I VP + Grav. Fields

The Initial Value Problem and the Dynamics of Gravitational Fields[1])

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This lecture will survey some of the recent advances that have been made in the dynamics of general relativity and other classical relativistic field theories. In addition, we shall indicate a few open problems that appear to be of basic interest.

1. Existence and Uniqueness Theorems for Geometrodynamics

The basic existence and uniqueness theorem states that Cauchy data on a spacelike hypersurface determines uniquely (up to spacetime diffeomorphisms) a piece of spacetime (filled with whatever matter or other fields are under consideration) containing the hypersurface. Moreover, it makes sense to look at the maximal development of such Cauchy data, just as it makes sense to look at maximal integral curves of ordinary differential equations.

The rigorous theory developing results of this type begins with Choquet-Bruhat [1948], [1952]. The subject as it existed up until about 1972 is adequately presented in Hawking and Ellis [1973]. Some of the key developments since then are as follows, in more or less chronological order:

(a) Fischer and Marsden [1972a] show how to write the evolution equations as a first-order symmetric hyperbolic system.

(b) Müller-zum-Hagen and Seifert [1977] study the characteristic initial value problem.

(c) Hughes, Kato and Marsden [1977] prove a conjecture of Hawking and Ellis, showing that the equations are well posed, with the metric in H^s, $s > 2.5$. (See also Fischer and Marsden [1979a].)

(d) For asymptotically flat spacetimes, Choquet-Bruhat and Christodoulou [1980] and Christodoulou [1980] prove well-posedness in the weighted Sobolev spaces of Nirenberg-Walker and Cantor ("SNWC spaces"; see Cantor [1979]). The crucial point here is to allow Hilbert spaces (cf. McOwen [1979]).

(e) Christodoulou and O'Murchadha [1980] solve the boost problem in SNWC spaces; i.e. they show that the piece of spacetime generated by the initial data is large enough at spatial infinity to include boosts. The methods may allow also for capturing a piece of Ω.

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Some open problems for 1.

(i) *Gauge Problem:* Current existence theory is based on the harmonic gauge of Choquet-Bruhat. Is there a direct proof valid in any gauge?

(ii) *Global Problem:* Find a gauge (such as the constant mean curvature gauge, i.e. $\varkappa$-gauge) in which global existence holds. Recent work of Christodoulou, Eardley and Moncrief on Yang-Mills fields and Maxwell Klein-Gordon fields gives one hope that the gravitational problem may be solvable. (See Segal [1979], Moncrief [1980a, b, c] and Eardley and Moncrief [1980]). For globally hyperbolic spacetimes with a compact Cauchy surface (cosmological case) global existence in a constant mean curvature gauge implies that the evolution in that gauge captures the entire spacetime; see below and Marsden and Tipler [1980]. For non-compact Cauchy surfaces, this need not be true; see Eardley and Smarr [1979].

(iii) *Boundaries and Gravitational Shocks.* Try to lower s even below 2.5 in the Cauchy problem to allow for jump discontinuities in the second derivatives of the metric ($s = 2.5$ is the crucial value, below which jumps are allowed). This would allow gravitational shocks. The solutions are presumably non-unique if $s < 2.5$ and the physically correct ones are picked out by some kind of entropy condition, as one does in gas dynamics. Can recent advances in geometric optics and Fourier integral operators (Guillemin and Sternberg [1977]) be used in the study of gravitational waves and shocks?

2. Hamiltonian Structures

The Hamiltonian formalism in general relativity goes back to Choquet-Bruhat, Dirac, Bergmann and Arnowitt-Deser and Misner. This formalism (referred to commonly as the ADM formalism) is found in, for example, Misner, Thorne and Wheeler [1973].

This formalism can be exhibited as follows (Fischer-Marsden [1976, 1979]). Let a slicing of spacetime $(V, {}^{(4)}g)$ be given, based on a 3-manifold M. This slicing determines a curve $g(\lambda)$ of Riemannian metrics on M and a curve of symmetric tensor densities $\pi(\lambda)$ (the conjugate momentum). Let the slicing have a lapse function N and a shift vector field X. Einstein's vacuum equations $\mathrm{Ein}\,({}^{(4)}g) = 0$ (the Einstein tensor formed from ${}^{(4)}g$) are then equivalent to the evolution equations in adjoint form

$$\frac{\partial}{\partial\lambda}\begin{pmatrix} g \\ \pi \end{pmatrix} = \mathbb{J} \circ D\Phi(g, \pi)^* \begin{pmatrix} X \\ N \end{pmatrix}; \qquad \mathbb{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}$$

together with the constraints

$$\Phi(g, \pi) = 0$$

where $\Phi(g, \pi) = \big(\mathscr{H}(g, \pi), \mathscr{J}(g, \pi)\big)$ is the super energy-momentum.

The choice of slicing is a gauge choice, and one may wish to determine it along with the dynamics. In particular, the constant mean curvature gauge is especially interesting, as we have already noted. As was indicated in Professor Wheeler's lecture and in Qadir and Wheeler [1980], this gauge has the property that its spacelike hypersurfaces tend to avoid singularities. If one can show that the mean curvature $\varkappa$ runs from $-\infty$ to ∞ and M is compact (closed universes) then the foliation fills out the whole Cauchy development and in fact this development is a "Wheeler universe" (see Tipler and Marsden [1980]).

Some other developments of interest in the Hamiltonian formalism are:

(a) The Hamiltonian and symplectic structures are investigated directly from the four dimensional point of view in Kijowski and Szczyrba [1976], and Kijowski and Tulczyjew [1979].

(b) There has been development of the idea that the constraint $\Phi = 0$ is the same as the vanishing of the Noether current generated by the gauge group of relativity i. e. all diffeomorphisms of V (equalling the identity at infinity for open universes); for relativity, see Fischer and Marsden [1972b], for gauge theories, see Cordero and Teitelboim [1976], Moncrief [1977] and Arms [1978].

(c) The Poincaré group at infinity or the BMS group have Noether currents of interest as well, (although we do not set them zero) such as the ADM energy-momentum tensor or the BMS energy-momentum tensor; see Regge-Teitelboim [1974] and Ashtekar and Streuble [1980].

(d) How $\varkappa$-slicings fit together with the BMS group and gravitational radiation has been investigated by Stumbles [1980]. (For related information on $\varkappa$-slicings, see Choquet-Bruhat, Fischer and Marsden [1979], Eardley and Smarr [1979], Marsden and Tipler [1980] and Treibergs [1980] and references therein).

(e) Teitelboim [1977] and Pilati [1978] have investigated the geometrodynamics of supergravity. Bao [1981] has put it onto the adjoint form above.

Some open problems for 2.

(i) Find sufficient conditions on a relativistic field theory with a given gauge group to ensure that the constraints in a Dirac analysis will be the zero level of the corresponding Noether current. (This is true for all the examples mentioned above).

(ii) How is the classical Noether constraint "color charge $= 0$" for Yang-Mills fields on Minkowski space related to quark confinement? (See Arms, Marsden and Moncrief [1980] for some discussion).

(iii) Is it true that the long time dynamics for a typical relativistic field theory is chaotic? Is the Kolmogorov-Arnold-Moser theory relevant? (Recently, Barrow has embarked on a very enlightening investigation of Misner's mixmaster model from the point of view of chaotic dynamical systems).

3. Spaces of Solutions and Linearization Stability

Let V be a fixed four manifold and let $\mathscr{E}$ be the set of all globally hyperbolic Lorentz metrics $g = {}^{(4)}g$ that satisfy the vacuum Einstein equations $\mathrm{Ein}\,(g) = 0$ on V (plus some additional technical smoothness conditions). Let $g_0 \in \mathscr{E}$ be a given solution. We ask: what is the structure of $\mathscr{E}$ in the neighborhood of g_0?

There are two basic reasons why this question is asked. First of all, it is relevant to the problem of finding solutions to the Einstein equations in the form of a perturbation series:

$$g(\lambda) = g_0 + \lambda h_1 + \frac{\lambda^2}{2} h_2 + \cdots$$

where λ is a small parameter. If $g(\lambda)$ is to solve $\mathrm{Ein}\,\big(g(\lambda)\big) = 0$ identically in λ then clearly h_1 must satisfy the *linearized Einstein equations*:

$$\mathrm{D}\,\mathrm{Ein}\,(g) \cdot h_1 = 0$$

where $\mathrm{D}\,\mathrm{Ein}\,(g)$ is the derivative of the mapping $g \mapsto \mathrm{Ein}\,(g)$. For such a perturbation series to be possible, is it sufficient that h_1 satisfy the linearized Einstein equations, i.e. is h_1 necessarily a direction of *linearized stability*? We shall see that in general the answer is no, unless drastic additional conditions hold. The second reason why the structure of $\mathscr{E}$ is of interest is in the problem of quantization of the Einstein equations. Whether one quantizes by means of direct phase space techniques (due to Dirac, Segal, Souriau and Kostant in various forms) or by Feynman path integrals, there will be difficulties near places where the space of classical solutions is such that the linearized theory is *not* a good approximation to the nonlinear theory.

The dynamical formulation mentioned in § 2 is crucial to the analysis of this problem. Indeed, the essence of the problem reduces to the study of structure of the space of solutions of the constraint equations $\Phi(g, \pi) = 0$.

As we shall see, the answer to these questions is this: $\mathscr{E}$ has a conical or quadratic singularity at g_0 if and only if there is a nontrivial Killing field for g_0 that belongs to the gauge group generating $\Phi = 0$ (thus, the flat metric on $T^3 \times \mathbb{R}$ has such Killing fields, but the Minkowski metric has none.) When $\mathscr{E}$ has such a singularity, we speak of a bifurcation in the space of solutions.

(a) Brill and Deser [1973] considered perturbations of the flat metric on $T^3 \times \mathbb{R}$ and discovered the first example of trouble in perturbation theory. They found, by going to a second order perturbation analysis, that they had to readjust the first order perturbations in order to avoid inconsistencies at second order. This was the first hint of a conical structure for $\mathscr{E}$ near solutions with symmetry.

(b) Fischer and Marsden [1973] found general sufficient conditions for $\mathscr{E}$ to be a manifold in terms of the Cauchy data for vacuum spacetimes.

(c) Choquet-Bruhat and Deser [1973] proved a version of the theorem that $\mathscr{E}$ is a manifold near Minkowski space, which was later improved by Choquet-Bruhat, Fischer and Marsden [1979].

(d) Moncrief [1975a] showed that the sufficient conditions derived by Fischer and Marsden for the compact case where equivalent to the requirement that (V, g_0) have no Killing fields. This then led to the link between symmetries and bifurcations.

(e) Moncrief [1975b] discovered the general splitting of gravitational perturbations generalizing Deser's [1967] decomposition. The further generalization to momentum maps (general Noether currents) was found by Arms, Fischer and Marsden [1975]. This then applies to other examples such as gauge theory and also gives York's decomposition (York [1974]) as special cases.

(f) D'Eath [1976] obtained the basic linearization stability results for Robertson-Walker universes.

(g) Moncriet [1976] discovered the spacetime significance of the second order conditions that arise when one has a Killing field and identified them with conserved quantities of Taub [1970]. Arms and Marsden [1979] showed that the second order conditions for compact spacelike hypersurfaces are nontrivial conditions.

(h) The description of the conical singularity in $\mathscr{E}$ near a spacetime with symmetries is due to Fischer, Marsden and Moncrief [1980] for one Killing field and to Arms, Fischer, Marsden and Moncrief [1981] in the general case.

(i) Moncrief [1977], Coll [1975] and Arms [1977, 1979] obtained the basic results for pure gauge theories and electromagnetism and gauge theories coupled to gravity.

(j) An abstract theory valid for arbitrary momentum maps was developed by Arms, Marsden and Moncrief [1980].

(k) Moncrief [1978] investigated the quantum analogues of linearization stabilities. Using $T^3 \times \mathbb{R}$, he shows that, unless such conditions are imposed, the correspondence principle is violated.

For vacuum gravity, let us state one of the main results in the cosmological case: suppose g_0 has a *compact* spacelike hypersurface $M \subset V$. (Actually we require the existence of at least one of constant mean curvature for technical reasons). Let S_{g_0} be the Lie group of isometries of g_0 and let k be its dimension.

Theorem.

1. $k = 0$, then $\mathscr{E}$ is a smooth manifold in a neighborhood of g_0 with tangent space at g_0 given by the solutions of the linearized Einstein equations.
2. If $k > 0$ then $\mathscr{E}$ is *not* a smooth manifold at g_0. A solution h_1 of the linearized equations is tangent to a curve in $\mathscr{E}$ if and only if h_1 is such that Taub conserved quantities vanish; i.e. for every Killing field X for g_0,

$$\int_M X \cdot [D^2 \operatorname{Ein}(g_0) \cdot (h_1, h_1)] \cdot Z \, d\mu_M = 0$$

where Z is the unit normal to the hypersurface M and "." denotes contraction with respect to the metric g_0.

All explicitly known solutions possess symmetries, so while 1. is "generic", 2. is what occurs in examples. This theorem gives a complete answer to the perturbation question: a perturbation series is possible if and only if all the Taub quantities vanish.

Let us give a brief abstract indication of why such second order conditions should come in. Suppose X and Y are Banach spaces and $F: X \to Y$ is a smooth map. Suppose $F(X_0) = 0$ and $x(\lambda)$ is a curve with $x(0) = x_0$ and $F(x(\lambda)) \equiv 0$. Let $h_1 = x'(0)$ so by the chain rule $DF(x_0) \cdot h_1 = 0$. Now suppose $DF(x_0)$ is not surjective and in fact suppose there is a linear functional $l \in Y^*$ orthogonal to its range: $\langle l, DF(x_0) \cdot u\rangle = 0$ for all $u \in X$. By differentiating $F(x(\lambda)) = 0$ twice at $\lambda = 0$, we get

$$D^2F(x_0) \cdot (h_1, h_1) + DF(x_0) \cdot x''(0) = 0.$$

Applying l gives

$$\langle l, D^2F(x_0) \cdot (h_1, h_1)\rangle = 0$$

which are necessary second order conditions that must be satisfied by h_1.

It is by this general method that one arrives at the Taub conditions. The issue of whether or not these conditions are sufficient is much deeper requiring extensive analysis and bifurcation theory (for $k = 1$ the Morse lemma is used, while for $k > 1$ the Kuranishi deformation theory is needed see Kuranishi [1965], Atiyah, Hitchin and Singer [1978] and § 4 below).

Some open problems for 3.

(i) Is the above phenomenon a peculiarity about vacuum gravity or is there an abstract theorem applicable to a broad class of relativistic field theories? The examples which have been and are being worked out suggest that the latter is the case. Good examples are the Yang-Mills equations for gauge theory (Moncrief [1977], Arms [1979]) the Einstein-Dirac equations (cf. Nelson and Teitelboim [1978]), the Einstein-Euler equations (Bao and Marsden [1981]) and supergravity (Pilati [1978], Bao [1981]). In each of these examples there is a gauge group playing the role of the diffeomorphism group of spacetime for vacuum gravity. This gauge group acts on the fields; when it fixes a field, it is a *symmetry* for that field. The relationship between symmetries of a field and singularities in the space of solutions of the classical equations is then as it is for vacuum gravity.

For this program to carry through, one first writes the four dimensional equations as Hamiltonian evolution equations plus constraint equations by means of the 3 + 1 procedures of Dirac. The constraint equations then must

1. *be the Noether conserved quantities for the gauge group and* 2. *satisfy some technical ellipticity conditions:* $(D\Phi)^*$ *must be an elliptic operator*. As is already been mentioned, for 1, it may be necessary to shrink the gauge group somewhat, especially for spacetimes that are not spatially compact. For example the isometries of Minkowski space do not belong to the gauge group generating the constraints but rather they generate the total energy-momentum vector of the

spacetime ... that this vector is time-like is the now famous positive energy problem ... see Brill and Deser [1978], Choquet/Bruhat, Fischer and Marsden [1979], Deser and Teitelboim [1976] and Schoen and Yau [1979, 1980].

(ii) In the space of solutions, the kernel of the symplectic form coincides with the infinitesimal gauge transformations (this follows from Moncrief's decomposition). Therefore, one can construct the space of true degrees of freedom, the quotient of $\mathscr{E}$ by the gauge group. Using Marsden-Weinstein [1974], one proves that this quotient is a smooth symplectic manifold near points where $\mathscr{E}$ is smooth. This leaves open the question: what is $\mathscr{E}$/(gauge group) like near points of symmetry, where $\mathscr{E}$ is singular?

(iii) How should one treat the Schwarzschild solution in the context of linearization stability? Do singularities in the space of solutions affect spacetime singularities in the sense of Hawking and Penrose? Do they affect Cauchy horizons?

4. Bifurcations of Momentum Maps

The role of the constraint equations as the zero set of the Noether conserved *quantity* of the gauge group leads one to investigate zero sets of the conserved quantities associated with symmetry groups rather generally. One goal is to begin answering question (i) in the previous section. This topic is of interest not only in relativistic field theories, but in classical mechanics too. For example the set of points in the phase space for n particles in $\mathbb{R}^3$ corresponding to zero total angular momentum is an interesting and complicated set, even for $n = 2$!

We shall present just a hint of the relationship between singularities and symmetries. The full story is a long one; one finally ends up with an answer similar to that in vacuum relativity. We refer to Arms, Marsden and Moncrief [1980] for additional details.

First we need a bit of notation (see Abraham and Marsden [1978], Chapter 4). Let M be a manifold and let a Lie group G act on M. Associated to each element ξ in the Lie algebra $\mathfrak{G}$ of G, we have a vector field ξ_M naturally induced on M. We shall denote the action by $\Phi : G \times M \to M$ and we shall write $\Phi_g : M \to M$ for the transformation of M associated with the group element $g \in G$. This $\xi_M(x) = \frac{d}{dt}\,\Phi_{\exp(t\xi)}(x)|_{t=0}$.

Now let (P, ω) be a symplectic manifold, so ω is a closed (weakly) non-degenerate two-form on P and let Φ be an action of a Lie group G on P. Assume the action is symplectic: i.e. $\Phi_g^*\omega = \omega$ for all $g \in G$. A *momentum mapping* is a smooth mapping $J : P \to \mathfrak{G}^*$ such that

$$\langle dJ(x) \cdot v_x, \xi \rangle = \omega\big(\xi_P(x), v_x\big)$$

for all $\xi \in \mathfrak{G}$, $v_x \in T_xP$ where $dJ(x)$ is the derivative of J at x, regarded as a linear map of T_xP to $\mathfrak{G}^*$ and $\langle , \rangle$ is the natural pairing between $\mathfrak{G}$ and $\mathfrak{G}^*$.

A momentum map is *Ad*-equivariant* when the following diagram commutes for each $g \in G$:

$$\begin{tikzcd} P \arrow[r, "\Phi_g"] \arrow[d, "J"'] & P \arrow[d, "J"] \\ \mathfrak{G}^* \arrow[r, "\mathrm{Ad}^*_{g^{-1}}"'] & \mathfrak{G}^* \end{tikzcd}$$

where $\mathrm{Ad}^*_{g^{-1}}$ denotes the co-adjoint action of G on $\mathfrak{G}^*$. If J is Ad* equivariant, we call (P, ω, G, J) a *Hamiltonian G-space.*

Momentum maps represent the (Noether) conserved quantities associated with symmetry groups acting on phase space. This topic is of course a very old one, but it is only with more recent work of Souriau and Kostant that a deeper understanding has been achieved.

See Fischer and Marsden [1979] for the sense in which the map Φ described in § 2 is the momentum map associated with the group of diffeomorphisms of spacetime. See Moncrief [1977] and Arms [1979] for the corresponding result for gauge theory.

Let S_{x_0} = (the component of the identity of) $\{g \in G \mid gx_0 = x_0\}$, called the symmetry group of x_0. Its Lie algebra is denoted $\mathfrak{s}_x$, so

$$\mathfrak{s}_{x_0} = \{\xi \in \mathfrak{G} \mid \xi_P(x_0) = 0\}.$$

Let (P, ω, G, J) be a Hamiltonian G-space. If $x_0 \in P$, $\mu_0 = J(x_0)$ and if

$$dJ(x_0) : T_x P \to \mathfrak{G}^*$$

is surjective (with split kernel), then locally $J^{-1}(\mu_0)$ is a manifold and $\{J^{-1}(\mu) \mid \mu \in \mathfrak{G}^*\}$ forms a regular local foliation of a neighbourhood of x_0. Thus, when $dJ(x_0)$ fails to be surjective, the set of solutions of $J(x) = 0$ could fail to be a manifold.

Theorem. $dJ(x_0)$ is surjective if and only if $\dim S_{x_0} = 0$; i.e. $\mathfrak{s}_{x_0} = \{0\}$.

Proof. $dJ(x_0)$ fails to be surjective if there is a $\xi \neq 0$ such that $\langle dJ(x_0) \cdot v_{x_0}, \xi\rangle = 0$ for all $v_{x_0} \in T_{x_0}P$. From the definition of momentum map, this is equivalent to $\omega_{x_0}(\xi_P(x_0), v_{x_0}) = 0$ for all v_{x_0}. Since ω_{x_0} is non-degenerate, this is, in turn equivalent to $\xi_P(x_0) = 0$; i.e. $\mathfrak{s}_{x_0} \neq \{0\}$.

One then goes on to study the structure of $J^{-1}(\mu_0)$ when x_0 *has* symmetries, by investigating second order conditions and using methods of bifurcation theory. It turns out that, as in relativistic field theories, $J^{-1}(\mu_0)$ has quadratic singularities characterized by the vanishing of second order conditions. The connection is not an accident since the structure of the space of solutions of a relativistic field theory is determined by the vanishing of the momentum map associated with the gauge group of that theory.

Some open problems for 4.

(i) All the results obtained so far on spaces of solutions are local. What is the global structure? Is there a global Morsetype theory for momentum maps?

(ii) Much current work on Yang-Mills fields and the twistor program for gravity utilize a Euclideanized viewpoint. Some routine calculations show that in such a context the connection between symmetries and bifurcations is lost. (In particular, the symmetries discussed by Rebbi and Jackiw [1976] are *not* related to Euclidean linearization instabilities.) What has become of the difficulties with perturbation series and quantization encountered in the Lorentz context?

(iii) Bifurcation theory exploits connections between symmetry and bifurcation to study phenomena like pattern formation. See for example, Golubitsky and Schaeffer [1979] and Sattinger [1980]. Can one use this theory in relativity to study physical consequences of breaking the symmetry of a solution of a relativistic field theory?

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