4. The initial value problem and the dynamical formulation of general relativity

ARTHUR E. FISCHER AND JERROLD E. MARSDEN†

In this chapter we discuss some of the inter-relationships between the initial value problem, the canonical formalism, linearization stability and the space of gravitational degrees of freedom. In the last decade, these topics have experienced a resurgence of interest as more advanced mathematical methods and viewpoints have begun to show the intimate relationships among these topics. At present, the literature regarding these areas of general relativity is a rapidly expanding body of knowledge.

Our purpose here is to present the current state of affairs from our own point of view. We shall use geometric methods developed by the authors to establish various connections between the above–mentioned topics. The main tools we shall use to develop this material are nonlinear functional analysis, an adjoint formalism for Hamiltonian field theories, and infinite-dimensional symplectic geometry. As we shall see, these tools and the topics we shall consider are naturally related.

For a more complete picture of the current state of affairs, the reader is urged to consult Choquet-Bruhat (1962), Arnowitt, Deser and Misner (1962), Hawking and Ellis (1973), Misner, Thorne and Wheeler (1973), Hanson, Regge and Teitelboim (1976), Kuchař (1976, 1977), Müller zum Hagen and Seifert (1976) and Choquet-Bruhat and York (1979).

Section 4.1 develops the Hamiltonian formalism for the dynamics of general relativity, usually called the ADM (Arnowitt, Deser and Misner) formalism. This is done using invariant concepts and the adjoint formalism developed by the authors. We show how to write the Einstein dynamical system explicitly in the compact form

$$\text{Evolution equations} \quad \begin{pmatrix} \dfrac{\partial g}{\partial \lambda} \\[2ex] \dfrac{\partial \pi}{\partial \lambda} \end{pmatrix} = J \circ [D\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix},$$

$$\text{Constraint equations} \quad \Phi(g, \pi) = 0.$$

† Research for this paper was partially supported by the National Science Foundation, USA, and the Science Research Council, UK.

This form of the equations is useful in understanding linearization stability and the space of gravitational degrees of freedom. We sketch how this adjoint formalism can be extended to all field theories which are minimally coupled to gravity.

The adjoint formalism leads naturally to the study of the constraint manifold in section 4.2; the main result in this section tells which points are manifold (regular) points and which are bifurcation (singular) points. We also show (using the adjoint formalism) that the constraint submanifold is in involution under the dynamical equations. The equations used to establish this result are equivalent to the Dirac canonical commutation relations.

With the dynamical formalism in hand, we discuss existence, uniqueness, and stability for the Cauchy problem in sections 4.3 and 4.4. In section 4.3 we summarize the general theory of hyperbolic initial value problems that we shall need for relativity. We give an abstract approach which gives as special cases existence and uniqueness results for first-order symmetric hyperbolic systems, second-order hyperbolic systems, or combinations of these systems. The theorems we present yield the sharpest known results with regard to differentiability. When applied in section 4.5, these theorems give the sharpest results regarding the existence and uniqueness theorems for the Cauchy problem of the empty space field equations (theorems 4.23 and 4.27). Some remarks are made to show how the abstract theory is applied to fields coupled to gravity.

Although considerable progress in the initial value problem has been made, the basic open problem of relating dynamical singularities (non-existence of 'all-time' solutions to the evolution equations) to singularities in the Hawking–Penrose sense remains unsolved.

Section 4.5 combines sections 4.2 and 4.4 to give conditions under which first-order perturbation theory is and is not valid and shows that perturbation series must be readjusted to be made consistent whenever a Killing vector is present. Necessary second-order conditions are given for a perturbation to be integrable. These results are due to the joint work of V. Moncrief and the authors.

Finally, section 4.6 discusses the elimination of gauges by a general reduction procedure for Hamiltonian systems. An application of this general procedure is then used to show that the space of gravitational degrees of freedom is generically an infinite-dimensional symplectic manifold. Thus, generically, the set of empty space geometries is an infinite-dimensional gravitational phase space without singularities. Our general formalism is also applicable to fields minimally coupled to gravity, and with little extra effort it can also be shown that the space of degrees of freedom for fields and gravity is also generically a symplectic manifold.

For more information regarding the topics presented here, the reader may consult Arms (1977*a*, *b*), Arms, Fischer and Marsden (1975), Choquet-Bruhat, Fischer and Marsden (1978), Fischer and Marsden (1978*a*, *b*, *c*), Fischer, Marsden and Moncrief (1978), Hughes, Kato and Marsden (1976), Kato (1977), Marsden and Weinstein (1974), Moncrief (1975*a*, *b*, *c*, 1976, 1977) and Weinstein (1977).

The authors thank J. Arms, Y. Choquet-Bruhat, K. Kuchař, V. Moncrief, R. Palais, R. Sachs, and A. Taub for their helpful advice and S. Hawking and W. Israel for their kind invitation to contribute this chapter.

4.1 Canonical formalism

We begin by recalling the four-dimensional Lagrangian formalism in classical field theory for fields coupled to gravity. Following this we treat the '3+1' or dynamical approach.

The notation is as follows: V_4 is a smooth 4-manifold with connected, oriented, paracompact and Hausdorff included in the term 'manifold'. TV_4 denotes its tangent bundle. We also let

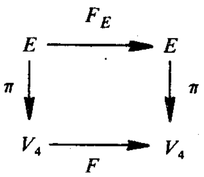
$L(V_4)=$ all smooth Lorentz metrics with signature $(-+++)$;

$S_2(V_4)=$ all smooth 2-covariant symmetric tensor fields on V_4.

Now let E be a vector bundle over V_4 with projection $\pi: E\to V_4$ and let its C^∞ sections be denoted $C^\infty(E)$. Often we take $E=T^r_s(V_4)$, the bundle of tensors with r contravariant indices and s covariant indices. However, it is important not to restrict one's attention merely to tensor field theories or else important field theories, such as Yang–Mills fields, will be excluded; see for example Hermann (1975), Arms (1977*b*), and references therein. Strictly speaking, Yang–Mills fields require the use of an affine bundle (bundle of connections on a principal bundle over V_4), but this does not affect the formalism in any significant way.

In coordinates, we write the components of $\varphi\in C^\infty(E)$ as φ_A where A denotes a set of multi-indices.

Let $\mathscr{D}(V_4)$ denote the (orientation-preserving) diffeomorphisms of V_4. For 'natural bundles' any $F\in\mathscr{D}(V_4)$ extends functorially to a bundle diffeomorphism F_E: $E \to E$ covering F; i.e., the diagram



commutes and $(F \circ G)_E = F_E \circ G_E$. For $E = T^r_s(V_4)$, F_E is the usual transformation of tensors under F. Then *pull back* by F is defined on sections of E, and is given by

$$F^*: C^\infty(E) \to C^\infty(E); \qquad \varphi \mapsto F_E^{-1} \circ \varphi \circ F = F^*\varphi$$

and its inverse, *push forward*, is defined by

$$F_*\varphi = F_E \circ \varphi \circ F^{-1}.$$

For the bundles associated with Yang–Mills fields, one also has transformation by an infinite-dimensional gauge group in addition to the notion of pull back and push forward by $\mathscr{D}(V_4)$.

Note that E may be a Whitney sum $E_1 \oplus E_2 \oplus \cdots E_k$ for k types of fields, so that our formalism is appropriate for interacting fields.

Let Ω denote the bundle of densities over V_4 (i.e., 4-forms) and write E^* for the *dual bundle* over V_4 whose fiber at x is

$$E_x^* \otimes \Omega_x$$

where E_x^* represents the vector space dual to E_x. Thus E^* is a bundle of vector densities over V_4. For example, if $E = T^r_s(V_4)$, $E^* = T^s_r(V_4) \otimes \Omega$ is the bundle of tensor densities of type $\binom{s}{r}$.

We have a natural L_2-pairing between $C^\infty(E)$ and $C^\infty(E^*)$ given by

$$(\varphi, \psi \otimes \mathrm{d}\mu)_{L_2} = \int_{V_4} \psi(\varphi)\, \mathrm{d}\mu,$$

where $\psi(\varphi)\,\mathrm{d}\mu = \psi(\varphi) \otimes \mathrm{d}\mu$, and we are assuming $\psi(\varphi)$ is $\mathrm{d}\mu$-integrable. We speak of $C^\infty(E^*)$ as the *natural L_2 dual of $C^\infty(E)$*.

Let E and F be two bundles over V_4 and A: $C^\infty(E) \to C^\infty(F)$ be a linear operator. The *natural adjoint* of A is defined by

$$A^*: C^\infty(F^*) \to C^\infty(E^*), \qquad (A^*(\tilde{\psi}), \varphi)_{L_2} = (\tilde{\psi}, A\varphi)_{L_2}.$$

for $\tilde{\psi} \in C^\infty(F^*)$, $\varphi \in C^\infty(E)$. We tacitly assume A^* exists.

If A is a differential operator, A^* will be computed in the usual way by integration by parts to give the adjoint differential operator. In general, A^* can be interpreted in the sense of unbounded operators (Kato, 1966).

For a bundle E over V_4 with dual bundle E^*, we take E to be the dual bundle of E^* so that $(E^*)^* = E$. Thus, in the example of $E = T^r_s(V_4)$, $E^* = T^s_r(V_4) \otimes \Omega$, and $(E^*)^* = T^r_s(V_4)$. Thus, with this convention, if

$$A: C^\infty(E) \to C^\infty(F^*),$$

then

$$A^*: C^\infty(F) \to C^\infty(E^*).$$

Consider now a Lagrangian density for a field theory coupled to gravity:

$$\mathscr{L}: L(V_4) \times C^\infty(E) \to C^\infty_{\mathrm{d}}(V_4),$$

where $C^\infty_{\mathrm{d}}(V_4) = \Omega$ is the bundle of scalar densities over V_4. Write $\mathscr{L}(g, \varphi) = \mathscr{L}_{\mathrm{grav}}(g) + \mathscr{L}_{\mathrm{fields}}(g, \varphi)$ where $\mathscr{L}_{\mathrm{grav}}(g) = (1/16\pi) R(g)\, \mathrm{d}\mu(g)$ and where $R(g)$ is the scalar curvature of g and $\mathrm{d}\mu(g) = (-\det g_{\alpha\beta})^{1/2}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ is the volume element associated with $g \in L(V_4)$; later we shall designate such a g as ${}^{(4)}g$.

If we demand that the action integral

$$S(g, \varphi) = \int_{\mathscr{D}} [\mathscr{L}_{\mathrm{grav}}(g) + \mathscr{L}_{\mathrm{fields}}(g, \varphi)]\, \mathrm{d}\mu(g)$$

be stationary for any bounded open region $\mathscr{D} \subset V_4$ with smooth boundary and for any variation h of g and variation ψ of φ vanishing on the boundary, we get

$$0 = \int_{\mathscr{D}} [\mathrm{D}\mathscr{L}_{\mathrm{grav}}(g) \cdot h + \mathrm{D}_g \mathscr{L}_{\mathrm{fields}}(g, \varphi) \cdot h + D_\varphi \mathscr{L}_{\mathrm{fields}}(g, \varphi) \cdot \psi]\, \mathrm{d}\mu(g)$$

for all h, ψ vanishing on $\partial\mathscr{D}$, where D, D_g, D_φ denote the (Fréchet) derivatives, and partial derivatives with respect to g and φ, respectively. Note that the variation h is in $S_2(V_4)$, the space of symmetric 2-covariant tensor fields on V_4, and ψ is in $C^\infty(E)$.

In terms of natural adjoints, this condition becomes the Euler–Lagrange equations:

$$[\mathrm{D}\mathscr{L}_{\mathrm{grav}}(g)]^* \cdot 1 + [\mathrm{D}_g \mathscr{L}_{\mathrm{fields}}(g, \varphi)]^* \cdot 1 = 0$$

and

$$[\mathrm{D}_\varphi \mathscr{L}_{\mathrm{fields}}(g, \varphi)]^* \cdot 1 = 0$$

where 1 is the constant function 1 in the space of real-valued functions, the dual space to the densities $C_d^\infty(V_4)$. These equations are equivalent to the usual way of writing the Euler–Lagrange equations (if $\mathscr{L}_{\text{fields}}$ are assumed to depend on the k-jet of g, φ):

$$\frac{\delta \mathscr{L}_{\text{grav}}}{\delta g}+\frac{\delta \mathscr{L}_{\text{fields}}}{\delta g}=0$$

$$\frac{\delta \mathscr{L}_{\text{fields}}}{\delta \varphi}=0.$$

Now, as in Lichnerowicz (1961), we have

$$\mathrm{D}R(g)\cdot h=\Delta \operatorname{tr} h+\delta\delta h-h\cdot \operatorname{Ric}(g)$$

where

$$\Delta = \text{Laplace–Beltrami operator on scalars; } \Delta f=-f_{;\alpha}{}^{;\alpha}$$

$$\operatorname{tr} = \text{trace; } \operatorname{tr} h = h^{\alpha}{}_{\alpha}$$

$$\delta h = -\operatorname{div} h = -h_{\alpha}{}^{\beta}{}_{;\beta}$$

$$\delta\delta h = \text{double divergence} = h^{\alpha\beta}{}_{;\alpha;\beta}$$

$$\operatorname{Ric}(g) = \text{Ricci tensor of } g = R_{\alpha\beta}$$

and

$$\mathrm{D}[\mathrm{d}\mu(g)]\cdot h=\tfrac{1}{2}(\operatorname{tr} h)\,\mathrm{d}\mu(g).$$

Thus,

$$\mathrm{D}\mathscr{L}_{\text{grav}}(g)\cdot h=\frac{1}{16\pi}[\Delta \operatorname{tr} h+\delta\delta h-\operatorname{Ein}(g)\cdot h]\,\mathrm{d}\mu(g),$$

where

$$\operatorname{Ein}(g)=\operatorname{Ric}(g)-\tfrac{1}{2}gR(g)$$

is the Einstein tensor of g (i.e., $G_{\alpha\beta}=R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R$). Since the integral of $\Delta \operatorname{tr} h+\delta\delta h$ vanishes for variations h that vanish on $\partial\mathscr{D}$, it follows that

$$[\mathrm{D}\mathscr{L}_{\text{grav}}(g)]^*\cdot 1=-\frac{1}{16\pi}[\operatorname{Ein}(g)]^{\#}\,\mathrm{d}\mu(g)$$

where $^{\#}$ means indices raised by g.

We let (see Hawking and Ellis, 1973, section 3.3)

$$\mathscr{T}(g,\varphi)=2\frac{\delta \mathscr{L}_{\text{fields}}}{\delta g}=2[\mathrm{D}_g\mathscr{L}_{\text{fields}}(g,\varphi)]^*\cdot 1\in S^2_d(V_4),$$

where $S_d^2(V_4) = S^2(V_4) \otimes \Omega$ denotes the space of 2-contravariant symmetric tensor densities on V_4.

Let $T(g, \varphi) = \mathcal{T}(g, \varphi)^*$, the dual tensor in $S_2(V_4)$ induced by the metric. Thus, $\mathcal{T} = T^{\#} \, d\mu(g)$ and T is the usual symmetric energy–momentum tensor associated with $\mathcal{L}_{\text{fields}}(g, \varphi)$.

The field equations now read:

$$\text{Ein}\,(g) = 8\pi T(g, \varphi) \quad (\text{i.e., } G_{\mu\nu} = 8\pi T_{\mu\nu})$$

and

$$\frac{\delta \mathcal{L}_{\text{fields}}}{\delta \varphi} = 0.$$

If one wishes to obtain field theories which are well posed, there are severe restrictions on the possible choices of $\mathcal{L}_{\text{fields}}$. For example, if we have a tensor theory and $\mathcal{L}_{\text{fields}}$ depends on derivatives of g, e.g. on the covariant derivative $\nabla\varphi$ of φ, then T in general will depend on second derivatives of g and on second derivatives of the fields φ. Likewise the equations for the fields will depend on second derivatives of the metric as well as second derivatives of the fields. In such circumstances, one may not have a well-defined system of hyperbolic equations (see Kuchař, 1976). Thus, one usually requires *minimal coupling*, i.e., $\mathcal{L}_{\text{fields}}$ depends only on point values of g. For the scalar, electrodynamic, and Yang–Mills fields, where $\mathcal{L}_{\text{fields}} = (1/16\pi) F \cdot F \, d\mu(g)$ and $F = dA + [A, A] =$ {curvature of a connection field A}, this presents no difficulties, as these systems are minimally coupled. For minimally coupled tensor field theories, one is able to classify the *natural* differential operators that may occur; see Palais (1959), Nijenhuis (1951) and Terng (1976).

Now we turn our attention to the Dirac-ADM dynamical formulation. We develop this material using modern symplectic geometry and an intrinsic version of the Dirac theory of constraints; see Abraham and Marsden (1978). Since gravitation plays a distinguished role, we shall discuss it first. Then we shall make some comments on the case of fields coupled to gravity.

As above, let V_4 be a four-dimensional manifold with Lorentzian metric ${}^{(4)}g$ which is oriented and time-oriented. We write ${}^{(4)}g$ to avoid confusion with Riemannian metrics g to be introduced later. Let M be a *compact* oriented three-dimensional manifold,† and let $i: M \to V_4$ be an

† The Hamiltonian formalism for the non-compact case is rather different. See Regge and Teitelboim (1974) and Choquet-Bruhat, Fischer and Marsden (1978). The existence and uniqueness theory discussed in sections 4.3 and 4.4 is valid in either case.

embedding of M such that the embedded manifold $i(M)=\Sigma$ is spacelike; i.e., the pull back $i^*({}^{(4)}g)=g$ is a Riemannian metric on M. Let $C^\infty_{\text{space}}(M; V, {}^{(4)}g)$ denote the set of all such spacelike embeddings. As in Ebin and Marsden (1970), this is a smooth manifold. Let k denote the second fundamental form of the embedding, defined at $m \in M$, for X, $Y \in T_mM$, by the usual formula

$$k_m(X, Y) = -{}^{(4)}g \circ i(m) \cdot ((T_m i \cdot Y), {}^{(4)}\nabla_{(T_m i \cdot X)}{}^{(4)}Z_\Sigma \circ i(m))$$

where ${}^{(4)}Z_\Sigma \circ i(m)$ is the forward-pointing unit timelike normal to Σ at $i(m)$. Thus $k_{ij} = -Z_{i;j}$ (where ';' denotes covariant differentiation using ${}^{(4)}g$; covariant differentiation using g is denoted with a vertical bar).

Let $\pi = \pi' \otimes \mathrm{d}\mu(g)$ be a 2-contravariant tensor density, whose tensor part π' is defined by $\pi' = [(\mathrm{tr}\, k)g - k]^\#$, where $^\#$ indicates the contravariant form of a covariant tensor with indices raised by g; similarly, $^\flat$ denotes the covariant form of a contravariant tensor. In the Hamiltonian formulation of Arnowitt, Deser and Misner, k plays the role of a velocity variable and π is its canonical momentum. Note that $\pi^{\text{ours}} = \pi^{\text{ADM}}\mathrm{d}^3x$. When we discuss the space of gravitational degrees of freedom in section 4.6, it is useful to know that if $(V_4, {}^{(4)}g)$ is globally hyperbolic with a Cauchy surface diffeomorphic to M, then any spacelike embedding of M in V_4 is also a Cauchy surface (see Hawking and Ellis, 1973; and Budic, Isenberg, Lindblom and Yasskin, 1978).

Now suppose we have a curve in $C^\infty_{\text{space}}(M; V_4, {}^{(4)}g)$; i.e., a curve i of spacelike embeddings of M into $(V_4, {}^{(4)}g)$. The λ-derivative of this curve defines a one-parameter family of vector fields ${}^{(4)}X_{\Sigma_\lambda}$ on the embedded hypersurfaces by the equation

$$\frac{\mathrm{d}i_\lambda}{\mathrm{d}\lambda} = {}^{(4)}X_{i_\lambda} = {}^{(4)}X_{\Sigma_\lambda} \circ i_\lambda : M \to TV_4$$

(see figure 4.1). The normal and tangential projections of ${}^{(4)}X_{\Sigma_\lambda}$ define a curve of functions $N_\lambda = {}^{(4)}X_\perp : M \to \mathbb{R}$ and vector fields ${}^{(4)}X_\parallel = X_\lambda : M \to TM$ on M by the equation

$${}^{(4)}X_{\Sigma_\lambda} \circ i_\lambda(m) = {}^{(4)}X_\perp(\lambda, m){}^{(4)}Z_{\Sigma_\lambda} \circ i_\lambda(m) + T_m i_\lambda \cdot {}^{(4)}X_\parallel(\lambda, m)$$

where ${}^{(4)}Z_{\Sigma_\lambda}$ is the forward-pointing unit timelike normal to Σ_λ. If $N_\lambda > 0$, then the map

$$F : I \times M \to V_4; \quad (\lambda, m) \mapsto i_\lambda(m)$$

is a diffeomorphism of $I \times M$ onto a tubular neighborhood of $i_0(M) = \Sigma_0$, if the interval $I = (-\beta, \beta)$ is chosen small enough. In this case, we call

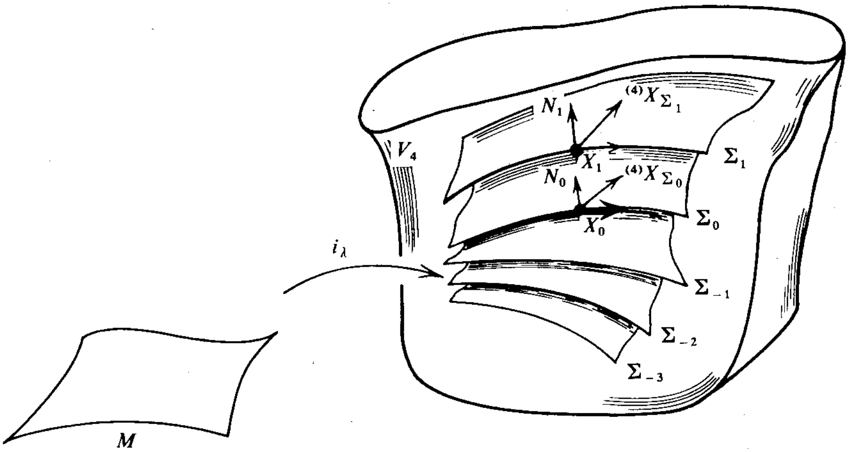


Figure 4.1. Spacelike embeddings of M into V_4 with the normal and tangential decomposition of the generator $^{(4)}X_{\Sigma_\lambda}$.

either the curve i_λ or the embedded spacelike hypersurfaces $\Sigma_\lambda = i_\lambda(M)$ a *slicing* of V_4.

The functions N_λ and the vector fields X_λ are the *lapse functions* and *shift vector fields* of Arnowitt, Deser and Misner (1962) and Wheeler (1964).

Using $F: I \times M \to V_4$ as a coordinate system for a tubular neighborhood of Σ_0 in V_4, coordinates (x^i), $i = 1, 2, 3$ on M, and $(x^\alpha) = (\lambda, x^i)$, $\alpha = 0, 1, 2, 3$ as coordinates on $I \times M$, the pulled back metric $F^{*(4)}g$ is

$$(F^{*(4)}g)_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -(N^2 - X_i X^i)\,\mathrm{d}\lambda^2 + 2X_i\,\mathrm{d}x^i\,\mathrm{d}\lambda + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

where $g_{ij} = (g_\lambda)_{ij}$, and $g_\lambda = i_\lambda^{*}\,{}^{(4)}g$.

Let k_λ be the curve of second fundamental forms for the embedded hypersurfaces $\Sigma_\lambda = i_\lambda(M)$, and let π_λ be their associated canonical momenta.

The following theorem contains the basic geometrodynamical equations due to Lichnerowicz (1944), Choquet-Bruhat (1952), Dirac (1959, 1964), and Arnowitt, Deser and Misner (1962).

Theorem 4.1

Let the vacuum Einstein field equations Ein $(^{(4)}g) = 0$ *hold on* V_4. *Then for each one-parameter family of spacelike embeddings* $\{i_\lambda\}$ *of* V_4, *the induced metrics* g_λ *and momentum* π_λ *on* Σ_λ *satisfy the following* *equations*:

$$\text{Evolution equations}\quad \begin{cases} \dfrac{\partial g}{\partial \lambda} = 2N[(\pi') - \tfrac{1}{2}g(\operatorname{tr}\pi')] + L_X g, \\[2ex] \dfrac{\partial \pi}{\partial \lambda} = -2N[\pi' \times \pi' - \tfrac{1}{2}(\operatorname{tr}\pi')\pi']\,\mathrm{d}\mu(g) \end{cases}$$

$$+\tfrac{1}{2}Ng^{\#}[\pi' \cdot \pi' - \tfrac{1}{2}(\operatorname{tr}\pi')^2]\,\mathrm{d}\mu(g)$$

$$-N[\operatorname{Ric}(g) - \tfrac{1}{2}R(g)g]^{\#}\,\mathrm{d}\mu(g)$$

$$+(\operatorname{Hess} N + g\Delta N)^{\#}\,\mathrm{d}\mu(g) + L_X\pi,$$

$$\text{Constraint equations}\quad \begin{cases} \mathcal{H}(g, \pi) = [\pi' \cdot \pi' - \tfrac{1}{2}(\operatorname{tr}\pi')^2 - R(g)]\,\mathrm{d}\mu(g) = 0, \\ \mathcal{J}(g, \pi) = 2(\delta_g \pi) = -2\pi_i{}^j{}_{|j} = 0. \end{cases}$$

Conversely, if i_λ is a slicing of $(V_4, {}^{(4)}g)$ such that the above evolution and constraint equations hold, then ${}^{(4)}g$ satisfies the (empty space) field equations.

Our notation in the theorem is as follows: $(\pi' \times \pi')^{ij} = (\pi')^{ik}(\pi')_k{}^j$; $\pi' \cdot \pi' = (\pi')^{ij}(\pi')_{ij}$; $\operatorname{Hess} N = N_{|i|j}$; $\Delta N = -g^{ij}N_{|i|j}$; and $L_X\pi = (L_X\pi')\,\mathrm{d}\mu(g) + \pi'(\operatorname{div} X)\,\mathrm{d}\mu(g)$ is the Lie derivative of the tensor density $\pi = \pi'\,\mathrm{d}\mu(g)$; note, $L_X\,\mathrm{d}\mu(g) = (\operatorname{div} X)\,\mathrm{d}\mu(g)$. The Ricci tensor $R_{\mu\nu}$ of ${}^{(4)}g$ is denoted $\operatorname{Ric}({}^{(4)}g)$ and that of g by $\operatorname{Ric}(g)$; $R(g)$ is the scalar curvature of g. We write $\operatorname{Ein}(g) = \operatorname{Ric}(g) - \tfrac{1}{2}R(g)g$, the Einstein tensor of g.

A sketch of the proof of theorem 4.1 is given after theorem 4.3.

The twelve first-order evolution equations for (g, π) correspond to the six second-order equations ${}^{(4)}G_{ij} = 0$, while the other four Einstein equations ${}^{(4)}G^{00} = 0$ and ${}^{(4)}G^0{}_i = 0$ appear as the constraint equations. More explicitly, in coordinates determined by a slicing i_λ, ${}^{(4)}Z_\Sigma$ has components ${}^{(4)}Z_\alpha = (-N, 0)$. If we define the 'perpendicular–perpendicular' and 'perpendicular–parallel' projections of the Einstein tensor by

$${}^{(4)}G_{\perp\perp} = Z_\alpha Z_\beta {}^{(4)}G^{\alpha\beta} = N^2\,{}^{(4)}G^{00}$$

and

$${}^{(4)}G^{\perp}{}_i = -Z_\alpha {}^{(4)}G^\alpha{}_i = N\,{}^{(4)}G^0{}_i,$$

then

$$\mathcal{H}(g, \pi) = -2\,{}^{(4)}G_{\perp\perp}\,\mathrm{d}\mu(g)$$

and

$$\mathcal{J}(g, \pi)_i = 2\,{}^{(4)}G^{\perp}{}_i \, \mathrm{d}\mu(g).$$

The evolution equations of this theorem are well posed, as is shown in section 4.4.

In the formulation of theorem 4.1, the lapse and shift are regarded as freely specifiable. In the 'thin sandwich' formulation, one regards g and $\dot{g}$ as Cauchy data, expresses π as a function of $(\dot{g}, N, X)$ and solves for N and X from the constraint equations

$$\mathcal{H}(g, \pi(\dot{g}, N, X)) = 0$$

$$\mathcal{J}(g, \pi(\dot{g}, N, X)) = 0;$$

see Misner, Thorne and Wheeler (1974). Upon linearizing, it is easy to see that this is not an elliptic system, so even if it is solvable, there will be some technical problems; in particular, regularity must fail. Thus, the thin sandwich formulation is rejected by most workers. For other difficulties with the thin sandwich formulation, see Christodoulou and Francaviglia (1978).

It is important to recognize various combinations of terms in the ADM evolution equations as Lie derivatives, and we have done so in the way theorem 4.1 is written. It is also useful to write the quadratic algebraic part of $\partial\pi/\partial\lambda$ as

$$S_g(\pi, \pi) = -2\{\pi' \times \pi' - \tfrac{1}{2}(\mathrm{tr}\,\pi')\pi'\}\,\mathrm{d}\mu(g) + \tfrac{1}{2}g^{\#}\{\pi' \cdot \pi' - \tfrac{1}{2}(\mathrm{tr}\,\pi')^2\}\,\mathrm{d}\mu(g).$$

This is the spray of the DeWitt metric, i.e., the terms in the evolution equation quadratic in π' (see below, and Fischer and Marsden, 1972*a*). Thus the terms in the evolution equation for π may be interpreted as follows:

$$\frac{\partial \pi}{\partial \lambda} = NS_g(\pi, \pi) - N\,\mathrm{Ein}\,(g)^{\#}\,\mathrm{d}\mu(g) + (\mathrm{Hess}\,N + g\,\Delta N)^{\#}\,\mathrm{d}\mu(g) + L_X\pi.$$

geodesic spray of the DeWitt metric	forcing term of the scalar curvature potential	'tilt' term due to non-constant N	'shift' term due to nonzero shift

See DeWitt (1967), Fischer and Marsden (1972*a*), and Kuchař (1976) for more information regarding the geometric interpretation of this equation.

In order to understand these equations in terms of a symplectic structure on a cotangent bundle, we must introduce the following spaces. Let $\mathcal{M}$ denote the space of C^∞ Riemannian metrics on M, and $\mathcal{D}=\mathcal{D}(M)$ the diffeomorphism group of M. We let $\mathcal{M}^{s,p}$ with $s>n/p$ denote the Riemannian metrics of Sobolev class $W^{s,p}$; the diffeomorphisms and other maps and tensors of class $W^{s,p}$ being denoted similarly. For ease of notation, however, we shall restrict to the C^∞ case in this section.

Let $T\mathcal{M}\approx\mathcal{M}\times S_2$ denote the tangent bundle of $\mathcal{M}$, where, as above, S_2 is the space of C^∞ 2-covariant symmetric tensor fields on M, and S^2_{d} is the space of C^∞ 2-contravariant symmetric tensor densities on M. Define $T^*\mathcal{M}\approx\mathcal{M}\times S^2_{\mathrm{d}}=\{(g,\pi)|g\in\mathcal{M},\ \pi\in S^2_{\mathrm{d}}\}$. We shall refer to $T^*\mathcal{M}$ as the 'L_2-cotangent bundle to $\mathcal{M}$'. For $k\in T_g\mathcal{M}\approx S_2$, $\pi\in T^*_g\mathcal{M}\approx S^2_{\mathrm{d}}$, there is a natural L_2-pairing

$$(\pi,k)_{L_2}=\int_M \pi\cdot k,$$

as explained above. Thus $T^*\mathcal{M}$ as defined is a subbundle of the 'true' contangent bundle. Since $T^*\mathcal{M}$ is open in $S_2\times S^2_{\mathrm{d}}$, the tangent space of $T^*\mathcal{M}$ at $(g,\pi)\in T^*\mathcal{M}$ is $T_{(g,\pi)}(T^*\mathcal{M})\approx S_2\times S^2_{\mathrm{d}}$.

We now show that $T^*\mathcal{M}$ carries a natural symplectic structure in which the evolution equations of the theorem are Hamiltonian. In order to include the lapse function and shift vector field into this scheme, it is necessary to develop the notion of a generalized Hamiltonian system.

On $T^*\mathcal{M}$ we define the globally constant symplectic structure

$$\Omega=\Omega_{(g,\pi)}:\ T_{(g,\pi)}(T^*\mathcal{M})\times T_{(g,\pi)}(T^*\mathcal{M})\to\mathbb{R}$$

as follows: for $(h_1,\omega_1),(h_2,\omega_2)\in T_{(g,\pi)}(T^*\mathcal{M})=S_2\times S^2_{\mathrm{d}}$,

$$\Omega_{(g,\pi)}((h_1,\omega_1),(h_2,\omega_2))=\int_M \omega_2\cdot h_1-\omega_1\cdot h_2.$$

Let

$$J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}:\ S^2_{\mathrm{d}}\times S_2\to S_2\times S^2_{\mathrm{d}}$$

be defined by

$$(\omega,h)\mapsto J\begin{pmatrix}\omega\\ h\end{pmatrix}=\begin{pmatrix}h\\ -\omega\end{pmatrix},$$

so that

$$J^{-1}=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}:\ S_2\times S^2_{\mathrm{d}}\to S^2_{\mathrm{d}}\times S_2,\ (h,\omega)\mapsto(-\omega,h).$$

Then

$$\Omega((h_1, \omega_1), (h_2, \omega_2)) = \int_M \left\langle J^{-1}\begin{pmatrix} h_1 \\ \omega_1 \end{pmatrix}, (h_2, \omega_2) \right\rangle.$$

We shall return to J shortly.

Let $C^\infty = C^\infty(M; \mathbb{R})$ denote the smooth real-valued functions on M;

C^∞_d = smooth scalar densities on M;

$\mathscr{X}$ = smooth vector fields on M;

and

Λ^1_d = smooth one-form densities on M.

Consider the functions

$$\mathscr{H}: T^*\mathscr{M} \to C^\infty_\mathrm{d}; \quad (g, \pi) \mapsto \mathscr{H}(g, \pi) = [\pi' \cdot \pi' - \tfrac{1}{2}(\mathrm{tr}\, \pi')^2 - R(g)]\, \mathrm{d}\mu(g);$$

$$\mathscr{J} = 2\delta: T^*\mathscr{M} \to \Lambda^1_\mathrm{d}; \quad (g, \pi) \mapsto 2(\delta_g \pi) = -2\pi_i{}^j{}_{|j};$$

and

$$\Phi = (\mathscr{H}, \mathscr{J}): T^*\mathscr{M} \to C^\infty_\mathrm{d} \times \Lambda^1_\mathrm{d}; \quad (g, \pi) \mapsto (\mathscr{H}(g, \pi), \mathscr{J}(g, \pi)).$$

At this point it is necessary to compute the derivatives of $\mathscr{H}$, $\mathscr{J}$, and Φ and their natural adjoints. The results are collected in the following.

Proposition 4.2

Letting $(g, \pi) \in T^*\mathscr{M}$, $(h, \omega) \in T_{(g,\pi)}(T^*\mathscr{M}) = S_2 \times S^2_\mathrm{d}$ *and* $(N, X) \in C^\infty \times \mathscr{X}$, *the derivatives of* $\mathscr{H}$, $\mathscr{J}$, Φ

$$\mathrm{D}\mathscr{H}(g, \pi): S_2 \times S^2_\mathrm{d} \to C^\infty_\mathrm{d},$$

$$\mathrm{D}\mathscr{J}(g, \pi): S_2 \times S^2_\mathrm{d} \to \Lambda^1_\mathrm{d},$$

$$\mathrm{D}\Phi(g, \pi): S_2 \times S^2_\mathrm{d} \to C_\mathrm{d} \times \Lambda^1_\mathrm{d}$$

and their natural adjoints

$$[\mathrm{D}\mathscr{H}(g, \pi)]^*: C^\infty \to S^2_\mathrm{d} \times S_2,$$

$$[\mathrm{D}\mathscr{J}(g, \pi)]^*: \mathscr{X} \to S^2_\mathrm{d} \times S_2,$$

$$[\mathrm{D}\Phi(g, \pi)]^*: C^\infty \times \mathscr{X} \to S^2_\mathrm{d} \times S_2$$

are given as follows:

$$\mathrm{D}\mathcal{H}(g,\pi)\cdot(h,\omega)=-S_g(\pi,\pi)\cdot h+[\mathrm{Ein}\,(g)\cdot h-(\delta\delta h+\Delta\,\mathrm{tr}\,h)]\,\mathrm{d}\mu(g)$$
$$+2[(\pi')-\tfrac{1}{2}(\mathrm{tr}\,\pi')g]\cdot\omega;$$

$$[\mathrm{D}\mathcal{H}(g,\pi)]^*\cdot N=\{[-NS_g(\pi,\pi)+(N\,\mathrm{Ein}\,(g)-(\mathrm{Hess}\,N+g\,\Delta N))^\sharp]$$
$$\otimes\mathrm{d}\mu(g),2N[(\pi')^\flat-\tfrac{1}{2}(\mathrm{tr}\,\pi')g]\};$$

$$\mathrm{D}\mathcal{J}(g,\pi)\cdot(h,\omega)=-2[\omega_i{}^j{}_{|j}+h_{ik}\pi^{kj}{}_{|j}+\pi^{jl}(h_{ij|l}-\tfrac{1}{2}h_{jl|i})];$$

$$[\mathrm{D}\mathcal{J}(g,\pi)]^*\cdot X=(-L_X\pi,L_Xg);$$

$$\mathrm{D}\Phi(g,\pi)\cdot(h,\omega)=(\mathrm{D}\mathcal{H}(g,\pi)\cdot(h,\omega),\mathrm{D}\mathcal{J}(g,\pi)\cdot(h,\omega));$$

and

$$[\mathrm{D}\Phi(g,\pi)]^*.(N,X)=[\mathrm{D}\mathcal{H}(g,\pi)]^*\cdot N+[\mathrm{D}\mathcal{J}(g,\pi)]^*\cdot X$$
$$=\{[-NS_g(\pi,\pi)+(N\,\mathrm{Ein}\,(g)-(\mathrm{Hess}\,N+g\,\Delta N))^\sharp]$$
$$\otimes\mathrm{d}\mu(g)-L_X\pi,2N[(\pi')^\flat-\tfrac{1}{2}(\mathrm{tr}\,\pi')g+L_Xg]\}.$$

The proof is a slightly long, but straightforward, computation.

As is shown in Arnowitt, Deser and Misner (1962), the evolution equations of theorem 4.1 are Hamilton's equations with Hamiltonian $N\mathcal{H}+X\cdot\mathcal{J}$, i.e.,

$$\frac{\partial g}{\partial\lambda}=\frac{\delta}{\delta\pi}(N\mathcal{H}+X\cdot\mathcal{J}),$$

$$\frac{\partial\pi}{\partial\lambda}=-\frac{\delta}{\delta g}(N\mathcal{H}+X\cdot\mathcal{J}).$$

Using the symplectic structure on $T^*\mathcal{M}$ defined by

$$J=\begin{pmatrix}0&I\\-I&0\end{pmatrix}:S_\mathrm{d}^2\times S_2\to S_2\times S_\mathrm{d}^2,\,(\omega,h)\mapsto J\begin{pmatrix}\omega\\h\end{pmatrix}=\begin{pmatrix}h\\-\omega\end{pmatrix},$$

and the correspondence

$$\left(\frac{\delta}{\delta g}(N\mathcal{H}+X\cdot\mathcal{J}),\frac{\delta}{\delta\pi}(N\mathcal{H}+X\cdot\mathcal{J})\right)=[\mathrm{D}\Phi(g,\pi)]^*\cdot\begin{pmatrix}N\\X\end{pmatrix},$$

the Hamiltonian equations in theorem 4.1 can be written in a very compact way.

Theorem 4.3

The Einstein system, defined by the evolution equations and constraint equations of theorem 4.1 can be written as

$$\text{Evolution equations} \quad \frac{\partial}{\partial\lambda}\begin{pmatrix} g \\ \pi \end{pmatrix} = J \circ [D\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix},$$

$$\text{Constraint equations} \quad \Phi(g, \pi) = (\mathscr{H}(g, \pi), \mathscr{J}(g, \pi)) = 0,$$

where (N, X) are the lapse function and shift vector field associated with the slicing, and where $[D\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}$ is given by proposition 4.2.

Sketch of proof of theorems 4.1 and 4.3. The Lagrangian density which generates the empty space Einstein equations is

$$\mathscr{L}_{\text{grav}}({}^{(4)}g) = \frac{1}{16\pi} R({}^{(4)}g)\, d\mu({}^{((4)}g)$$

where $d\mu({}^{(4)}g) = (-\det {}^{(4)}g)^{1/2}\, d^4x = N(\det g)^{1/2}\, d^3x\, d\lambda = N\, d\lambda\, d\mu(g)$. A computational part of the proof, which we shall not do, is to show that $\mathscr{L}_{\text{grav}}$ can be written in the following $(3+1)$-dimensional form (see equation 7-3.13 in Arnowitt, Deser and Misner, 1962, and equations 21–90 in Misner, Thorne and Wheeler, 1974)

$$\begin{aligned} 16\pi\mathscr{L}_{\text{grav}}({}^{(4)}g) &= NR({}^{(4)}g)\, d\mu(g)\, d\lambda \\ &= \left[\pi^{ij}\frac{\partial g_{ij}}{\partial\lambda} - N\mathscr{H}(g, \pi) - X \cdot \mathscr{J}(g, \pi)\right] d\lambda \\ &\quad -2[\pi^i{}_j X^j - \tfrac{1}{2}X^i \operatorname{tr}\pi + (\operatorname{grad} N)^i\, d\mu(g)]_{,i}\, d\lambda \\ &\quad -\left(\frac{\partial}{\partial\lambda}\operatorname{tr}\pi\right) d\lambda. \end{aligned}$$

Here i_λ is a slicing of V_4 so that V_4 can be identified with $I \times M$. Note that our $\pi = \pi'\, d\mu(g) = \pi'(\det g)^{1/2}\, d^3x = \pi^{\text{ADM}}\, d^3x$ contains the d^3x term to complete $(\det g)^{1/2}$ to a volume element on M. Similarly, the volume element $d\mu({}^{(4)}g)$ conains $d^4x = d^3x\, d\lambda$, explaining the overall multiplicative factor $d\lambda$.

Set $\beta = \beta^i = -2[\pi^i{}_j X^j - \frac{1}{2}X^i \operatorname{tr}\pi + (\operatorname{grad} N)^i\, d\mu(g)]$, a vector density on M; note that $\beta^i{}_{,i} = \beta^i{}_{|i} = \operatorname{div}\beta$. The action for gravity can be written as

$$16\pi S_{\text{grav}}({}^{(4)}g) = 16\pi \int_{V_4} \mathscr{L}_{\text{grav}}({}^{(4)}g)$$

$$= 16\pi \int_I d\lambda \int_M \left[\pi \cdot \frac{\partial g}{\partial \lambda} - N\mathscr{H}(g, \pi) - X \cdot \mathscr{J}(g, \pi)\right]$$

$$+ 16\pi \int_I d\lambda \int_M \left(\operatorname{div} \beta - \frac{\partial}{\partial \lambda} \operatorname{tr} \pi\right).$$

Integrating the div β term to zero on M, and dropping the total time derivative term

$$\int_{I=[a,b]} d\lambda \int_M \frac{\partial}{\partial \lambda} \operatorname{tr} \pi = \int_M (\operatorname{tr} \pi)_{\lambda=b} - \int_M (\operatorname{tr} \pi)_{\lambda=a}$$

as a constant that will not enter into the variation of S_{grav}, we have

$$16\pi S_{\text{grav}}({}^{(4)}g) = 16\pi \int_I d\lambda \int_M \left(\pi \cdot \frac{\partial g}{\partial \lambda} - N\mathscr{H} - X \cdot \mathscr{J}\right)$$

$$= 16\pi \int_I d\lambda \int_M \left[\pi \cdot \frac{\partial g}{\partial \lambda} - \Phi(g, \pi) \cdot \begin{pmatrix} N \\ X \end{pmatrix}\right].$$

Varying the action with respect to ${}^{(4)}g$ in the direction ${}^{(4)}h$ which vanishes on $\{a\} \times M$ and $\{b\} \times M$ induces a variation (h, ω) of (g, π) which also vanishes on each end manifold $\{a\} \times M$ and $\{b\} \times M$. Thus, taking the extremum of the action for an arbitrary variation (h, ω) vanishing on the end manifolds $\{a\} \times M$ and $\{b\} \times M$ gives

$$0 = 16\pi \, dS_{\text{grav}}({}^{(4)}g) \cdot {}^{(4)}h = 16\pi \int_I d\lambda \int_M \left(\omega \cdot \frac{\partial g}{\partial \lambda} + \pi \cdot \frac{\partial h}{\partial \lambda}\right)$$

$$- 16\pi \int_I d\lambda \int_M \left\langle D\Phi(g, \pi) \cdot (h, \omega), \begin{pmatrix} N \\ X \end{pmatrix}\right\rangle$$

$$= 16\pi \int_I d\lambda \int_M \left(\omega \cdot \frac{\partial g}{\partial \lambda} - \frac{\partial \pi}{\partial \lambda} \cdot h\right) + 16\pi \left[\int_M (\pi \cdot h)_{\lambda=b} - \int_M (\pi \cdot h)_{\lambda=a}\right]$$

$$- 16\pi \int_I d\lambda \int_M \left\langle (h, \omega), [D\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}\right\rangle$$

$$= 16\pi \int_I d\lambda \int_M \left\langle (h, \omega), \left[\left(-\frac{\partial \pi}{\partial \lambda}, \frac{\partial g}{\partial \lambda}\right) - [D\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}\right]\right\rangle$$

where the term involving the total time derivative $\int_I d\lambda \int_M (\partial/\partial\lambda)(\pi \cdot h)$ integrates to zero in the λ-variable by virtue of the vanishing of h on the end manifolds. Since the variation (h, ω) was arbitrary, we conclude that

$$\left(-\frac{\partial \pi}{\partial \lambda}, \frac{\partial g}{\partial \lambda}\right) = [\mathrm{D}\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}$$

so that

$$J\begin{pmatrix} \dfrac{-\partial \pi}{\partial \lambda} \\ \dfrac{\partial g}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial g}{\partial \lambda} \\ \dfrac{\partial \pi}{\partial \lambda} \end{pmatrix} = J \circ [\mathrm{D}\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}. \quad \blacksquare$$

We now give a few additional details on the Hamiltonian structure of the adjoint equation in theorem 4.3.

Let $F: T^*\mathcal{M} \to \mathbb{R}$ be a real-valued function on $T^*\mathcal{M}$ that comes from a density $\mathcal{F}: T^*\mathcal{M} \to C_{\mathrm{d}}^{\infty}$; i.e.,

$$F(g, \pi) = \int_M \mathcal{F}(g, \pi).$$

Then the *Hamiltonian vector field* of F,

$$X_F: T^*\mathcal{M} \to T(T^*\mathcal{M})$$

is defined by

$$\mathrm{d}F(g, \pi) \cdot (h, \omega) = \Omega(X_F(g, \pi), (h, \omega))$$

where Ω is the symplectic structure on $T^*\mathcal{M}$.

Proposition 4.4

The Hamiltonian vector field X_F is given by

$$X_F(g, \pi) = J \circ [\mathrm{D}\mathcal{F}(g, \pi)]^* \cdot 1.$$

Proof. $$\Omega(X_F(g, \pi), (h, \omega)) = -\int \langle X_F(g, \pi), J^{-1}(h, \omega) \rangle,$$

and so

$$\begin{aligned} \mathrm{d}F(g, \pi) \cdot (h, \omega) &= \int \mathrm{D}\mathcal{F}(g, \pi) \cdot (h, \omega) \\ &= \int \langle [\mathrm{D}\mathcal{F}(g, \pi)]^* \cdot 1, (h, \omega) \rangle \\ &= -\int \langle J \circ [D\mathcal{F}(g, \pi)]^* \cdot 1, J^{-1}(h, \omega) \rangle \quad (J^* = -J) \\ &= \Omega\{J \circ [\mathrm{D}\mathcal{F}(g, \pi)]^* \cdot 1, (h, \omega)\}. \end{aligned} \quad \blacksquare$$

In particular, if $F=\int N\mathcal{H}+X\cdot\mathcal{J}=\int\langle(N,X),\Phi\rangle$, then

$$X_F(g,\pi)=J\circ[\mathrm{D}(N\mathcal{H}+X\cdot\mathcal{J})]^*\cdot 1$$
$$=J\circ[\mathrm{D}\Phi(g,\pi)]^*\cdot\begin{pmatrix}N\\X\end{pmatrix}$$

showing that the Einstein evolution equations are Hamilton's equations on the symplectic manifold $T^*\mathcal{M}$ with Hamiltonian density $N\mathcal{H}+X\cdot\mathcal{J}$.

Now suppose $F_1, F_2\colon T^*\mathcal{M}\to\mathbb{R}$ are real-valued functions on $T^*\mathcal{M}$ that arise from densities $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Then their Poisson bracket,

$$\{F_1,F_2\}\colon T^*\mathcal{M}\to\mathbb{R},$$

is defined by

$$\{F_1,F_2\}(g,\pi)=\Omega(X_{F_1}(g,\pi),X_{F_2}(g,\pi)),$$

where X_F is the Hamiltonian vector field for F.

Proposition 4.5

The Poisson bracket $\{F_1, F_2\}$ defined above is given by

$$\{F_1,F_2\}(g,\pi)=\int\langle[\mathrm{D}_g\mathcal{F}_1(g,\pi)]^*\cdot 1,[\mathrm{D}_\pi\mathcal{F}_2(g,\pi)]^*\cdot 1\rangle$$
$$-\int\langle[\mathrm{D}_\pi\mathcal{F}_1(g,\pi)]^*\cdot 1,\mathrm{D}_g\mathcal{F}_2(g,\pi)\cdot 1\rangle.$$

Proof.

$$\{F_1,F_2\}(g,\pi)=\Omega(X_{F_1}(g,\pi),X_{F_2}(g,\pi))$$
$$=-\int\langle X_{F_1}(g,\pi),J^{-1}\circ X_{F_2}(g,\pi)\rangle$$
$$=-\int\langle J\circ[\mathrm{D}\mathcal{F}_1(g,\pi)]^*\cdot 1,J^{-1}\circ J\circ[\mathrm{D}\mathcal{F}_2(g,\pi)]^*\cdot 1\rangle$$
$$=-\int\langle[\mathrm{D}\mathcal{F}_1(g,\pi)]^*\cdot 1,J^*\circ[\mathrm{D}\mathcal{F}_2(g,\pi)]^*\cdot 1\rangle$$
$$=\int\langle[\mathrm{D}\mathcal{F}_1(g,\pi)]^*\cdot 1,J\circ[\mathrm{D}\mathcal{F}_2(g,\pi)]^*\cdot 1\rangle$$

$$= \int \left\langle ([\mathrm{D_g}\mathscr{F}_1(g,\pi)]^* \cdot 1, [\mathrm{D}_\pi\mathscr{F}_1(g,\pi)]^* \cdot 1) \times \begin{pmatrix} [\mathrm{D}_\pi\mathscr{F}_2(g,\pi)]^* \cdot 1 \\ -[\mathrm{D_g}\mathscr{F}_2(g,\pi)]^* \cdot 1 \end{pmatrix} \right\rangle$$

$$= \int \langle [\mathrm{D_g}\mathscr{F}_1(g,\pi)]^* \cdot 1, [D_\pi\mathscr{F}_2(g,\pi)]^* \cdot 1\rangle$$

$$- \int \langle [\mathrm{D}\pi\mathscr{F}_1(g,\pi)]^* \cdot 1, [D_g\mathscr{F}_2(g,\pi)]^* \cdot 1\rangle.$$

This result may be written in 'physics notation' as

$$\{F_1, F_2\} = \int \left(\frac{\delta\mathscr{F}_1}{\delta g}\frac{\delta\mathscr{F}_2}{\delta\pi} - \frac{\delta\mathscr{F}_1}{\delta\pi}\frac{\delta\mathscr{F}_2}{\delta g} \right).$$

Now consider the case when $F_2 = \int N\mathscr{H} + X \cdot \mathscr{J}$. Then, from the proof above,

$$\{F, N\mathscr{H} + X \cdot \mathscr{J}\}(g,\pi) = \int \langle [D\mathscr{F}(g,\pi)]^* \cdot 1, J \circ [D(N\mathscr{H} + X \cdot \mathscr{J})]^* \cdot 1\rangle$$

$$= \int \left\langle [\mathrm{D}\mathscr{F}(g,\pi)]^* \cdot 1, J \circ [\mathrm{D}\Phi(g,\pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix} \right\rangle$$

$$= \int \mathrm{D}\mathscr{F}(g,\pi) \cdot \begin{pmatrix} \dfrac{\partial g}{\partial\lambda} \\ \dfrac{\partial\pi}{\partial\lambda} \end{pmatrix}$$

$$= \int \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathscr{F}(g,\pi)$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda}F(g,\pi).$$

What this means is the following. Let $(g(\lambda), \pi(\lambda))$ be a solution of the Einstein evolution equations with lapse and shift $N(\lambda)$, $X(\lambda)$. Let $F(\lambda) = F(g(\lambda), \pi(\lambda))$. Then

$$\frac{\mathrm{d}F}{\mathrm{d}\lambda} = \{F, N\mathscr{H} + X \cdot \mathscr{J}\}.$$

Thus, as expected, a Poisson bracket with the Hamiltonian $\int N\mathscr{H} + X \cdot \mathscr{J}$ generates λ-derivatives of $F(g(\lambda), \pi(\lambda))$ where $(g(\lambda), \pi(\lambda))$ is the flow with initial data $(g(0), \pi(0))$ and lapse and shift $(N(\lambda), X(\lambda))$.

Actually, the form of the Einstein equations as they appear in theorem 4.3 can be extended to include field theories coupled to gravity.

This extended form is at the basis of a covariant formulation of Hamiltonian systems (Kuchař, 1976*a, b, c*; Fischer and Marsden, 1976, 1978*a, b*). For example, the canonical formulation of the covariant scalar wave equation $\Box\phi = m^2\phi + F'(\phi)$ on a spacetime $(V_4 = I \times M, {}^{(4)}g)$ in terms of a general lapse and shift is as follows.

Consider the Hamiltonian

$$\mathcal{H}(g; \phi, \pi) = \{\tfrac{1}{2}[(\pi'_\phi)^2 + |\nabla\phi|^2 + m^2\phi^2] + F(\phi)\}\, \mathrm{d}\mu(g)$$

for the scalar field (the background metric is considered as implicitly given for this example). We can construct a 2-contravariant symmetric tensor density $\mathcal{T}$ obtained by varying $\mathcal{H}(g; \phi, \pi_\phi)$ with respect to g:

$$\mathcal{T} = -2[\mathrm{D}_g\mathcal{H}(g; \phi, \pi_\phi)]^* \cdot 1,$$

and a one-form density $\mathcal{J}(\phi, \pi_\phi)$ from the relationship

$$\int \langle X, \mathcal{J}(\phi, \pi_\phi)\rangle = \int \langle \pi_\phi, L_X\phi\rangle,$$

so that $\mathcal{J}(\phi, \pi_\phi) = \pi_\phi \cdot \mathrm{d}\phi$. This condition expresses $\mathcal{J}$ as the conserved quantity for the coordinate invariance group on M (Fischer and Marsden, 1972). If we set $\Phi = (\mathcal{H}, \mathcal{J})$, then the Hamiltonian equations of motion for ϕ in a general slicing of the spacetime with lapse N and shift X are

$$\frac{\partial}{\partial\lambda}\begin{pmatrix}\phi\\ \pi_\phi\end{pmatrix} = J \circ [\mathrm{D}\Phi(g; \phi, \pi_\phi)]^* \cdot \begin{pmatrix}N\\ X\end{pmatrix},$$

exactly as for general relativity. A computation shows that this system is equivalent to the covariant scalar wave equation given above. Here $\mathrm{D}\Phi(g; \phi, \pi_\phi)$ is the derivative of Φ with respect to the scalar field and its canonical momentum π_ϕ.

If we couple the scalar field with gravity by regarding the scalar field as a source, the equation for the gravitational momentum $\partial\pi/\partial\lambda$ in theorems 4.1 and 4.3 is altered by the addition of the term $\frac{1}{2}N\mathcal{T}$, and the equation for $\partial g/\partial\lambda$ is unchanged. The constraint equations become

$$\mathcal{H}_{\text{grav}}(g, \pi) + \mathcal{H}_{\text{scalar}}(g; \phi, \pi_\phi) = 0 \quad \text{and} \quad \mathcal{J}_{\text{grav}}(g, \pi) + \mathcal{J}_{\text{scalar}}(\phi, \pi_\phi) = 0.$$

More generally, if one considers the total Hamiltonian $\mathcal{H}_{\mathrm{T}} = \mathcal{H}_{\text{grav}} + \mathcal{H}_{\text{fields}}$ and a total universal flux tensor $\mathcal{J}_{\mathrm{T}} = \mathcal{J}_{\text{grav}} + \mathcal{J}_{\text{fields}}$, and $\Phi_{\mathrm{T}} = (\mathcal{H}_{\mathrm{T}}, \mathcal{J}_{\mathrm{T}})$, and if the non-gravitational fields are non-derivatively (minimally) coupled to the gravitational fields, the general form of the coupled equations is

$$\frac{\partial}{\partial\lambda}\begin{pmatrix} g \\ \pi \\ \phi_A \\ \pi^A \end{pmatrix} = J\circ[D\Phi_T(g, \pi; \phi_A, \pi^A)]^* \cdot \begin{pmatrix} N \\ X \\ \psi \end{pmatrix},$$

$$\Phi(g, \pi; \phi_A, \pi^A) = (\Phi_T(g, \pi; \phi_A, \pi^A), \Phi_{deg}(g, \pi; \phi_A, \pi^A)) = 0.$$

Here, ϕ_A represents all non-gravitational dynamical fields, π^A their conjugate momenta, and $\Phi_{deg} = 0$ represents additional constraints due to degeneracies in ${}^{(4)}\mathcal{L}$, and ψ are the corresponding non-dynamical (degenerate) fields. These results provide a unified covariant Hamiltonian formulation of general relativity coupled to other Lagrangian field theories and in fact allow the empty space case to be extended formally to the non-derivative coupling case. The proof that the description of fields coupled to gravity can be given in the 'L_2-adjoint formulation' as above is based on the work of Kuchař (1976*a*, *b*, *c*; 1977), who, in his milestone series of papers, gives in detail the canonical formulation for covariant field theories, a study initiated by Dirac (see Dirac, 1964, and the references therein). We refer to Arms (1977*a*, *b*) for the realization of this formulation for Yang–Mills fields.

The formalism of this section can be extended to the case where M is non-compact. This case has many technical problems, but there is one basic difference: the fall-off rate for asymptotically flat metrics is not fast enough to allow integration by parts. This has led Regge and Teitelboim (1974) to conclude that the proper Hamiltonian actually generating the evolution equations contains an additional surface integral term corresponding to the mass. Thus, in the asymptotically flat case, the mass can be interpreted as the 'true' generator of the evolution equation after the constraints $\Phi = 0$ are imposed. These ideas are discussed in Choquet-Bruhat, Fischer and Marsden (1978).

4.2 The constraint manifold

Let $C_{\mathcal{H}} = \{(g, \pi) \in T^*\mathcal{M} | \mathcal{H}(g, \pi) = 0\}$ denote the set of solutions of the Hamiltonian constraint and let $C_\delta = \{(g, \pi) \in T^*\mathcal{M} | \mathcal{J}(g, \pi) = -2\pi_i{}^j{}_{|j} = 0\}$ denote the set of solutions of the divergence constraint. Thus $\mathcal{C} = \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_\delta \subset T^*\mathcal{M}$ is the constraint set for the vacuum Einstein system.

Two important facts about $\mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_{\delta}$ are that the constraints are maintained by the evolution equations for any choice of lapse function and shift vector field, and that generically, $\mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_{\delta}$ is a smooth submanifold of $T^*\mathscr{M}$.

From the spacetime point of view, maintenance of the constraints is equivalent to the contracted Bianchi identities, differential identities generated by the covariance of the four-dimensional field equations. This maintenance in time of the constraints is necessary for the consistency of the evolution and constraint equations.

The manifold nature of $\mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_{\delta}$, while of intrinsic interest, is the key to understanding the linearization stability of the field equations, as we shall see.

We begin by noting that the Hamiltonian and momentum functions are covariant with respect to the infinite-dimensional gauge group $\mathscr{D}(M)$ of diffeomorphisms of M. That is, for any $\eta \in \mathscr{D}(M)$ and $(g, \pi) \in T^*\mathscr{M}$,

$$\mathscr{H}(\eta^* g, \eta^* \pi) = \eta^* \mathscr{H}(g, \pi),$$

and

$$\mathscr{J}(\eta^* g, \eta^* \pi) = \eta^* \mathscr{J}(g, \pi),$$

and hence

$$\Phi(\eta^* g, \eta^* \pi) = \eta^* \Phi(g, \pi).$$

Here η^* denotes the usual pull back of tensors.

If η_λ is a curve in $\mathscr{D}(M)$ with $\eta_0 =$ identity, and we define the vector field X by

$$X = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \eta_\lambda\right)_{\lambda=0},$$

then differentiating the relations above in λ and evaluating at $\lambda = 0$ gives the infinitesimal version of covariance:

$$\mathrm{D}\mathscr{H}(g, \pi) \cdot (L_X g, L_X \pi) = L_X[\mathscr{H}(g, \pi)],$$

and

$$\mathrm{D}\mathscr{J}(g, \pi) \cdot (L_X g, L_X \pi) = L_X[\mathscr{J}(g, \pi)],$$

and hence

$$\mathrm{D}\Phi(g, \pi) \cdot (L_X g, L_X \pi) = L_X[\Phi(g, \pi)].$$

Similar identities are generated by the gauge invariance of Yang–Mills fields.

The next theorem computes the rate of change of $\mathcal{H}$ and $\mathcal{J}$ along a solution of the evolution equations for a general lapse and shift. The infinitesimal covariance accounts for the Lie derivatives in the resulting formulae.

Theorem 4.6

For an arbitrary lapse $N(\lambda)$ and shift $X(\lambda)$, let $(g(\lambda), \pi(\lambda))$ be a solution of the Einstein evolution equations

$$\frac{\partial}{\partial\lambda}\begin{pmatrix} g \\ \pi \end{pmatrix} = J \circ [\mathrm{D}\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}.$$

Then $(\mathcal{H}(\lambda), \mathcal{J}(\lambda)) = \{\mathcal{H}(g(\lambda), \pi(\lambda)), \mathcal{J}(g(\lambda), \pi(\lambda))\}$ satisfies the following system of equations:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\lambda} = \frac{1}{N}\,\mathrm{div}\,(N^2\mathcal{J}) + L_X\mathcal{H}$$

and

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}\lambda} = (\mathrm{d}N)\mathcal{H} + L_X\mathcal{J}.$$

If, for some λ_0 in the domain of existence of the solution, $(g(\lambda_0), \pi(\lambda_0)) = (g_0, \pi_0) \in \mathscr{C}_{\mathcal{H}} \cap C_\delta$ (that is, $\Phi(g_0, \pi_0) = 0$), *then $(g(\lambda), \pi(\lambda)) \in \mathscr{C}_{\mathcal{H}} \cap \mathscr{C}_\delta$ for all λ for which the solution exists.*

Remark. It follows (from uniqueness theorems in the next section) that if a solution of the evolution equations intersects $\mathscr{C}_{\mathcal{H}} \cap \mathscr{C}_\delta$, it must lie wholly within $\mathscr{C}_{\mathcal{H}} \cap \mathscr{C}_\delta$.

Proof. The infinitesimal covariance of $\mathcal{H}$ is used as follows:

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{H}(g, \pi)}{\mathrm{d}\lambda} &= \mathrm{D}\mathcal{H}(g, \pi) \cdot \left(\frac{\partial g}{\partial\lambda}, \frac{\partial \pi}{\partial\lambda}\right) \\
&= \mathrm{D}\mathcal{H}(g, \pi) \cdot \left\{J \circ [\mathrm{D}\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}\right\} \\
&= \mathrm{D}\mathcal{H}(g, \pi) \cdot \{J \circ [(\mathrm{D}\mathcal{H}(g, \pi))^* \cdot N + (\mathrm{D}\mathcal{J}(g, \pi))^* \cdot X]\} \\
&= \mathrm{D}\mathcal{H}(g, \pi) \cdot \{J \circ [(\mathrm{D}\mathcal{H}(g, \pi))^* \cdot N + (-L_X\pi, L_Xg)]\} \\
&= \mathrm{D}_g\mathcal{H}(g, \pi) \cdot \{[\mathrm{D}_\pi\mathcal{H}(g, \pi)]^* \cdot N\} \\
&\quad - \mathrm{D}_\pi\mathcal{H}(g, \pi) \cdot \{[\mathrm{D}_g\mathcal{H}(g, \pi)]^* \cdot N\} + L_X\mathcal{H}(g, \pi).
\end{aligned}$$

The first two terms in the expression for $\partial\mathcal{H}/\partial\lambda$ involve a rather tedious computation. The results are

$$\mathrm{D}_g\mathcal{H}\cdot[(\mathrm{D}_\pi\mathcal{H})^*\cdot N]-\mathrm{D}_\pi\mathcal{H}\cdot[(\mathrm{D}_g\mathcal{H})^*\cdot N]=-\frac{1}{N}\delta[N^2\delta(2\pi)].$$

Thus we arrive at

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\lambda}&=-\frac{1}{N}\delta[N^2\delta(2\pi)]+L_X\mathcal{H}\\&=\frac{1}{N}\operatorname{div}(N^2\mathcal{J})+L_X\mathcal{H}.\end{aligned}$$

The evolution equation for $\mathcal{J}(g,\pi)$ follows from infinitesimal covariance of $\Phi(g,\pi)$ as follows:

Let $Y\in\mathcal{X}$ be any vector field on M (independent of λ). Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\int\langle Y,\mathcal{J}(g,\pi)\rangle&=\int\left\langle Y,\frac{\mathrm{d}\mathcal{J}(g,\pi)}{\mathrm{d}\lambda}\right\rangle\\
&=\int\left\langle Y,\mathrm{D}\mathcal{J}(g,\pi)\cdot\left(\frac{\partial g}{\partial\lambda},\frac{\partial\pi}{\partial\lambda}\right)\right\rangle\\
&=\int\left\langle Y,\mathrm{D}\mathcal{J}(g,\pi)\cdot\left\{J\circ[\mathrm{D}\Phi(g,\pi)]^*\cdot\begin{pmatrix}N\\X\end{pmatrix}\right\}\right\rangle\\
&=-\int\left\langle\mathrm{D}\Phi(g,\pi)\cdot\{J\circ[\mathrm{D}\mathcal{J}(g,\pi)]^*\cdot Y\},\begin{pmatrix}N\\X\end{pmatrix}\right\rangle\qquad(J^*=-J)\\
&=-\int\langle\mathrm{D}\Phi(g,\pi)\cdot(L_Yg,L_Y\pi),(N,X)\rangle\\
&=-\int\langle L_Y\Phi(g,\pi),(N,X)\rangle\quad\text{(infinitesimal covariance of }\Phi)\\
&=-\int NL_Y\mathcal{H}(g,\pi)+\langle X,L_Y\mathcal{J}(g,\pi)\rangle\\
&=\int(L_YN)\mathcal{H}+\int\langle L_YX,\mathcal{J}\rangle\quad\text{(integration by parts)}\\
&=\int Y(\mathrm{d}N)\mathcal{H}-\int\langle L_XY,\mathcal{J}\rangle\\
&=\int Y(\mathrm{d}N)\mathcal{H}+\int\langle Y,L_X\mathcal{J}\rangle.\end{aligned}$$

Since Y is arbitrary,

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}\lambda}=(\mathrm{d}N)\mathcal{H}+L_X\mathcal{J}.\qquad\blacksquare$$

In terms of the Poisson brackets introduced in the previous section, we can rewrite theorem 4.6 as follows.

Theorem 4.7

Given $N_1, N_2: M \to \mathbb{R}$, $X_1, X_2: M \to TM$, and

$$F_1 = \int (N_1 \mathcal{J} + X_1 \cdot \mathcal{J}): T^*\mathcal{M} \to \mathbb{R}$$

$$F_2 = \int (N_2 \mathcal{H} + X_2 \cdot \mathcal{J}): T^*\mathcal{M} \to \mathbb{R},$$

then

$$\{F_1, F_2\} = \int (L_{X_1} N_2 - L_{X_2} N_1)\mathcal{H} + \int \langle (N_1 \operatorname{grad} N_2 - N_2 \operatorname{grad} N_1), \mathcal{J}\rangle + \langle L_{X_1} X_2, \mathcal{J}\rangle,$$

and, in particular,

$$\left\{\int N_1 \mathcal{H}, \int N_2 \mathcal{H}\right\} = \int \langle N_1 \operatorname{grad} N_2 - N_2 \operatorname{grad} N_1, \mathcal{J}\rangle$$

$$\left\{\int N\mathcal{H}, \int X \cdot \mathcal{J}\right\} = -\int (L_X N)\mathcal{H}$$

$$\left\{\int X_1 \cdot \mathcal{J}, \int X_2 \cdot \mathcal{J}\right\} = \int \langle L_{X_1} X_2, \mathcal{J}\rangle.$$

The verification that these relations are equivalent to theorem 4.6 is straightforward. We refer to these relationships as the *Dirac canonical commutation relations.*

The following infinitesimal version of theorem 4.1 will be important in understanding and interpreting a splitting due to Moncrief (1976), and in understanding the construction leading to the space of gravitational degrees of freedom (section 4.6).

Proposition 4.8

Let $(g, \pi) \in \mathscr{C}_{\mathcal{H}} \cap \mathscr{C}_{\delta}$. Then

$$\operatorname{range}\{J \circ [\mathrm{D}\Phi(g, \pi)]^*\} \subset \ker \mathrm{D}\Phi(g, \pi).$$

Proof. *Let* $(h, \omega) \in \operatorname{range}\{J \circ [\mathrm{D}\Phi(g, \pi)]^*\}$, and $(N, X) \in C^\infty \times \mathscr{X}$ be such that $(h, \omega) = J \circ [\mathrm{D}\Phi(g, \pi)]^* \cdot (N, X)$. Let $(N(\lambda), X(\lambda))$ be an arbitrary lapse and shift such that $(N(0), X(0)) = (N, X)$. Let $(g(\lambda), \pi(\lambda))$ be the solution to the evolution equations with lapse and shift $(N(\lambda), X(\lambda))$ and with initial data $(g, \pi) \in \mathscr{C}_{\mathcal{H}} \cap \mathscr{C}_{\delta}$. Since $\Phi(g, \pi) = 0$, by theorem 4.6,

$\Phi(g(\lambda), \pi(\lambda)) = 0$ for all λ for which the solution exists. Hence,

$$0 = \frac{d}{d\lambda}\Phi(g(\lambda), \pi(\lambda))\Big|_{\lambda=0} = D\Phi(g(\lambda), \pi(\lambda)) \cdot \left(\frac{\partial g(\lambda)}{\partial \lambda}, \frac{\partial \pi(\lambda)}{\partial \lambda}\right)\Big|_{\lambda=0}$$

$$= D\Phi(g(\lambda), \pi(\lambda))$$

$$\cdot \left\{J \circ [D\Phi(g(\lambda), \pi(\lambda))]^* \cdot \begin{pmatrix} N(\lambda) \\ X(\lambda) \end{pmatrix}\right\}\Big|_{\lambda=0}$$

$$= D\Phi(g, \pi) \cdot \left\{J \circ [D\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}\right\}$$

$$= D\Phi(g, \pi) \cdot (h, \omega).$$

Hence, $(h, \omega) \in \ker D\Phi(g, \pi)$. ■

We now examine the manifold structure of the constraint set $\mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_{\delta}$. We introduce the following conditions on $(g, \pi) \in T^*\mathscr{M}$:

$C_{\mathscr{H}}$: If $\pi = 0$, then g is not flat;

C_{δ}: If for $X \in \mathscr{X}(M)$, $L_X g = 0$ and $L_X \pi = 0$, then $X = 0$;

C_{tr}: $\mathrm{tr}\, \pi'$ is a constant on M.

We consider the constraints one at a time; first, the Hamiltonian constraint.

Proposition 4.9

Let $(g, \pi) \in \mathscr{C}_{\mathscr{H}}$ satisfy condition $C_{\mathscr{H}}$. Then $\mathscr{C}_{\mathscr{H}}$ is a C^∞ submanifold of $T^*\mathscr{M}$ in a neighborhood of (g, π) with tangent space

$$T_{(g,\pi)}\mathscr{C}_{\mathscr{H}} = \ker D\mathscr{H}(g, \pi).$$

The proof relies on some facts about elliptic operators and Sobolev spaces. We briefly recall the relevant facts (see Palais, 1965; Berger and Ebin, 1969, for proofs).

Let Ω be an open bounded region of $\mathbb{R}^n$ with smooth boundary. For any C^∞ function f from $\mathbb{R}^n$ to $\mathbb{R}^m$, we define the $W^{s,p}(\Omega, \mathbb{R}^m)$ norm of f to be

$$\|f\|_{W^{s,p}} = \sum_{0 \leq \alpha \leq s} \|D^\alpha f\|_{L_p(\Omega)}$$

where D^α is the total derivative of f of order α and $\| \;\|_{L_p(\Omega)}$ denotes the usual L_p norm on Ω: $\|g\|_{L_p(\Omega)}=(\int_\Omega |g(x)|^p\, dx)^{1/p}$. By definition, $W^{s,p}(\Omega, \mathbb{R}^m)$ is the completion of $C^\infty(\Omega, \mathbb{R}^m)=\{$restrictions of C^∞ functions on $\mathbb{R}^n$ to $\Omega\}$ with respect to this norm.

We shall shorten $W^{s,p}(\Omega, \mathbb{R}^m)$ and similar expressions to $W^{s,p}$ when there is little chance of confusion.

For a compact manifold M with no boundary and a vector bundle E over M, $W^{s,p}(E)$ shall denote the space of all sections of E that are of class $W^{s,p}$ in some (and hence every) covering of M by charts. For real-valued functions we shall just write $W^{s,p}$, but for other tensor bundles we shall make up special notations for $W^{s,p}(E)$, such as $\mathcal{M}^{s,p}$ for the $W^{s,p}$ space of Riemannian metrics.

In case $p=2$ the spaces $W^{s,p}$ are denoted H^s. In this case, and only in this case, do we get Hilbert spaces.

Now suppose we have two vector bundles E and F, over the same manifold M, and a linear differential operator D of order k,

$$\mathrm{D}: C^\infty(E) \to C^\infty(F).$$

A linear differential operator of order k is a map such that for given charts on E and F (and hence for all charts), the operator takes the form $\mathrm{D}=\sum_{|\alpha|<k} a_\alpha(x)\mathrm{D}^\alpha$, where $\mathrm{D}^\alpha=\partial^{|\alpha|}/\partial x_1{}^{\alpha_1}, \ldots, \partial x_n{}^{\alpha_n}$ is a partial derivative in a chart U for M, $\alpha=(\alpha_1, \ldots, \alpha_n)$, $|\alpha|=\sum_{i=1}^n \alpha_i$, and $a_\alpha(x)$ is a linear function from the model space for the fiber E_x to the model space for the fiber F_x over $x \in U$. We can regard D as a map between Sobolev spaces:

$$\mathrm{D}: W^{s+k,p} \to W^{s,p}.$$

D has an L_2-adjoint D^* defined as usual by the equation

$$(\mathrm{D}f, g)_{L_2}=(f, \mathrm{D}^*g)_{L_2}; \text{ that is, } \int_M \langle \mathrm{D}f, g\rangle\, \mathrm{d}\mu = \int_M \langle f, \mathrm{D}^*g\rangle\, \mathrm{d}\mu,$$

where $\mathrm{d}\mu$ is some preferred volume element such as that associated with a metric: $\mathrm{d}\mu(g)=[\det(g_{ij})]^{1/2}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, and $\langle , \rangle$ is an inner product on the fibers. This structure is not needed if one uses natural adjoints.

A differential operator D is *elliptic* if it has injective (principal) symbol. For each x in M and for each $\xi \in T_x^*M$ = the fiber of the cotangent bundle, the *symbol* $\sigma_\xi(\mathrm{D})$ is a linear map from the fiber E_x to the fiber F_x. In the expression of D in charts, $\sigma_\xi(\mathrm{D})$ is obtained by substituting the components of $\xi \in T_x^*M$ for the corresponding partial derivatives in the terms involving the highest-order derivatives. Thus, for each coordinate on F_x, $\sigma_\xi(x)$ is a homogeneous kth degree polynomial in the components of ξ. For example, the symbol of the ordinary Laplacian $\nabla^2 = \sum_{i=1}^2 \partial^2/\partial x_i^2$ is $\sigma_\xi(\nabla^2) = \|\xi\|^2$.

For elliptic operators we have the following basic splitting theorem.

Fredholm alternative*: *Theorem 4.10

If either D *or* D^* *is elliptic, then* $W^{s,p}(F) = \text{range } D \oplus \ker D^*$, *where the sum is an* L_2 *orthogonal direct sum.*

Proof of proposition 4.9. Consider the map $\mathscr{H}: T^*\mathscr{M} \to C_d^\infty$; $(g, \pi) \mapsto \mathscr{H}(g, \pi)$. We shall show that under condition $C_{\mathscr{H}}$,

$$D\mathscr{H}(g, \pi): T_{(g,\pi)}(T^*\mathscr{M}) = S_2 \times S_d^2 \to T_{\mathscr{H}(g,\pi)}C_d^\infty = C_d^\infty$$

is surjective with splitting kernel so that $\mathscr{H}$ is a submersion at (g, π). Using Sobolev spaces and the implicit function theorem, and then passing to the C^∞ case via a regularity argument, it follows that $\mathscr{C}_{\mathscr{H}} = \mathscr{H}^{-1}(0)$ is a smooth submanifold in a neighborhood of (g, π).

From theorem 4.10 it follows that $D\mathscr{H}(g, \pi)$ is surjective provided that its L_2-adjoint

$$[D\mathscr{H}(g, \pi)]^*: C^\infty \to S_d^2 \times S_2$$

$$[DH(g, \pi)]^* \cdot N = \{-NS_g(\pi, \pi) + [N\,\text{Ein}\,(g) - \text{Hess}\,N - g\,\Delta N]^\# \,d\mu(g),$$
$$2N[(\pi')^\flat - \tfrac{1}{2}(\text{tr}\,\pi')g]\}$$

is injective and has injective symbol.

The symbol of $[D\mathscr{H}(g, \pi)]^*$ is

$$\sigma_\xi[D\mathscr{H}(g, \pi)]^* = [(-\xi \otimes \xi + g\|\xi\|^2)^\# \,d\mu(g), 0]:$$
$$\mathbb{R} \to ((T_x^*M \otimes T_x^*M)_{\text{sym}}\,d\mu(g), (T_xM \otimes T_xM)_{\text{sym}})$$

for $\xi \in T_x^*M$. For $s \in \mathbb{R}$, $\xi \neq 0$, $(-\xi \otimes \xi + g\|\xi\|^2)s = 0$ implies, by taking the trace, $2\|\xi\|^2 s = 0$ so $s = 0$, so that the symbol is injective.

Any $N \in \ker [D\mathscr{H}(g, \pi)]^*$ satisfies

(i) $-NS_g(\pi, \pi) + [N\,\text{Ein}\,(g) - \text{Hess}\,N - g\,\Delta N]^\# \,d\mu(g) = 0$

(ii) $2N[(\pi') - \tfrac{1}{2}(\text{tr}\,\pi')g] = 0$.

Taking the trace of (ii) gives $N(\text{tr}\,\pi') = 0$ and so from (ii) again $N\pi = 0$. Thus, from (i),

(iii) $N\,\text{Ein}\,(g) - \text{Hess}\,N - g\,\Delta N = 0$.

From the trace of (iii)

$$2\,\Delta N+\tfrac{1}{2}R(g)N=0.$$

However, from $\mathscr{H}(g,\pi)=0$ and $N\pi=0$, it follows that $NR(g)=0$. Hence

$$\Delta N=0$$

and so $N=$ constant.

If $\pi\neq 0$, then $N\pi=0$ implies $N=0$, since N is constant. Thus $[\mathrm{D}\mathscr{H}(g,\pi)]^*$ is injective and hence $\mathrm{D}\mathscr{H}(g,\pi)$ is surjective.

If $\pi=0$, then from (iii), $N\,\mathrm{Ein}\,(g)=0$ since N is constant and so $N\,\mathrm{Ric}\,(g)=0$. Thus, if $N\neq 0$, then $\mathrm{Ric}\,(g)=0$ and hence g is flat, since $\dim M=3$. But a flat g and $\pi=0$ is ruled out by condition $C_{\mathscr{H}}$. Hence $N=0$, and again $\mathrm{D}\mathscr{H}(g,\pi)$ is surjective. ■

Proposition 4.11

If $(g,\pi)\in\mathscr{C}_\delta=\{(g,\pi)|\mathscr{J}(g,\pi)=0\}\subset T^*\mathscr{M}$ satisfies condition C_δ, then $\mathscr{C}_\delta$ is a smooth submanifold of $T^*\mathscr{M}$ in a neighborhood of (g,π) with tangent space

$$T_{(g,\pi)}\mathscr{C}_\delta=\ker\,[\mathrm{D}\mathscr{J}(g,\pi)].$$

Proof. The adjoint of the derivative of $\mathscr{J}(g,\pi)$ is given by

$$[\mathrm{D}\mathscr{J}(g,\pi)]^*\cdot X=(-L_X\pi,L_Xg).$$

The symbol is injective (from its injectivity in the second component alone). The kernel of $[\mathrm{D}\mathscr{J}(g,\pi)]^*$ is $\{X|L_X\pi=0,\ L_Xg=0\}$ so that injectivity of $[\mathrm{D}\mathscr{J}(g,\pi)]^*$ is exactly condition C_δ. The result then follows by the implicit function theorem as in proposition 4.9. ■

To show that the intersection $\mathscr{C}=\mathscr{C}_{\mathscr{H}}\cap\mathscr{C}_\delta$ is a submanifold of $T^*\mathscr{M}$, we need additional restrictions because there may be points at which the intersection is not transversal. At this point it is necessary to assume that (g,π) satisfies the condition $\mathrm{tr}\,\pi'=$ constant.

Theorem 4.12

Let $(g,\pi)\in\mathscr{C}_{\mathscr{H}}\cap\mathscr{C}_\delta$ satisfy the conditions $C_{\mathscr{H}}$, C_δ, and C_{tr}. Then the constraint set $\mathscr{C}=\mathscr{C}_{\mathscr{H}}\cap\mathscr{C}_\delta$ is a C^∞ submanifold of $T^*\mathscr{M}$ in a neighborhood of (g,π) with tangent space

$$T_{(g,\pi)}\mathscr{C}=\ker\mathrm{D}\Phi(g,\pi)$$

where $\Phi=(\mathscr{H},\mathscr{J})$.

Proof. We want to show $D\Phi(g, \pi) = (D\mathscr{H}(g, \pi), D\mathscr{J}(g, \pi))$ is surjective for $(g, \pi) \in \mathscr{C}$ and satisfying the given conditions. The adjoint is

$$[D\Phi(g, \pi)]^*: C^\infty \times \mathscr{X} \to S_d^2 \times S_2,$$

$$(N, X) \mapsto [D\Phi(g, \pi)]^* \cdot (N, X) = [D\mathscr{H}(g, \pi)]^* \cdot N + [D\mathscr{J}(g, \pi)]^* \cdot X.$$

For $\xi \in T_x^*M$, $\xi \neq 0$, the symbol of this map, $\sigma_\xi[D\Phi(g, \pi)]^*$, $\xi \in T_x^*M$, may be shown to be injective, as above (see, however, remarks on various types of ellipticity in Fischer and Marsden, 1975*b*). Thus it remains to show that $[D\Phi(g, \pi)]^*$ is injective. Let $(N, X) \in \ker [D\Phi(g, \pi)]^*$. Then, from the formula for $[D\Phi(g, \pi)]^*$, we have

(i) $-NS_g(\pi, \pi) + [N \operatorname{Ein}(g) - (\operatorname{Hess} N + g\,\Delta N)]^\# \, d\mu(g) - L_X\pi = 0$

and

(ii) $2N[(\pi') - \frac{1}{2}(\operatorname{tr} \pi')g] + L_X g = 0.$

Taking the trace of (i) and (ii), we get:

(iii) $-\dfrac{N}{2}\mathscr{H}(g, \pi) + 2(\Delta N)\, d\mu(g) + \operatorname{tr} L_X\pi = 0$

and

(iv) $-N \operatorname{tr} \pi' + 2 \operatorname{div} X = 0.$

Now $\operatorname{tr} L_X\pi = X \cdot d \operatorname{tr} \pi - \pi \cdot L_X g + (\operatorname{div} X)(\operatorname{tr} \pi)$, since $L_X\pi = (L_X\pi') \otimes d\mu(g) + \pi' \otimes (\operatorname{div} X)\, d\mu(g)$ (in coordinates, $(L_X\pi)^{ij} = X^k \pi^{ij}{}_{|k} - \pi^{ik} X^j{}_{|k} - \pi^{jk} X^i{}_{|k} + X^k{}_{|k}\pi^{ij}$).

Since $\mathscr{H}(g, \pi) = 0$, (iii) reduces to

(v) $2(\Delta N)\, d\mu(g) + X \cdot d \operatorname{tr} \pi - \pi \cdot L_X g + (\operatorname{div} X)(\operatorname{tr} \pi) = 0.$

Using (ii) and (iv) to eliminate $L_X g$ and $\operatorname{div} X$, respectively, in (v) gives

(vi) $2(\Delta N) + X \cdot d \operatorname{tr} \pi' - \pi' \cdot L_X g + (\operatorname{div} X)(\operatorname{tr} \pi')$

$$= 2\,\Delta N + 2N\pi' \cdot [(\pi')^\flat - \tfrac{1}{2}(\operatorname{tr} \pi')g] + \frac{N}{2}(\operatorname{tr} \pi')(\operatorname{tr} \pi') + X \cdot d \operatorname{tr} \pi'$$

$$= 2\,\Delta N + 2N\pi' \cdot \pi' + \frac{N}{2}(\operatorname{tr} \pi')^2 + X \cdot d \operatorname{tr} \pi'$$

$$= 2\,\Delta N + 2N[\pi' \cdot \pi' - \tfrac{1}{4}(\operatorname{tr} \pi')^2] + X \cdot d \operatorname{tr} \pi' = 0.$$

Now note that the coefficient of N, namely $P(\pi', \pi') = \pi' \cdot \pi' - \frac{1}{4}(\operatorname{tr} \pi')^2 = [\pi' - \frac{1}{2}(\operatorname{tr} \pi')g] \cdot [\pi' - \frac{1}{2}(\operatorname{tr} \pi')g]$, is positive-definite.

Thus, if tr π' is a constant, (vi) becomes

$$2\,\Delta N + 2P(\pi', \pi')N = 0$$

which implies $N = 0$ unless $\pi' = 0$, in which case N = constant. In this case, from (i), Ein $(g) = 0$ and so Ric $(g) = 0$, i.e., g is flat since dim $M = 3$. However, the case $(g_F, 0)$, where g_F is flat, is excluded by condition $C_{\mathcal{H}}$. Thus, $\pi' \neq 0$ and $N = 0$. Then, by (i) and (ii), $L_X g = 0$ and $L_X \pi = 0$, which, by condition $C\delta$, implies $X = 0$. Thus $(N, X) = (0, 0)$ and so $[D\Phi(g, \pi)]^*$ is injective, under conditions $C_{\mathcal{H}}$, C_δ, and C_{tr}. The result of the theorem then follows by the implicit function theorem. ■

Remark. That one must impose the condition tr π' is a constant to show that the intersection $\mathscr{C}_{\mathcal{H}} \cap \mathscr{C}_\delta$ is a manifold is an annoying feature of the analysis. One might suspect that under conditions $C_{\mathcal{H}}$ and C_δ alone, the system (i) and (ii) is injective. The difficulty is that in the system, say (vi) and (ii) for (N, X), the $X \cdot \mathrm{d}$ tr π' coupling term seems to be sufficient to prevent one from showing uniqueness for this system. The results of Moncrief, discussed in section 4.5, will shed light on this point.

In Choquet-Bruhat, Fischer and Marsden (1978), and Marsden and Tipler (1979) the existence of hypersurfaces with tr π' equal to a constant is discussed. Thus these preferred hypersurfaces will be the place to check conditions $C_{\mathcal{H}}$ and C_δ.

4.3 The abstract Cauchy problem and hyperbolic equations

This section summarizes the general theory for hyperbolic initial-value problems that we shall need for relativity. The complete proofs are technical and lengthy, so only the ideas will be given. The papers of Kato (1975*a*, *b*, 1977) and Hughes, Kato and Marsden (1977) can be consulted for details. The present abstract approach is preferred since it gives as special cases both first-order symmetric and second-order hyperbolic, or combinations of these systems. Moreover, it yields the sharpest known results with regard to differentiability.

We shall begin with the linear case, then treat the nonlinear. We give a result on differentiability of the time t map for later use and then explain how the results apply to hyperbolic systems.

It is necessary to assume the reader is familiar with linear semigroup theory; see, for example, Hille and Phillips (1967), Yosida (1974), or Marsden and Hughes (1978).

If X is a Banach space, $G(X, M, \beta)$ denotes the set of generators A of C_0 semigroups $e^{tA} = U(t)$ on X satisfying

$$\|U(t)\| \leq M e^{t\beta}, \quad t \geq 0;$$

i.e., by the Hille–Yosida theorem, $\lambda - A$ is one-to-one and onto X and

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \beta)^n}, \quad n = 1, 2, \ldots, \lambda > \beta.$$

If $M = 1$, we say A is *quasi-accretive* or $U(t)$ is *quasi-contractive*. This is the class of linear semigroups of importance to us. We recall that for $\varphi \in D(A)$, the domain of A, $U(t)\varphi = \varphi(t)$ lies in $D(A)$ as well and satisfies the evolution equation

$$\frac{d}{dt}\varphi(t) = A\varphi(t), \tag{4.1}$$

where $\varphi(\cdot)$ is regarded as a map of $[0, \infty)$ to X for purposes of computing the time derivative.

Let X, Y be Banach spaces, $Y \subset X$ with the inclusion continuous and dense. Let $U(t, s)$ be a family of bounded operators on X defined for $0 \leq s \leq t \leq T$; here $[0, T]$ is a conveniently chosen time interval; T could be arbitrarily large. Let $A(t)$ be a family of linear generators on X, $Y \subset D(A(t))$, $0 \leq t \leq T$. We call $U(t, s)$ a family of (strong) *evolution operators* for A if

(i) $U(s, s) = 1$ and $(t, s) \mapsto U(t, s)$ is strongly continuous in X;
(ii) $U(t, s)U(s, r) = U(t, r)$, $0 \leq r \leq s \leq t \leq T$;
(iii) $U(t, s)$ is a bounded operator of Y to Y and is strongly continuous in (t, s);
(iv) $(\partial/\partial t)U(t, s)\varphi = A(t)U(t, s)\varphi$, $\varphi \in Y$ (forward differential equation) and each side is strongly continuous in (t, s) with values in $B(Y, X)$ (the bounded operators from Y to X) and $\partial/\partial t$ is taken in the X-norm.

If we differentiate (ii) with respect to s at $s = r$, and use (iv), we formally get the backwards differential equation:

$$\frac{\partial}{\partial s} U(t, s)\varphi = -U(t, s)A(s)\varphi$$

for $\varphi \in Y$. If we write (iv) as an integral equation in time, this is easy to prove; write

$$U(t, s)\varphi = \varphi + \int_s^t A(\tau)U(\tau, s)\varphi \, d\tau$$

and use the identity

$$\frac{1}{h}[U(t,s+h)\varphi-U(t,s)\varphi]$$

$$=U(t,s+h)\left[\frac{\varphi-U(s+h)\varphi}{h}\right]+\frac{1}{h}[U(s+h,s)\varphi-\varphi]$$

$$-\frac{1}{h}\int_s^{s+h}A(\tau)U(\tau,s+h)\varphi\,d\tau.$$

A family $A(t)\in G(X,M,\beta)$ (for M,β fixed) is called *stable* if for any $s_j\geqslant 0$ and $0\leqslant t_1\leqslant\cdots\leqslant t_k\leqslant T$,

$$\exp[s_kA(t_k)]\exp[s_{k-1}A(t_{k-1})]\ldots\exp[s_1A(t_1)]$$
$$\leqslant M\exp[\beta(s_1+\ldots+s_k)]$$

or, equivalently,

$$\|(\lambda-A(t_k))^{-1}\cdots(\lambda-A(t_1))^{-1}\|\leqslant\frac{M}{(\lambda-\beta)^k},\quad\lambda>\beta.$$

If $A(t)\in G(X,1,\beta)$, then $A(t)$ is clearly stable. If we let X_t denote X with a new norm $\|\,\|_t$ depending on t in an exponential fashion:

$$\|\varphi\|_t\leqslant\|\varphi\|_s\ e^{c|t-s|},\quad s,t\in[0,T]$$

and if $A(t)\in G(X_t,1,\beta)$, then $A(t)$ is stable in X_{t_0} with $M=e^{2cT\beta}$; see Kato (1970, Proposition 3.4). The same reference, Proposition 3.5, shows that a bounded perturbation of a stable family is stable.

In the following theorem, $L_*^\infty([0,T];B(Y,X))$ denotes the (equivalence class of) *strongly* measurable essentially bounded functions from $[0,T]$ to $B(Y,X)$ and $\mathrm{Lip}_*([0,T];B(Y,X))$ denotes the strong indefinite integrals of functions in $L_*^\infty([0,T];B(Y,X))$.

Theorem 4.13 (Kato, 1973)

Assume

(i) *$A(t)$ is a stable family of generators in X, $0\leqslant t\leqslant T$;*

(ii) *$Y\subset X$, with continuous dense inclusion and $D(A)\supset Y$;*

(iii) *there is a family $S(t)$: $Y\to X$ of isomorphisms (onto) such that*

$$S(t)A(t)S(t)^{-1}=A(t)+B(t),$$

where $B(t) \in B(X)$, a bounded operator on X, and

(*a*) $t \mapsto S(t)$ *lies in* $\mathrm{Lip}_*([0, T]; B(Y, X))$

(*b*) $t \mapsto B(t)$ *lies in* $L^\infty_*([0, T]; B(X))$;

(iv) $t \mapsto A(t) \in B(Y, X)$ *is norm continuous.*

Then there is a unique family of strong evolution operators for A.

See Kato (1973) for the proof.

The case where the domain of $A(t)$ is constant in time is much simpler. Here we assume $D(A(t)) = Y$ and that $A \in \mathrm{Lip}_*([0, T]; B(Y, X))$. Then (iii) will hold with $B = 0$ and

$$S(t) = \lambda - A(t), \quad \lambda > \beta.$$

However, for the hyperbolic problems we wish to consider, the domains need not be constant. The constant domain case was the subject of the original work of Kato (1966); see also Yosida (1974).

The inhomogeneous problem

$$\frac{\partial u}{\partial t} = A(t)u + f(t), \quad u(0) = \varphi$$

can be treated by a clever trick of Kato (1977). Namely, we suspend the equation on $X \times \mathbb{R}$ and consider the equivalent homogeneous problem

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ k \end{pmatrix} = \hat{A}(t)\begin{pmatrix} u \\ k \end{pmatrix}, \quad u(0) = \varphi, k(0) = 1$$

where

$$\hat{A}(t) = \begin{pmatrix} A(t) & f(t) \\ 0 & 0 \end{pmatrix}.$$

Then the theorem above may be applied to $\hat{A}$.

In many nonlinear problems it is often convenient to consider associated (time-dependent) linear Cauchy problems and for these, the theorem above is applicable.

To illustrate how the theorem applies we consider the two cases that mainly concern us, namely first-order symmetric hyperbolic and second-order hyperbolic systems. We shall treat these on $\mathbb{R}^m$, but due to the hyperbolicity of the equations, the results can be localized and therefore applied to compact manifolds as well; see Hawking and Ellis (1973).

We first consider first-order symmetric hyperbolic systems of Friedrichs (1954); see also Fischer and Marsden (1972*b*) and Kato

(1975*a*, *b*, 1977). The form is

$$a_0(t,x)\frac{\partial u}{\partial t}=\sum_{j=1}^{m} a_j(t,x)\frac{\partial u}{\partial x^j}+a(t,x)u, \tag{4.2}$$

where $u(t,x)\in \mathbb{R}^N$, a_j, a are real. We assume

(i) there are constant matrices a_j^{∞}, a^{∞} such that

$$a_j-a_j^{\infty},\ a-a^{\infty}\in C([0,T],H^0(\mathbb{R}^m)\cap L^{\infty}([0,T],H^s(\mathbb{R}^m)),\qquad j=0,1,\ldots,m$$

$$a_0-a_0^{\infty}\in \mathrm{Lip}\,([0,T],H^{s-1}(\mathbb{R}^m)).$$

Here $H^s(\mathbb{R}^m)$ is the usual Sobolev space on $\mathbb{R}^m$ (with range unspecified) and $s>(m/2)+1$.

(ii) a_j are symmetric matrices;

(iii) $a_0(t,x)\geqslant cI$ for some $c>0$.

Theorem 4.14

Under these conditions, the hypotheses of theorem 4.13 *are satisfied with*

$$X=L^2(\mathbb{R}^m)=H^0(\mathbb{R}^m)$$

$$Y=H^{s'}(\mathbb{R}^m),\quad 1\leqslant s'\leqslant s$$

$$S(t)=(1-\Delta)^{s'/2}$$

$$A(t)=a_0(t,\cdot)^{-1}\left[\sum_{j=1}^{m} a_j(t,\cdot)\frac{\partial}{\partial x^j}+a(t,\cdot)\right]$$

(*the closure of this operator on* C_0^{∞}), *i.e.*, (4.2) *generates a strong evolution system in* L^2 *which maps* $H^{s'}$ *to* $H^{s'}$ (*regularity*).

Warning. The domain of $A(t)$ need not be $H^1(\mathbb{R}^m)$; *e.g.*, *the* a_j may vanish.

The idea of the proof is as follows. If we put on X the energy norm

$$\|\varphi\|_t^2=\int_{\mathbb{R}^m}[a_0(t,x)\varphi(x)]\cdot\varphi(x)\,\mathrm{d}x,$$

we find that

$$A(t)\in G(X_t,1,\beta)$$

with

$$\beta = \sup_{t,x} \left| a(t, x) - \frac{1}{2} \sum_{j=1}^{m} \frac{\partial a_j}{\partial x^j}(t, x) \right|,$$

which is finite by the Sobolev inequalities. The key idea here is the estimate

$$\|[\lambda - A(t)]^{-1}\|_t \geqslant \frac{1}{(\lambda - \beta)},$$

which, by the Schwarz inequality, is implied by

$$\langle [\lambda - A(t)]\varphi, \varphi \rangle_t \geqslant (\lambda - \beta) \|\varphi\|_t^2.$$

The latter is readily proved using integration by parts and the symmetry of a_j. The stability of $A(t)$ results from the fact that the norms $\| \, \|_t$ depend exponentially on t. The hardest part is to prove that

$$B(t) = [S, A(t)] S^{-1}$$

is a bounded operator in X, where $[\,,]$ is the commutator. One writes the commutator out explicitly; the key estimate boils down to an estimate on the commutator

$$\left[S, \frac{\partial a_j}{\partial x^i} \right].$$

The required estimates on this commutator use a lengthy but relatively straightforward series of Sobolev-type estimates. Details may be found in Kato (1975*b*) for $s' = 1$, the general case being similar.

Remark. Results of this type for (4.2) already appear in early work of Friedrichs (1954) and Courant and Hilbert (1962). However, sharp differentiability hypotheses, which are crucial for nonlinear problems, were never clearly spelled out. An intermediate attempt was given in Fischer and Marsden (1972*b*) and the formulation was then sharpened and clarified by Kato (1975*a*). The present unified scheme, suggested by Hughes, Kato, and Marsden (1977) is due to Kato.

Next, we consider second-order hyperbolic systems. The form is

$$a_{00}(t, x) \frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2 u}{\partial x^i \, \partial x^j} + 2 \sum_{i=1}^{m} a_{0i}(t, x) \frac{\partial^2 u}{\partial t \, \partial x^i} + a_0(t, x) \frac{\partial u}{\partial t}$$

$$+ \sum_{i=1}^{m} a_i(t, x) \frac{\partial u}{\partial x^i} + a(t, x) u \qquad (4.3)$$

where, again, $u(t, x) \in \mathbb{R}^N$, $a_{\alpha\beta}$, a_α, a are $N \times N$ matrix functions and we assume: $s > \frac{1}{2}m + 1$ and

(i) there are constant matrices $a^\infty_{\alpha\beta}$, a^∞_α, a^∞ such that

$$a_{\alpha\beta} - a^\infty_{\alpha\beta},\ a - a^\infty \in \mathrm{Lip}\,([0, T]; H^{s-1}(\mathbb{R}^m)) \subset L^\infty([0, T]; H^s(\mathbb{R}^m));$$

(ii) $a_{\alpha\beta}$ is symmetric;
(iii) $a_{00}(t, x) \geqslant cI$ for some $c > 0$;
(iv) strong ellipticity; there is an $\varepsilon > 0$ such that

$$\sum_{i,j=1}^{M} a_{ij}(t, x)\xi_i\xi_j \geqslant \varepsilon\left(\sum_{j=1}^{m} \xi_j^2\right)$$

(a matrix inequality) for all $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$.

Theorem 4.15

Under these conditions, the hypotheses of theorem 4.13 *are satisfied with*

$$X = H^1(\mathbb{R}^m) \times H^0(\mathbb{R}^m),$$
$$Y = H^{s'+1}(\mathbb{R}^m) \times H^{s'}(\mathbb{R}^m), \quad 1 \leqslant s' \leqslant s,$$
$$S = (1-\Delta)^{s'/2} \times (1-\Delta)^{s'/2},$$

$$A(t) = \begin{pmatrix} 0 & I \\ a_{00}^{-1}\left[\sum a_{ij}\dfrac{\partial^2}{\partial x^i\,\partial x^j} + \sum a_j\dfrac{\partial}{\partial x^j} + a\right] & a_{00}^{-1}\left[2\sum a_{0j}\dfrac{\partial}{\partial x^j} + a_0\right]\end{pmatrix}$$

(*the closure of this operator on* C_0^∞), *i.e.*, (4.3) *generates a strong evolution system in* X *which maps* Y *to* Y.

Here we have written (4.3) in the usual way as a system in $(u, \dot{u})$, first order in time.

One uses the norm

$$\|(\varphi, \dot{\varphi})\|_t^2 = \int_{\mathbb{R}^m} \left[\sum_{i,j=1}^{m} a_{ij}(t, x)\frac{\partial\varphi}{\partial x^i} \cdot \frac{\partial\varphi}{\partial x^j} + c\varphi \cdot \varphi + a_{00}(t, x)\dot{\varphi} \cdot \dot{\varphi}\right] \mathrm{d}x$$

where the constant c is chosen sufficiently large. By Gårdings inequality, this gives an equivalent norm on X (this uses strong ellipticity). It is then straightforward to get the estimate

$$\|[\lambda - A(tS)]^{-1}\|_t \geqslant \frac{1}{(\lambda - \beta)}$$

by showing, as before, that

$$\langle[\lambda - A(t)]u, u\rangle_t \geq (\lambda - \beta)\|u\|_t^2.$$

One can also show, as in Yosida (1974) that $\lambda - A(t)$ is one-to-one and onto X, so $A(t) \in G(X_t, 1, \beta)$, and, as above, $\|\cdot\|_t$ varies exponentially with t, so $A(t)$ is a stable family.

Again, the proof of boundedness of $B(t)$ requires estimates on commutators; for details see Hughes, Kato and Marsden (1977).

For later use in the nonlinear problem (and in lemma 4.22), it is crucial to have sharp differentiability assumptions on the coefficients as stated here.

Remark. Since the abstract theorem includes both (4.2) and (4.3) as special cases, it is clear that coupled systems of such equations can be handled in a similar way. This is important for certain types of matter fields coupled to the gravitational field.

Now we turn to the nonlinear problem. As above, let X and Y be Banach spaces, with Y densely and continuously included in X. Let $W \subset Y$ be open, let $T > 0$ and let $G: [0, T] \times W \to X$ be a given mapping. A nonlinear evolution equation has the form

$$\dot{u}(t) = G(t, u(t)), \quad \text{where } \dot{u} = \frac{\mathrm{d}u}{\mathrm{d}t}. \tag{4.4}$$

If $s \in [0, T]$ and $\Phi \in W$ are given, a solution curve (or integral curve) of G with value ϕ at s is a map $u(\cdot) \in C^0([s, T], W) \cap ([s, T], X)$ such that (4.4) hold on $[s, T]$ and $u(s) = \phi$.

If these solution curves exist and are unique for ϕ in an open set $U \subset W$, we can define evolution operators $F_{t,s}$: $U \to W$ that map $u(s) = \phi$ to $u(t)$. We say (4.4) is *well posed* (or is *Cauchy stable*) if $F_{t,s}$ is continuous (in the Y-topology on U and W) for each t, s satisfying $0 \leqslant s \leqslant t \leqslant T$. We remark that joint continuity of $F_{t,s}(\phi)$ in (t, s, ϕ) follows under general hypotheses (Chernoff and Marsden, 1974). Furthermore, if one has well-posedness for short time intervals, it is easy to obtain it for the maximally extended flow.

Well-posedness can be difficult to establish in specific examples, especially for 'hyperbolic' ones. The continuity of $F_{t,s}$ from Y to Y cannot in general be replaced by stronger smoothness conditions such as Lipschitz or even Hölder continuity; a simple example showing this, namely $\dot{u} + uu_x = 0$ in $Y = H^{s+1}, X = H^s$ on $\mathbb{R}$, is given in Kato (1975*a*). A discussion of these smoothness questions is given below.

The most thoroughly studied nonlinear evolution equations are those giving rise to nonlinear contraction semigroups generated by monotone operators (Brezis, 1973). These sometimes have evolution operators defined on all of X. This is not typical of hyperbolic problems, where $F_{t,s}$ may be defined only in Y, may be continuous from Y to Y, be differentiable from Y to X, and be Y-locally Lipschitz from X to X, without being X-locally Lipschitz from X to X or Y-locally Lipschitz from Y to Y, as is shown by the example above.

Specializing (4.4), we shall consider the quasi-linear abstract Cauchy problem

$$\dot{u} = A(t, u)u + f(t, u), \quad 0 \leqslant t \leqslant T, \quad u(0) = \varphi \tag{4.5}$$

where u takes values in X and $A(t, u)$ is an (unbounded) linear operator depending on the unknown u in a nonlinear fashion. We include f for completeness, although it can be omitted by using Kato's suspension trick mentioned above.

Here are our assumptions.

We start from four (real) Banach spaces

$$Y \subset X \subset Z' \subset Z,$$

with all the spaces reflexive and separable and the inclusions continuous and dense. We assume that

(Z′) Z' is an interpolation space between Y and Z; thus if $U \in B(Y) \cap B(Z)$, then $U \in B(Z')$ with $\|U\|_{Z'} \leqslant c \max\{\|U\|_Y, \|U\|_Z\}$; $B(Y)$ denotes bounded operators on Y.

Let $N(Z)$ be the set of all norms in Z equivalent to the given one $\|\ \|_Z$. Then $N(Z)$ is a metric space with the distance function

$$\mathrm{d}(\|\ \|_\mu, \|\ \|_\nu) = \log \max\{\sup_{0 \neq z \in Z} \|z\|_\mu/\|z\|_\nu, \sup_{0 \neq z \in Z} \|z\|_\nu/\|z\|_\mu\}.$$

We now introduce four functions, A, N, S, and f on $[0, T] \times W$, where $T > 0$ and W is an open set in Y, with the following properties:

For all $t, t', \ldots, \in [0, T]$ for all $w, w', \ldots \in W$, there is a real number β and there are positive numbers $\lambda_N, \mu_N, \ldots$ such that the following conditions hold.

(Ñ) $N(t, w) \in N(Z)$, with

$\mathrm{d}(N(t, w), \|\ \|_Z) \leqslant \lambda_N$,

$\mathrm{d}(N(t', w'), N(t, w)) \leqslant \mu_N(|t' - t| + \|w' - w\|_X)$.

(S) $S(t, w)$ is an isomorphism of Y onto Z, with
$\|S(t, w)\|_{Y,Z} \leq \lambda_S$, $\|S(t, w)^{-1}\|_{Z,Y} \leq \lambda'_S$,
$\|S(t', w') - S(t, w)\|_{Y,Z} \leq \mu_S(|t' - t| + \|w' - w\|_X)$.

(A1) $A(t, w) \in G(Z_{N(t,w)}, 1, \beta)$, where $Z_{N(t,w)}$ denotes the Banach space Z with norm $N(t, w)$. This means that $A(t, w)$ is a C_0-generator in Z such that $\|e^{\tau A(t,w)}z\| \leq e^{\beta\tau}\|z\|$ for all $\tau \geq 0$ and $z \in Z$.

(A2) $S(t, w)A(t, w)S(t, w)^{-1} = A(t, w) + B(t, w)$, where
$B(t, w) \in B(Z)$, $\|B(t, w)\|_Z \leq \lambda_B$.

(A3) $A(t, w) \in B(Y, X)$, with $\|A(t, w)\|_{Y,X} \leq \lambda_A$ and
$\|A(t, w') - A(t, w)\|_{Y,Z'} \leq \mu_A\|w' - w\|_{Z'}$
and with $t \mapsto A(t, w) \in B(Y, Z)$ continuous in norm.

(f1) $f(t, w) \in Y$, $\|f(t, w)\|_Y \leq \lambda_f$, $\|f(t, w') - f(t, w)\|_{Z'} \leq \mu_f\|w' - w\|_Z$,
and $t \mapsto f(t, w) \in Z$ is continuous.

Remarks. (i) If $N(t, w) = \text{const} = \| \ \|_Z$, condition (N) is redundant. If $S(t, w) = \text{const} = S$, condition (S) is trivial. If both are assumed, and $X = Z' = Z$, we have the case of Kato (1975*b*).

(ii) In most applications we can choose $Z' = Z$ and/or $Z' = X$.

(iii) The paper of Hughes, Kato and Marsden (1977) had an additional condition (A4) which was then shown to be redundant in Kato (1977).

Theorem 4.16

Let (Z′), (N), (S), (A1) *to* (A3), *and* (f1) *be satisfied. Then there are positive constants ρ' and $T' \leq T$ such that if $\phi \in Y$ with $\|\phi - y_0\|_Y \leq \rho'$, then* (4.5) *has a unique solution u on* $[0, T']$ *with*

$$u \in C^0([0, T']; W) \cap C^1([0, T']; X).$$

Here ρ' depends only on λ_N, λ_S, λ'_S, and $R = \text{dist}\,(y_0, Y \backslash W)$, while T' may depend on all the constants $\beta, \lambda_N, \mu_N, \ldots$ and R. When ϕ varies in Y subject to $\|\phi - y_0\|_Y \leq \rho'$, the map $\phi \mapsto u(t)$ is Lipschitz continuous in the Z'-norm, uniformly in $t \in [0, T']$.

To establish well-posedness, we have to strengthen some of the assumptions. We assume the following conditions:

(B) $\|B(t, w') - B(t, w)\|_Z \leq \mu_B\|w' - w\|_Y$.

(f2) $\|f(t, w') - f(t, w)\|Y \leq \mu'_f\|w' - w\|_Y$.

Theorem 4.17

Let (Z′), (N), (S), (A1) *to* (A3), (B), (f1), *and* (f2) *be satisfied, where* $S(t, w)$ *is assumed to be independent of* w. *Then there is a positive constant* $T'' \leq T'$ *such that when* ϕ *varies in* Y *subject to* $\|\phi - y_0\|_Y \leq \rho'$, *the map* $\phi \mapsto u(t)$ *given by theorem* 4.16 *is continuous in the* Y*-norm, uniformly in* $t \in [0, T'']$.

Remark. As in Kato (1975*b*), one can prove a similar continuity theorem when not only the initial value ϕ but also the functions N, A, and f are varied, i.e., the solution is 'stable' when the equations themselves are varied. It appears, on the other hand, that the variation of S is rather difficult to handle.

The theorem thus guarantees the existence of (locally defined) maps

$$F_{t,s}: Y \to Y$$

which are continuous in all variables. We have

$$F_{s,s} = \mathrm{Id}$$
$$F_{t,s} \circ F_{s,r} = F_{t,r}$$

as in the linear case. We speak of $F_{t,s}$ as the *evolution operators generated by the equation* (4.5). The general notion of evolution operators for (4.4) is defined in an analogous manner.

The idea behind the proof of theorem 4.17 is to fix a curve $v(t)$, $v(0) = \phi$ in Y and to let $u(t)$ be the solution of the 'frozen coefficient problem'

$$\dot{u} = A(t, v)u + f(t, v), \quad u(0) = \phi$$

which is guaranteed by theorem 4.13. This defines a map Φ: $v \mapsto u$ and we look for a fixed point of Φ. In a suitable function space and for T' sufficiently small, Φ is in fact a contraction, so has a unique fixed point.

However, it is not so simple to prove that u depends continuously on ϕ and detailed estimates from the linear theory are needed. The proof more or less has to be delicate since the dependence on ϕ is not locally Lipschitz in general. For details of these proofs, we refer to Kato (1975*b*, 1977) and Hughes, Kato and Marsden (1977).

The continuous dependence of the solution on ϕ leads us naturally to investigate if it is smooth in any sense. This is important for studying the relationship between nonlinear theories and their linearization. The following results are taken from some unpublished notes of Dorroh and Marsden.

First, we give the notion of differentiability appropriate for the generator G of (4.4). Let X and Y be Banach spaces with $Y \subset X$ continuously and densely included. Let $U \subset Y$ be open and $f: U \to X$ be a given mapping. We say f is *α-differentiable* if for each $x \in U$ there is a bounded linear operator $\mathrm{D}f(x)$: $Y \to X$ such that

$$\frac{\|f(x+h)-f(x)-\mathrm{D}f(x)\cdot h\|_X}{\|h\|_X} \to 0$$

as $\|h\|_Y \to 0$. If f is α-differentiable and $x \mapsto \mathrm{D}f(x) \in B(Y, X)$ is norm continuous, we call f *C^1 α-differentiable.* Notice that this is *stronger* than C^1 in the Fréchet sense. If f is α-differentiable and

$$\|f(x+h)-f(x)-\mathrm{D}f(x)\cdot h\|_X / \|h\|_X$$

is uniformly bounded for x and $x+h$ in some T neighborhood of each point, we say that f is *locally uniformly α-differentiable.*

Most concrete examples can be checked using the following proposition.

Proposition 4.18

Suppose f: $U \subset Y \to X$ is of class C^2, and locally in the Y topology

$$x \mapsto \frac{\|\mathrm{D}^2 f(x)(h, h)\|_X}{\|h\|_Y \|h\|_X}$$

is bounded. Then f is locally uniformly C^1 α-differentiable.

This follows easily from the identity

$$f(x+h)-f(x)-\mathrm{D}f(x)\cdot h = \int_0^1 \int_0^1 \mathrm{D}^2 f(x+sth)(h, h)\,\mathrm{d}s\,\mathrm{d}t.$$

Next, we turn to the appropriate notion for the evolution operators.

A map g: $U \subset Y \to X$ is called *β-differentiable* if it is α-differentiable and $\mathrm{D}g(x)$, for each $x \in U$, extends to a bounded operator X to X.

β-differentiable maps obey a chain rule. For example, if $g_1: Y \to Y$, $g_2: Y \to Y$ and each is β-differentiable (as maps of Y to X) and are continuous from Y to Y, then $g_2 \circ g_1$ is β-differentiable with, of course,

$$D(g_2 \circ g_1)(x) = Dg_2[g_1(x)] \circ Dg_1(x).$$

The proof of this fact is routine. In particular, one can apply the chain rule to $F_{t,s} \circ F_{s,r} = F_{t,r}$ if each $F_{t,s}$ is β-differentiable. Differentiating this in s at $s = r$ gives the backwards equation for $x \in Y$:

$$\frac{\partial}{\partial s} F_{t,s}(x) = -DF_{t,s}(x) \cdot G(x).$$

Then differentiation in r at $r = s$ gives

$$DF_{t,s}(x) \cdot G(x) = G[F_{t,s}(x)],$$

the flow invariance of the generator.

We leave it to the reader to supply rigorous proofs of these claims following the hint from the linear case.

For the following theorem we assume these hypotheses: $Y \subset X$ is continuously and densely included and $F_{t,s}$ is a continuous evolution system on an open subset $D \subset Y$ and the X-infinitesimal generator $G(t)$ of $F_{t,s}$ has domain† D. Also, we assume:

(H_1) $G(t)$: $D \subset Y \to X$ is locally uniformly C^1 α-differentiable. Its derivative is denoted $D_x G(t, x)$ and is assumed strongly continuous in t.

(H_2) For $x \in D$, $s \geq 0$, let $T_{x,s}$ be the lifetime of x beyond s, i.e., $\sup\{t \geq s | F_{t,s}(x)$ is defined$\}$. Assume there is a strongly continuous linear evolution system $\{U^{x,s}(\tau, \sigma): 0 \leq \sigma \leq \tau \leq T_{x,s}\}$ in X whose X-infinitesimal generator is an extension of $\{D_x G(t, F_{t,s}x) \in B(Y, X);\ 0 \leq t \leq T_{x,s}\}$; i.e., if $y \in Y$,

$$\frac{\partial}{\partial \tau} U^{x,s}(\tau, \sigma) \cdot y \Big|_{\tau=0} = D_x G(\tau, F_{\tau,s}(x)) \cdot y.$$

Theorem 4.19 (J. R. Dorroh)

Under the hypotheses above, $F_{t,s}$ is β-differentiable at x and in fact,

$$DF_{t,s}(x) = U_{x,s}(t, s).$$

† As in the linear case, $G(t)$ may have an extension to a larger domain, but we are only interested in $G(t)$ on D here.

Proof. Define $\varphi_t(x, y) = \varphi(t, x, y)$ by

$$G(t, x) - G(t, y) = \mathrm{D}_x G(t, y) \cdot (x - y) + \|x - y\|_X \varphi_t(x, y)$$

(or zero if $x = y$) and notice that by local uniformity, $\|\varphi(t, x, y)\|_X$ is uniformly bounded if x and y are Y-close. By joint continuity of $F_{t,s}(x)$, for $0 < t < T_{x,s}$, $\|\varphi(t, F_{t,s}y, F_{t,s}x)\|_X$ is bounded for $0 \leqslant s \leqslant T$ if $\|x - y\|_Y$ is sufficiently small.

By construction, we have the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} F_{t,s}(x) = G[F_{t,s}(x)], \quad 0 \leqslant s \leqslant t \leqslant T_{x,s}, \; x \in D.$$

Let

$$w(t, s) = F_{t,s}(y) - F_{t,s}(x)$$

so that

$$\frac{\partial w(t, s)}{\partial t} = G(t, F_{t,s}(y)) - G(t, F_{t,s}(x))$$

$$= \mathrm{D}_x G(t, F_{t,s}(x)) w(t, s) + \|w(t, s)\|_X \varphi(t, F_{t,s}y, F_{t,s}x).$$

Since $\mathrm{D}_x G(t, F_{t,s}x) \cdot w(t, s)$ is continuous in t, s with values in X, and writing $U = U_{x,x}$ we have the backwards differential equation:

$$\frac{\partial}{\partial \sigma} U(t, \sigma) w(\sigma, s) = U(t, \sigma) \frac{\partial w(\sigma, s)}{\partial \sigma} - U(t, \sigma) \mathrm{D}_x G(\sigma, F_{\sigma,s}(x)) \cdot w(\sigma, s)$$

$$= U(t, \sigma) \cdot \|w(\sigma, s)\|_X \varphi(\sigma, F_{\sigma,s}(y), F_{\sigma,s}(x)).$$

Hence, integrating from $\sigma = s$ to $\sigma = t$,

$$w(t, s) = U(t, s)(y - x) + \int_s^t U(t, \sigma) \|w(\sigma, s)\|_X \varphi(\sigma, F_{\sigma,s}(y), F_{\sigma,s}(x))\, \mathrm{d}\sigma.$$

Let $\|U(\tau, \sigma)\|_{X,X} \leqslant M$, and $\|\varphi[\sigma, F_{\sigma,s}(y), F_{\sigma,s}(x)]\|_X \leqslant M_2$, $0 \leqslant s \leqslant \sigma \leqslant \tau \leqslant T$. Thus, by Gronwall's inequality,

$$\|w(t, s)\|_x \leqslant M_1 \, \mathrm{e}^{M_1 M_2} T \|y - x\|_X = M_3 \|y - x\|_X.$$

In other words,

$$\frac{\|F_{t,s}(y) - F_{t,s}(x) - U(t, s)(y - x)\|_X}{\|y - x\|_X} \leqslant M_1 M_3 \int_s^t \|\varphi(\sigma, F_{\sigma,s}(y), F_{\sigma,s}(x))_X \, \mathrm{d}\sigma.$$

From the bounded convergence theorem, we conclude that $F_{t,s}$ is β-differentiable at x and $\mathrm{D}F_{t,s}(x) = U(t, s)$. ($\varphi(t, F_{t,s}(y), F_{t,s}(x))$ is strongly measurable in s since $\varphi(x, y)$ is continuous for $x \neq y$.) ■

This completes our description of the abstract nonlinear theory. Next, we state how the nonlinear existence and uniqueness theorem applies to quasi-linear equations of types (4.2) and (4.3).

First, we consider the first-order case:

$$a_0(t, x, u)\frac{\partial u}{\partial t} = \sum_{j=1}^{m} a_j(t, x, u)\frac{\partial u}{\partial x^j} + a(t, x, u). \tag{4.6}$$

We assume

(i) $s > \frac{1}{2}m + 1$ and a_α, a are of class C^{s+1} in the variables t, x, u (possibly locally defined in u);

(ii) the linear conditions (i), (ii), (iii) of theorem 4.14 hold locally uniformly in u.

Theorem 4.20

Under these conditions, theorems 4.16, 4.17 *and* 4.19 *hold for* (4.6), *i.e.*, (4.6) *generates a unique local evolution system* $F_{t,s}$ *in* $X = H^{s-1}(\mathbb{R}^m)$ *with* $Y = H^s(\mathbb{R}^m)$ *and* $Z = Z' = L^2(\mathbb{R}^m)$; $F_{t,s}$ *maps* Y *to* Y *continuously and, for* t, s *fixed, is* β*-differentiable as a map of* Y *to* X.

The full details of the proof require a lengthy discussion of Sobolev space estimates to verify the hypotheses, but it is relatively straightforward. See Kato (1975*a*, *b*) for details. We note that one may also choose $X = Z = Z' = L^2(\mathbb{R}^m)$, $Y = H^s(\mathbb{R}^m)$, but the choices in theorem 4.20 are appropriate for theorem 4.19. Again, length and their technical nature preclude giving details of how theorem 4.19 applies to (4.6). It is again a semi-routine Sobolev space exercise.

For the second-order case, we proceed as follows. Consider

$$a_{00}(t, s, u, \nabla u)\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{m} a_{ij}(t, x, u, \nabla u)\frac{\partial^2 u}{\partial x^i\,\partial x^j} + 2\sum_{j=1}^{m} a_{0i}(t, x, u, \nabla u)\frac{\partial^2 u}{\partial t\,\partial x^i} + a(t, x, u, \nabla u). \tag{4.7}$$

Here

$$\nabla u = \left(\frac{\partial u}{\partial x^1}, \ldots, \frac{\partial u}{\partial x^m}, \frac{\partial u}{\partial t}\right).$$

We assume

(i) $a_{\alpha\beta}$, a are of class C^{s+1} in all variables (possibly locally defined in u);

(ii) the linear conditions (i), (ii), (iii), (iv) of theorem 4.15 hold locally uniformly in u.

Theorem 4.21

(i) *If* $s > \frac{1}{2}m + 1$, *theorems* 4.16, 4.17 *and* 4.19 *hold for* (4.7) *with*

$$X = H^s(\mathbb{R}^m) \times H^{s-1}(\mathbb{R}^m),$$
$$Z = Z' = H^1(\mathbb{R}^m) \times H^0(\mathbb{R}^m),$$
$$Y = H^{s+1}(\mathbb{R}^m) \times H^s(\mathbb{R}^m),$$

i.e., (4.7) *generates a unique local evolution system* $F_{t,s}: Y \to Y$ *which is continuous and for fixed t, s is* β*-differentiable from Y to X.*

(ii) *If* $a_{\alpha\beta}$ *do not depend on* ∇u, *then the same conclusions hold with* $s > \frac{1}{2}m$.

For details of the proof, see Hughes, Kato and Marsden (1977).

As we shall see in the next section, case (ii) is the case relevant for general relativity. Note that if $m = 3$, solutions $(u, \dot{u})$ will lie in $Y = H^r \times H^{r-1}$ where $r > 2.5$. For example, in this case, (4.7) gives a well-posed problem for u in H^3. (Notice that u is only C^1 in this case and need not be C^2.) For hyperbolic systems, theorems 4.20 and 4.21 are the sharpest known results, although these problems have been considered by a large number of authors,† such as Choquet-Bruhat (1952, 1962), Courant-Hilbert (1962), Dionne (1962), Frankl (1937), Krzyzanski and Schauder (1934), Leray (1953), Lichnerowicz (1967), Lions (1969), Petrovskii (1937), Schauder (1935), and Sobolev (1939).

4.4 The Cauchy problem for relativity

We shall begin with the vacuum problem and then go on to consider gravity coupled to other fields. We begin by reviewing the classic work of Lichnerowicz (1944) and Choquet-Bruhat (1952) and the introduction of harmonic coordinates. We shall be brief since this is described in Choquet-Bruhat (1962) and in Hawking and Ellis (1973). Our main result is that for H^s spacetimes with $s > 2.5$, there is a satisfactory existence theorem 4.23 and uniqueness theorem 4.27 for

† For relativity, some partial results in H^3 were indicated by Hawking and Ellis (1973, p. 251).

the Cauchy problem. These results are the sharpest presently known for the Cauchy problem.

We work on $\mathbb{R}^4$ for simplicity and because of the hyperbolicity, with no essential loss of generality. For empty space relativity, one searches for a Lorentz metric $g_{\mu\nu}(t, x^i)$ whose Ricci curvature $R_{\mu\nu}$ is zero; i.e., $g_{\mu\nu}(t, x^i)$ must satisfy the system

$$R_{\mu\nu}\left(t, x^i, g_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x^\alpha}, \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}\right) = -\frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} - \frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu}$$
$$+ \frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\alpha\mu}}{\partial x^\beta \partial x^\nu}$$
$$+ H_{\mu\nu}\left(g_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)$$
$$= 0,$$

where $H_{\mu\nu}(g_{\mu\nu}, \partial g_{\mu\nu}/\partial x^\alpha)$ is a rational combination of $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial x^\alpha$ with denominator det $g_{\mu\nu} \neq 0$. Note that the contravariant tensor $g^{\mu\nu}$ is a rational combination of the $g_{\mu\nu}$ with denominator det $g_{\mu\nu} \neq 0$.

Let $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ be the Einstein tensor, where $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature. Then, as is well known, $G^0{}_\mu$ contains only first-order time derivatives of $g_{\mu\nu}$. Thus $G^0{}_\mu(0, x^i)$ can be computed from the Cauchy data $g_{\mu\nu}(0, x^i)$ and $\partial g_{\mu\nu}(0, x^i)/\partial t$ alone, and therefore $G^0{}_\mu(0, x^i) = 0$ is a necessary condition on the Cauchy data in order that a spacetime $g_{\mu\nu}(t, x^i)$ have the given Cauchy data and satisfy $G_{\mu\nu} = 0$, which is equivalent to $R_{\mu\nu} = 0$.

The existence part of the Cauchy problem for the system $R_{\mu\nu} = 0$ is as follows.

Let $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ *be Cauchy data of class* $(H^s(\Omega), H^{s-1}(\Omega))$, $s \geq 3$, *such that* $\mathring{G}^0{}_\mu(x^i) = 0$. *Let* Ω_0 *be a proper subdomain*, $\bar{\Omega}_0 \subset \Omega$. *Find an* $\varepsilon > 0$ *and a spacetime* $g_{\mu\nu}(t, x^i)$, $|t| < \varepsilon$, $(x^i) \in \Omega_0 \subset \Omega$ *such that*

(i) $g_{\mu\nu}(t, x^i)$ *is* H^s *jointly in* $(t, x^i) \in (-\varepsilon, \varepsilon) \times \Omega_0$;

(ii) $(g_{\mu\nu}(0, x^i), \partial g_{\mu\nu}(0, x^i)/\partial t) = (\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$;

(iii) $g_{\mu\nu}(t, x^i)$ *has zero Ricci curvature.*

The system $R_{\mu\nu} = 0$ is a quasi-linear system of ten second-order partial differential equations for which the highest-order terms involve mixing of the components of the system. As it stands, there are no known theorems about partial differential equations which can be applied to resolve the Cauchy problem. However, as was first noted by

Lanczos (1922) (and in fact by Einstein himself (1916*b*) for the linearized equations) the Ricci tensor simplifies considerably in harmonic coordinates, i.e., in a coordinate system for which the contracted Cristoffel symbols vanish, $\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0$. In fact, an algebraic computation shows that

$$R_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha\,\partial x^\beta} + \frac{1}{2}g_{\mu\alpha}\frac{\partial \Gamma^\alpha}{\partial x^\nu} + \frac{1}{2}g_{\nu\alpha}\frac{\partial \Gamma^\alpha}{\partial x^\mu} + H_{\mu\nu}$$

so that in a coordinate system for which $\Gamma^\mu = 0$,

$$R_{\mu\nu} = R^{(\mathrm{h})}_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha\,\partial x^\beta} + H_{\mu\nu}.$$

The operator $-\frac{1}{2}g^{\alpha\beta}(\partial^2/\partial x^\alpha\,\partial x^\beta)$ operates the same way on each component of the system $g_{\mu\nu}$ so that there is no mixing in the highest-order derivatives. Thus the normalized system $R^{(\mathrm{h})}_{\mu\nu} = 0$ is considerably simpler than the full system. In fact, the system $R^{(\mathrm{h})}_{\mu\nu} = 0$ has only simple characteristics so that $R^{(\mathrm{h})}_{\mu\nu} = 0$ is a strictly hyperbolic system.

The importance of the use of harmonic coordinates and of the system $R^{(\mathrm{h})}_{\mu\nu} = 0$ is based on the fact that it is sufficient to solve the Cauchy problem for $R^{(\mathrm{h})}_{\mu\nu} = 0$; this remarkable fact, discovered by Choquet-Bruhat (1952), is based on the observation that the condition $\mathring{\Gamma}^\mu(x^i) \equiv \mathring{g}^{\alpha\beta}(x^i)\mathring{\Gamma}^\mu_{\alpha\beta}(x^i) = 0$ is propagated off the hypersurface $t = 0$ for solutions $g_{\mu\nu}$ of $R^{(\mathrm{h})}_{\mu\nu} = 0$. This is established in the next lemma.

Lemma 4.22

Let $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ *be of Sobolev class* (H^s, H^{s-1}) *on* Ω, $s > \frac{1}{2}n + 1$, $n = 3$, *and suppose that* $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ *satisfies*

(i) $\mathring{\Gamma}^\mu(x^i) = 0$,

(ii) $\mathring{G}^0_\mu(x^i) = 0$.

If $g_{\mu\nu}(t, x)$, $|t| < \varepsilon$, $x \in \Omega_0$, Ω_0 *a proper subdomain,* $\bar{\Omega}_0 \subset \Omega$, *is an* H^s-*solution of*

$$R^{(\mathrm{h})}_{\mu\nu} = -\tfrac{1}{2}g^{\alpha\beta}(\partial^2 g_{\mu\nu}/\partial x^\alpha\,\partial x^\beta) + H_{\mu\nu} = 0,$$

$$(g_{\mu\nu}(0, x), \partial g_{\mu\nu}(0, x)/\partial t) = (\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i)),$$

then $\Gamma^\mu(t, x^i) = 0$ *for* $|t| < \varepsilon$, $x \in \Omega_0$.

Proof. Let $g_{\mu\nu}(t, x^i)$ satisfy (i), (ii), and $R^{(\mathrm{h})}_{\mu\nu} = 0$. Then a straightforward computation shows that $\Gamma^\mu(t, x^i) = g^{\alpha\beta}(t, x^i)\Gamma^\mu_{\alpha\beta}(t, x^i)$ satisfies

$\partial\Gamma^\mu(0, x^i)/\partial t = 0$. From $G^{\mu\nu}{}_{;\nu} = 0$ and $R^{(h)}_{\mu\nu} = 0$, Γ^μ is shown to satisfy the system of linear equations

$$g^{\alpha\beta}\frac{\partial^2\Gamma^\mu}{\partial x^\alpha\,\partial x^\beta} + A^{\beta\mu}_\alpha\left(g_{\mu\nu}, g^{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)\frac{\partial\Gamma^\alpha}{\partial x^\beta} = 0.$$

This linear system is of the form (4.3) for which a uniqueness and existence theorem holds. Thus, by theorem 4.15, $\Gamma^\mu(0, x^i) = 0$ and $\partial\Gamma^\mu(0, x^i)/\partial t = 0$ imply $\Gamma^\mu(t, x^i) = 0$. ■

According to the lemma, an H^s-solution of $R^{(h)}_{\mu\nu} = 0$ with prescribed Cauchy data is also a solution of. $R_{\mu\nu} = 0$ (since $\Gamma^\mu(t, x) = 0 \Rightarrow R^{(h)}_{\mu\nu} = R_{\mu\nu}$), provided that the Cauchy data satisfies (i) $\mathring{\Gamma}^\mu = 0$ and (ii) $\mathring{G}^0{}_\mu = 0$. As mentioned above, (ii) is a necessary condition on the Cauchy data for a solution $g_{\mu\nu}(t, x)$ to satisfy $R_{\mu\nu} = 0$. If (i) is not satisfied, then a set of Cauchy data can be found whose evolution under $R^{(h)}_{\mu\nu} = 0$ leads to an H^s-spacetime which, by an H^{s+1}-coordinate transformation, gives rise to a spacetime with the original Cauchy data (see theorem 4.26 below and Fischer and Marsden, 1972*b*).

From theorem 4.21 we conclude that Cauchy data of class (H^s, H^{s-1}) has an H^s-time evolution for $s > 2.5$ and Cauchy stability holds.

We can also prove this result by reducing the strictly hyperbolic system $R^{(h)}_{\mu\nu} = 0$ to a quasi-linear symmetric hyperbolic first-order system. This will be outlined below.

Theorem 4.23

Let Ω be an open bounded domain in $\mathbb{R}^3$ with Ω_0 a proper subdomain, $\bar{\Omega}_0 \subset \Omega$, and let $(\mathring{g}_{\mu\nu}(x), \mathring{k}_{\mu\nu}(x))$, $(x^i) \in \Omega$, $0 \leq \mu$, $\nu \leq 3$, $1 \leq i \leq 3$, be of Sobolev class (H^s, H^{s-1}), $s > 2.5$. Suppose that $\mathring{\Gamma}^\mu(x^i) = 0$ and $\mathring{G}^0{}_\mu(x) = 0$. Then there exists an $\varepsilon > 0$ and a unique Lorentz metric $g_{\mu\nu}(t, x)$, $|t| < \varepsilon$, $(x^i) \in \Omega_0$ such that

(i) $g_{\mu\nu}(t, x^i)$ is jointly of class H^s;
(ii) $R^{(h)}_{\mu\nu}(t, x^i) = 0$;
(iii) $(g_{\mu\nu}(0, x^i), \partial g_{\mu\nu}(0, x^i)/\partial t) = (\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$.

From lemma 4.22, this $g_{\mu\nu}(t, x^i)$ also satisfies $R_{\mu\nu}(t, x^i) = 0$. Moreover, $g_{\mu\nu}(t, x^i)$ depends continuously on $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ in the (H^s, H^{s-1}) topology. If $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ is of class (C^∞, C^∞) on Ω, then $g_{\mu\nu}(t, x^i)$ is C^∞ for all t for which the solution exists.

See below for a discussion of solutions on all of $\mathbb{R}^3$ with spatial asymptotic conditions.

We have already indicated how this follows directly from theorem 4.21 and lemma 4.22. To give another proof using theorem 4.20, we reduce the system $R^{(h)}_{\mu\nu}=0$ to a first-order system by introducing the ten new unknowns $k_{\mu\nu}=\partial g_{\mu\nu}/\partial t$ and the thirty new unknowns $g_{\mu\nu,i}=\partial g_{\mu\nu}/\partial x^i$ and considering the quasi-linear first-order system of fifty equations:

$$\partial g_{\mu\nu}/\partial t = k_{\mu\nu},$$

$$g^{ij}\left(\frac{\partial g_{\mu\nu,i}}{\partial t}\right) = g^{ij}\frac{\partial k_{\mu\nu}}{\partial x^i}, \tag{4.8}$$

$$-g^{00}\frac{\partial k_{\mu\nu}}{\partial t} = 2g^{0j}\frac{\partial k_{\mu\nu}}{\partial x^j} + g^{ij}\frac{\partial g_{\mu\nu,i}}{\partial x^j} - 2H_{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu}).$$

We are considering $H_{\mu\nu}$ as a polynomial in $g_{\mu\nu,i}$ and $k_{\mu\nu}$, and rational in $g_{\mu\nu}$ with denominator $\det g_{\mu\nu}\neq 0$. At first, we extend our initial data to all of $\mathbb{R}^3$, say, to equal the Minkowski metric outside a compact set, and consider the system (4.8) on $\mathbb{R}^3$. Note that the Cauchy data need not satisfy the constraints $G^0{}_\mu=0$ during the transition.

The matrix g^{ij} has inverse $g_{jk}-(g_{j0}g_{k0}/g_{00})$, i.e., $g^{ij}[g_{jk}-(g_{j0}g_{k0}/g_{00})]=\delta^i_k$, so that the second set of thirty equations can be inverted to give

$$\partial g_{\mu\nu,i}/\partial t = \partial k_{\mu\nu}/\partial x^i. \tag{4.9}$$

For $g_{\mu\nu}$ of class C^2, (4.9) implies

$$g_{\mu\nu,i} = \partial g_{\mu\nu}/\partial x^i,$$

so that the system (4.8) is equivalent to $R^{(h)}_{\mu\nu}=0$.

Let

$$u = \begin{pmatrix} g_{\mu\nu} \\ g_{\mu\nu,i} \\ k_{\mu\nu} \end{pmatrix}$$

be a fifty-component column vector, where $g_{\mu\nu,i}$ is listed as

$$\begin{pmatrix} g_{00,1} \\ \vdots \\ g_{33,1} \\ \vdots \\ g_{00,3} \\ \vdots \\ g_{33,3} \end{pmatrix}$$

Let $0^{10}=10\times 10$ zero matrix, $I^{10}=10\times 10$ identity matrix, and let $A^0(u)=A^0(g_{\mu\nu}, g_{\mu\nu}, k_{\mu\nu})$ and $A^j(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu})$ be the 50×50 matrices given by

$$A^0(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu})=\begin{pmatrix} I^{10} & 0^{10} & 0^{10} & 0^{10} & 0^{10} \\ 0^{10} & g^{11}I^{10} & g^{12}I^{10} & g^{13}I^{10} & 0^{10} \\ 0^{10} & g^{12}I^{10} & g^{22}I^{10} & g^{23}I^{10} & 0^{10} \\ 0^{10} & g^{13}I^{10} & g^{23}I^{10} & 0^{10} & 0^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & -g^{00}I^{10} \end{pmatrix}$$

$$A^j(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu})=\begin{pmatrix} 0^{10} & 0^{10} & 0^{10} & 0^{10} & 0^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & g^{j1}I^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & g^{j2}I^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & g^{j3}I^{10} \\ 0^{10} & g^{1j}I^{10} & g^{2j}I^{10} & g^{3j}I^{10} & 2g^{j0}I^{10} \end{pmatrix}$$

and let $B(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu})$ be the fifty-component column vector given by

$$B(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu})=\begin{pmatrix} k_{\mu\nu} \\ 0^{30} \\ -2H_{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu}) \end{pmatrix}$$

where 0^{30} is the thirty-component zero column vector.

Note that $A^0(u)$ and $A^i(u)$ are symmetric, and that $A^0(u)$ is positive-definite if $g_{\mu\nu}$ has Lorentz signature. A direct verification shows that the first-order quasi-linear symmetric hyperbolic system

$$A^0(u)(\partial u/\partial t)=A^i(u)(\partial u/\partial x^i)+B(u)$$

is just the system (4.8). From theorem 4.20 we conclude that for Cauchy data

$$\mathring{u}(x^i)=\begin{pmatrix} \mathring{g}_{\mu\nu}(x^i) \\ \mathring{g}_{\mu\nu,i}(x^i) \\ \mathring{k}_{\mu\nu}(x^i) \end{pmatrix}$$

of Sobolev class H^{s-1}, $s-1>\frac{1}{2}n+1$, there exists an $\varepsilon>0$ and a solution

$$u(t, x^i)=\begin{pmatrix} g_{\mu\nu}(t, x^i) \\ g_{\mu\nu,i}(t, x^i) \\ k_{\mu\nu}(t, x^i) \end{pmatrix}$$

of class H^{s-1}. By Sobolev's lemma, $u(t, x^i)$ is also of class C^2, and so, by the second set of equations of (4.5), $g_{\mu\nu,i} = \partial g_{\mu\nu}/\partial x^i$. Since $(g_{\mu\nu,i}, k_{\mu\nu}) = (\partial g_{\mu\nu}/\partial x^i, \partial g_{\mu\nu}/\partial t)$ is of class H^{s-1}, $g_{\mu\nu}(t, x^i)$ is in fact of class H^s. The continuous dependence of the solutions on the initial data follows from the general theory.

To recover the result for the domain Ω from the result for $\mathbb{R}^n$, we can use the standard domain of dependence arguments; see Courant and Hilbert (1962).

Since Ω is bounded, $(\overset{\circ}{g}_{\mu\nu}, \overset{\circ}{k}_{\mu\nu})$ of class C^∞ implies that the solution is in the intersection of all the Sobolev spaces and hence is C^∞; again, we are using the general regularity result about symmetric hyperbolic systems.

From lemma 4.22, the $g_{\mu\nu}(t, x^i)$ so found satisfy the field equations $R_{\mu\nu} = 0$.

While the second-order approach gives $s > 2.5$, e.g., $s = 3$ (see theorem 4.21(ii)) the first-order approach as it stands only gives $s > 3.5$, e.g., $s = 4$. It can be refined, but it requires a knowledge of the special structure of the equations and ellipticity. For these reasons, the second-order methods seem more attractive.

For the case of asymptotic conditions, some care must be exercised. Spacetimes which are spatially like $1/r$ will not be of class H^s. Fix a background spacetime $\overset{b}{g}_{\alpha\beta}$ with prescribed fall-off to the Minkowski metric at ∞. For example, a specified mass will determine the coefficient of $1/r$; $\overset{b}{g}_{\alpha\beta}$ could be a Schwarzschild-type solution with the singularity at $r = 0$ smoothed out.

We let our variables be $u_{\alpha\beta} = g_{\alpha\beta} - \overset{b}{g}_{\alpha\beta}$ and solve for $u_{\alpha\beta}$. Although $g_{\alpha\beta}$ will not be in H^s itself, $u_{\alpha\beta}$ will be.

Assume the following conditions on $\overset{b}{g}_{\alpha\beta}$:

$$\overset{b}{g}_{\alpha\beta} \in C_b^{s+1}(\mathbb{R}^3, \mathbb{R}), \quad \dot{\overset{b}{g}}_{\alpha\beta} \in H^s(\mathbb{R}^3, \mathbb{R})$$

and (4.10)

$$\frac{\partial \overset{b}{g}_{\alpha\beta}}{\partial x^i} \in H^s(\mathbb{R}^3, \mathbb{R}), \quad 0 \leqslant \alpha, \beta \leqslant 3, \quad 1 \leqslant i \leqslant 3.$$

In the variables $u_{\alpha\beta}$, the equations (4.8) are of the form (4.7).

The coefficients of the second-order terms do not involve derivatives of u, so only $s > \frac{1}{2}n$ is required.

Let us write $H^s_{\overset{b}{g}_{\alpha\beta}}$ for the space of $g_{\alpha\beta}$ such that $g_{\alpha\beta} - \overset{b}{g}_{\alpha\beta} \in H^s$, topologized accordingly. Then theorem 4.21 yields:

Theorem 4.24

Let (4.10) *hold. Then, for* $s > 1.5$ *and initial data in a ball about* $(g^b_{\alpha\beta}, \dot{g}^b_{\alpha\beta})$ *in* $H^{s+1}_{g^b_{\alpha\beta}} \times H^s_{g^b_{\alpha\beta}}$, (4.8) *have a unique solution in the same space for a time interval* $[0, T']$, $T' > 0$. *The solution depends continuously on the initial data in this space (i.e., it is well posed or 'Cauchy stable') and smoothly in the sense of theorem* 4.19.

Thus, with the asymptotic conditions subtracted off, $H^3 \times H^2$ initial data generates a piece of H^3 spacetime in a way which depends continuously on the initial data. If T' is allowed to be large, the Lorentz character of $g_{\alpha\beta}$ could be lost or a singularity could develop.

A by-product of the proof is regularity; i.e., if existence holds in $H^{s+1} \times H^s$ on $[0, T']$ and the initial data is smoother, then so is the solution on the *same* interval $[0, T']$. Thus C^∞ initial data gives C^∞ solutions.

An interesting problem is to determine whether or not the spacetime generated by initial data satisfying (4.10) is large enough to include asymptotic boosts. An examination of the proofs shows that the time of existence increases at least logarithmically at spatial infinity, so the proofs as they stand do not seem to give an affirmative answer.

We now show that any two H^s-spacetimes, $s > 2.5$, which are Ricci flat and which have the same Cauchy data are related by an H^{s+1}-coordinate transformation. The key idea is to show that any H^s-spacetime when expressed in harmonic coordinates is also of class H^s. This in turn is based on an old result of Sobolev (1963); namely, that solutions to the wave equation with (H^s, H^{s-1}) coefficients preserve (H^{s+1}, H^s) Cauchy data, a result implied by theorem 4.15. We can give an alternative proof of this result using the well-known result that any single second-order hyperbolic equation can be reduced to a system of symmetric hyperbolic equations (see Fischer and Marsden, 1972*b*). The result follows:

Lemma 4.25

Let $s > 2.5$ *and* $(\psi_0(x), \dot{\psi}_0(x))$ *be of Sobolev class* (H^{s+1}, H^s) *on* $\mathbb{R}^3$. *Then there exists a unique* $\psi(t, x)$ *of class* H^{s+1} *that satisfies*

$$g^{\mu\nu}(t, x)\left(\frac{\partial^2 \psi}{\partial x^\mu \, \partial x^\nu}\right) + b^\mu(t, x)\left(\frac{\partial \psi}{\partial x^\mu}\right) + c(t, x)\psi = 0$$

$$\left(\psi(0, x), \frac{\partial \psi(0, x)}{\partial t}\right) = (\psi_0(x), \dot{\psi}_0(x)),$$

where $g^{\mu\nu}(t, x)$ is a Lorentz metric of class H^s, $b^\mu(t, x)$ is a vector field of class H^{s-1}, and $c(t, x)$ is of class H^{s-1}.

We can now prove that when one transforms an H^s-spacetime to harmonic coordinates, it stays H^s.

Theorem 4.26

Let $g_{\mu\nu}(t, x)$ be an H^s-spacetime, $s>2.5$. Then there exists an H^{s+1}-coordinate transformation $\bar{x}^\lambda(x^\mu)$ such that

$$\bar{g}_{\mu\nu}(\bar{x}^\lambda)=\frac{\partial x^\alpha}{\partial \bar{x}^\mu}(\bar{x}^\lambda)\frac{\partial x^\beta}{\partial \bar{x}^\nu}(\bar{x}^\lambda)g_{\alpha\beta}[x^\mu(\bar{x}^\lambda)]$$

is an H^s-spacetime with $\bar{\Gamma}^\mu(\bar{t}, \bar{x})=\bar{g}^{\alpha\beta}\bar{\Gamma}^\mu_{\alpha\beta}(\bar{t}, \bar{x})=0$.

Proof. To find $\bar{x}^\lambda(x^\mu)$ consider the wave equation

$$\Box\psi=-g^{\alpha\beta}\left(\frac{\partial^2\psi}{\partial x^\alpha\,\partial x^\beta}\right)+g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\left(\frac{\partial\psi}{\partial x^\mu}\right)=0,$$

and let $\bar{t}(t, x)$ be the unique solution of the wave equation with Cauchy data $t(0, x)=0$, $\partial\bar{t}(0, x)/\partial t=1$, and let $\bar{x}^i(t, x)$ be the unique solution of the wave equation with Cauchy data

$$\bar{x}^i(0, x)=x^i,\quad \frac{\partial\bar{x}^i}{\partial t}(0, x)=0.$$

For $g_{\mu\nu}$ of class H^s, Γ^μ is of class H^{s-1}, so $\bar{t}(t, x)$ and $\bar{x}(t, x)$ are H^{s+1}-functions and in fact by the inverse function theorem for H^s-functions (Ebin, 1970), $(\bar{t}(t, x), \bar{x}(t, x))$ is an H^{s+1} diffeomorphism in a neighborhood of $t=0$.

Since $\Box\,\bar{x}^\mu(t, x)=0$ is an invariant equation,

$$\Box\,\bar{x}^\mu=-\bar{g}^{\alpha\beta}\frac{\partial^2\bar{x}^\mu}{\partial\bar{x}^\alpha\,\partial\bar{x}^\beta}+\bar{g}^{\alpha\beta}\bar{\Gamma}^\nu_{\alpha\beta}\frac{\partial\bar{x}^\mu}{\partial\bar{x}^\nu}=\bar{g}^{\alpha\beta}\bar{\Gamma}^\mu_{\alpha\beta}=0$$

in the barred coordinate system, so $\bar{x}^\mu$ is a system of harmonic coordinates. ■

Remark. This theorem may be regarded as a special case of the general theory of harmonic maps (Eells and Sampson, 1964).

As a simple consequence of lemma 4.25 we have the following uniqueness result for the Einstein equations:

Theorem 4.27

Let $g_{\mu\nu}(t, x)$ and $\bar{g}_{\mu\nu}(t, x)$ be two Einstein flat H^s-spacetimes with $s > 2.5$ and such that $(g_{\mu\nu}(0, x), \partial g_{\mu\nu}(0, x)/\partial t) = (\bar{g}_{\mu\nu}(0, x), \partial \bar{g}_{\nu\nu}(0, x)/\partial t)$. Then $g_{\mu\nu}(t, x)$ and $\bar{g}_{\mu\nu}(t, x)$ are related by an H^{s+1}-coordinate change in a neighborhood of $t = 0$.

Proof. From lemma 4.25 there exist H^{s+1}-coordinate transformations $y^\mu(x^\alpha)$ and $\bar{y}^\mu(x^\alpha)$ such that the transformed metrics

$$(\partial x^\alpha/\partial y^\mu)(\partial x^\beta/\partial y^\nu)g_{\alpha\beta} \text{ and } (\partial x^\alpha/\partial \bar{y}^\mu)(\partial x^\beta/\partial \bar{y}^\nu)\bar{g}_{\alpha\beta}$$

satisfy $R^{(h)}_{\mu\nu} \doteq 0$. Since the Cauchy data for $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are equal, the transformed metrics also have the same Cauchy data. By uniqueness,

$$(\partial x^\alpha/\partial y^\mu)(\partial x^\beta/\partial y^\nu)g_{\alpha\beta} = (\partial x^\alpha/\partial \bar{y}^\mu)(\partial x^\beta/\partial \bar{y}^\nu)\bar{g}_{\alpha\beta}.$$

Since the composition of H^{s+1}-coordinate changes is also H^{s+1}, $\bar{g}_{\alpha\beta}$ is related to $g_{\alpha\beta}$ by an H^{s+1}-coordinate change in a neighbourhood of $t = 0$. ■

The local existence and uniqueness theorems 4.23 and 4.27 can be globalized in the same spirit that one studies maximal integral curves for systems of ordinary differential equations. This leads to the following theorem of Choquet-Bruhat and Geroch (1969).

Theorem 4.28

Fix a compact manifold M and let $(g_0, \pi_0) \in \mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_{\delta} = \mathscr{C}$ (the solutions of the constraint equations). Then there is a spacetime $(V_4, {}^{(4)}g_0)$ and a spacelike embedding i_0: $M \to V_4$ such that:

(i) Ein $({}^{(4)}g_0) = 0$;

(ii) *the metric and conjugate momentum induced on $\Sigma_0 = i_0(M)$ is* (g_0, π_0);

(iii) Σ_0 *is a Cauchy surface*;†

(iv) $(V_4, {}^{(4)}g_0)$ *is maximal (i.e., cannot be properly and isometrically embedded in another spacetime with properties* (i), (ii), *and* (iii)).

† So that $(V_4, {}^{(4)}g)$ is globally hyperbolic (Hawking and Ellis, 1973, Proposition 6.6.3), and hence any compact spacelike hypersurface is Cauchy (Budic, Isenberg, Lindblom and Yasskin, 1977).

This spacetime $(V_4, {}^{(4)}g_0)$ *is unique in the sense that if we have another* $(V'_4, {}^{(4)}g'_0)$ *with* (i)–(iv) *holding, there is a unique diffeomorphism* F: $V_4 \to V'_4$ *such that*

(i) $F^* \, {}^{(4)}g'_0 = {}^{(4)}g_0$ (F is isometric) *and*

(ii) $F \circ i_0 = i'_0$.

The proof is conveniently available in Hawking and Ellis (1973). The uniqueness of F uses the fact that an isometry is determined by its action on a frame at a point. The linearized version of this result is needed in the next section (see Fischer and Marsden, 1978*a*, for details).

Theorem 4.29

Let $(V_4, {}^{(4)}g_0)$ *be a vacuum spacetime, i.e.,* $\text{Ein}\,({}^{(4)}g_0) = 0$ *with a compact Cauchy surface* $\Sigma_0 = i_0(M)$ *and with induced metric and canonical momentum* $(g_0, \pi_0) \in \mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_{\delta}$. *Let* $(h_0, \omega_0) \in S_2 \times S^2_d$ *satisfy the linearized constraint equations, i.e.,*

$$\mathrm{D}\Phi(g_0, \pi_0) \cdot (h_0, \omega_0) = 0.$$

Then there exists an ${}^{(4)}h_0 \in S_2(V_4)$ *such that*

$$\mathrm{D}\,\mathrm{Ein}\,({}^{(4)}g_0) \cdot {}^{(4)}h_0 = 0$$

and such that the linearized Cauchy data induced by ${}^{(4)}h_0$ *on* Σ_0 *is* (h_0, ω_0).

If ${}^{(4)}h'_0$ *is another such solution, there is a unique vector field* ${}^{(4)}X$ *on* V_4 *such that*

$${}^{(4)}h'_0 = {}^{(4)}h_0 + L_{{}^{(4)}X}\, {}^{(4)}g_0$$

and ${}^{(4)}X$ *and its derivative vanish on* Σ_0.

Remarks (*a*). The linearized Cauchy data is defined in the same manner as the (g, π) are defined. In fact, if ${}^{(4)}g(\rho)$ is a curve of Lorentz metrics tangent to ${}^{(4)}h$ at ${}^{(4)}g_0$, then

$$(h_0, \omega_0) = \left(\left.\frac{\partial g(\rho)}{\partial \rho}\right|_{\rho=0}, \; \left.\frac{\partial \pi(\rho)}{\partial \rho}\right|_{\rho=0} \right)$$

where $(g(\rho), \pi(\rho))$ are the Cauchy data induced on Σ_0 from ${}^{(4)}g(\rho)$.

(*b*). One can view harmonic coordinates as a technical tool in which to verify the abstract theory in section 4.3. However, once this is done, well-posedness follows in any gauge. For example, one can give a coordinate-free treatment of hyperbolic systems (see Marsden, Ebin and Fischer, 1972 p. 247). Furthermore, for numerical calculations, work of Smarr and others indicates that maximal slices or slices of constant mean curvature may be more useful than harmonic coordinates.

The abstract theory given in section 4.3 (see theorem 4.16) applies to fields coupled to gravity as well as to pure gravity. There are several points to be noted however (cf. Hawking and Ellis, 1963, Section 7.7).

(i) The fields should be minimally coupled to gravity so the hyperbolic character of the equations for the gravitational field is not destroyed.

(ii) The energy–momentum tensor must be a smooth function (not necessarily polynomial) of ${}^{(4)}g$, ${}^{(4)}\varphi$.

(iii) For fixed ${}^{(4)}g$, the (linearized) matter equations should be well posed. This is needed so that hypothesis (A1) of theorem 4.16 can be verified.†

The other conditions of theorem 4.16 are of a technical nature, but cannot be ignored (they sharpen and are needed to verify condition (*b*), p. 254, of Hawking and Ellis, 1973). For examples of coupled systems and existence theory done by direct methods, see Choquet-Bruhat (1962).

For systems coupled to gravity, the results on uniqueness and global Cauchy developments given above for the vacuum equations carry over in routine fashion.

4.5 Linearization stability of the vacuum Einstein equations

Linearization stability concerns the validity of first-order perturbation theory. The idea is the following. Suppose we have a differentiable function F and points x_0 and y_0 such that $F(x_0)=y_0$. A standard procedure for finding other solutions to the equation $F(x)=y_0$ near x_0 is to solve the linearized equation $DF(x_0)\cdot h=0$ and assert that $x=x_0+\rho h$ is, for small ρ, an approximate solution to $F(x)=y_0$. Technically, this assertion may be stated as follows: there exists a curve of exact solutions $x(\rho)$ for small ρ such that $F(x(\rho))=y_0$, $x(0)=x_0$, and $x'(0)=h$. If this assertion is valid, we say F is *linearization stable* at x_0. It is easy to give examples where the assertion is false. For instance, in two dimen-

† As noted by Hawking and Ellis (1973), this can be roughly described by saying that 'the null cones of the matter equations coincide with (as in the Einstein–Maxwell system) or lie within the null cone of the spacetime metric'.

sions $F(x_1, x_2) = x_1^2 + x_2^2 = 0$ has no solutions other than (0, 0), although the linearized equation $DF(0, 0) \cdot (h, k) = 0$ has many solutions. Thus it is a non-vacuous question whether or not an equation is linearization stable at some given solution. Intuitively, linearization stability means that first-order perturbation theory is valid near x_0 and there are no spurious directions of perturbation.

The question of linearization stability is important for relativity. In the literature it was often assumed that solutions to the linearized equations do in fact approximate solutions to the exact equations. However, Brill and Deser (1973) indicated that for the flat three-torus, with zero extrinsic curvature, there are solutions to the linearized constraint equations which are not approximated by a curve of exact solutions. They gave a second-order perturbation argument to show that subject to the condition $\operatorname{tr} \pi = 0$, there are no other nearby solutions to the constraint equations, except essentially trivial modifications, even though there are many non-trivial solutions to the linearized equations (see Fischer and Marsden, 1975*a*, for a complete proof). It is analogous to and is proved by techniques similar to the following *Isolation Theorem* in geometry (Fischer and Marsden, 1975b).†

Theorem 4.30

If M is compact and g_F is a flat metric on M, then there is a neighborhood U_{g_F} of g_F in the space of metrics $\mathcal{M}$ such that any metric g in the neighborhood U_{g_F} with $R(g) \geq 0$ is flat.

The proof amounts to a version of the Morse lemma adapted to infinite-dimensional spaces with special attention needed because of the coordinate invariance of the scalar curvature map.

The results on linearization stability are due, independently, to Choquet-Bruhat and Deser (1972) for flat space, and Fischer and Marsden (1973*a*, 1974, 1975*a*) for the general case of empty spacetimes with a compact hypersurface. The methods used are rather different. O'Murchadha and York (1974*a*) generalized the Choquet-Bruhat and Deser method to the case of spacetimes with a compact hypersurface; see Choquet-Bruhat and York (1979). Results for Robertson–Walker spacetimes were proved by D'Eath (1975) and results

† This result has recently been globalized by Schoen and Yau as a special case of their solution to the mass problem in relativity. For example, they prove that on the three-torus, any metric with $R(g) \geq 0$ is flat.

for gauge theories coupled to gravity have been obtained by Arms (1977). The flat space result is:

Theorem 4.31

Near Minkowski space, the Einstein empty space equations Ein $({}^{(4)}g)=0$ *are linearization stable.*

In this theorem, one must use suitable function spaces with asymptotic conditions and asymptotically flat spacetimes. We will only consider the compact case in this article; see Choquet-Bruhat, Fischer and Marsden (1978) for the non-compact case.

We begin by defining linearization stability for the empty space Einstein equations.

Let Ein $({}^{(4)}g_0)=0$. An *infinitesimal deformation* of ${}^{(4)}g_0$ is a solution ${}^{(4)}h \in S_2(V_4)$ of the linearized equations

$$\mathrm{D\,Ein}\,({}^{(4)}g_0)\cdot{}^{(4)}h=0.$$

The Einstein equations are *linearization stable at* ${}^{(4)}g_0$ (or ${}^{(4)}g_0$ is *linearization stable*) if for every infinitesimal deformation ${}^{(4)}h$ of ${}^{(4)}g_0$, there exists a C^1 curve ${}^{(4)}g(\rho)$ of exact solutions to the empty space field equations (on the same V_4),

$$\mathrm{Ein}\,[{}^{(4)}g(\rho)]=0,$$

such that ${}^{(4)}g(0)={}^{(4)}g_0$ and $\partial{}^{(4)}g(0)/\partial\rho={}^{(4)}h_0$.

This definition has to be qualified slightly to be strictly accurate. Namely, for any compact set $D\subset V_4$, we only require ${}^{(4)}g(\rho)$ to be defined for $|\rho|<\varepsilon$ where ε may depend on D. The reason for this is that ${}^{(4)}g(\rho)$ will be developed from a curve of Cauchy data $(g(\rho),\pi(\rho))$ and so ${}^{(4)}g(\rho)$ will be uniformly close to ${}^{(4)}g_0$ on compact sets for $|\rho|<\varepsilon$, but not on all of V_4 in general.

Since we are fixing the hypersurface topology M here, all Cauchy developments lead topologically to the same spacetime $V_4\approx\mathbb{R}\times M$, so fixing V_4 is not a serious restriction. Topological perturbations are, of course, another story.

Using the linearized dynamical Einstein system, linearization stability of the Einstein equations is equivalent to linearization stability of the constraint equations, as we shall see below. In fact, linearization stability of a well-posed hyperbolic system of partial differential equations is equivalent to linearization stability of any nonlinear constraints present.

In terms of the linearized map $D\Phi(g, \pi)$, we can give necessary and sufficient conditions for the constraint equations

$$\Phi(g, \pi) = 0$$

to be linearization stable at (g_0, π_0); that is, if $(h, \omega) \in S_2 \times S_d^2$ satisfies the linearized equations

$$D\Phi(g_0, \pi_0) \cdot (h, \omega) = 0,$$

then there exists a differentiable curve $(g(\rho), \pi(\rho)) \in T^*\mathcal{M}$ of exact solutions to the constraint equations

$$\Phi(g(\rho), \pi(\rho)) = 0$$

such that $(g(0), \pi(0)) = (g_0, \pi_0)$ and

$$\left(\frac{\partial g(0)}{\partial \rho}, \frac{\partial \pi(0)}{\partial \rho}\right) = (h, \omega).$$

The main result follows:

Theorem 4.32

Let $\Phi = (\mathcal{H}, \mathcal{J}): T^*\mathcal{M} \to C_d^\infty \times \Lambda_d^1$ be defined as in section 4.2 *so $\mathscr{C}_{\mathcal{H}} \cap \mathscr{C}_{\delta} = \Phi^{-1}(0)$. Let $(g_0, \pi_0) \in \mathscr{C}_{\mathcal{H}} \cap \mathscr{C}_{\delta}$. The following conditions are equivalent*:

(i) *the constraint equations*

$$\Phi(g, \pi) = 0$$

are linearization stable at (g_0, π_0);

(ii) $D\Phi(g_0, \pi_0): S_2 \times S_d^2 \to C_d^\infty \times \Lambda_d^1$ *is surjective*;

(iii) $[D\Phi(g_0, \pi_0)]^*: C^\infty \times \mathscr{X} \to S_d^2 \times S_2$ *is injective.*

Remark. In section 4.2 we listed some sufficient conditions in order for (ii) to be valid, namely the conditions $C_{\mathcal{H}}$, C_δ, and C_{tr}.

Proof of theorem 4.32. In section 4.2 we showed that $[D\Phi(g_0, \pi_0)]^*$ is elliptic. Thus, the equivalence of (ii) and (iii) is an immediate consequence of the Fredholm alternative.

(ii) implies (i). The kernel of $D\Phi(g_0, \pi_0)$ splits by the Fredholm alternative. Thus the implicit function theorem implies that near (g_0, π_0), $\Phi^{-1}(0)$ is a smooth manifold. Here one must use the Sobolev spaces and pass to C^∞ by a regularity argument, as in Fischer and Marsden (1975*b*). Since any tangent vector to a smooth manifold is tangent to a curve in the manifold, (i) results.

(i) implies (iii). This is less elementary and will just be sketched. Assume (i) and that $[\mathrm{D}\Phi(g_0, \pi_0)]^* \cdot (N, X) = 0$, but $(N, X) \neq 0$. We will derive a contradiction by showing that there is a necessary second-order condition on first-order deformations (h, ω) that must be satisfied in order for the deformation to be tangent to a curve of exact solutions to the constraints. Thus, let (h, ω) be a solution to the linearized equations, and let $(g(\rho), \pi(\rho))$ be a curve of exact solutions of

$$\Phi(g(\rho), \pi(\rho)) = 0 \tag{4.11}$$

through (g_0, π_0) and tangent to (h, ω). Differentiating (4.11) twice and evaluating at $\rho = 0$ gives

$$\mathrm{D}\Phi(g_0, \pi_0) \cdot (g''(0), \pi''(0)) + \mathrm{D}^2\Phi(g_0, \pi_0) \cdot ((h, \omega), h, \omega)) = 0 \tag{4.12}$$

where

$$g''(0) = \frac{\partial^2 g(0)}{\partial \rho^2} \quad \text{and} \quad \pi''(0) = \frac{\partial^2 \pi(0)}{\partial \rho^2}.$$

Contracting (4.12) with (N, X) and integrating over M, the first term of (4.12) gives

$$\int \langle (N, X), \mathrm{D}\Phi(g_0, \pi_0) \cdot (g''(0), \pi''(0)) \rangle$$
$$= \int \langle [\mathrm{D}\Phi(g_0, \pi_0)]^* \cdot (N, X), (g''(0), \pi''(0)) \rangle = 0,$$

since $(N, X) \in \ker [\mathrm{D}(g_0, \pi_0)]^*$.

Thus the first term of (4.12) drops out, leaving the necessary condition

$$\int \langle (N, X), \mathrm{D}^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega)) \rangle = 0, \tag{4.13}$$

which must hold for all $(h, \omega) \in \ker \mathrm{D}\Phi(g_0, \pi_0)$. An argument like that in Bourguignon, Ebin and Marsden (1975) can be used to show that (4.13) is a non-trivial condition (see Arms and Marsden, 1979, and Fischer, Marsden and Moncrief, 1978). ■

The procedure for finding a second-order condition when linearization stability fails is quite general. See Fischer and Marsden (1975*a*, *b*) for other applications.

From the linearization stability of the constraint equations, we can deduce linearization stability of the spacetime, and vice versa, as follows.

Theorem 4.33

Let $(V_4, {}^{(4)}g_0)$ *be a vacuum spacetime which is the maximal development of Cauchy data* (g_0, π_0) *on a compact hypersurface* $\Sigma_0 = i_0(M)$.

Then the Einstein equations on V_4,

$$\text{Ein}\,({}^{(4)}g) = 0,$$

are linearization stable at ${}^{(4)}g_0$ *if and only if the constraint equations*

$$\Phi(g, \pi) = 0$$

are linearization stable at (g_0, π_0).

In particular, if conditions $C_{\mathscr{H}}$, C_δ, *and* C_{tr} *hold for* (g_0, π_0), *then the Einstein equations are linearization stable.*

Proof. Assume first that the constraint equations are linearization stable. Let ${}^{(4)}h_0$ be a solution to the linearized equations at ${}^{(4)}g_0$ and let (h_0, ω_0) be the induced deformation of (g, π) on Σ_0. Now (h_0, ω_0) satisfies the linearized constraint equations. By assumption, there is a curve $(g(\rho), \pi(\rho)) \in \mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_\delta$ tangent to (h_0, ω_0) at (g_0, π_0).

By the existence theory for the Cauchy problem, there is a curve ${}^{(4)}g(\rho)$ of maximal solutions on $V_4 \simeq \mathbb{R} \times M$ of Ein $({}^{(4)}g(\rho)) = 0$ and with Cauchy data $(g(\rho), \pi(\rho))$. By theorems 4.19 and 4.24 ${}^{(4)}g(\rho)$ will be, for a given choice of lapse and shift, a smooth function of ρ in the sense of theorem 4.19 or in the usual C^∞ sense. As earlier, for any compact set $D \subset V_4$ and $\varepsilon > 0$, there is a $\delta > 0$ such that ${}^{(4)}g(\rho)$ is within ε of ${}^{(4)}g_0$ (using any standard topology) on D.

Using the uniqueness results for the linearized and full Einstein system, one can transform the curve ${}^{(4)}g(\rho)$ by diffeomorphisms so that ${}^{(4)}h_0$ is its tangent at $\rho = 0$. See Fischer and Marsden (1978*a*) for details. ■

Moncrief (1975*a*) has proved that for $(g, \pi) \in \mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_\delta$, the map $[D\Phi(g, \pi)]^*$ is injective if and only if a spacetime ${}^{(4)}g$ generated by (g, π) has no (non-trivial) Killing vector fields ${}^{(4)}Y$ (i.e., $L_{{}^{(4)}Y}\,{}^{(4)}g = 0$ implies ${}^{(4)}Y = 0$); together with theorems 4.32 and 4.33, Moncrief's result then gives necessary and sufficient conditions for a spacetime with compact Cauchy spacelike hypersurfaces to be linearization stable.

Moncrief's result still does not give necessary and sufficient conditions for $[D\Phi(g, \pi)]^*$ to be injective in terms of the (g, π) (the conditions $C_{\mathscr{H}}$, C_{δ}, and $\mathfrak{C}_{tr}$ are sufficient but not necessary), but bypasses the condition tr $\pi' =$ constant, apparently rendering it much less important.

Theorem 4.34 (Moncrief, 1975a)

Let ${}^{(4)}g$ *be a solution to the empty space field equations* $\mathrm{Ein}\,({}^{(4)}g) = 0$. *Let* $\Sigma_0 = i_0(M)$ *be a compact Cauchy hypersurface with induced metric* g_0 *and canonical momentum* π_0. *Then* $\ker\,[D\Phi(g_0, \pi_0)]^*$ *(a finite-dimensional vector space) is isomorphic to the space of Killing vector fields of* ${}^{(4)}g$. *In fact,*

$$(Y_{\perp}, Y_{\parallel}) \in \ker\,[D\Phi(g_0, \pi_0)]^*$$

if and only if there exists a Killing vector field ${}^{(4)}Y$ *of* ${}^{(4)}g$ *whose normal and tangential components to* Σ_0 *are* $Y_{\perp}$ *and* $Y_{\parallel}$.

See Coll (1977) and Fischer and Marsden (1978*a*) for alternative proofs to the one given by Moncrief.

As an important corollary of this result, we observe that *the condition* $\ker\,[D\Phi(g_0, \pi_0)]^* = \{0\}$ *is hypersurface independent* (since it is equivalent to the absence of Killing vector fields, which is hypersurface independent). The condition is also obviously unchanged if we pass to an isometric spacetime.

Putting all this together yields the main linearization stability theorem.

Theorem 4.35

Let ${}^{(4)}g_0$ *be a solution of the vacuum field equations* $\mathrm{Ein}\,({}^{(4)}g_0) = 0$. *Assume that the spacetime* $(V_4, {}^{(4)}g_0)$ *has a compact Cauchy surface* Σ_0.

Then the Einstein equations on V_4

$$\mathrm{Ein}\,({}^{(4)}g) = 0$$

are linearization stable at ${}^{(4)}g_0$ *if and only if* ${}^{(4)}g_0$ *has no Killing vector fields.*

We conclude this section by briefly examining the case in which ${}^{(4)}g_0$ is not linearization stable. The goal is to find necessary and sufficient conditions on a solution ${}^{(4)}h$ of the linearized equations so that ${}^{(4)}h$ is tangent to a curve of exact solutions through ${}^{(4)}g_0$. The necessary conditions will be derived; for sufficiency, see Fischer, Marsden and Moncrief (1978).

In theorem 4.32 we showed that if ${}^{(4)}h$ is tangent to a curve of exact solutions and $(N, X) \in \ker [D\Phi(g_0, \pi_0)]^*$, then

$$\int_{\Sigma_0} \langle (N, X), D^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega)) \rangle = 0.$$

Following Moncrief (1976), we can re-express this second-order condition in terms of the spacetime, just as the condition $\ker D\Phi(g_0, \pi_0) = \{0\}$ was so re-expressed. See Fischer and Marsden (1978*a*) and Fischer, Marsden and Moncrief (1978) for alternative proofs.

***Theorem 4.36** (**Moncrief, 1976**).*

Let $\mathrm{Ein}\,({}^{(4)}g_0) = 0$, *and let* ${}^{(4)}h \in S_2(V_4)$ *satisfy the linearized equations*

$$D\,\mathrm{Ein}\,({}^{(4)}g_0) \cdot {}^{(4)}h = 0.$$

Let ${}^{(4)}Y$ *be a Killing vector field of* ${}^{(4)}g_0$ *(so that* ${}^{(4)}g_0$ *is linearization unstable). Let* Σ_0 *be a compact Cauchy hypersurface and let* $(Y_\perp, Y_\parallel)$ *be the normal and tangential components of* ${}^{(4)}Y$ *on* Σ_0. *Then a necessary second-order condition for* ${}^{(4)}h$ *to be tangent to a curve of exact solutions is*

$$\begin{aligned}\int_{\Sigma_0} &\langle D^2\,\mathrm{Ein}\,({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h), \quad ({}^{(4)}Y_{\Sigma_0}, {}^{(4)}Z_{\Sigma_0}) \rangle \, d\mu(g_0) \\ &= \int_{\Sigma_0} \langle (Y_\perp, Y_\parallel), D^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega)) \rangle = 0.\end{aligned} \tag{4.14}$$

If $\mathrm{Ein}\,({}^{(4)}g_0) = 0 = D\,\mathrm{Ein}\,({}^{(4)}g_0) \cdot {}^{(4)}h_0$, then $D^2\,\mathrm{Ein}\,({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)$ has zero divergence (Taub, 1970). Thus, if ${}^{(4)}Y$ is a Killing vector field, then the vector field

$${}^{(4)}W = {}^{(4)}Y \cdot [D^2\,\mathrm{Ein}\,({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)]$$

also has zero divergence. Thus the necessary second-order condition

$$\int_{\Sigma_0} \langle {}^{(4)}W, {}^{(4)}Z_{\Sigma_0} \rangle \, d\mu(g_0) = 0$$

on first-order deformations is independent of the Cauchy hyper-surface on which it is evaluated. The integral of ${}^{(4)}W$ over a Cauchy hypersurface then represents a conserved quantity for the gravitational field, constructed from a solution ${}^{(4)}h$ of the linearized equations and from a Killing vector field ${}^{(4)}Y$. The interesting and important feature of this conserved quantity of Taub, as shown by theorem 4.36, is that unless it is zero, the first-order solution ${}^{(4)}h$ from which ${}^{(4)}W$ was constructed is not tangent to any curve of exact solutions. Thus, for spacetimes which are not linearization stable, Taub's conserved quantity plays the central role in testing whether or not perturbations ${}^{(4)}h$ are spurious (i.e., are not tangent to any curve of exact solutions).

4.6 The space of gravitational degrees of freedom

We now review some results of symplectic geometry that provide a basis for a unified description of the various splittings that occur in general relatively (Arms, Fischer and Marsden, 1975). These results are based on a general reduction of phase spaces for which there is an invariant Hamiltonian system under some group action (Marsden and Weinstein, 1974). A further application of these results leads to the construction of the symplectic space of gravitational degrees of freedom (Fischer and Marsden, 1978*b*).†

Background references for the material in this section are Abraham and Marsden (1978), Chernoff and Marsden (1974), and Marsden (1974).

Let P be a manifold and Ω a symplectic form on P; that is, Ω is a closed (weakly) non-degenerate two-form. For relativity, P will be $T^*\mathcal{M}$ and Ω will be the canonical symplectic form J^{-1}, as described in section 4.1.

Let G be a topological group which acts canonically on P; that is, for each $g \in G$, the action of g on P, $\Phi_g : p \mapsto g \cdot p$, preserves Ω. Assume there is a *moment* Ψ for the action. This means the following: Ψ is a map from P to $\mathfrak{g}^*$, the dual to the Lie algebra $\mathfrak{g} = T_eG$ of G, such that

$$\Omega(\xi_P(p), v_p) = \langle \mathrm{d}\Psi(p) \cdot v_p, \xi \rangle$$

for all $\xi \in \mathfrak{g}$, where ξ_P is the corresponding infinitesimal generator (Killing form) on P, and $v_p \in T_pP$. Another way to define Ψ is to require that for each ξ, the map $p \mapsto \langle \Psi(p), \xi \rangle$ be an energy function for the Hamiltonian vector field ξ_P. This concept of a moment is an important

† It should be noted that in the case of compact Cauchy surfaces, the space of gravitational degrees of freedom has had all of the dynamical degrees of freedom factored out. For some purposes this may be undesirable and a less severe identification may be wanted. (See York, 1972, and Fischer and Marsden, 1977.)

geometrization of the various conservation theorems of classical mechanics and field theory, including Noether's theorem.

It is easy to prove that if H is a Hamiltonian function on P with corresponding Hamiltonian vector field X_H, i.e., $\mathrm{d}H(p) \cdot v = \Omega_p(X_H(p), v)$, or equivalently, $i_{X_H}\Omega = \mathrm{d}H$, and if H is invariant under G, then Ψ is a constant of the motion for X_H; i.e., if F_t is the flow of X_H, then $\Psi \circ F_t = \Psi$.

As an example, consider a group G acting on a configuration space Q. This action lifts to a canonical action on the phase space T^*Q. The moment in this case is given by

$$\langle \Psi(\alpha_q), \xi \rangle = \langle \xi_Q(q), \alpha_q \rangle,$$

where α_q belongs to T^*Q. If G is the set of translations or rotations, Ψ is linear or angular momentum, respectively. As expected, Ψ is a vector, and the transformation property required of this vector is equivariance of the moment under the co-adjoint action of G on $\mathfrak{g}$; that is, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\Phi_g} & P \\ \downarrow & & \downarrow \\ \mathfrak{g}^* & \xrightarrow[\mathrm{Ad}^*_{g^{-1}}]{} & \mathfrak{g}^* \end{array}$$

must commute. We shall consider only equivariant moments.

There are several classical theorems concerning reduction of phase spaces. In celestial mechanics, there is Jacobi's elimination of the node, which states that in a rotationally invariant system, we can eliminate four of the variables and still have a Hamiltonian system in the new variables. Another classical theorem of Hamiltonian mechanics states that the existence of k first integrals in involution allows a reduction of $2k$ variables in the phase space. Both of these theorems follow from a theorem of Marsden and Weinstein (1974) on the reduction of phase space.

To construct this reduced space, let $\mu \in \mathfrak{g}^*$ and set

$$G_\mu = \{g \in G | \mathrm{Ad}^*{}_{g^{-1}}\mu = \mu\}.$$

Consider $\Psi^{-1}(\mu) = \{p | \Psi(p) = \mu\}$. The equivariance condition implies that G preserves $\Psi^{-1}(\mu)$, so we can consider $P_\mu = \Psi^{-1}(\mu)/G_\mu$. In the case that $\Psi^{-1}(\mu)$ is a manifold (e.g., μ is a regular value) and G acts freely and properly on this manifold, we have:

Theorem 4.37

P_μ inherits a natural symplectic structure from P, and a Hamiltonian system on P which is invariant under the canonical action of G projects naturally to a Hamiltonian system on P_μ.

In Jacobi's elimination of the node, G is SO(3), so $\mathfrak{g}$ is $\mathbb{R}^3$ and the co-adjoint action is the usual one. Thus the isotropy subgroup G_μ of a point μ in $\mathbb{R}^3$ is S^1. If n is the dimension of P, then $\Psi^{-1}(\mu)$ is the solution set for three equations so the dimension of $\Psi^{-1}(\mu)/G_\mu$ is $n-3-1=n-4$. For k first integrals in involution, G is a k-dimensional abelian group, so the co-adjoint action is trivial and $G_\mu = G$. Thus the dimension of $\Psi^{-1}(\mu)/G$ is $n-2k$. Another known theorem that follows from theorem 4.37 is the Kostant–Kirillov theorem which states that the orbit of a point μ of $\mathfrak{g}^*$ under the adjoint action is a symplectic manifold.

Now we shall show how to obtain a general splitting theorem for symplectic manifolds, one piece of which is tangent to the reduced space P_μ (Arms, Fischer and Marsden, 1975). This includes the splitting theorems for symmetric tensors as a special case.

A splitting theorem for a symplectic manifold P requires a positive-definitive but possibly only weakly non-degenerate metric, or other such structure to give a dualization. This is so that orthogonal complements may be defined. Suppose we know, say from the Fredholm theorem, that

$$T_pP = \text{range } (T_p\Psi)^* \oplus \ker T_p\Psi$$

(here $(T_p\Psi)^*$ is the usual L_2-adjoint). Of course, in finite dimensions this is automatic. Define

$$\alpha_p: \mathfrak{g}_\mu \to T_pP;\ \xi \mapsto \xi_P(p)$$

where $\mathfrak{g}_\mu$ is the Lie algebra of G_μ. Suppose we also have the splitting

$$T_pP = \text{range } \alpha_P \oplus \ker \alpha_P^*.$$

There is a general compatibility condition between these two splittings, namely range $\alpha_p \subset \ker T_p\Psi$, which follows readily from equivariance. In fact,

$$\text{range } \alpha_p = T_p(G\cdot p) \cap \ker T_p\Psi.$$

This compatibility condition implies the finer splitting:

$$T_pP = \text{range } (T_p\Psi)^* \oplus \text{range } \alpha_p \oplus (\ker T_p\Psi \cap \ker \alpha_p^*), \qquad (4.15)$$

i.e.,

$$T_pP = \text{range}\ (T_p\Psi)^* \oplus T_p\ (G_\mu \cdot p) \oplus \ker\ Tp\Psi/[Tp(G_\mu \cdot p)].$$

Note that the third summand is the tangent space to P_μ. The geometric picture is given in figure 4.2. For the purposes of this figure we number the

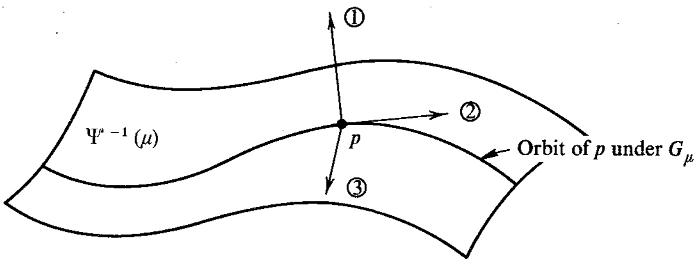


Figure 4.2. The geometry of a general symplectic decomposition.

summands in the previous decomposition as

$$T_pP = ① \oplus ② \oplus ③,$$

where

① belongs to range $(T_p\Psi)^*$, the orthogonal complement of the tangent space to the level $\Psi^{-1}(\mu)$;
② belongs to range α_p, the tangent space to the orbit of p under G_μ;
③ is in $(\ker T_p\Psi \cap \ker \alpha_P^*)$, and is the part of the decomposition which is tangent to the reduced symplectic manifold.
② and ③ together are $\ker T_p\Psi$, the tangent space to $\Psi^{-1}(\mu)$.

A basic splitting of Moncrief (1975*b*) can be viewed as a special case of this result. We choose $P = T^*\mathcal{M}$ and the 'group' is $G = C^\infty_{\text{space}}(M;\ V_4,\ {}^{(4)}g)$, the spacelike embeddings of M to Cauchy hypersurfaces in $(V_4,\ {}^{(4)}g)$, an Einstein flat spacetime which is the maximal development with respect to some Cauchy hypersurface $\Sigma \subset V_4$.

Although G is not a group, it is enough like a group for the analysis to work.† G 'acts' on (g, π) as follows (see figure 4.3). Let $(V_4,\ {}^{(4)}g,\ i_0)$, $\text{Ein}\ ({}^{(4)}g) = 0$, be a maximal development which has (g_0, π_0) as Cauchy data on an embedded Cauchy hypersurface $\Sigma_0 = i_0(M)$, $i_0: M \to V_4$ (i_0 is like an origin for C^∞_{space}). Then $i \in C^\infty_{\text{space}}(M;\ V_4,\ {}^{(4)}g)$ maps (g_0, π_0) to

† One uses the more general reduction procedure described in Weinstein (1977) and Abraham and Marsden (1978).

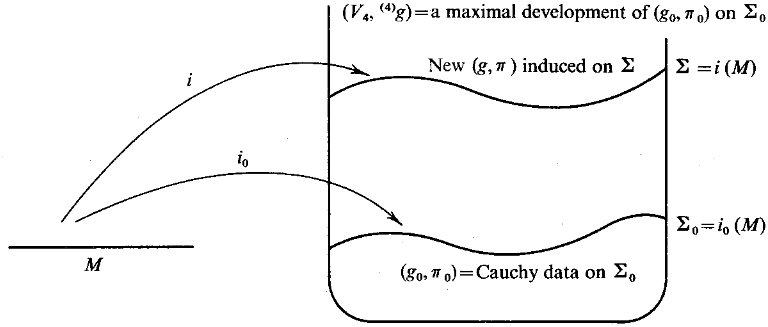


Figure 4.3. Representation of the 'action' of the space of embeddings on the space of Cauchy data.

the (g, π) induced on the hypersurface $\Sigma = i(M)$. The set of all such (g, π) define the orbit of (g_0, π_0) in $\mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_\delta$. These orbits are disjoint, and so define an equivalence relation, $\sim$, in $\mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_\delta$.

Although this is not an action (since C^∞_{space} is not a group), it has well-defined orbits and the symplectic analysis above applies (Fischer and Marsden, 1978*b*). Using the adjoint form of the Einstein evolution system, the moment of 'this action' on a tangent vector ${}^{(4)}X_\Sigma \in T_{i_0} C^\infty_{\text{space}}(M; V_4, {}^{(4)}g)$ with lapse N and shift X is computed to be

$$\Psi_{(g,\pi)}({}^{(4)}X) = \int N\mathscr{H}(g, \pi) + X \cdot \mathscr{J}(g, \pi).$$

Here the ${}^{(4)}X_\Sigma$ or the (N, X) can be thought of as belonging to the 'Lie algebra' of C^∞_{space}.

Since $\Psi^{-1}(0)$ is precisely the constraint set $\mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_\delta$, we choose $\mu = 0$, so $G_\mu = G$. From the equations of motion, we find that

$$\alpha_{(g,\pi)}: \mathfrak{g} \to T_{(g,\pi)}(T^*\mathscr{M})$$

is given by

$$(N, X) \mapsto J \circ [D\Phi(g, \pi)]^* \cdot \begin{pmatrix} N \\ X \end{pmatrix},$$

so the symplectic decomposition (4.15) becomes

$$T_{(g,\pi)}T^*\mathscr{M} = \{\text{range } [D\Phi(g, \pi)]^*\}^* \oplus \text{range } \{J \circ [D(g, \pi)]^*\}$$
$$\oplus \ker D\Phi(g, \pi) \cap [\ker D\Phi(g, \pi) \circ J]^*$$

which is Moncrief's splitting. Elements of the first summand infinitesimally deform (g, π) to Cauchy data which do not satisfy the constraint equations. Elements of the second summand infinitesimally deform

(g, π) to Cauchy data that generate an isometric spacetime, and elements of the third summand infinitesimally deform (g, π) in the direction of new Cauchy data that generate a non-isometric solution to the empty space field equation; see figure 4.4 and compare with figure 4.2.

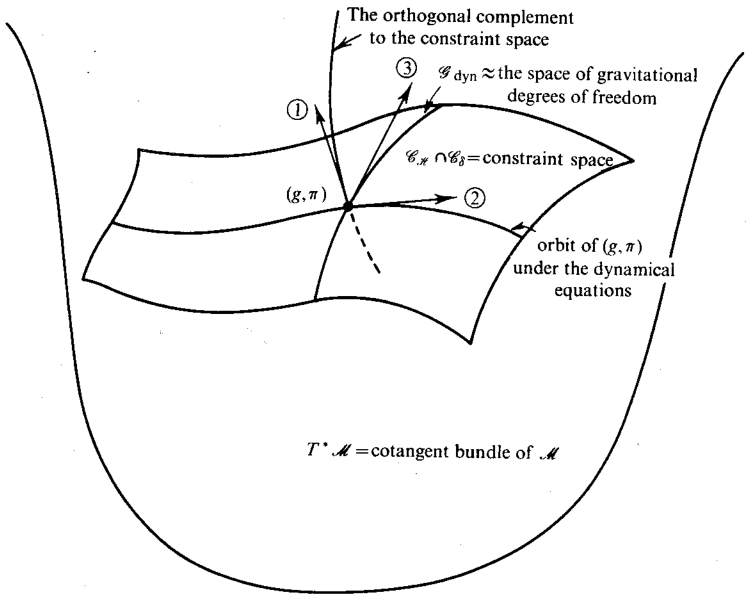


Figure 4.4. Symplectic decomposition applied to the Einstein equations to construct the space of gravitational degrees of freedom.

This third summand represents the tangent space to the reduced space $P_\mu \approx \mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_\delta / \sim$. This quotient by the equivalence relation described above is naturally isomorphic to the space of gravitational degrees of freedom,

$$\mathscr{G}(V_4) = \mathscr{E}(V_4)/\mathscr{D}(V_4),$$

namely the set of maximal solutions to the vacuum Einstein equations

$$\mathscr{E}(V_4) = \{ {}^{(4)}g \,|\, \text{Ein}\,({}^{(4)}g) = 0, \text{ and such that } (V_4, {}^{(4)}g) \text{ is the maximal development of the Cauchy data on some Cauchy hypersurface} \}$$

modulo the spacetime diffeomorphism group $\mathscr{D}(V_4)$. This is the space of isometry classes of empty space solutions of the Einstein equations, or the space of gravitational degrees of freedom since the coordinate gauge group has been factored out.

We call the representation of $\mathscr{G}(V_4)$ described here the *dynamical representation* since one uses the canonical formulation to define $P_\mu \approx$

$\mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_{\delta}/\sim$. See York (1971), Choquet-Bruhat and York (1979), and Fischer and Marsden (1977) for a conformal representation of $\mathscr{G}(V_4)$.

As we have emphasized, in the case of compact hypersurfaces, one identifies all the (g, π) which occur on slicings in an Einstein flat maximal spacetime. In the non-compact case one does *not* do this, as is explained in Regge and Teitelboim (1974) and Choquet-Bruhat, Fischer and Marsden (1978).

A few further remarks are in order regarding the decomposition of $T_{(g,\pi)}(T^*\mathcal{M})$.

Set $\mathfrak{g}_{(g,\pi)} = \ker \mathrm{D}\Phi(g, \pi) \cap [\ker \mathrm{D}\Phi(g, \pi) \circ J]^*$, the third summand in the decomposition above. The summand $\mathfrak{g}_{(g,\pi)}$ generalizes the classical transverse traceless (TT) decomposition of Deser (1965) and Brill and Deser (1968). Indeed for $\pi = 0$ and $R(g) = 0$, Moncrief's decomposition reduces to two copies of the Berger and Ebin (1972) splitting. If moreover, Ric $(g) = 0$ (so that g is flat), we regain the original Brill–Deser splitting.

Now suppose $(h, \omega) \in \mathscr{G}_{(g,\pi)}$. Then (h, ω) satisfies the following equations:

$$\mathrm{D}\Phi(g, \pi) \cdot (h, \omega) = 0, \tag{4.16}$$

and

$$[\mathrm{D}\Phi(g, \pi) \circ J] \cdot (h, \omega)^* = \mathrm{D}\Phi(g, \pi) \cdot ((\omega')^\flat, -h^\# \,\mathrm{d}\mu(g)) = 0. \tag{4.17}$$

Written out in terms of the constraint functions $\mathscr{H}$ and $\mathscr{J}$, these equations are

$$\begin{aligned} \mathrm{D}\mathscr{H}(g, \pi) \cdot (h, \omega) &= 0, \\ \mathrm{D}\mathscr{H}(g, \pi) \cdot ((\omega')^\flat, -h^\# \,\mathrm{d}\mu(g)) &= 0, \\ \mathrm{D}\mathscr{J}(g, \pi) \cdot (h, \omega) &= 0, \\ \mathrm{D}\mathscr{J}(g, \pi) \cdot ((\omega')^\flat, -h^\# \,\mathrm{d}\mu(g)) &= 0. \end{aligned}$$

These equations, eight conditions on twelve functions of three variables, formally leave four functions of three variables as parameters of the space $\mathscr{G}_{(g,\pi)}$. Formally, $\mathscr{G}_{(g,\pi)}$ is the tangent space to the space of gravitational degrees of freedom, which is parametrized by four functions of three variables.

Moreover, there is a certain 'symplectic symmetry' in the summand $\mathscr{G}_{(g,\pi)}$, reflected in (4.16) and (4.17) above: if $(h, \omega) \in \mathscr{G}_{(g,\pi)}$, then $J \circ (h, \omega)^*$ is also in $\mathscr{G}_{(g,\pi)}$. We shall refer to this symmetry as J-invariance of $\mathscr{G}_{(g,\pi)}$.

Proposition 4.38

The (weak) symplectic form Ω on $S_2 \times S_d^2$ naturally induces a weak symplectic form Ω' on any J-invariant subspace of $S_2 \times S_d^2$. In particular, $\mathcal{G}_{(g,\pi)}$ is a (weak) symplectic linear space.

Proof. The symplectic form Ω on $S_2 \times S_d^2$ defined by

$$\Omega((h_1, \omega_1), (h_2, \omega_2)) = \int_M \langle J^{-1}(h_1, \omega_1), (h_2, \omega_2)\rangle$$

defines by the same formula an antisymmetric bilinear form Ω' on $\mathcal{G}_{(g,\pi)}$ (or any other J-invariant subspace of $S_2 \times S_d^2$). One has to show Ω' is non-degenerate. Thus suppose for $(h_1, \omega_1) \in \mathcal{G}_{(g,\pi)}$,

$$\int_M \langle J^{-1}(h_1, \omega_1), (h_2, \omega_2)\rangle = 0$$

for all $(h_2, \omega_2) \in \mathcal{G}_{(g,\pi)}$. Since $\mathcal{G}_{(g,\pi)}$ is J-invariant,

$$J \circ \begin{pmatrix} h_1 \\ \omega_1 \end{pmatrix}^* \in \mathcal{G}_{(g,\pi)}.$$

Thus letting

$$\begin{pmatrix} h_2 \\ \omega_2 \end{pmatrix} = -J \circ \begin{pmatrix} h_1 \\ \omega_1 \end{pmatrix}^*,$$

and since $J^* = -J$, we have

$$0 = -\int \langle J^{-1}(h_1, \omega_1), J \circ (h_1, \omega_1)^*\rangle = \int \langle (h_1, \omega_1), (h_1, \omega_1)^*\rangle$$

$$= \int_M (h_1 \cdot h_1 + \omega_1' \cdot \omega_1')\, d\mu(g).$$

Thus $(h_1, \omega_1) = 0$ so that Ω' is non-degenerate. ■

Proposition 4.38 is a special case of the following general result of symplectic geometry (see Weinstein, 1977).

Theorem 4.39

Let (V, Ω) be a (weak) symplectic vector space and $W \subset V$ a subspace. Let $W_\Omega^\perp = \{v \in V \mid \Omega(v, \omega) = 0$ for all $\omega \in W\}$ and assume W is co-isotropic, i.e., $W_\Omega^\perp \subset W$. Then $W/W_\Omega^\perp$ is, in a natural way, a (weak) symplectic vector space.

Proof. Denote an element of $W/W_\Omega^\perp$ by $w+W_\Omega^\perp$. Define Ω on the quotient by $\Omega(w_1+W_\Omega^\perp, w_2+W_\Omega^\perp)=\Omega(w_1, w_2)$. Since $\Omega(w_i, W_\Omega^\perp)=0$, $i=1, 2$ this is well defined. On the other hand, if $\Omega(w_1+W_\Omega^\perp, w_2+W_\Omega^\perp)=0$ for all w_2, $w_2 \in W_\Omega^\perp$, then $w_2+W_\Omega^\perp$ is the zero element of the quotient. ■

Proposition 4.38 follows as a corollary by letting $V=S_2\times S_d^2=T_{(g,\pi)}T^*\mathcal{M}$, Ω as given, and $W=\ker D\Phi(g,\pi)$. Then,

$$\begin{aligned} W_\Omega^\perp &= \{(h,\omega)|\Omega((h,\omega), W)=0\} \\ &= \{(h,\omega)|J^{-1}(h,\omega) \text{ is orthogonal to } \ker D\Phi(g,\pi)\} \\ &= \{(h,\omega)|J^{-1}(h,\omega)\in \text{range } [D\Phi(g,\pi)]^*\} \\ &= \text{range } \{J\circ[D\Phi(g,\pi)]^*\}\subset W. \end{aligned}$$ ■

The symplectic structure on $\mathcal{G}$ described above may be important for the problem of quantizing gravity. The symplectic structure presented here is probably implicit in the work of Bergmann (1958), Dirac (1959), and DeWitt (1967). The present formulation, however, allows one to be rather precise and geometric. First of all, it may allow one to use the Segal or Kostant–Souriau quantization formalism to carry out a full quantization or a semi-classical quantization. Secondly, the approach presented here enables one to show that near metrics ${}^{(4)}g$ in $\mathscr{E}(V_4)$ with no isometries (and hence no spacetime Killing vector fields), $\mathcal{G}=\mathscr{E}(V_4)/\mathscr{D}(V_4)$ is a smooth manifold and is locally isomorphic, in a natural way, to $\mathscr{C}_{\mathscr{H}}\cap\mathscr{C}_\delta/\sim$, and thus carries a canonical symplectic structure.† Thus, in the neighborhood of Einstein flat spacetimes without Killing vector fields, the space $\mathcal{G}=\mathscr{E}(V_4)/\mathscr{D}(V_4)$ of gravitational degrees of freedom is itself a symplectic manifold, or if you prefer, a gravitational *phase space* without singularities, each element of which represents an empty space geometry. Note that $\mathcal{G}$ is (generically) a symplectic manifold even though it is not a cotangent bundle. We conjecture that $\mathcal{G}$ can actually be stratified into symplectic manifolds, similar to the stratification of superspace; see Fischer (1970) and Bourguignon (1975). The singularities in $\mathcal{G}$ occur near spacetimes with symmetries, and these are of a conical nature (Fischer, Marsden and Moncrief, 1978). Moncrief has emphasized in his 1978 Gravity Research

† An interesting point here is that $\mathscr{C}_{\mathscr{H}}\cap\mathscr{C}_\delta$, although (generically) a submanifold of $T^*\mathcal{M}$, does not have a natural symplectic structure induced from $T^*\mathcal{M}$, since the tangent space of $\mathscr{C}_{\mathscr{H}}\cap\mathscr{C}_\delta$ is not J-invariant. One must pass to the quotient manifold $\mathscr{C}_{\mathscr{H}}\cap\mathscr{C}_\delta/\sim$ in order to get a symplectic structure induced by the symplectic structure of $T^*\mathcal{M}$.

Foundation essay that these singularities have an important effect on quantization procedures (for instance, for de Sitter spacetimes).

The methods we have employed to analyze gravity, being based on the L_2-adjoint formalism, carry over directly to fields minimally coupled to gravity, and in particular to Yang–Mills fields. In the latter case, one divides out not by $\mathscr{D}(V_4)$, but by the larger group of equivariant bundle diffeomorphisms (i.e. gauge transformations covering diffeomorphisms of spacetime). Using the methods presented here, we can show that the space of degrees of freedom for fields and gravity, for fields minimally coupled to gravity, is, generally, a symplectic manifold; see Fischer and Marsden (1978*b*, *c*).

Finally we remark that we hope that the geometric methods presented here help to unfold some of the inter-relationships that exist between general relativity, differential geometry, functional analysis, nonlinear partial differential equations, infinite-dimensional dynamical systems, symplectic geometry, and the theory of singularities. Certainly, all of these areas of mathematics (and others) will have to make their contribution to the study of gravitational theory before the final analysis is in.