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Applications of the Blowing-Up Construction and Algebraic Geometry to Bifurcation Problems

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A generalization of the Morse lemma to vector-valued functions is proved by a blowing-up argument. This is combined with a theorem from algebraic geometry on the number of real solutions of a system of homogeneous equations of even degree to yield a new bifurcation theorem. Bifurcation in a one- or multi-parameter problem is guaranteed if the leading term is of even degree (it is often two) and satisfies a regularity condition. Applications are given to nonlinear eigenvalue problems and to the Hopf bifurcation.

INTRODUCTION

This paper presents new bifurcation theorems which combine a generalization of the Morse lemma to vector-valued functions with a theorem in algebraic geometry on the number of real solutions of a system of homogeneous equations of even degree. The results are applied to specific bifurcation problems having a multiple eigenvalue or several parameters.

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To facilitate a comparison with other results, we shall state a special case of our results. (The following special hypotheses are by no means necessary.) Let L be an elliptic self-adjoint operator on a suitable Banach space Y of functions, with another suitable Banach space of functions $X =$ domain $L \subset Y$. Suppose $R: X \to Y$ is a smooth map with $R(0)=0$ and $DR(0)=0$. Let λ_0 be an eigenvalue of L of multiplicity n and consider the bifurcation problem

$$f(x, \lambda) = Lx + (\lambda - \lambda_0)x + R(x) = 0, \tag{1}$$

where $x \in X$ and $\lambda \in \mathbb{R}$. Perhaps the best known result concerning (1) is the theorem of Krasnoselski (see Nirenberg [31]). This theorem states (assuming the domain and boundary conditions are such that L^{-1} exists and is compact, and $L^{-1}R$ is compact) that if n is odd, then $(0,0)$ is a bifurcation point. In other words, there are solutions near $(0,0) \in X \times \mathbb{R}$ other than the trivial solutions $(0, \lambda)$. The proof uses the Leray–Schauder degree. For $n=1$ other proofs are available, such as use of the implicit function theorem (Crandall and Rabinowitz [7]) or the Morse lemma (Nirenberg [31] and Berger [4]). For general n, there are also results due to McLeod and Sattinger [28] and Alexander [3] that provide sufficient conditions for $(0,0)$ to be a bifurcation point.

Let X_1 denote the kernel of $L - \lambda_0 I$, and let X_1 be spanned by an orthonormal basis $u_1, \ldots, u_n$. Write $x \in X_1$ as $x = \sum x_i u_i$ with $x_i \in \mathbb{R}$. Let $D^2R(0)$ be the second derivative of R at 0 and write

$$\langle D^2R(0)(u_j, u_k), u_i \rangle = C_i^{jk}$$

so that for each i, C_i^{jk} is symmetric in j, k. Now make the following *regularity* hypothesis:

(R) *For each nonzero* $(x, \lambda) \in X_1 \times \mathbb{R}$ *satisfying*

$$\lambda x_i + \sum_{j,k} C_i^{jk} x_j x_k = 0, \qquad i = 1, \ldots, n \tag{2}$$

the $(n+1) \times n$ *matrix*

$$\left[\sum_j C_i^{jk} x_j + \lambda \delta_i^k, x_i \right], \tag{3}$$

where x_i *denotes the* $1 \times n$ *column vector of* x_i*'s and* δ_i^k *is the* $n \times n$ *identity, has (maximal) rank* n.

Our result applied to (1) states that *under assumption* (R), $(0,0)$ *is a bifurcation point.* In fact, there exists an odd number l, $1 \leq l \leq 2^n - 1$, of nontrivial solutions to (2), and each solution is a direction of bifurcation for

the original problem (1). For $n=1$, condition (R) holds automatically, so $(0,0)$ is a bifurcation point with one bifurcation direction.

The regularity assumption is used in two ways. First, by means of the implicit function theorem and a blowing-up argument, solutions of (1) near $(0,0)$ are put in one to one correspondence with solutions of (2). Second, algebraic geometry is used to prove that (2) has nontrivial solutions. The crucial point is to obtain *real* solutions. Bezout's theorem deals with complex solutions, so modifications must be made in order to draw real conclusions.

The importance of condition (R) as a criterion for bifurcation is that, at least in principle, it can be checked by certain algebraic conditions on the coefficients C_i^{jk} in (3); see Section 2.8, Remark 4. The work of McLeod and Sattinger, applied to (1), shows that under assumption (R), solutions of (1) near $(0,0)$ are in one-to-one correspondence with solutions of (2), so that bifurcation occurs if (2) has nontrivial solutions. Our result says in addition that (2) must have nontrivial solutions. The work of Alexander applied to (1) reduces to Krasnoselski's theorem. Our result, by contrast, sometimes guarantees bifurcation for (1) when n is even. A simple example is $n=2$, $C_1^{11} = C_2^{22} = 1$, and the other $C_i^{jk} = 0$. The work of McLeod and Sattinger is discussed in 3.2, that of Alexander in 2.8, Remark 3.

Our original motivation in studying the blowing-up procedure for finite-dimensional problems was to extend the procedure to non-Fredholm maps. Such extensions, along with an application to the structure of the set of metrics on S^n with a given scalar curvature, are planned for future publications.

The content of the paper is as follows: Section 1 studies the structure of the zero set of a C^l map $g: \mathbb{R}^n \to \mathbb{R}^m$ satisfying $g(0)=0$, $Dg(0)=0$, and a regularity condition on $B(x,y) = D^2g(0)(x,y)$, namely, that at each nonzero solution of $Q(x) = (1/k!)\, B(x,x) = 0$, the linear map $y \mapsto B(x,y)$ is surjective. The zero sets of g and of Q are related by a homeomorphism ϕ with certain differentiability properties. There is a similar result if $g(0)=0$, $Dg(0)=0, \ldots, D^{k-1}g(0)=0$, and $D^kg(0)$ satisfies a regularity condition. Our proof involves blowing up the singularity. The proof is such that ϕ is equivariant relative to any given orthogonal group action on $\mathbb{R}^n$ and $\mathbb{R}^m$ for which g is equivariant. This compatibility with respect to group action is important in many problems and is used in our discussion of the Hopf bifurcation. A comparison of our theorem with the closely related results of Magnus [21, 22, 23], Szulkin [37], and Kuo [18] is given. We also relate the theorem to the theory of contact equivalence.

Using the Liapunov–Schmidt procedure, the results of Section 1 are used in Section 2 to study the set of solutions of a bifurcation problem

$$f: X \times \mathbb{R}^p \to Y$$

near $(0,0)$ under the assumptions that $(0, \lambda)$ is a solution for all λ near 0 and

$D_x f(0,0)$ is Fredholm of index zero. The structure of the solution set is reduced to a study of the solution of a system of homogeneous algebraic equations. Assuming these homogeneous equations are of even degree and satisfy a regularity condition, algebraic geometry is used to show that there are always nontrivial solutions. In case $p = 1$ (one parameter) it is shown that there is an odd number of nontrivial solution branches.

In Section 3 applications are made to a nonlinear eigenvalue problem and to the Hopf bifurcation. For the nonlinear eigenvalue problem we give a bifurcation criterion that is related to the work of McLeod and Sattinger. For the Hopf bifurcation we follow Crandall and Rabinowitz [8] and show that the methods here are applicable to the standard Hopf situation. Our approach gives a more geometric perspective to their approach and, like a recent method of Golubitsky and Langford [12], explicitly uses the natural $SO(2)$ symmetry in the problem. A point to notice is that the condition of regularity of the second derivative on its zero set applies to both of these apparently diverse bifurcation problems.

Section 4 discusses the genericity of the hypotheses we make. From the results presented, it is reasonable to expect our hypotheses to be applicable to most one-parameter bifurcation problems. They also hold in certain several-parameter problems such as the Hopf bifurcation.

Although the paper has some of the spirit of generic bifurcation theory (Chow *et al.* [5, 6] and Golubitsky and Schaeffer [13]), no attempt at an unfolding theory is made in the context developed here.

1. Blowing-Up in Finite Dimensions

This section shows that the zero set of a finite-dimensional mapping near a singular point is determined by the first nonzero term in its Taylor series, provided that term satisfies a certain regularity assumption. There are many theorems of this type available in the literature, and a comparison will be made below. Our primary motivations are to establish a context suitable for the following section and for an infinite-dimensional version, which will be the subject of another publication.

We begin with some notation. Suppose X and Y are Banach spaces and B: $X \times \cdots \times X$ (k times) $\to Y$ is a continuous symmetric k-multilinear mapping, where $k \geqslant 2$ is an integer. The *k-form associated* to B is the map $Q: X \to Y$ defined by

$$Q(x) = \frac{1}{k!} B(x, x, \ldots, x).$$

The derivative of Q at x is the linear map

$$DQ(x)u = \frac{1}{(k-1)!} B(x, x, \ldots, x, u)$$

from X to Y.

1.1. Definition. We say that Q is *regular* at a point x if $DQ(x)$ is surjective. We say that Q is *regular on its zero set* if it is regular at each nonzero $x \in Q^{-1}(0)$.

Notice that if $Q^{-1}(0) = \{0\}$, Q is automatically regular on its zero set. We also observe that if S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $Q: \mathbb{R}^n \to \mathbb{R}^m$ is regular on its zero set, then $Q^{-1}(0) \cap S^{n-1}$ is a real analytic manifold of dimension $n - m - 1$, and $Q^{-1}(0)$ is the cone on $Q^{-1}(0) \cap S^{n-1}$.

1.2. *Examples*

Example 1. Let $f: M \to \mathbb{R}$ be a C^2 real-valued function on a manifold M and have a critical point at $x_0 \in M$. The quadratic form $Q(v) = \frac{1}{2} D^2 f(x_0)(v, v)$ is regular at all $v \neq 0$ if and only if x_0 is a nondegenerate critical point.

Example 2. Figure 1 shows the zero set of a quadratic map (i.e., $k = 2$) $Q: \mathbb{R}^3 \to \mathbb{R}^2$ that is regular on its zero set. Here $Q = (Q_1, Q_2)$, where each of Q_1 and Q_2 has index one. $Q^{-1}(0) = Q_1^{-1}(0) \cap Q_2^{-1}(0)$ is the intersection of two cones through the origin that meet transversally away from the origin.

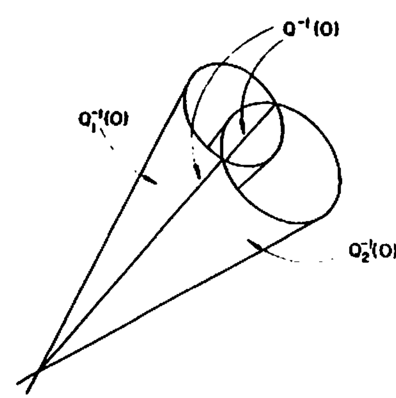

Figure 1

Here is a case that will be useful in Example 3.1. Letting the variables in $\mathbb{R}^3$ be denoted (z_1, z_2, λ), consider

$$Q_1(z_1, z_2, \lambda) = az_1^2 + 2bz_1z_2 + bz_2^2 + 2z_1\lambda,$$
$$Q_2(z_1, z_2, \lambda) = bz_1^2 + 2bz_1z_2 + az_2^2 + 2z_2\lambda,$$

where $a \neq 0$ and $b \neq 0$ and let $Q = (Q_1, Q_2)$. Let $\alpha = (a-b)(a-5b)$. By explicitly solving the equation $Q = 0$, one can check that $Q^{-1}(0)$ consists of two, three, or four lines according to whether α is negative, zero, or positive. Then Q is regular on its zero set for $\alpha \neq 0$. For no values of a and b is Q regular at every nonzero point, since DQ fails to be surjective along the line $z_2 = z_1$, $\lambda = (b-a)z_1$.

EXAMPLE 3. Let $Q: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be quadratic and write $Q((x, y), (\lambda, \mu)) = (Q_1((x, y), (\lambda, \mu)), Q_2((x, y), (\lambda, \mu)))$. Suppose $Q((0, 0), (\lambda, \mu)) = 0$ for all (λ, μ) and the 2×2 matrix $\partial(Q_1, Q_2)/\partial(x, y)((0, 0), (\lambda, \mu))$ has a nonzero determinant for each $(\lambda, \mu) \neq (0, 0)$. Then Q is regular at each nonzero point. This may be proved by explicitly writing out the matrices involved. This example is relevant for Hopf bifurcation; see Example 3.3.

The main result on blowing up in finite dimensions is as follows:

1.3. THEOREM. *Let k and l be integers, $2 \leqslant k < l$. Suppose $g: \mathbb{R}^n \to \mathbb{R}^m$ is of class C^l and $g(0) = 0$, $Dg(0) = 0, \ldots, D^{k-1}g(0) = 0$. Let Q be the k-form associated to $D^kg(0)$ and assume that Q is regular on its zero set. Then*

(1) *There are neighborhoods U_1, U_2, containing $0 \in \mathbb{R}^n$ and a C^1 diffeomorphism $\phi: U_1 \to U_2$, which is C^l away from 0, such that*

(a) *$\phi(Q^{-1}(0) \cap U_1) = g^{-1}(0) \cap U_2$, and*
(b) *$\phi(0) = 0$ and $D\phi(0) = $ identity.*

(2) *If Q is regular at each nonzero point, then ϕ can be chosen so that, in addition,*

(c) *$g(\phi(x)) = Q(x)$ for all $x \in U_1$.*

Remark 1. Conclusion (a) implies that g and Q have homeomorphic zero sets. According to (b) this homeomorphism is induced by a C^1 diffeomorphism of the ambient space that is close to the identity map near 0.

Remark 2. From (b) notice that if $Q(v) = 0$, where $v \neq 0$, then the line $l(t) = tv$ in $Q^{-1}(0)$ is mapped by ϕ into the C^1 curve $\phi(l(t))$ in the zero set of g, which is also tangent to v at $t = 0$. We thus speak of each $v \in Q^{-1}(0)$, $v \neq 0$ as a direction of bifurcation. In the language of linearization stability, the direction v is also called *integrable*; see Fischer and Marsden [11]. The existence of a curve $x(t)$ in $g^{-1}(0)$ tangent to v at $t = 0$ may be proved

directly by writing $x(t) = tv + t^2u(t)$ and solving for $u(t)$ by the implicit function theorem.

Remark 3. For $m = 1$, $k = 2$, and Q regular on its zero set, this theorem follows from the Morse lemma.

Remark 4. If one does not demand (c), then one can arrange things so $U_1 = U_2$ and ϕ preserves the norm.

Proof. By Taylor's theorem we can write

$$g(x) = Q(x) + h(x)(x, \ldots, x),$$

where $h(x)$ is k-multilinear from $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ to $\mathbb{R}^m$, is C^{l-k} in x, is C^l in x away from zero, and satisfies $h(0) = 0$. Now "blow up" g by defining $\tilde{g}$: $S^{n-1} \times \mathbb{R} \to \mathbb{R}^m$ as

$$\tilde{g}(x, r) = \frac{g(rx)}{r^k} = Q(x) + h(rx)(x, \ldots, x).$$

Thus $\tilde{g}$ is C^{l-k} on $S^{n-1} \times \mathbb{R}$ and is C^l away from $r = 0$.

Let V be a neighborhood of $Q^{-1}(0) \cap S^{n-1}$ in S^{n-1} such that $DQ(x)$ is surjective for all $x \in \bar{V}$. Let $E = \{(x, r, \psi) \in V \times \mathbb{R} \times \mathbb{R}^n \mid \psi \in [\operatorname{Ker} DQ(x)]^\perp\}$. Thus E is a vector bundle over $V \times \mathbb{R}$ with fibre at (x, r) equal to $[\operatorname{Ker} DQ(x)]^\perp$. Define $G: E \to \mathbb{R}^m$ by

$$G(x, r, \psi) = Q(x + \psi) - \tilde{g}(x, r).$$

The diffeomorphism ϕ will be constructed using an implicit solution $\psi(x, r)$ of the equation $G(x, r, \psi) = 0$. We note that $G(x, 0, 0) = 0$ for all $x \in V$, and

$$\frac{\partial G}{\partial \psi}(x, 0, 0) = DQ(x) \mid \operatorname{Ker} DQ(x)^\perp$$

is an isomorphism for all $x \in V$.

From the implicit function theorem and compactness of $\bar{V}$, it follows that there is an $\varepsilon > 0$ and a C^{l-k} map ψ defined on $V \times (-\varepsilon, \varepsilon)$, $\psi(x, r) \in \operatorname{Ker} DQ(x)^\perp$, such that $G(x, r, \psi(x, r)) = 0$, $\psi(x, 0) = 0$ for all $x \in V$, and ψ is C^l away from $r = 0$.

Using a standard partition of unity argument, we can find a neighborhood W of $Q^{-1}(0) \cap S^{n-1}$, $\bar{W} \subset V$, and a map $\chi: S^{n-1} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that χ agrees with ψ on $W \times (-\varepsilon, \varepsilon)$ and $\chi(x, 0) = 0$ for all $x \in S^{n-1}$.

From $Q(x+\chi(x,r))=\tilde{g}(x,r)$, $x\in W$, $|r|<\varepsilon$, we have the identity $r^kQ(x+\chi(x,r))=g(rx)$, or, equivalently,

$$g(x)=Q\left(x+\|x\|\,\chi\left(\frac{x}{\|x\|},\|x\|\right)\right) \tag{4}$$

valid for x in $\mathbb{R}^n$ with $x/\|x\|\in W$ and $0<\|x\|<\varepsilon$.

Define the map α from the ε-ball $B_\varepsilon(0)$ to $\mathbb{R}^n$ by

$$\begin{aligned}\alpha(x)&=x+\|x\|\,\chi\left(\frac{x}{\|x\|},\|x\|\right), && x\neq 0,\\ &=0, && x=0.\end{aligned} \tag{5}$$

Clearly, α is C^l away from 0 and continuous at 0. We claim that α is C^1 at 0 and $D\alpha(0)=$ identity. To show this it suffices to show that $\lim_{x\to 0}D\alpha(x)=$ identity. Using $\|x\|=\langle x,x\rangle^{1/2}$, we compute that, for $x\neq 0$ and $v\in\mathbb{R}^n$, $D\alpha(x)v = v + \langle x/\|x\|,\ v\rangle\chi(x/\|x\|,\ \|x\|) + (\partial\chi/\partial x)(x/\|x\|,\ \|x\|)\cdot(v-\langle x/\|x\|,v\rangle(x/\|x\|))+\langle x,v\rangle(\partial\chi/\partial r)(x/\|x\|,\|x\|)$. Fix v with $\|v\|=1$ and let $x\to 0$. Since $\chi(x/\|x\|,0)=0$ identically we can use compactness of S^{n-1} to conclude that $D\alpha(x)v\to v$ as $x\to 0$ uniformly in x and v.

Thus, after shrinking ε if necessary, α is a C^1 diffeomorphism of $B_\varepsilon(0)$ to $\alpha(B_\varepsilon(0))$. We claim that for ε small enough, $\alpha(g^{-1}(0)\cap B_\varepsilon(0))=Q^{-1}(0)\cap\alpha(B_\varepsilon(0))$. In fact the identity (4) shows that for $x/\|x\|\in W$, $\alpha(x)\in Q^{-1}(0)$ if and only if $x\in g^{-1}(0)$. Thus all we need show is that for $x/\|x\|\notin W$ and $\|x\|$ sufficiently small, $\alpha(x)\notin Q^{-1}(0)$. For $x/\|x\|\notin W$, $Q(x/\|x\|)$ is bounded away from 0. Since $\chi(x/\|x\|,r)\to 0$ as $r\to 0$ uniformly in $x/\|x\|$, for $x/\|x\|\notin W$ and $\|x\|$ sufficiently small $Q(x/\|x\|+\chi(x/\|x\|,\|x\|))\neq 0$. It follows that $\alpha(x)\notin Q^{-1}(0)$ for $x/\|x\|\notin W$ and $\|x\|$ sufficiently small.

Let $\phi=\alpha^{-1}$. Conclusion 1 of the theorem is immediate, and conclusion 2 holds if in our initial construction of ψ we take $V=S^{n-1}$ and then let $\chi=\psi$. ∎

We now make a few remarks on the proof of Theorem 1.3 and extensions of the theorem. Following this we shall compare the result with others in the literature.

1.4. *Remarks on Theorem* 1.3

Remark 1. That the zero sets are homeomorphic (by a homeomorphism that is a C^l diffeomorphism away from the origin) may be proved by various alternative methods. One of these involves directly projecting $\tilde{g}^{-1}(0)$ onto $Q^{-1}(0)$ near $S^{n-1}\times\{0\}$. Both sets are manifolds of dimension $n-m$ that intersect $S^{n-1}\times\{0\}$ transversally along $(Q^{-1}(0)\cap S^{n-1})\times\{0\}$. The idea is skektched in Fig. 2. However, the proof we gave produces a diffeomorphism of the ambient space, not merely of the zero sets.

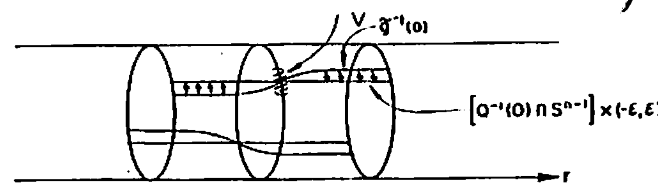

FIGURE 2

Remark 2. At the expense of losing a derivative away from 0, one can construct the diffeomorphism as the time one map of a flow; see Magnus [22]. The same loss of a derivative occurs in Moser's [29] proof of Darboux's theorem and the equivalence of volume elements, the technique of which Magnus uses. (This last remark is due to R. Douady). Using the Whitney properties of the remainder, the theorem is valid for $k=l$ and $\tilde{g}$ is actually C^l on $S^{n-1}\times\mathbb{R}$. (See Tuan and Ang [41]).

Remark 3. The map ϕ constructed in the proof of Theorem 1.3 is in general no better than C^1 at 0, even when g is algebraic. (For the Morse lemma it is well known that ϕ can be chosen to be C^{l-2}.) To show this, we shall show that $\alpha=\phi^{-1}$ need not have a second derivative at 0. Let $g(x)=Q(x)+H(x)$, Q (resp. H) a homogeneous polynomial map of degree two (resp. three), and assume Q is regular at every nonzero x. Then $\tilde{g}(x,r)=Q(x)+rH(x)$. The map ψ: $S^{n-1}\times\mathbb{R}\to\mathbb{R}^n$ is defined by

$$Q(x+\psi)-(Q(x)+rH(x))=0 \tag{6}$$

$$\psi\in\operatorname{Ker}DQ(x)^\perp. \tag{7}$$

After expanding $Q(x+\psi)$, (6) becomes.

$$DQ(x)\psi+Q(\psi)-rH(x)=0.$$

Take the partial derivative with respect to r:

$$DQ(x)\frac{\partial\psi}{\partial r}(x,r)+DQ(\psi(x,r))\frac{\partial\psi}{\partial r}(x,r)-H(x)=0.$$

When $r=0$, $\psi=0$, so since $DQ(0)=0$, we have

$$DQ(x)\frac{\partial\psi}{\partial r}(x,0)=H(x). \tag{8}$$

From (7), $(\partial\psi/\partial r)(x,r)$ must belong to $\operatorname{Ker}DQ(x)^\perp$, so (8) implies

$$\frac{\partial\psi}{\partial r}(x,0)=[DQ(x)\,|\,\operatorname{Ker}DQ(x)^\perp]^{-1}H(x). \tag{9}$$

From (6) we compute

$$\alpha(tx) = tx + t\|x\|\,\psi\left(\frac{x}{\|x\|}, t\|x\|\right), \qquad t \geqslant 0.$$

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \alpha(tx) = 2\|x\|^2 \frac{\partial\psi}{\partial r}\left(\frac{x}{\|x\|}, 0\right)$$

$$= 2\|x\|^2 \left[DQ\left(\frac{x}{\|x\|}\right) \Big| \operatorname{Ker} DQ\left(\frac{x}{\|x\|}\right)^{\perp}\right]^{-1} H\left(\frac{x}{\|x\|}\right)$$

by (9). Let $R(x)$ denote this last expression.

If α has a second derivative at 0, then $D^2\alpha(0)(x, x) = (d^2/dt^2)|_{t=0}\,\alpha(tx) = R(x)$. Thus $R(x)$ must be a homogeneous polynomial map of degree two. Clearly, $R(x)$ *is* homogeneous of degree two, but in general it is *not* a polynomial. For example, suppose $n = m + 1$ and $H^{-1}(0)$ consists of 3^m lines, which is possible by Bezout's theorem. Then $R^{-1}(0)$ includes 3^m isolated lines (isolated as points in $\mathbb{R}P^{n-1}$), which is impossible for a degree 2 polynomial map.

Remark 4. There is a generalization which allows the components of g to have different k's. Specifically, suppose $g: \mathbb{R}^n \to \mathbb{R}^m$, $g = (g_1,..., g_m)$ and g_i is of class C^{l_i} with derivatives up to order $k_i - 1$ vanishing at zero, where $2 \leqslant k_i < l_i$. Let $Q_i(x) = (1/k_i!)\, D^{k_i}g_i(0)(x,..., x)$ and $Q = (Q_1,..., Q_m)$. If Q is regular on its zero set, then the conclusions of Theorem 1.3 hold with $l = \min(l_1,..., l_m)$. The proof is essentially the same. This generalization is suggested by Szulkin [37].

Remark 5. The hypotheses of Theorem 1.3 imply that the origin is an isolated critical point in the zero set, i.e., there are no nearby points where $g(x) = 0$ and $Dg(x)$ fails to be surjective. Problems in which the origin is not an isolated critical point are also of interest; see, for instance, Shearer [36] and Hale and Taboas [14]. Theorem 1.3 can be applied to some problems of this type as follows: Assume that the set of critical points of g forms a smooth manifold C. Then 1.3 may be applicable to g restricted to a transverse subspace to C. Under these circumstances one obtains a parametrized version of 1.3. This is the idea of Shearer [36]. The parametrized Morse lemma for a real-valued function with a nondegenerate critical manifold is also proved this way. For another example, see Ratiu [32, p. 263].

Problems with symmetry where the critical point is not fixed by the group action often have the character just described. However, if the group action leaves the critical point fixed, Remark 6 following shows that 1.3 may be applicable in an invariant way. (The former case can be reduced to the latter by use of a slice for the action.)

Remark 6. Let Γ be a compact Lie group acting orthogonally on $\mathbb{R}^n$ and $\mathbb{R}^m$ and suppose in Theorem 1.3 that g is equivariant (or covariant), i.e., $g(\gamma x) = \gamma g(x)$ for all $x \in \mathbb{R}^n$ and $\gamma \in \Gamma$. Then ϕ can be chosen to commute with the action Γ. The key step in showing this is to show that χ can be chosen equivariant with respect to the actions of Γ on $S^{n-1} \times \mathbb{R}$ and $\mathbb{R}^n$. (The action on Γ on $S^{n-1} \times \mathbb{R}$ is $\gamma(x, r) = (\gamma x, r)$.) To construct χ in an equivariant manner, first choose V to be invariant under the action of Γ. The reader may check that $\psi: V \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is then equivariant. (The proof uses the fact that $\gamma(\operatorname{Ker} DQ(x)^{\perp}) = \operatorname{Ker} DQ(\gamma x)^{\perp}$ and uniqueness of $\psi(x, r)$.) Extend ψ to $\hat{\chi}: S^{n-1} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ as described in the proof, taking care that the set W such that $\hat{\chi}$ and χ agree on $W \times (-\varepsilon, \varepsilon)$ is invariant. Now define $\psi(x, r) = \int_\Gamma \gamma^{-1}\hat{\chi}(\gamma x, r)\, d\gamma$, where $d\gamma$ is Haar measure on Γ.

The hypothesis of regularity is compatible with the presence of a symmetry group whose action leaves the critical point (in our case $0 \in \mathbb{R}^n$) fixed. We shall see an explicit example for the Hopf bifurcation in Example 3.3.

Remark 7. A number of problems can be put in a form suitable for the application of 1.3 by means of a scaling transformation. For example, if $\hat{g}(x, \lambda, \varepsilon) = x^3 - \lambda x - \varepsilon +$ higher order terms, then $g(x, \mu, \nu) = x^3 - \mu^2 x - \nu^3 +$ higher order terms has $k = 3$ and 1.3 applies. The general method of scaling using Newton diagrams is explained in, for example, Sattinger [35]. This trick will be used in Section 3 in our discussion of the relationship with the work of McLeod and Sattinger [28].

Remark 8. As we have mentioned, one of our goals in a later publication is to prove an infinite-dimensional version of Theorem 1.3. However, using the Liapunov–Schmidt procedure one can derive a Banach space result for Fredholm maps directly from Theorem 1.3. This is described in the following paragraphs.

Let X and Y be Banach spaces and $f: X \to Y$ a C^l map, $l \geqslant 1$, with $f(0) = 0$. The problem that will now be discussed is to describe the structure of $f^{-1}(0)$ near $0 \in X$. If $Df(0)$ is surjective and if the kernel of $Df(0)$, denoted $\operatorname{Ker} Df(0)$, has a closed complement, then the implicit function theorem implies that $f^{-1}(0)$ near 0 is a submanifold of X diffeomorphic to an open subset of $\operatorname{Ker} Df(0)$. If $Df(0)$ is not surjective, the Liapunov–Schmidt procedure, which will now be recalled, can often be used. Assume $\operatorname{Ker} Df(0) = X_1$ has a closed complement X_2, so that $X = X_1 \oplus X_2$; and assume $\operatorname{Im} Df(0) = Y_2$, the image of $Df(0)$, is closed with closed complement Y_1, so that $Y = Y_1 \oplus Y_2$. Let P be the projection of Y to Y_2. For $x \in X$, write $x = x_1 + x_2$, $x_1 \in X_1$ and $x_2 \in X_2$. By the implicit function theorem, $P \circ f(x_1 + x_2) = 0$ defines $x_2 = u(x_1)$. If $g: X_1 \to Y_2$ is (locally) defined by $g(x_1) = (I - P) \circ f(x_1 + u(x_1))$, then $f^{-1}(0)$ is the graph of u over $g^{-1}(0)$.

An equivalent procedure is as follows: Since f is transversal to Y_2, $M = F^{-1}(Y_1) = (P \circ f)^{-1}(0)$ is a manifold in a neighborhood of 0, tangent to X_1. Then $f^{-1}(0)$ is the zero set of $(I - P) \circ f$ restricted to M. The map u is such that its graph is M. The tangency of M to X_1 is equivalent to $Du(0) = 0$, which can be checked by implicit differentiation.

If $Df(0)$ is Fredholm (i.e., the kernel is finite dimensional and the image has finite-dimensional complement) then the Liapunov–Schmidt procedure will enable us to obtain the following consequence of 1.3:

1.5. COROLLARY. *Using the above notation, assume that $Df(0)$ is Fredholm and let $2 \leqslant k < l$. For $1 \leqslant j < k$ assume $D^j f(0)$ restricted to $X_1 \times \cdots \times X_1$ (j times) is 0. Let $B = (I - P) \circ D^k f(0) \mid X_1 \times \cdots \times X_1$ (k times). Assume that Q (the k-form associated to B) is regular on $Q^{-1}(0)$. Then there are neighborhoods V_1 and V_2 of 0 in X and a C^1 diffeomorphism $\Phi: V_1 \to V_2$, that is C^l away from 0 such that*

(a) $\Phi(Q^{-1}(0) \cap V_1) = f^{-1}(0) \cap V_2$;

(b) $\Phi(0) = 0$ *and* $D\Phi(0) =$ *identity.*

Proof. The derivatives of g can be computed by implicit differentiation to be $Dg(0) = 0, \ldots, D^{k-1}g(0) = 0$, and $D^k g(0) = (I - P)\, D^k f(0) \mid X_1 \times \cdots \times X_1$, so that 1.3 can be applied to g. If $\phi: U_1 \to U_2$ is the C^1 diffeomorphism given by 1.3, then we can choose $V_1 = U_1 \times X_2$, $V_2 = U_2 \times X_2$ and $\Phi(x_1, x_2) = (\phi(x_1), u(\phi(x_1)) + x_2)$. Properties (a) and (b) of ϕ are inherited by Φ since u is C^l and $Du(0) = 0$. ∎

1.6. *Remarks*

Remark 1. In 1.5 note that we do not assert that $f(\Phi(x)) = Q(x)$, even if Q is regular on all nonzero vectors.

Remark 2. The reduced map g may satisfy the hypotheses of Theorem 1.3 although the hypotheses of 1.5 do not hold. Corollary 1.5 just gives a convenient way of sometimes checking that g satisfies the hypotheses of 1.3.

Remark 3. Suppose X and Y come equipped with (not necessarily complete) inner products $\langle\ ,\ \rangle$ and suppose there is a compact Lie group Γ acting orthogonally on X and on Y, i.e., in such a way as to preserve the inner products (e.g., $\langle \gamma x_1, \gamma x_2 \rangle = \langle x_1, x_2 \rangle$). Suppose, furthermore, that f is equivariant, i.e. $f(\gamma x) = \gamma f(x)$ for all $x \in X$, and that $X_1 = \operatorname{Ker} Df(0)$ and $Y_2 = \operatorname{Im} Df(0)$ have closed orthogonal complements. Then $P \circ f(x)$ and hence $u(x_1)$ are equivariant. Thus, since $(I - P) \circ f(x)$ is equivariant, so is $g(x)$. (The remark that the reduced mapping g is equivariant is standard and

elementary; see Sattinger [35].) By 1.4, Remark 6, ϕ can be chosen to be equivariant. Consequently, $\Phi: X \to X$ constructed above will be equivariant as well.

Next we discuss the relationship between Theorem 1.3 and other approaches in the literature.

First, we recall some of the history of Theorem 1.3. Magnus [21] gave a result related to Theorem 1.3 but which differs in its technical conclusions and has a complicated proof. He gave a simpler proof in [22, 23]; see 1.4, Remark 2. Szulkin [37] has proved a generalization to maps that are perturbations of homogeneous (not necessarily polynomial) maps. He also allows the different g_i to begin with homogeneous terms of different degrees; see 1.4, Remark 3. An earlier version of the work of the present authors was presented in Marsden [24].

Second, we discuss the relationship with the work of Kuo [18]. A polynomial map $z: \mathbb{R}^n \to \mathbb{R}^m$ of degree r, with $z(0) = 0$, is called *variety-sufficient* (abbreviated to v-sufficient) if, for any two C^r maps f and g: $\mathbb{R}^n \to \mathbb{R}^m$ which both have r-jet z, the germs of $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic. If $v_1, \ldots, v_p$ are vectors in $\mathbb{R}^n$ define $h_i =$ distance of v_i to span $\{v_j \mid j \neq i\}$ and define $d(v_1, \ldots, v_m) = \min\{h_1, \ldots, h_m\}$. If $f: \mathbb{R}^n \to \mathbb{R}^m$, $f(0) = 0$, $d > 0$, $w > 0$ define $H_d(f; w) = \{x \in \mathbb{R}^n \mid |f(x)| \leqslant w|x|^d\}$. Kuo shows that if there is a neighborhood U of 0 in $\mathbb{R}^n$, $\varepsilon > 0$, $w > 0$ such that

$$d(\nabla z_1, \ldots, \nabla z_m) \geqslant \varepsilon |x|^{r-1}$$

in $H_r(z; w) \cap U$ (where $z = (z_1, \ldots, z_m)$), then z is v-sufficient. It can be shown that if z is *homogeneous* of degree r and if it is *regular* on its zero set, then the preceeding inequality holds for suitable ε and w. This shows at least the part of Theorem 1.3 which asserts the existence of an homeomorphism between the zero sets of g and Q. Note that Theorem 1.3 asserts in addition that the homeomorphism is actually induced by a diffeomorphism of the ambient space whose derivative is the identity at zero. The proof of 1.3 provides another proof of a special case of Kuo's result and also gives additional information about the differentiability of the homeomorphism.

Third, we relate Theorem 1.3 to the theory of contact equivalence, which has been used in bifurcation theory in recent years. See, for example, Golubitsky and Schaeffer [13]. Let f and g be two germs of C^∞ mappings $(\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$. They are called *contact equivalent* if there exists a C^∞ diffeomorphism germ $\eta: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a C^∞ germ $\psi: (\mathbb{R}^n \times \mathbb{R}^m, (0, 0)) \to (\mathbb{R}^m, 0)$ such that for each fixed $x \in \mathbb{R}^n$, the germ $y \to \psi(x, y)$ is a diffeomorphism germ and $f(x) = \psi(x, g \circ \eta(x))$.

Let $\mathscr{E}_x$ be the ring of germs of C^∞ functions at 0 in $\mathbb{R}^n$, $\mathscr{M}_x$ the maximal ideal in $\mathscr{E}_x$ of functions vanishing at the origin, and $\mathscr{E}_y$ and $\mathscr{M}_y$ the corresponding ring and ideal on $\mathbb{R}^m$. Let $f = (f_1, \ldots, f_m)$ and $I(f) =$ ideal in

$\mathscr{E}_x$ generated by $f_1, \dots, f_m$. (So $I(f) = f^*(\mathscr{M}_y)\mathscr{E}_x$.) It is proved in Mather [26] that contact equivalence of f and g is equivalent to the existence of a diffeomorphism germ $\phi: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\phi^*(I(g)) = I(f)$. (If f and g were analytic this would mean that the germs at 0 of the varieties $f^{-1}(0)$ and $g^{-1}(0)$ would be isomorphic in the sense of the theory of analytic varieties.)

The germ f is called *k-determining relative to contact equivalence* if every g, whose k-jet is equal to the k-jet of f, is contact equivalent to f. In Mather [27] f k-determining relative to contact equivalence is shown to imply the inclusion

$$\mathscr{M}_x \left\{ \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\} + [I(f)]^m \supseteq [\mathscr{M}_x^{k+1}]^m,$$

where each term is an $\mathscr{E}_x$-submodule of $\mathscr{E}_x^m = \mathscr{E}_x \oplus \cdots \oplus \mathscr{E}_x$ (m times).

Now suppose f is a homogeneous polynomial map $\mathbb{R}^n \to \mathbb{R}^m$ of degree k. By the preceding inclusion, if f is k-determining relative to contact equivalence, then f is regular on its zero set. Magnus [22] remarks that Theorem 1.3 can be thought of as a rough converse to this statement. That there is no true converse can be seen as follows.

In Section 4 it will be shown that, in a precise sense, for any (n, m, k), almost every kth degree homogeneous polynomial mapping $(\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ is regular on its zero set. In contrast, a counting argument, shows that for most (n, m, k) *no* kth degree homogeneous polynomial map $\mathbb{R}^n \to \mathbb{R}^m$ can be k-determining relative to contact equivalence. In fact, if H_j denotes the space of jth degree homogeneous polynomials functions on $\mathbb{R}^n$ and if f were such a map, then from the preceeding inclusion the inequality $\dim[\mathscr{M}_x^{k+1}]^m \cap [H_{k+1}]^m \leqslant \dim \mathscr{M}_x\{\partial f/\partial x_1, \dots, \partial f/\partial x_n\} \cap [H_{k+1}]^m + \dim[I(f)]^m \cap [H_{k+1}]^m$ would follow. But $\dim([\mathscr{M}_x^{k+1}]^m \cap [H_{k+1}]^m) = m\binom{n+k}{k+1}$, $\dim \mathscr{M}_x\{\partial f/\partial x_1, \dots, \partial f/\partial x_n\} \cap [H_{k+1}]^m \leqslant \frac{1}{2}n^2(n+1)$ and $\dim[I(f)]^m \cap [H_{k+1}]^m \leqslant nm^2$. For most (n, m, k) the inequality $nm^2 + \frac{1}{2}n^2(n+1) \geqslant m\binom{n+k}{k+1}$ does not hold. For instance, Example 3.1 has n replaced by $n+1$, $m = n$, and $k = 2$ and the preceeding inequality fails when the n of Example 3.1 is $\geqslant 5$.

2. Applications to Bifurcation Theory

This section will discuss the following situation: Let X and Y be Banach spaces and $f: X \times \mathbb{R}^p \to Y$ a C^l map, $l \geqslant 3$. ($\mathbb{R}^p$ is thought of as the parameter space.) A classical example is when $p = 1$, $X \subset Y$ and f has the form

$$f(x, \lambda) = Lx - \lambda x + F(x, \lambda),$$

where L is a bounded linear operator from X to Y and $\|F(x, \lambda)\|_Y = O(\|x\|_X^2)$ for $\|x\|_X$ and λ small.

2.1. Definition. Suppose $\lambda \mapsto (x_0(\lambda), \lambda)$ is a p-dimensional manifold of solutions to $f(x, \lambda) = 0$ (i.e., $f(x_0(\lambda), \lambda) = 0$), then $(x_0, \lambda_0) = (x_0(\lambda_0), \lambda_0)$ is called a *bifurcation point* if every neighborhood of (x_0, λ_0) in $X \times \mathbb{R}^p$ contains a solution (x, λ) with $x \neq x_0(\lambda)$.

The criterion for bifurcation we shall give combines our work on blowing up in the previous section with results from algebraic geometry. First we reduce the question of whether (x_0, λ_0) is a bifurcation point to the question of whether a set of algebraic equations $Q(x, \lambda) = 0$ has solutions with $x \neq 0$. Then algebraic geometry is used to establish the existence of nontrivial solutions.

We make the following hypotheses on the C^l map $f: X \times \mathbb{R}^p \to Y$.

(H1) $x_0: \mathbb{R}^p \to X$ is a C^l map defined in a neighborhood Λ of $\lambda_0 \in \mathbb{R}^p$ such that $f(x_0(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$.

(H2) $D_x f(x_0, \lambda_0): X \to Y$ is Fredholm with kernel X_1 and range Y_2, where $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ ($D_x f$ is the partial derivative of f with respect to the X variable).

(H3) $D_x f(x_0, \lambda_0)$ has index zero; i.e., $\dim X_1 = \dim Y_1 = n$.

(H4) $(I - P)\, D^2 f(x_0, \lambda_0) \,|\, (X_1 \times \mathbb{R}^p)^2 = 0, \dots, (I - P)\, D^{k-1} f(x_0, \lambda_0) \,|\, (X_1 \times \mathbb{R}^p)^{k-1} = 0$, where k is an integer, $2 \leqslant k < l$, and P is the projection onto Y_2. (If $k = 2$, condition (H4) is vacuous.)

(H5) Let $B = (I - P)\, D_x^k f(x_0, \lambda_0) \,|\, (X_1 \times \mathbb{R}^p)^k$ and let Q be the corresponding k-form. Assume Q is regular on its zero set.

(H6) k is even.

2.2. Theorem. *Under hypotheses* (H1)–(H6), (x_0, λ_0) *is a bifurcation point.*

Before giving the proof we shall make some remarks.

2.3. *Remarks*

Remark 1. The solutions $(u, v) \in X_1 \times \mathbb{R}^p$ of $Q(u, v) = 0$ give the directions of bifurcation. See 1.3, Remark 2.

Remark 2. Problems involving bifurcation at simple eigenvalues are covered by the case $\dim X_1 = \dim Y_1 = 1$, $p = 1$. The standard hypotheses, such as described in Crandall and Rabinowitz [7] or Nirenberg [31] imply these with $k = 2$.

Remark 3. The case $\dim X_1 = \dim Y_1 = 2$, $p = 1$, is in Szulkin [37, p. 241].

Remark 4. Problems involving bifurcation at eigenvalues of multiplicity n are covered by the case $\dim X_1 = \dim Y_1 = n$. Theorem 2.2 then includes the results obtained by McLeod and Sattinger [28]. They have various conditions on partial derivatives which, after rescaling, can all be combined into a single statement of regularity of the second derivative on its zero set. This is detailed in Section 3.

Remark 5. Notice that there are no restrictions on the multiplicity $\dim X_1 = \dim Y_1$ or the number p of parameters. However, the hypotheses can only be satisfied for certain triples (n, p, k); see Section 4.

Remark 6. If $p = 1$, then there are an odd number ($\leqslant k^n - 1$) of nontrivial solution branches bifurcating from (x_0, λ_0). (See 2.7.)

Remark 7. Conditions (H4) and (H5) can be replaced by other hypotheses; see 1.5, Remark 2.

Remark 8. Both Dancer and Magnus have pointed out to us that Theorem 2.2 (under similar but not the same hypotheses) can be proved using degree theory. (See Dancer [9] and Magnus [19]).

To prove Theorem 2.2 we first use the Liapunov–Schmidt procedure to define $u(x_1, \lambda)$ implicitly by

$$Pf(x_1 + u(x_1, \lambda), \lambda) = 0.$$

We then define

$$g: X_1 \times \mathbb{R}^p \to Y_1$$

by

$$g(x_1, \lambda) = (I - P) f(x_1 + u(x_1, \lambda), \lambda)$$

and seek the zeros of g. The zeros of f are the set $\{(x_1 + u(x_1, \lambda), \lambda) \mid g(x_1, \lambda) = 0\}$ so the results for f can be read off those for g.

Implicit differentiation of the preceeding two equations shows that the x_1 derivatives of g at (x_0, λ_0) vanish up to order $k - 1$, and that

$$D^k g(x_0, \lambda_0) = (I - P)\, D^k f(x_0, \lambda_0) \mid (X_1 \times \mathbb{R}^p)^k.$$

Therefore, it suffices to prove a bifurcation theorem for the map g. We can assume without loss of generality that $X_1 = Y_1 = \mathbb{R}^n$ and $x_0(\lambda) \equiv 0$. Theorem 2.2 therefore follows from the next result.

2.4. Theorem. *Let $2 \leqslant k < l$. Suppose $g: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is of class C^l, $g(0) = 0, \ldots, D^{k-1} g(0) = 0$ and $g(0, \lambda) = 0$ identically. Let Q be the k-form associated to $D^k g(0)$ and assume Q is regular on $Q^{-1}(0)$. If k is even, then $(0, 0)$ is a bifurcation point. More precisely,*

$$Q^{-1}(0) = (\{0\} \times \mathbb{R}^p) \cup C,$$

where C is the cone from $(0, 0)$ to a nonempty $(p-1)$-dimensional analytic submanifold of S^{n+p-1} that does not intersect $\{0\} \times \mathbb{R}^p$. There are neighborhoods U_1 and U_2 of $(0, 0)$ in $\mathbb{R}^{n+p}$ and a C^1 diffeomorphism $\phi: U_1 \to U_2$ with $\phi(0, 0) = (0, 0)$, $D\phi(0, 0) = $ identity, that is C^l away from $(0, 0)$; and (1) $\phi(Q^{-1}(0) \cap U_1) = g^{-1}(0) \cap U_2$, (2) $\phi \mid (\{0\} \times \mathbb{R}^p) \cap U_1$ is the identity, and (3) $\phi \mid (C \cap U_1) \backslash \{0, 0\}$ is a C^l diffeomorphism onto the manifold $M = g^{-1}(0) \cap U_2 \backslash (\{0\} \times \mathbb{R}^p)$. (See Fig. 3.)

Theorem 2.4 follows from 1.3 provided it can be shown that $Q^{-1}(0)$ contains such a nontrivial cone C. (The assertion that the ϕ supplied by 1.3 is the identity on $\{0\} \times \mathbb{R}^p$ follows from an examination of the proofs.) The next theorem, of interest in itself, will study $Q^{-1}(0)$ in order to complete the proof of 2.4 and hence of 2.2.

2.5. Theorem. *If $Q: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a k-form regular on $Q^{-1}(0)$, k is even, and $Q \mid \{0\} \times \mathbb{R}^p = 0$, then $Q^{-1}(0)$ contains a p-dimensional cone which intersects $\{0\} \times \mathbb{R}^p$ only at $(0, 0)$.*

The fact that $Q^{-1}(0)$ is the cone on a $(p-1)$-dimensional submanifold follows from the regularity of Q on $Q^{-1}(0)$. The heart of the matter concerns the existence of a nontrivial connected component. This is established in the following development.

Let $P^k(\mathbb{R}^{n+p}, \mathbb{R}^n)$ (resp. $P^k(\mathbb{C}^{n+p}, \mathbb{C}^n)$) denote the space of homogeneous polynomial maps of degree k from $\mathbb{R}^{n+p}$ to $\mathbb{R}^n$ (resp. $\mathbb{C}^{n+p}$ to $\mathbb{C}^n$). Let $R^k =$

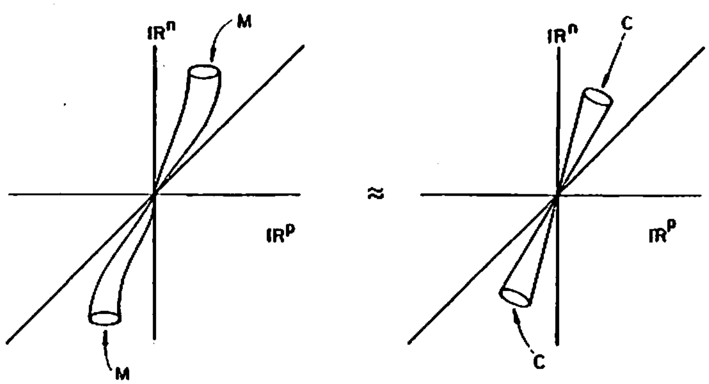

Figure 3

$\{\hat{Q} \in P^k(\mathbb{R}^{n+p}, \mathbb{R}^n) \mid \hat{Q}$ is regular on $\hat{Q}^{-1}(0) \subset \mathbb{R}^{n+p}\}$ and let $R^k_{\mathbb{C}} = \{\hat{Q} \in P^k(\mathbb{C}^{n+p}, \mathbb{C}^n) \mid \hat{Q}$ is regular on $\hat{Q}^{-1}(0) \subset \mathbb{C}^{n+p}\}$. $P^k(\mathbb{R}^{n+p}, \mathbb{R}^n)$ is naturally regarded as a subset of $P^k(\mathbb{C}^{n+p}, \mathbb{C}^n)$.

2.6. LEMMA. *$R^k_{\mathbb{C}} \cap P^k(\mathbb{R}^{n+p}, \mathbb{R}^n)$ is a nonempty Zariski open subset of $P^k(\mathbb{R}^{n+p}, \mathbb{R}^n)$.*

Proof. In $\mathbb{C}P^{n+p-1} \times P^k(\mathbb{C}^{n+p}, \mathbb{C}^n)$ consider the algebraic subset $V = \{([z], \hat{Q}) \mid \hat{Q}(z) = 0$ and rank $D\hat{Q}(z) < n\}$. Let π denote the projection of $\mathbb{C}P^{n+p-1} \times P^k(\mathbb{C}^{n+p}, \mathbb{C}^n)$ onto the second factor. By the main theorem of elimination theory (see Mumford [30]), $\pi(V)$ is an algebraic subset of $P^k(\mathbb{C}^{n+p}, \mathbb{C}^n)$. Observe that $\pi(V)$ is the complement of $R^k_{\mathbb{C}}$. The set $R^k_{\mathbb{C}}$ will now be shown to be nonempty by explicit construction. For example, let the $n \times (n+p)$ matrix (a_{ij}) be chosen so that all $n \times n$ minors have rank n. Define $\hat{Q}_i(z_1, ..., z_{n+p}) = \sum_{j=1}^{n+p} a_{ij} z_j^k$ and define $\hat{Q} = (\hat{Q}_1, ..., \hat{Q}_n)$. Then $\hat{Q}$ is in $R^k_{\mathbb{C}}$. The lemma now follows from the fact that $P^k(\mathbb{R}^{n+p}, \mathbb{R}^n)$ is Zariski dense in $P^k(\mathbb{C}^{n+p}, \mathbb{C}^n)$ (or one observes that the above example provides an element of $R^k_{\mathbb{C}} \cap P^k(\mathbb{R}^{n+p}, \mathbb{R}^n)$ if all the a_{ij} are real). ∎

We now show that $Q^{-1}(0) \neq \{0\} \times \mathbb{R}^p$. For any homogeneous polynomial map $h: \mathbb{R}^{n+p} \to \mathbb{R}^n$, let V_h denote the variety defined by h in the real projective space $\mathbb{R}P^{n+p-1}$ and let $V_h^{\mathbb{C}}$ denote the variety defined by h in $\mathbb{C}P^{n+p-1}$. Suppose that in fact $Q^{-1}(0) = \{0\} \times \mathbb{R}^p$. Then V_Q is the linear space $\mathbb{R}P^{p-1} \hookrightarrow \mathbb{R}P^{n+p-1}$. The complementary linear space $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+p-1}$ intersects V_Q transversally in one point. Choose, by 2.6, $\hat{Q} \in R^k_{\mathbb{C}} \cap P^k(\mathbb{R}^{n+p}, \mathbb{R}^n)$ so close to Q that $\mathbb{R}P^n$ intersects $V_{\hat{Q}}$ transversally at one point. By Bezout's theorem (see Mumford [30]) a generic n-dimensional linear space W in $\mathbb{C}P^{n+p-1}$ meets $V^{\mathbb{C}}_{\hat{Q}}$ in k^n points and hence an even number of points. The space W can be assumed defined by linear equations with real coefficients and can be taken so close to $\mathbb{R}P^n$ that W also intersects $V_{\hat{Q}}$ transversally at one point. Since both $\hat{Q}$ and the equations defining W are real, complex conjugation must preserve $V^{\mathbb{C}}_{\hat{Q}} \cap W$. Hence the nonreal points of $V^{\mathbb{C}}_{\hat{Q}} \cap W$ occur in conjugate pairs so the number of points of $V_{\hat{Q}} \cap W$ must be even. This contradiction completes the proof. ∎

The following result has a similar proof.

2.7. THEOREM. *Let $2 \leq k < l$. Suppose $g: \mathbb{R}^{n+1} \to \mathbb{R}^n$ is of class C^l and $g(0) = 0$, $Dg(0) = 0, ..., D^{k-1}g(0) = 0$. Let Q be the k-form associated to $D^k g(0)$ and assume Q is regular on $Q^{-1}(0)$ and k is even. Then $Q^{-1}(0)$ consists of an even number ($\leq k^n$) of lines through 0. There is a neighborhood U of 0 in $\mathbb{R}^{n+1}$ such that $g^{-1}(0) \cap U$ consists of the same number of C^1 curves through 0, each tangent to a different one of the lines in $Q^{-1}(0)$.*

Note that 2.7 implies: if in addition $g(0, \lambda) \equiv 0$, $(0, \lambda) \in \mathbb{R}^n \times \mathbb{R}$, then $(0, 0)$ is a bifurcation point with an odd number $\leq k^n - 1$ of lines bifurcating from $\{0\} \times \mathbb{R}$. Of course the same thing will be true about the original map f from which g came by the Liapunov–Schmidt procedure.

2.8. *Remarks*

Remark 1. It is instructive to consider the special case $n = 1$, $p = 1$ for 2.7, i.e., bifuration at simple eigenvalues. Then Q is a real-valued k-form in two variables x and y and can be factored $Q(x, y) = ay^r \prod_{i=1}^{k-r} (x - \lambda_i y)$. Regularity on the zero set implies $r = 1$ or 0 and the λ_i are distinct. Since Q is real the complex λ_i occur in conjugate pairs so this leaves an even number of real linear factors. In the special case of $k = 2$ this shows the existence of one nontrivial solution branch. This latter fact also follows from the fact that $(0, 0)$ is a nondegenerate critical point of index 1 for $g: \mathbb{R}^2 \to \mathbb{R}$.

Remark 2. In Theorem 2.4 we considered $g: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ rather than $g: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ for the following reasons: If $m < n$, then (assuming regularity of Q), $(0, 0)$ is trivially a bifurcation point since $Q^{-1}(0) \backslash \{(0, 0)\}$ is a manifold of dimension $n - m + p$, so $Q^{-1}(0)$ must contain more than $\{0\} \times \mathbb{R}^p$. If $m > n$, then the regularity condition is impossible to satisfy.

Remark 3. Alexander [3] has studied maps $f: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ such that $f(0, \lambda) = 0$ and $(\partial f/\partial x)(0, \lambda)$ is invertible for $0 < \|\lambda\| < \varepsilon$. He has obtained a condition on the map $\lambda \mapsto (\partial f/\partial x)(0, \lambda)$ from a deleted neighborhood of 0 in $\mathbb{R}^p$ to $GL(n)$ that guarantees that $(0, 0)$ is a bifurcation point. Such a map determines an element of $\pi_{p-1}(GL(n))$,. which stabilizes for large n to an element $\gamma \in \pi_{p-1}(GL)$. The homotopy groups of GL are given by

p mod 8	1	2	3	4	5	6	7	8
$\pi_{p-1}(GL)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$.

Alexander's result states that if $p \equiv 1$ or 2 mod 8 and $\gamma \neq 0$, then $(0, 0)$ is a bifurcation point; if $p = 4k$ and γ is not divisible by a certain number b_k, then $(0, 0)$ is a bifurcation point; in all other cases $(0, 0)$ is not necessarily a bifurcation point.

It is interesting to note that the hypotheses of Theorem 2.4 may be satisfied when Alexander's condition is not. For example, let $g(x_1, x_2, \lambda) = (x_1\lambda + x_1^2 + \cdots, x_2\lambda + x_2^2 + \cdots)$ and assume $g(0, 0, \lambda) \equiv 0$. Bifurcation is guaranteed by Theorem 2.4. However, the map

$$\lambda \to \frac{\partial g}{\partial(x_1, x_2)}(0, 0, \lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \cdots$$

determines the zero element of $\pi_0(GL)$ (since $\det\left[\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right]$ is positive for all $\lambda \neq 0$; or see Alexander [3, Result (2)]). Thus bifurcation is not guaranteed by Alexander's result.

The map $h(x_1, x_2, \lambda) = (x_1\lambda + x_2^3, x_2\lambda - x_1^3)$ is a standard example of a problem with no bifurcation at $(0, 0)$ (see Nirenberg [31, p. 82]). The reader can verify that the hypotheses of Theorem 2.4 as well as Alexander's hypotheses fail, as they must.

Remark 4. In principle the condition that Q be regular on its zero set can be checked without identifying the zero set. Thus Theorems 2.2 and 2.4 give verifiable conditions that guarantee bifurcation.

Observe that Q fails to be regular on its zero set if and only if there is an $x \in \mathbb{R}^{n+p}$ such that (a) $x \neq 0$, (b) $Q(x) = 0$, and (c) each $n \times n$ submatrix of $DQ(x)$ has determinant 0. Let $Q(x_1,..., x_{n+p}) = (\sum_I a_{1I}x^I,..., \sum_I a_{nI}x^I)$, where $I = (i_1,..., i_{n+p})$ is a multi-index with $i_1 + \cdots + i_{n+p} = \deg Q$. According to the Tarski–Seidenberg theorem (Seidenberg [34]), there is a finite collection of systems $\{\psi_\alpha\}$ of polynomial equations and inequalities in the variables a_{iI}, with rational coefficients, such that there is an $x \in \mathbb{R}^{n+p}$ satisfying (a)–(c) for given $(a_{1I},..., a_{nI})$ if and only if $(a_{1I},..., a_{nI})$ satisfies at least one of the systems ψ_α. The systems ψ_α are, at least in theory, effectively constructible.

The first example in Section 3 will give, in this spirit, a sufficient condition for bifurcation at a multiple eigenvalue for a common nonlinear eigenvalue problem.

Remark 5. Theorem 2.4 can be generalized to allow g as in 1.4, Remark 4, with at least one k_i even. Then bifurcation is proved by essentially the same argument.

Remark 6. E. N. Dancer points out that the estimate ($\leqslant k^n$) on the number of lines can also be proved using complex degree theory, cf. Dancer [10].

3. EXAMPLES

3.1. EXAMPLE. Consider the following equation for a scalar function $u(x)$:

$$f(u, \lambda) = \Delta u - (\lambda + \lambda_0)\, g(u) = 0 \qquad \text{in } \Omega$$

with boundary conditions

$$u = 0 \qquad \text{on } \partial\Omega,$$

where Ω is a bounded region in $\mathbb{R}^N$ with smooth boundary, Δ is the Laplacian, u belongs to a suitable Banach space (say H_0^{s+2}, so that f: $H_0^{s+2} \times \mathbb{R} \to H^s$), $\lambda \in \mathbb{R}$, λ_0 is an eigenvalue of Δ, $g(0) = 0$, and $g'(0) = 1$. In

Nirenberg [31] the problem of when $(0, 0)$ is a bifurcation point is studied (see also Berger [4]). It is shown that if λ_0 has multiplicity one, then $(0, 0)$ is a bifurcation point. By the Liapunov–Schmidt procedure the problem is reduced to studying the zeros of a real-valued function of two variables to which the Morse lemma can be applied.

If the multiplicity is $n > 1$ the Liapunov–Schmidt procedure leads to n functions each of $n + 1$ variables, so the Morse lemma is not applicable. Instead conditions can be given so that Theorem 2.2 with $k = 2$ and $p = 1$ is applicable. Thus $(0, 0)$ will be guaranteed to be a bifurcation point.

Let $\operatorname{Ker}(D_u f(0, 0)) = \operatorname{Ker}(\Delta - \lambda_0 I)$ be spanned by the L^2 orthonormal functions $u_1,..., u_n$. The second derivative is given by $D^2 f(0, 0)((u, \lambda), (v, \rho)) = -\lambda_0 g''(0)\, uv - u\rho - v\lambda$. Let $u = z_1 u_1 + \cdots + z_n u_n$. The quadratic map that must, according to Section 2, be considered is $Q = (Q_1(z_1,..., z_n, \lambda),..., Q_n(z_1,..., z_n, \lambda))$, where

$$-Q_i(z_1,..., z_n, \lambda) = \lambda_0 g''(0) \int_\Omega u_i |z_1 u_1 + \cdots + z_n u_n|^2 + 2\lambda z_i.$$

Here is a specific case: *consider*

$$\Delta u - (\lambda - 10)\, g(u) = 0$$

on $\Omega = [0, \pi] \times [0, \pi]$ *in* $\mathbb{R}^2$ with $u = 0$ *on* $\partial\Omega$. Assume $g(0) = 0$, $g'(0) = 1$ *and* $g''(0) \neq 0$. *Then* $u = 0$, $\lambda = 0$ *is a bifurcation point with* $l = 3$ *branches of nontrivial solutions.*

Indeed, we take $n = 2$, $u_1 = \sin 3x \sin y$, $u_2 = \sin x \sin 3y$. (Normalizing u_1 and u_2 is not necessary here.) Let $a \equiv \int u_1^3 = \int u_2^3 = \frac{16}{27}$ and $b \equiv \int u_1^2 u_2 = \int u_1 u_2^2 = \frac{68}{75}$. Then Q_1 and Q_2 have the form of 1.2, Example 2, the result of which gives $l = 3$ as stated.

Explicit calculations like the above specific case are not always easy to carry out. Thus we seek a computable condition for regularity of Q on its zero set, since this implies, by 2.2, bifurcation.

Let $M(z_1,..., z_n)$ be the $n \times n$ symmetric matrix whose (ij) entry is $\sum_m (\int_\Omega u_i u_j u_m)\, z_m$. Then $(z_1,..., z_n, \lambda) \in Q^{-1}(0)$ if and only if either $(z_1,..., z_n)$ is an eigenvector of $\lambda_0 g''(0)\, M(z_1,..., z_n)$ corresponding to the eigenvalue -2λ or $(z_1,..., z_n) = (0,..., 0)$. At $(0, 0,..., 0, \lambda)$, $\lambda \neq 0$, Q is always regular. At any other $(z_1,..., z_n, \lambda) \in Q^{-1}(0)$, Q is regular if $-\lambda$ is not an eigenvalue of $\lambda_0 g''(0)\, M(z_1,..., z_n)$. It follows that a sufficient condition for regularity is that the matrix $\lambda_0 g''(0)\, M(z_1,..., z_n)$ not have eigenvalues λ and 2λ. (If $\lambda = 0$ this is taken to mean the dimension of the corresponding eigenspace is one.) In principal this condition can be tested using resultants as follows: First one obviously needs $\lambda_0 g''(0) \neq 0$. Next let

$$\det(M(z_1,..., z_n) - XI) = X^n + A_1 X^{n-1} + \cdots + A_n.$$

Observe that A_i is an homogeneous polynomial in the variables $z_1,\dots, z_n$ of degree i. The condition that twice an eigenvalue is not an eigenvalue becomes the condition that the polynomials

$$X^n + A_1 X^{n-1} + \cdots + A_n \cdot$$
$$\cdot\ (2^n - 1) X^{n-1} + (2^{n-1} - 1) A_1 X^{n-2} + \cdots + A_{n-1}$$

do not have a common root. This condition is equivalent to the nonvanishing of the resultant

$$\left|\begin{array}{ccccccc}
1 & A_1 & A_2 & \cdots & A_n & & \\
 & 1 & A_1 & \cdots & & A_n & \\
 & & & \vdots & & & \\
 & & 1 & A_1 & A_2 & \cdots & A_n \\
(2^n-1) & (2^{n-1}-1)A_1 & \cdots & A_{n-1} & & & \\
 & (2^n-1) & (2^{n-1}-1)A_1 & \cdots & A_{n-1} & & \\
 & & & \vdots & & & \\
 & & & (2^n-1) & (2^{n-1}-1)A_1 & \cdots & A_{n-1}
\end{array}\right|.$$

This resultant is a homogeneous polynomial in $z_1,\dots, z_n$ of degree $n(n-1)$. Thus a sufficient condition for regularity is that this form be positive or negative definite. To illustrate: Let $n=2$, $a=\int_\Omega u_1^3$, $b=\int_\Omega u_2 u_1^2$, $c=\int_\Omega u_2^2 u_1$, and $d=\int_\Omega u_2^3$. Then the form in question is the quadratic form

$$-2((a+c)z_1 + (b+d)z_2)^2 + 9\{(az_1 + bz_2)(cz_1 + dz_2) - (bz_1 + cz_2)^2\}$$
$$= Az_1^2 + 2Bz_1 z_2 + Cz_2^2$$

and the sufficient condition for bifurcation is simply that $AC - B^2 > 0$.

Finally, it should be recalled that when the regularity can be verified, not only is $(0, 0)$ guaranteed to be a bifurcation point but the number of curves bifurcating off from the trivial one is an odd number l, where $l \leqslant 2^n - 1$. To actually compute l and the directions of the bifurcating curves would require a more detailed study of Q, as we did in the specific case above. In that specific case, $b = c$, $a = d$, and

$$AC = \{2a^2 - 5ab + 11b^2\}^2, \qquad B^2 = \tfrac{1}{4}\{13b^2 - 5a^2 + 8ab\}^2$$

With $a = \frac{16}{27}$ and $b = \frac{68}{75}$ one finds $AC > B^2$ so the test using resultants is effective in this case. More generally *one can check that if* $\lambda_0 = -(2r+1)^2 - (2s+1)^2$, $r \neq s$, *then* λ_0 *is an eigenvalue of* Δ *of multiplicity two and* $AC > B^2$. Thus bifurcation is assured in this case by Theorem 2.2.

3.2. Example. Example 3.1 falls into the class of nonlinear eigenvalue problems studied by McLeod and Sattinger [28]. Theorems 2.4 and 2.7 can be applied quite easily to these problems, as follows:

After application of the Liapunov–Schmidt procedure, their nonlinear eigenvalue problems yield equations of the form

$$h(x, \lambda) = \lambda Lx + Q(x) + R(x, \lambda).$$

Here $h\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, L is linear, Q is homogeneous of degree $k \geqslant 2$, and R satisfies $R(0, 0) = 0$, $DR(0, 0) = 0$, $(\partial^2 R/\partial x\, \partial\lambda)(0, 0) = 0$, $(\partial^2 R/\partial x^2)(0, 0) = 0,\dots, (\partial^k R/\partial x^k)(0, 0) = 0$. McLeod and Sattinger assume (1) L is invertible, (2) $Q^{-1}(0) = 0$, and (3) each solution of $Lx \pm Q(x) = 0$ is regular, (i.e., if $Lx \pm Q(x) = 0$, then $y \mapsto Ly \pm DQ(x)(y)$ is surjective). They conclude that each zero of $Lx \pm Q(x) = 0$ gives rise to a one parameter bifurcating branch of solutions. This result can be derived from 1.3 by first rescaling by setting $\lambda = \varepsilon^{k-1}$ (when k is even) or $\lambda = \pm\varepsilon^{k-1}$ (when k is odd). Then the new function $g(x, \varepsilon) = h(x, \varepsilon^{k-1})$ (or $h(x, -\varepsilon^{k-1})$) consists of a homogeneous polynomial of degree k plus higher order terms. Assumptions (1)–(3) imply that this homogeneous polynomial is regular on its zero set. Note that 2.4 guarantees bifurcation when k is even, i.e., 2.4 guarantees that the equation $Lx \pm Q(x) = 0$ has nontrivial solutions.

3.3. *Hopf Bifurcation*

We will now show how the finite-dimensional Hopf bifurcation theorem can be derived from Theorem 2.4. The fact that the hypotheses of this theorem are satisfied follows easily from the treatment of Crandall and Rabinowitz [8] of Hopf bifurcation. Their approach will now briefly be recalled.

Consider the differential equation

$$\frac{du}{dt} + f(\mu, u) = 0, \qquad (*)$$

where $f \in C^3(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(\mu, 0) = 0$ for all $\mu \in \mathbb{R}$. Equation $(*)$ has the family of equilibrium solutions $\{(\mu, 0) \mid \mu \in \mathbb{R}\}$.

Let $L_0 = (\partial f/\partial u)(0, 0)$ and assume

(1) $\pm i$ are algebraically simple eigenvalues of L_0 and $\pm ki \notin$ spectrum of L_0 for $k = 0, 2, 3,\dots$.

Let a be the eigenvector of L_0 with eigenvalue i. Standard arguments show that there exist C^2 functions $\beta(\mu)$ and $x(\mu)$ defined by $(\partial f/\partial u)(\mu, 0)\, x(\mu) = \beta(\mu)\, x(\mu)$, $\beta(0) = i$ and $x(0) = a$. We assume the *Hopf condition*:

(2) $\operatorname{Re} \beta'(0) \neq 0$,

i.e., the eigenvalues cross the imaginary axis transversally.

The problem is to find periodic solutions of (*) with period near 2π. This will be done by finding 2π-periodic solutions of

$$\frac{du}{d\tau}+(1+\rho)f(\mu,u)=0$$

that have ρ close to 0 and letting $t=(1+\rho)\tau$. Let $F(\rho,\mu,u)=(du/d\tau)+(1+\rho)f(\mu,u)$. F can be regarded as a C^3 mapping from $\mathbb{R}^2\times C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)$ to $C^0_{2\pi}(\mathbb{R},\mathbb{R}^n)$, where $C^s_{2\pi}(\mathbb{R},\mathbb{R}^n)$ denotes the space of C^s 2π-periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$ with the C^s sup norm topology. The problem now becomes to describe $F^{-1}(0)$ near $(0,0,0)$, which will be done using Theorems 2.2 and 2.4. Thus all small-amplitude periodic solutions of (*) with period near 2π will be found.

Clearly, $\operatorname{Ker}(DF(0,0,0))=\mathbb{R}^2\oplus\operatorname{Ker}((d/d\tau)+L_0)$. Here $\operatorname{Ker}((d/d\tau)+L_0)$ is two dimensional and is spanned by $\phi_0=\operatorname{Re}(e^{-i\tau}a)$ and $\phi_1=\operatorname{Im}(e^{-i\tau}a)$. Moreover, $\phi_0'=\phi_1$ and $\phi_1'=-\phi_0$. Therefore $L_0\phi_0=-\phi_1$ and $L_0\phi_1=\phi_0$.

Now $\operatorname{Ker}(-(d/d\tau)+L_0^*)$ is also two dimensional. A basis ψ_0, ψ_1 can be found with $\psi_0'=\psi_1$, $\psi_1'=-\psi_0$, and $\langle\phi_i,\psi_j\rangle=\delta_{ij}$, where $\langle g,h\rangle=\int_0^{2\pi}g(\tau)\,h(\tau)\,d\tau$. Then $\operatorname{Im}(DF(0,0,0))=\operatorname{Im}((d/d\tau)+L_0)=\{g\in C^0_{2\pi}(\mathbb{R},\mathbb{R}^n)\mid\langle g,\psi_i\rangle=0,i=0,1\}$.

Let $L_1=(\partial^2F/\partial\mu\,\partial u)(0,0)$. Then the following computations may easily be carried out:

$$\langle L_1\phi_0,\psi_0\rangle=\operatorname{Re}\beta'(0),\qquad\langle L_1\phi_1,\psi_0\rangle=\operatorname{Im}\beta'(0),$$
$$\langle L_1\phi_0,\psi_1\rangle=-\operatorname{Im}\beta'(0),\qquad\langle L_1\phi_1,\psi_1\rangle=\operatorname{Re}\beta'(0).$$

Also, $D^2F(0,0,0)((\rho,\mu,u),(\rho,\mu,u))=2\rho L_0u+2\mu L_1u+(\partial^2F/\partial u^2)(0,0,0)(u,u)$. Let P be orthogonal projection onto $\operatorname{Im}(DF(0,0,0))$. Write vectors in $\operatorname{Ker}(DF(0,0,0))$ as $(\rho,\mu,x_0\phi_0+x_1\phi_1)$ and use the basis ψ_0, ψ_1 for $\operatorname{Ker}(P)$. In these coordinates the quadratic map associated to $(I-P)\,D^2F(0,0,0)\mid\operatorname{Ker}(DF(0,0,0))\times\operatorname{Ker}(DF(0,0,0))$ is

$$Q(\rho,\mu,x_0,x_1)=\left(\rho\begin{bmatrix}0&1\\-1&0\end{bmatrix}+\mu\begin{bmatrix}\operatorname{Re}\beta'(0)&\operatorname{Im}\beta'(0)\\-\operatorname{Im}\beta'(0)&\operatorname{Re}\beta'(0)\end{bmatrix}\right)\begin{pmatrix}x_0\\x_1\end{pmatrix}+\tilde{Q}(x_0,x_1),$$

where $\tilde{Q}$ is a quadratic map $\mathbb{R}^2$ to $\mathbb{R}^2$. Using $\operatorname{Re}\beta'(0)\neq0$ one computes $(\partial Q/\partial(x_0,x_1))(\rho,\mu,0,0)$ is an isomorphism if $(\rho,\mu)\neq(0,0)$ and $(\partial Q/\partial(\rho,\mu))(\rho,\mu,x_0,x_1)$ is an isomorphism if $(x_0,x_1)\neq(0,0)$. Hence $DQ(\rho,\mu,x_0,x_1)$ is surjective for all $(\rho,\mu,x_0,x_1)\neq(0,0,0,0)$. (See also 1.2, Example 3.)

Application of Theorem 2.2 now yields the existence of a two-dimensional surface bifurcating from $(0,0,0)$ in $\mathbb{R}^2\times C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)$.

It is well known that *there is a unique family of nontrivial closed orbits bifurcating from* $(0,0,0)$, i.e., there are no other periodic orbits (with nearby periods) near $(0,0,0)$. (See, for instance Crandall and Rabinowitz [8]). We now give a simple proof of this assertion using some of the remarks on equivariance in Section 1.

The group $SO(2)$ (proper rotations of the plane) acts on $C^s_{2\pi}(\mathbb{R},\mathbb{R}^n)$ by $T_\theta(u(\tau))=u(\tau+\theta)$ and is a group of isometries for the L^2 inner product. Clearly, F is equivariant with respect to T_θ, i.e.,

$$T_\theta F(\rho,\mu,u)=F(\rho,\mu,T_\theta u).$$

From 1.6, Remark 3, the bifurcating mapping $(I-P)\circ F(\rho,\mu,x_0\phi_0+x_1\phi_1)$ is equivariant. Hence so is $\tilde{Q}$. But the action of $SO(2)$ on the space spanned by ϕ_0, ϕ_1 and ψ_0, ψ_1 is by rotation. Since there are no maps: $\mathbb{R}^2\to\mathbb{R}^2$ of even degree equivariant with respect to rotation through π, it follows that $\tilde{Q}=0$. Therefore $Q^{-1}(0)=\rho\mu$ plane $\cup\, x_0x_1$ plane. From 1.6, Remark 3, there is an equivariant diffeomorphism Φ: $\mathbb{R}^2\times C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)\to\mathbb{R}^2\times C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)$ defined near 0 taking $Q^{-1}(0)$ to $F^{-1}(0)$. By Theorem 2.4, we have $\Phi(\mathbb{R}^2\times\{0\})=$ identity. Hence $\Phi(x_0x_1\text{ plane})=$ nontrivial solutions of $F^{-1}(0)$. Consider the line $t\mapsto\Phi(t\phi_0)$ $(t\geqslant0)$. Any (ρ,μ,u) in $F^{-1}(0)$ with $u\neq0$ and (ρ,μ,u) sufficiently close to $(0,0,0)$ is $\Phi(0,0,x_0\phi_0+x_1\phi_1)$ for some x_0, x_1. Let T_θ be a rotation which transforms $x_0\phi_0+x_1\phi_1$ to $t\phi_0$. Then by equivariance, $T_\theta(\rho,\mu,u)=\Phi(t\phi_0)$. Hence the uniqueness assertion follows.

We note that while the preceeding methods give the structure of the set of periodic solutions near 2π, they do not give detailed information on the phase portraits near the periodic solutions without more work. For this center manifold theory can be used (see Ruelle and Takens [33], Marsden and McCracken [25] and Hassard *et al.* [15]). This theory also proves a stronger uniqueness theorem under stronger hypotheses: there are no other nearby periodic orbits of *any* period in if (1), $\pm i$ are the only eigenvalues on the imaginary axis.

We have not investigated if the methods of the present paper can be applied to more degenerate Hopf bifurcations such as bifurcation at multiple complex eigenvalues and to cases where assumption (2) fails. (See Takens [38], Kielhoffer [17], Golubitsky and Langford [12].)

4. Genericity of Regularity Conditions

This section will make precise and answer to some extent the question: How general or how restrictive are the regularity conditions of Theorems 1.3, 2.2, 2.4, 2.5, and 2.7?

4.1. Lemma. *Let M, N and P be finite-dimensional manifolds and let F: $M \times N \to P$ be C^∞. Let $A \subset P$ be a closed submanifold and assume F is everywhere transverse to A. Then there is a dense subset $\mathscr{A} \subset M$ such that if $a \in \mathscr{A}$, then $F \mid \{a\} \times N$ is transverse to A everywhere on N. If N is compact, then $\mathscr{A}$ is open.*

Proof. This follows from standard transversality theory. See, for instance Abraham and Robbin [1]. ∎

Let $L_s^k(\mathbb{R}^n, \mathbb{R}^m)$ denote the symmetric k-multilinear maps $\mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^m$. Given $B \in L_s^k(\mathbb{R}^n, \mathbb{R}^m)$ let $Q_B: \mathbb{R}^n \to \mathbb{R}^m$ be the associated k-form defined by $Q_B(x) = (1/k!)\, B(x,..., x)$. Let $\mathscr{C} = \{B \in L_s^k(\mathbb{R}^n, \mathbb{R}^m) \mid Q_B$ is regular on $Q_B^{-1}(0)\}$. (If $m \geqslant n$, then $B \in \mathscr{C}$ if and only if $Q_B(x) = 0$ implies $x = 0$.)

4.2. Theorem. *$\mathscr{C}$ is open and dense in $L_s^k(\mathbb{R}^n, \mathbb{R}^m)$.*

Proof. $B \in \mathscr{C}$ if and only if $Q_B \mid S^{n-1}$ is transverse to $0 \in \mathbb{R}^m$. Define F: $L_s^k(\mathbb{R}^n, \mathbb{R}^m) \times S^{n-1} \to \mathbb{R}^m$ by $F(B, x) = Q_B(x)$. If $C \in L_s^k(\mathbb{R}^n, \mathbb{R}^m)$ and $v \in T_x S^{n-1}$, then $DF(B, x)(C, v) = Q_C(x) + (1/(k-1)!)\, B(x,..., x, v)$. Now $C \to Q_C(x)$ is surjective so $DF(B, x)$ is surjective. Lemma 4.1 completes the proof. ∎

In fact the complement of $\mathscr{C}$ is contained in a proper algebraic subvariety of $L_s^k(\mathbb{R}^n, \mathbb{R}^m)$. The proof of this fact is virtually identical with the proof of Lemma 2.6.

Now let $\tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m)$ denote $\{B \in L_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m) \mid$ if $v_1,..., v_k \in \{0\} \times \mathbb{R}^p$, then $B(v_1,..., v_k) = 0\}$. Let $\tilde{\mathscr{C}} = \{B \in \tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m) \mid Q_B$ is regular on $Q^{-1}(0)\}$. Because of Theorems 2.2 and 2.4, it is reasonable to ask about the "size" of $\tilde{\mathscr{C}}$ as a subset of $\tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m)$.

Let $\bar{L}_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ denote $\{T \in L_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)) \mid$ if $\lambda \in \mathbb{R}^p - \{0\}$, then $T(\lambda,..., \lambda) \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ has rank $m\}$. Observe that $\bar{L}_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ is open in $L_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ since $T \in \bar{L}_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ if and only if $Q_T(S^{p-1})$ does not intersect the closed subset of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ consisting of maps of rank $< m$, which will be denoted by $\mathrm{Sing} \subset \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

4.3. Theorem. *The set $\tilde{\mathscr{C}}$ is*

(a) *open in $\tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m)$, and*

(b) *dense in $\tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m)$ if and only if $p \leqslant n - m + 1$.*

We shall prove this along with additional information in Lemmas 4.4 and 4.5.

4.4. Lemma. *The set $\tilde{\mathscr{C}}$ is*

(a) *open in $\tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m)$,*

(b) *empty if and only if $\bar{L}_s^{k-1}(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ is empty, and*

(c) *dense if and only if $\bar{L}_s^{k-1}(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ is dense in $L_s^{k-1}(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$.*

Proof. Since $B \in \tilde{\mathscr{C}}$ if and only if $Q_B \mid S^{n+p-1}$ is transverse to $0 \in \mathbb{R}^m$, it follows that $\tilde{\mathscr{C}}$ is open.

Identifying $\mathbb{R}^p$ with $\{0\} \times \mathbb{R}^p \subset \mathbb{R}^{n+p}$ and $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+p}$, define $\pi: \tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m) \to L_s^{k-1}(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ by $\pi(B)(\lambda_1,..., \lambda_{k-1}) = B(\lambda_1,..., \lambda_{k-1}, \cdot) \mid \mathbb{R}^n \times \{0\}$. The map π is surjective linear. Therefore $\pi^{-1}\bar{L}_s^{k-1}(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ is open and is dense if and only if $\bar{L}_s^{k-1}(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ is dense.

If $\mathscr{A} = \{B \in \tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m) \mid Q_B$ is regular on $Q_B^{-1}(0) \backslash (\{0\} \times \mathbb{R}^p)\}$, then $\tilde{\mathscr{C}} = \pi^{-1}\bar{L}_s^{k-1}(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)) \cap \mathscr{A}$. To complete the proof it remains only to show $\mathscr{A}$ is always dense in $\tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m)$. This can be done, as in 4.2, by defining $F: \tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^m) \times (S^{n+p-1} \backslash (\{0\} \times \mathbb{R}^p)) \to \mathbb{R}^m$ by $F(B, v) = Q_B(v)$ and showing $DF(B, v)$ is surjective everywhere. ∎

4.5. Lemma. (a) *$\bar{L}_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ is dense in $L_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ if and only if $p \leqslant n - m + 1$.*

(b) *There exist p, k, n, m with $p > n - m + 1$ such that $\bar{L}_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)) \neq \varnothing$.*

Proof. Using 4.1 one shows that there is an open dense set $\mathscr{B} \subset L_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ such that $T \in \mathscr{B}$ implies $Q_T \mid S^{p-1}$ is transverse to Sing. (Sing is a stratified set, not a manifold, but 4.1 still applies.) If $p \leqslant n - m + 1$, then since $\mathrm{codim}(\mathrm{Sing}) = n - m + 1$, $\mathscr{B} = \bar{L}_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$.

Now suppose $p > n - m + 1$. Choose L in the top-dimensional stratum of Sing and choose λ arbitrarily in S^{p-1}. It is then easy to construct $T \in L_s^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ such that $Q_T(\lambda) = L$ and Q_T is transverse to Sing at λ. Hence for T' near T, we have $Q_{T'}(S^{p-1}) \cap \mathrm{Sing} \neq \varnothing$.

An example where $p > n - m + 1$ and $\bar{L}^k(\mathbb{R}^p, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$ is not empty is: $k = 1$, $p = n = m = 2$, and $T(x, y) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$. ∎

Let us specialize to the case $m = n$ considered in Section 2. $\tilde{C}$ is open and dense in $\tilde{L}_s^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ by 4.3, so when $p = 1$ the assumptions of Theorem 2.2 should typically hold. When $p > 1$, $\tilde{C}$ is not dense in $\tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^n)$, so the assumptions of Theorem 2.2 will not typically hold. Nevertheless there are triples (n, p, k) with $p \geqslant 2$ for which $\tilde{C}$ is nonempty open in $\tilde{L}_s^k(\mathbb{R}^{n+p}, \mathbb{R}^n)$. For these triples (n, p, k) the assumptions of Theorem 2.2 should hold for a nonnegligible collection of problems. In fact, the Hopf bifurcation example has $n = p = k = 2$.

For $k = 2$ there is the following result: Write $n = (2a + 1)\, 2^b$, where a and

b are integers, and set $b = c + 4d$, where c and d are integers and $0 \leqslant c \leqslant 3$. Let $\rho(n) = 2^c + 8d$.

4.6. Proposition. *$\tilde{C} \subset \tilde{L}_s^2(\mathbb{R}^{n+p}, \mathbb{R}^n)$ is nonempty if and only if $p \leqslant \rho(n)$.*

Proof. According to 4.4, $\tilde{L}_s^2(\mathbb{R}^{n+p}, \mathbb{R}^n)$ is nonempty if and only if there exists a linear map $T: \mathbb{R}^p \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ such that $T\lambda$ is an isomorphism for every $\lambda \neq 0$. By Clifford algebra constructions one can produce such maps for all $p \leqslant \rho(n)$ (see Husemoller [16, Chap. 11]). On the other hand, the existence of such a $T: \mathbb{R}^p \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ implies that there exist $p - 1$ linearly independent tangent vector fields on S^{n-1}. (Let $e_1, \ldots, e_p$ be a basis for $\mathbb{R}^p$. We can assume $Te_p = \text{identity}$, for if it does not, we consider $\lambda \mapsto (Te_p)^{-1} T\lambda$. Define tangent vector fields v_i on $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ by $v_i(x) = (Te_i)x - \langle (Te_i)x, x \rangle x$. To show that the v_i are linearly independent it suffices to show that for $x \neq 0$, $\sum_{i=1}^{p-1} c_i(Te_i)x$ is not a multiple of x unless all $c_i = 0$. But $\sum_{i=1}^{p-1} c_i(Te_i)x + c_p x = T(\sum_{i=1}^{p} c_i e_i)x \neq 0$ unless $\sum_{i=1}^{p} c_i e_i = 0$, i.e., all $c_i = 0$.) Adams [2] has shown that there do not exist $\rho(n)$ linearly independent tangent vector fields on S^{n-1}, so we are done. ∎

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References

1. R. Abraham and J. Robbin, "Transversal Mappings and Flows," Benjamin, New York, 1968.
2. J. F. Adams, Vector fields on spheres, *Ann. of Math.* 75 (1962), 603–632.
3. J. C. Alexander, Bifurcation of zeroes of parametrized functions, *J. Funct. Anal.* 29 (1978), 37–53.
4. M. Berger, "Nonlinearity and Functional Analysis," Academic Press, New York, 1977.
5. S. Chow, J. Hale, and J. Mallet-Paret, Applications of generic bifurcation, I, *Arch. Rational Mech. Anal.* 59 (1975), 159–188.
6. S. Chow, J. Hale, and J. Mallet-Paret, Applications of generic bifurcation, II, *Arch. Rational Mech. Anal.* 62 (1976), 209–236.
7. M. Crandall and P. Rabinowitz, Bifurcation at simple eigenvalues, *J. Funct. Anal.* 8 (1971), 321–340.
8. M. Crandall and P. Rabinowitz, The Hopf bifurcation theorem in infinite dimensions, *Arch. Rational Mech. Anal.* 67 (1977), 53–72.

9. E. N. Dancer, Bifurcation theory in real Banach spaces, *Proc. London Math. Soc.* 23 (1971), 699–734.
10. E. N. Dancer, Bifurcation theory for analytic operators, *Proc. London Math. Soc.* 26 (1973), 359–384.
11. A. Fischer and J. Marsden, Linearization stability of nonlinear partial differential equations, *in* "Proc. Symposium on Pure Math.," Vol. 27, (1975) pp. 219–263, Amer. Math. Soc., Providence, R.I.
12. M. Golubitsky and W. Langford, Classification and unfoldings of degenerate Hopf bifurcations, *J. Diff. Eqns.* 41 (1981), 375–415.
13. M. Golubitsky and D. Schaeffer, A theory for imperfect bifurcation via singularity theory, *Comm. Pure and Appl. Math.* 22 (1979), 21–98.
14. J. K. Hale and P. Z. Taboas, Bifurcation near degenerate families, *J. Appl. Anal.* 11 (1980), 21–37.
15. B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, "Theory and Applications of Hopf bifurcation," London Math. Soc. Lect. Notes No. 41, Cambridge Univ. Press., London/New York, 1980.
16. D. Husemoller, "Fibre Bundles," Springer-Verlag, New York/Berlin, 1966.
17. H. Kielhöfer, Hopf bifurcation at multiple eigenvalues, *Arch. Rational Mech. Anal.* 69 (1979), 53–83.
18. T. C. Kuo, Characterization of v-sufficiency of jets, *Topology* 11 (1972), 115–131.
19. R. J. Magnus, Applications of topological degree to the theory of branching, Battelle Math. Report, No. 90, 1974.
20. R. J. Magnus, The reduction of a vector-valued function near a critical point, Battelle Math. Report, No. 93, 1975.
21. R. J. Magnus, On the local structure of the zero set of a Banach space valued mapping, *J. Funct. Anal.* 22 (1976), 58–72.
22. R. J. Magnus, "On Locally Transforming One Function into Another," 1978, preprint.
23. R. J. Magnus, Topological equivalence in bifurcation theory, Lecture Notes in Mathematics, No. 799, pp. 263–276, Springer-Verlag, New York/Berlin, 1980.
24. J. Marsden, Qualitative methods in bifurcation theory, *Bull. Amer. Math. Soc.* 84 (1978), 1125–1148.
25. J. E. Marsden and M. McCracken, "The Hopf Bifurcation and Its Applications," Appl. Math. Sciences, Vol. 19, Springer-Verlag, New York/Berlin, 1976.
26. J. Mather, Stability of C^∞ mappings, III: Finitely determined map germs, *Publ. Math. I.H.E.S.* 35 (1968), 127–156.
27. J. Mather, Stability of C^∞ mappings, IV: Classification of stable germs by R-algebras, *Publ. Math. I.H.E.S.* 37 (1969), 223–248.
28. J. B. McLeod and D. H. Sattinger, Loss of stability and bifurcation at a double eigenvalue, *J. Funct. Anal.* 14 (1973), 62–84.
29. J. Moser, On the volume elements on a manifold, *Trans. Amer. Math. Soc.* 120 (1965), 286–294.
30. D. Mumford, "Algebraic Geometry I, Complex Projective Varieties," Grundlehren der Math. Wiss. Vol. 221, Springer-Verlag, New York/Berlin, 1976.
31. L. Nirenberg, "Topics in Nonlinear Analysis," Courant Institute Lecture Notes, New York, 1974.
32. T. Ratiu, On the smoothness of the time t-map of the KdV equation and the bifurcation of the eigenvalues of Hill's operator, pp. 248–294, Lecture Notes in Mathematics, No. 755, Springer-Verlag, New York/Berlin, 1979.
33. D. Ruelle and F. Takens, On the nature of turbulence, *Comm. Math. Phys.* 20 (1971), 167–192.
34. A. Seidenberg, A new decision method for elementary algebra, *Ann. of Math.* 60 (1954), 365–674.

35. D. H. SATTINGER, "Group Theoretic Methods in Bifurcation Theory," Lecture Notes in Mathematics, No. 762, Springer-Verlag, New York/Berlin, 1979.
36. M. SHEARER, Secondary bifurcation near a double eigenvalue, *SIAM J. Math. Anal.* **11** (1980), 365–389.
37. A. SZULKIN, Local structure of the zero-sets of differentiable mappings and application to bifurcation theory, *Math. Scand.* **45** (1979), 232–242.
38. F. TAKENS, Unfolding of certain singularities of vector fields; generalized Hopf bifurcations, *J. Differential Equations* **14** (1973), 476–493.
39. P. RABIER, A generalization of the implicit function theorem for mappings from $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$ and its applications (preprint).
40. S. D. TALIAFERRO, Bifurcation at multiple eigenvalues and stability of bifurcating solutions (preprint).
41. V. T. TUAN AND D. D. ANG, A representation theorem for differentiable functions, *Proc. Am. Math. Soc.* **75** (1979), 343–350.

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