e\|^2$$

for some $\eta > 0$ *and all symmetric matrices* e.

For $\ell = 0$, any $\phi = Q \in SO(3)$ is a solution of (E). These are the trivial solutions. Let ℓ_0 be given and look at loads $\lambda\ell$ for ℓ near ℓ_0 and λ small. The kernel of the equations of linear elasticity, i.e. equations (E) linearized at $\phi = Q$ and with $\lambda = 0$ is $T_Q SO(3)$ the tangent space to $SO(3)$ at $Q \in SO(3)$. In fact, it follows from (H3) that $SO(3)$ is a non-degenerate critical manifold for V_0 and so every point of $SO(3)$ is a possible bifurcation point.

By using a variant of the Liapunov-Schmidt procedure one shows that there is a function $f_{\lambda\ell}: SO(3) \longrightarrow \mathbb{R}$ whose critical points are in one to one correspondence with solutions of (E) near $SO(3)$. By using Liusternick-Schnirelmann category one concludes that (E) has at least four solutions, one of which is a (local) minimum of the energy.

Some computation shows that $f_{\lambda\ell}$ has the form

$$f_{\lambda\ell}(Q) = -\lambda \langle Q\ell, \mathrm{Id} \rangle + \frac{\lambda^2}{2} \langle Q\ell_0, u_{Q\ell_0} \rangle + \text{h.o.t.}$$

(h.o.t. means 'higher order terms'), where $u_{Q\ell_0}$ satisfies the equations of linear elasticity with loads $Q\ell_0$ (projected to the equilibrated loads). The term $\langle Q\ell_0, \mathrm{Id} \rangle$ on $SO(3)$ will have degeneracies if ℓ_0 has some 'symmetries'. Let S_{ℓ_0} denote the set of critical points of the function $Q \longmapsto \langle Q\ell_0, \mathrm{Id} \rangle$. We classify the loads according to the topology of S_{ℓ_0} as shown in the accompanying table. The critical points of $f_{\lambda\ell}$ can be studied by a further Liapunov-Schmidt reduction to S_{ℓ_0}; after removing a factor of λ, one obtains a function $\tilde{f}: S_{\ell_0} \longrightarrow \mathbb{R}$,

$$\tilde{f}(Q) = -\langle Q\ell, \mathrm{Id} \rangle - \frac{\lambda}{2} \langle Q\ell_0, u_{Q\ell_0} \rangle + \text{h.o.t.}$$

whose critical points are in one to one correspondence with solutions of (E). A study of $\tilde{f}$ leads to the results on the number of solutions shown in the accompanying table.

Type of ℓ_0	S_{ℓ_0}	Number of Solutions = n
0	4 points	4
1	2 points $\cup$ $\mathbb{RP}^1$	$4 \leq n \leq 6$
2	1 point $\cup$ $\mathbb{RP}^2$	$4 \leq n \leq 14$
3	$\mathbb{RP}^1 \cup \mathbb{RP}^1$	$4 \leq n \leq 8$
4	$SO(3) \cong \mathbb{RP}^3$	$4 \leq n \leq 40$

The numb. of solutions is sharp (examples show that the maximum can be achieved) and assumes that the Betti form, defined by $\beta(Q) = \langle Q\ell_0, u_{Q\ell_0} \rangle$ has non-degenerate critical points.

The numbers in this table come from Bezout's theorem applied to an associated system of cubic polynomials on the double covering: the number of solutions branching from $\mathbb{RP}^s$ can be as many as $(3^{s+1}-1)/2$.

For loads ℓ near type 1 loads ℓ_0 and λ small, the solutions are arranged as follows. There are two unique solutions near the two points of S_{ℓ_0} and two, three or four near the set $\mathbb{RP}^1 \approx S^1$. These solutions on $\mathbb{RP}^1$ can bifurcate according to the astroid bifurcation, which is a symmetric marriage of four cusps. The stabilities can be explicitly calculated.

We now make some remarks to put the above results in perspective.

1. The role of the group SO(3) in this problem differs from that considered by several other authors (see the lectures of Schaeffer and Dancer in these proceedings) in the following two ways:

(a) the group SO(3) acts freely on the set of trivial solutions

and (b) the group SO(3) acts on the parameter space (the space of loads) as well as on the configuration space (the space of ϕ's).

Despite the differences there is the common feature that bifurcation points can be described and classified by their degree of symmetry, which is closely linked with their degree of degeneracy.

2. The techniques can be adapted to more degenerate situations. For example, for the problem of the Rivlin cube discussed in Schaeffer's lectures, one can show that for small normal loads and any isotropic material, there are homogeneous solutions in one to one correspondence with $\mathbb{RP}^2$.

3. As its name implies, linearization stability refers to the stability of solvability of nonlinear equations under the process of linearization. In elasticity, if (u_1, ℓ_1) is a solution of the linearized problem (with ℓ_1 equilibrated) one asks if there is a curve

$$\phi(\lambda) = \mathrm{Id} + \lambda u_1 + \lambda^2 u_2 + \text{h.o.t.},$$

$$\ell(\lambda) = \lambda \ell_1 + \lambda^2 \ell_2 + \text{h.o.t.}$$

satisfying the nonlinear equations. There is a well-known

obstruction to this at order λ^2, namely the Signorini compatibility conditions:

$$\int \ell_1 \times u_1 = 0 .$$

Even if the Betti form is degenerate, this can be shown to be the *only* obstruction. To obtain an accompanying perturbation scheme one needs a non-trivial refinement of the classical schemes proposed by Signorini and Truesdell and Noll.

4. There is an interesting parallel between the results quoted in the preceding remark and some recent results in classical relativistic field theory. The space of solutions of the Yang-Mills or Einstein Yang-Mills equations on a spatially compact spacetime has a quadratic singularity at each point with symmetry; that is at each solution with an isotropy group of dimension $\geqslant 1$ relative to the action of the appropriate gauge group. As one continually breaks symmetry by descending through the lattice of subgroups, one arrives at the generic solutions with at most discrete symmetries, at which point the solution set is a smooth manifold. When one attempts to write solutions in a perturbation series, there is a constraint at second order called the Taub conditions which is analogous to the Signorini conditions. Again it is somewhat remarkable that this is the only obstruction. However, despite this close analogy, the technical details apparently have little to do with one another. For details, references and related bifurcation properties of momentum maps (Noether quantities) in mechanics, see Arms, Marsden and Moncrief [1981].

5. Connections between symmetry and bifurcation are also prevalent in dynamics as well as in statics. We mention one example for illustration. The Lagrange top, i.e. the heavy top with two equal moments of inertial is an integrable system because of its symmetry. If this symmetry is broken by slightly altering the moments of inertia, there is a bifurcation from a homoclinic orbit to aperiodic solutions by the introduction of Smale horseshoes. See Holmes and Marsden [1982] for details.

All of these examples suggest that there must be a deep unified connection between symmetry and bifurcation which surfaces in different ways for different types of problems.

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REFERENCES

[1] Arms, J.M., Marsden, J.E., and Moncrief, V.: 1981, *Symmetry and bifurcation of momentum mappings*, Comm. Math. Phys. 78, pp.455-478.

[2] Holmes, P.J., and Marsden, J.E.: 1982, *Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups*, Indiana Univ. Math. J. (to appear).

[3] Chillingworth, D.R.J., Marsden, J.E., and Wan, Y.H.: 1982, *Symmetry and bifurcation in three dimensional elasticity I*, Arch. Rat. Mech. An. (to appear), *II* Arch. Rat. Mech. An. (to appear), *III* (preprint).

[4] Marsden, J.E., and Wan, Y.H.: 1982, *Linearization stability and Signorini series for the traction problem in elasto-statics* (preprint).