

"Isolated Gravitating Systems in General Relativity," Ed. J. Ehlers, Italian Physical Society (1979), pp. 322-395.

Topics in the Dynamics of General Relativity (*).

A. E. FISCHER and J. E. MARSDEN

Department of Mathematics, University of California - Santa Cruz and Berkeley, Cal.

Introduction.

These notes cover several interrelated topics in the dynamics of general relativity. The main thrust is to present informally the Hamiltonian approach of Dirac [1, 2] and of Arnowitt, Deser and Misner [3] in a new way and to investigate the results which can be obtained from this approach. We write the evolution equations in the compact Hamiltonian form

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} g \\ \pi \end{pmatrix} = J \circ [D\Phi(g, \pi)]^* \begin{pmatrix} N \\ X \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

N = lapse function, X = shift vector field, g = 3-metric on a spacelike hypersurface, π = conjugate momentum and $\Phi(g, \pi) = (\mathcal{H}(g, \pi), \mathcal{J}(g, \pi)) = 0$ are the constraint equations. This form of the equations is useful in understanding the Hamiltonian structure of the evolution equations, their relationship to the linearized constraint equations, recent splittings of Moncrief [4] and the space of true gravitational degrees of freedom.

Consideration of the map $D\Phi(g, \pi)^*$, the L_2 -adjoint of the derivative of Φ at (g, π) , first arose in the authors' investigations of linearization stability of Einstein's equations [5, 6], i.e. in the validity of first-order perturbation analysis. This and Moncrief's beautiful contributions [4, 7] are presented in sect. 5 and 6.

The presence of the matrix J in the equations and indeed their Hamiltonian

(*) This research was partially supported by the National Science Foundation of the United States, the Science Research Council of the United Kingdom, and the Department of Applied Mathematics and Theoretical Physics of Cambridge University.

nature suggests that the machinery of symplectic geometry be used. We show how the various splittings (*e.g.*, Deser's [8] transverse-traceless decomposition and Moncrief's generalization [4]) can all be obtained by using a universal construction on symplectic geometry, and how this construction gives insight into the space of gravitational degrees of freedom (see [9-14]).

This paper is a preliminary version of a more extensive work in preparation. We thank Prof. J. EHLERS for the invitation and opportunity to present a more current version of our work than is presently available, and for his many penetrating comments. Some of the material here was also presented at the Département de Mécanique, Université de Paris, in May and June 1975. We thank Prof. Y. CHOQUET-BRUHAT for her arrangements and many helpful comments, and J. M. ARMS whose lecture notes and comments were of great assistance.

We also acknowledge the valuable comments of P. D'EATH, G. GIBBONS, S. HAWKING, K. KUCHAR, V. MONCRIEF, R. SACHS, A. TAUB and A. WEINSTEIN, and the hospitality of Cambridge University for support during part of the preparation of this work.

1. - Sobolev spaces and decomposition theorems.

We shall take the general point of view of considering geometric objects such as the Ricci tensor $\text{Ric}(g)$ of a Riemannian metric g as functions defined on the space of all Riemannian metrics \mathcal{M} . Variational derivatives of these objects can be computed by using differential calculus on these function spaces.

Thus, before beginning geometrodynamics, it is useful to recall the basic function spaces and some of their key properties, which we shall need.

Let Ω be an open bounded region of \mathbb{R}^n with smooth boundary. For any C^∞ function f from \mathbb{R}^n to \mathbb{R}^m , we define the $W^{s,p}(\Omega, \mathbb{R}^m)$ norm of f to be

$$\|f\|_{W^{s,p}} = \left(\sum_{0 \leq \alpha \leq s} \|D^\alpha f\|_{L_p(\Omega)}^p \right)^{1/p},$$

where D^α is the total derivative of f of order α and $\|\cdot\|_{L_p(\Omega)}$ denotes the usual L_p norm on Ω :

$$\|g\|_{L_p(\Omega)} = \left(\int_\Omega |g(x)|^p dx \right)^{1/p}.$$

By definition, $W^{s,p}(\Omega, \mathbb{R}^m)$ is the completion of $C^\infty(\Omega, \mathbb{R}^m)$ (= restrictions of C^∞ functions on \mathbb{R}^n to Ω) with respect to this norm.

Note:

1) We consider $f \in C^\infty$ on \mathbb{R}^n rather than just on Ω because we wish to have differentiability on the boundary.

2) We shall shorten $W^{s,p}(\Omega, \mathbb{R}^m)$ and similar expressions to $W^{s,p}$ when there is little chance of confusion.

For a compact manifold M with no boundary and a vector bundle E over M , $W^{s,p}(E)$ shall denote the space of all sections of E that are of class $W^{s,p}$ in some (and hence every) covering of M by charts. For real-valued functions we shall just write $W^{s,p}$, but for other tensor bundles we shall make up special notations for $W^{s,p}(E)$ (see below).

In case $p = 2$ the spaces $W^{s,p}$ are denoted H^s . In this case, and only in this case, do we get Hilbert spaces.

The spaces H^s (not $W^{s,p}$ in general) are the basic spaces for existence theory for nonlinear hyperbolic equations. As we shall see, general relativity has equations of this type. For elliptic equations it is useful to allow p to be general.

The Sobolev spaces have the following properties (see, for example, [15, 16] or [17] for proofs):

1) *Sobolev embedding*: If $s > n/p + k$, where n is the dimension of M , then the inclusion of $W^{s,p}$ into C^k is a continuous, and in fact compact, embedding. The latter fact is called Rellich's theorem; this theorem also tells us that $W^{s,p}$ is compactly included in $W^{s',p}$ if $s > s'$.

2) *Multiplication*: If $s > n/p$ and $0 < \alpha < s$, then any pointwise bilinear map $\alpha \cdot \beta$ induces a multiplication $W^{s,p} \times W^{s-\alpha,p} \rightarrow W^{s-\alpha,p}$ which is continuous and hence C^∞ .

3) *Composition*: If the function f satisfies either of the conditions below, then the map $W^{s,p} \rightarrow W^{s,p}$; $g \mapsto g \circ f$ is C^k , $k > 0$. The conditions are

a) f is C^{k+1} , or

b) f is a diffeomorphism and is of class $W^{s+k,p}$, where $s > n/p + 1$.

Now suppose we have two vector bundles E and F , over the same manifold M , and a linear differential operator D of order k ,

$$D: C^\infty(E) \rightarrow C^\infty(F).$$

A linear differential operator of order k is a map such that, for given charts on E and F (and hence for all charts), the operator takes the form $D = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha$, where $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ is a partial derivative on the model space for M , $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \sum_{i=1}^n \alpha_i$, and $a_\alpha(x)$ is a linear function from the model space for the fiber E_x to the model space for the fiber F_x over $x \in M$. We can regard D as a map between Sobolev spaces:

$$D: W^{s+k,p} \rightarrow W^{s,p}.$$

D has an L_2 -adjoint D^* defined by the equation

$$(Df, g)_{L_1} = (f, D^*g)_{L_1},$$

that is,

$$\int_M \langle Df, g \rangle \mu = \int_M \langle f, D^*g \rangle \mu,$$

where μ is some preferred volume element such as that associated with a metric $\mu(g) = \sqrt{\det(g_{ij})} dx^1 \wedge \dots \wedge dx^n$, and \langle, \rangle is an inner product on the fibers. This structure is not needed if one uses spaces of tensor densities, *e.g.*, if D maps vector fields to 2-covariant densities, D^* would map 2-contravariant tensor fields to 1-form densities. These adjoints are called «natural adjoints» and will be discussed later.

In practice, one computes D^* by integration by parts, and in fact this leads to the proof that D^* exists and yields a local formula for it [15].

Definitions. An operator is *elliptic* if it has injective (principal) symbol. For each x in M and for each $\xi \in T_x^*M$ = the fiber of the cotangent bundle, the symbol $\sigma_k(D)$ is a linear map from the fiber E_x to the fiber F_x . In the expression of D in charts, $\sigma_k(D)$ is obtained by substituting the components of $\xi \in T_x^*M$ for the corresponding partial derivatives in the terms involving the highest-order derivatives. Thus for each co-ordinate on F_x , $\sigma_k(x)$ is a homogeneous k -th degree polynomial in the components of ξ . For example, the symbol of the ordinary Laplacian $\nabla^2 = \sum_{i=1}^n (\partial^2 / \partial x_i^2)$ is $\sigma_k(\nabla^2) = |\xi|^2$.

For elliptic operators we have an important splitting theorem.

Fredholm alternative theorem. If either D or D^* is elliptic, then $W^{s,p}(F) = \text{range } D \oplus \ker D^*$, where the sum is an L_2 orthogonal direct sum.

The proof of the Fredholm alternative uses the elliptic estimate

$$\|u\|_{W^{s+1,p}} \leq C(\|Du\|_{W^{s,p}} + \|u\|_{L^p}),$$

where $1 < q < \infty$, and Rellich's theorem ($W^{s,p}$ is compactly included in $W^{s',p}$ for $s > s'$), to show that an elliptic operator has a finite-dimensional kernel and a closed range. The L_2 case, where $s = 0$ and $p = 2$, then follows immediately from the defining equation for D^* ; the L_2 orthogonal complement for range D is $\ker D^*$, because

$$0 = \langle Df, g \rangle_{L_2} = \langle f, D^*g \rangle_{L_2} \quad \text{for all } f \in W^{s,p}(E) \text{ if and only if } D^*g = 0.$$

A regularity argument extends the result from L_2 to $W^{s,p}$; see, for instance, [18] for proofs and [19] for extensions.

Applications of the Sobolev space properties.

1) Suppose M is a compact manifold with no boundary. Let $S_2^{s,p}$ be the set of symmetric covariant two-tensors of class $W^{s,p}$ and $\mathcal{M}^{s,p}$ be the subset of Riemannian metrics, i.e. the positive definite symmetric tensors of class $W^{s,p}$. We choose $s > n/p$ so that, by the Sobolev embedding, $W^{s,p} \subset C^0$ and the set of Riemannian metrics is open in $S_2^{s,p}$. Thus the fiber of the tangent bundle to $\mathcal{M}^{s,p}$ will be the linear space $S_2^{s-2,p}$.

Consider the maps

$$\text{Ric}: \mathcal{M}^{s,p} \rightarrow S_2^{s-2,p}; g \mapsto \text{Ric}(g),$$

the Ricci tensor formed from g (in tensor notation $g_{ij} \mapsto R_{ij}(g)$), and

$$R: \mathcal{M}^{s,p} \rightarrow W^{s-2,p}; g \mapsto R(g),$$

the scalar curvature of g . The map R will be smooth if Ric is smooth, for R is the contraction of Ric ; symbolically, $R(g) = g^{-1} \cdot \text{Ric}(g) = g^{ij} R_{ij}$. Because differentiation is a continuous linear map between the spaces involved, the smoothness of Ric depends on the multiplications that occur in computing $\text{Ric}(g)$. The second-order derivatives appear linearly with components of g as coefficients, so, by the multiplication property for Sobolev spaces, $s > n/p$ suffices for the second-order terms. But the first-order derivatives appear quadratically, so $s > n/p + 1$ is necessary to make these maps C^∞ . Thus $\text{Ric}: \mathcal{M}^{s,p} \rightarrow S_2^{s-2,p}$ and $R: \mathcal{M}^{s,p} \rightarrow W^{s-2,p}$ are C^∞ if $s > n/p + 1$. The derivatives of these maps are given by a calculation of Lichnerowicz [20] which we shall study later. Sign conventions on the curvature tensor are an ever-present problem. We follow the conventions of [21].

2) Let $\mathcal{Q}^{s,p} = \{\eta | \eta \text{ and } \eta^{-1} \text{ are diffeomorphisms of } M \text{ of class } W^{s,p}\}$. If $s > n/p + 1$, then $\mathcal{Q}^{s,p} \subset W^{s,p}$ is open, so that $\mathcal{Q}^{s,p}$ is a C^∞ (Banach) manifold and the composition property for Sobolev spaces implies that composition is continuous and $\mathcal{Q}^{s,p}$ is a topological group.

3) The first step in the main decomposition theorems for 2-tensors that we shall give later is the *canonical decomposition*.

Canonical decomposition. Given a fixed C^∞ metric g on a compact manifold M , any symmetric 2-tensor h can be split into two parts,

$$h = \check{h} + L_X g,$$

where \check{h} has zero divergence, $\delta(\check{h}) = -(\check{h})_{i,j} = 0$, $X \in \mathcal{X}^{s,p} = W^{s,p}(TM)$ (the $W^{s,p}$ vector fields), $L_X g$ is the Lie derivative of g , $(L_X g)_{ij} = X_{ij} + X_{ji}$ ($i, j =$ co-

variant derivative with respect to g) and the two pieces are L_x orthogonal and unique (so X is unique up to Killing vector fields).

Proof. The proof is a straightforward application of the Fredholm alternative to the operator

$$\alpha_g: \mathcal{X}^{s,p} \rightarrow S_2^{s-1,p}; \quad X \mapsto L_X g.$$

The symbol of α_g is $\sigma_2(\alpha_g): V \mapsto \xi \otimes V^b + V^b \otimes \xi$, where $(V^b)_i = g_{ij} V^j = V_i$ is the covariant form of V . $\sigma_2(\alpha_g)$ is injective, for, if $\xi \neq 0$ and $\xi \otimes V^b + V^b \otimes \xi = 0$, then, by contraction, $\xi^i V_i = 0$, so $\xi^i \xi_i V_i + \xi^i V_i \xi_i = 0$, so $V_i = 0$. Thus α_g is elliptic. (Note that since the proof of the Fredholm alternative theorem includes a proof that the kernel of an elliptic operator is finite dimensional, we have as a corollary the classical result that the Killing vector fields are a finite-dimensional subset of $\mathcal{X}^{s,p}$.) Since

$$(L_X g, h)_{L_2} = \int_M (X_{i|j} + X_{j|i}) h^{ij} \mu(g) = \int_M (-2X_i h^{ij}) \mu(g) = (X, 2\delta h)_{L_2},$$

we get $\alpha_g^* = 2\delta$, and the splitting $h = \hat{h} + L_X g$ therefore follows. \square

The canonical splitting has a natural geometric interpretation as shown in fig. 1. Consider the action of the diffeomorphism group $\mathcal{D}^{s+1,p}$ on $\mathcal{M}^{s,p}$ for suitable choices of s, p , where a diffeomorphism η acts on a metric by push-

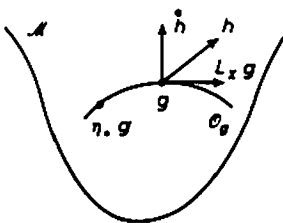


Fig. 1.

forward, i.e. by pull-back via its inverse: $g \mapsto (\eta^{-1})^* g = \eta_* g$; thus g transforms «in the same direction» as the point map η . This is only a C^0 action, but the orbit of g is a smooth manifold [22].

The infinitesimal generators of this group action at g are the symmetric two-tensors $-L_X g$, where X is a vector field of Sobolev class $W^{s+1,p}$. Thus the tensors $-L_X g$ are tangent to the orbit of g under this action. Any tangent vector to $\mathcal{M}^{s,p}$ at g is a symmetric two-tensor h that can be split in a part that is tangent to the orbit, $L_X g$ for some X , and a part \hat{h} that is L_X -perpendicular to the orbit O_g in the sense that $\int_M \langle \hat{h}, L_X g \rangle \mu(g) = 0$. If we want to consider the orbit space $\mathcal{M}^{s,p} / \mathcal{D}^{s+1,p}$ of $\mathcal{M}^{s,p}$ by the action of $\mathcal{D}^{s+1,p}$, \hat{h} is an obvious candidate for a tangent vector to the orbit space. Metrics $(\eta^{-1})^* g$

on the orbit of g are isometric to g and hence are geometrically equivalent to g , so \dot{h} represents infinitesimal directions of nonequivalent (i.e. nonisometric) geometric deformations.

The splitting theorems of Deser, Barbanace and York make a further decomposition of \dot{h} , as we shall see later.

2. - The Hamiltonian structure of geometrodynamics.

We shall begin by showing how the Arnowitt, Deser and Misner [3] Hamiltonian formulation of general relativity can be written in a compact form using the L_2 -adjoint operators of the linearized constraints. Here we shall restrict ourselves to space-times with a compact spacelike hypersurface. The noncompact case is rather different, as indicated in [23].

Let V_4 be a 4-dimensional manifold with Lorentzian metric ${}^{(4)}g$ which is oriented and time-oriented. Let M be a compact oriented 3-dimensional manifold, and let $i: M \rightarrow V_4$ be an embedding of M such that the embedded manifold $i(M) = \Sigma$ is spacelike; i.e. the pull-back $i^*({}^{(4)}g) = g$ is a Riemannian metric on M . Let $E^\infty(M, V_4, {}^{(4)}g)$ denote the set of all such spacelike embeddings. As in [24], this is a smooth manifold. Let k denote the second fundamental form of the embedding, defined at $m \in M$, for $X, Y \in T_m M$, by the usual formula

$$k_m(X, Y) = -{}^{(4)}g \circ i(m) \cdot ((T_m i \cdot Y), {}^{(4)}\nabla_{(T_m i \cdot X)} {}^{(4)}Z_\Sigma \circ i(m)),$$

where ${}^{(4)}Z_\Sigma \circ i(m)$ is the forward-pointing unit timelike normal to Σ at $i(m)$. Thus $k_{ij} = -Z_{ij}$, where ∇ denotes covariant differentiation using ${}^{(4)}g$. Covariant differentiation using g is denoted with a vertical bar.

Let $\pi = \pi' \otimes \mu(g)$ be a 2-contravariant tensor density, whose tensor part π' is defined by $\pi' = ((\text{tr } k)g - k)^\sharp$, where \sharp indicates the contravariant form of a covariant tensor with indices raised by $g^\sharp = g^{ij}$; similarly \flat denotes the covariant form of a contravariant tensor. In the Hamiltonian formulation of Arnowitt, Deser and Misner, k plays the role of a velocity variable and π is its canonical momentum in the DeWitt metric (see [25] for this latter interpretation). Note that $\pi^{\text{ADM}} = \pi^{\text{ADM}} d^3x$.

Now suppose we have a curve in $E^\infty(M, V_4, {}^{(4)}g)$, i.e. a curve i_λ of spacelike embeddings of M into $(V_4, {}^{(4)}g)$. The λ -derivative of this curve defines a 1-parameter family of vector fields ${}^{(4)}X_{i_\lambda}$ on the embedded hypersurfaces by the equation

$$\frac{di_\lambda}{d\lambda} = {}^{(4)}X_{i_\lambda} \circ i_\lambda: M \rightarrow TV_4$$

(see fig. 2). The normal and tangential projections of ${}^{(4)}X_{i_\lambda}$ define a curve of

functions $N_\lambda = {}^{(4)}X_\perp: M \rightarrow \mathbb{R}$ and vector fields ${}^{(4)}X_\lambda = -X_\lambda: M \rightarrow TM$ on M by the equation

$${}^{(4)}X_{\Sigma_\lambda} \circ i_\lambda(m) = {}^{(4)}X_\perp(\lambda, m) {}^{(4)}Z_{\Sigma_\lambda} \circ i_\lambda(m) + T_m i_\lambda \cdot {}^{(4)}X_\lambda(\lambda, m),$$

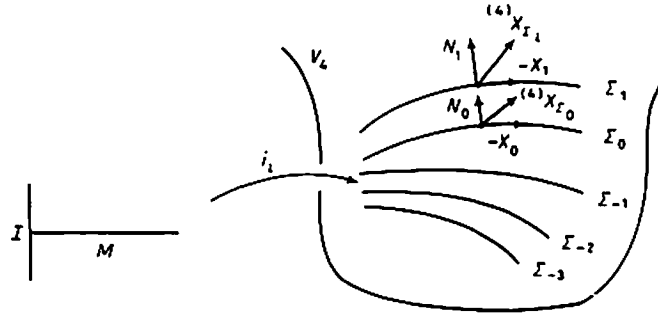


Fig. 2.

where ${}^{(4)}Z_{\Sigma_\lambda}$ is the forward-pointing unit timelike normal to Σ_λ . If $N_\lambda > 0$, then the map

$$F: I \times M \rightarrow V_4; (\lambda, m) \mapsto i_\lambda(m)$$

is a diffeomorphism of $I \times M$ onto a tubular neighborhood of $i_0(M) = \Sigma_0$, if the interval $I = (-\beta, \beta)$ is chosen small enough. In this case we call either the curve i_λ or the embedded spacelike hypersurfaces $\Sigma_\lambda = i_\lambda(M)$ a *slicing* of V_4 .

The functions N_λ and the vector fields X_λ are the *lapse functions* and *shift vector fields* of Arnowitt, Deser and Misner [3] and Wheeler [26] (see fig. 2). We have changed the sign of the shift vector field for various reasons, but basically our conventions give a shift vector field X_λ which generates a 1-parameter family of diffeomorphisms $f_\lambda: M \rightarrow M$, defined by $f_0 = id_M$, and $df_\lambda/d\lambda = X_\lambda \circ f_\lambda$, such that the new family of embeddings $\tilde{i}_\lambda = i_\lambda \circ f_\lambda$ has zero shift (and lapse $\bar{N}_\lambda = N_\lambda \circ f_\lambda$). To see this, note that

$$\begin{aligned} \frac{d\tilde{i}_\lambda}{d\lambda} &= \frac{di_\lambda}{d\lambda} \circ f_\lambda + T_{i_\lambda} \cdot \frac{df_\lambda}{d\lambda} = (N_\lambda {}^{(4)}Z_{\Sigma_\lambda} \circ i_\lambda) \circ f_\lambda - T_{i_\lambda} \cdot X_\lambda \circ f_\lambda + T_{i_\lambda} \cdot X_\lambda \circ f_\lambda = \\ &= (N_\lambda \circ f_\lambda) {}^{(4)}Z_{\Sigma_\lambda} \circ i_\lambda \circ f_\lambda = \bar{N}_\lambda {}^{(4)}Z_{\Sigma_\lambda} \circ \tilde{i}_\lambda. \end{aligned}$$

In the convention of Arnowitt, Deser and Misner [3], one has to consider the flow of $-X_{\text{shift}}^{\text{ADM}}$ to transform the shift away. As phrased in Wheeler's way (as in [21] eqs. (21)-(39) and p. 21-49), the 'perpendicular connector' between two neighboring hypersurfaces has components

$$(dt, -X_{\text{shift}}^{\text{ADM}}) = (dt, X_{\text{shift}}^{\text{ours}}).$$

For «dynamical» reasons for the change of sign of the shift vector field, see [25], p. 557.

Our change in sign of the shift forces us to change the sign of the divergence constraint (see below). This, however, gives an object which conforms with a certain universal tensor, constructed for general field theories independently by Kuchař [27] and FISCHER and MARSDEN [28]. For the scalar field this tensor is $-\pi_\varphi \nabla \varphi$ and for the electromagnetic field it is essentially $E \times B$. In Kuchař's conventions, where the shift is that of ADM, his universal tensor gives the negative of these objects.

Using $F: I \times M \rightarrow V_4$ as a co-ordinate system for a tubular neighborhood of Σ_0 in V_4 , co-ordinates (x^i) , $i = 1, 2, 3$, on M , and $(x^\alpha) = (\lambda, x^i)$, $\alpha = 0, 1, 2, 3$, as co-ordinates on $I \times M$, we can write the pulled-back metric $F^{*(4)}g$ in co-ordinates as

$$(F^{*(4)}g)_{\alpha\beta} dx^\alpha dx^\beta = -(N^2 - X_i X^i) d\lambda^2 - 2X_i dx^i d\lambda + g_{ij} dx^i dx^j,$$

where $g_{ij} = (g_\lambda)_{ij}$ and $g_\lambda = i_\lambda^{*(4)}g$.

Let k_λ be the curve of second fundamental forms for the embedded hypersurfaces $\Sigma_\lambda = i_\lambda(M)$, and let π_λ be their associated canonical momenta.

The following theorem contains the basic geometrodynamical equations due to DIRAC [1, 2, 29] and to ARNOWITT, DESER and MISNER (see [3] and references therein).

2.1. Theorem. — *Let the vacuum Einstein field equations $\text{Ein}^{(4)}g = 0$ hold on V_4 . Then for each one-parameter family of spacelike embeddings $\{i_\lambda\}$ of V_4 , the induced metrics g_λ and momentum π_λ on Σ_λ satisfy the following equations:*

$$\begin{aligned} & \frac{\partial g}{\partial \lambda} = 2N \left((\pi') - \frac{1}{2} g(\text{tr } \pi') \right) - L_X g, \\ \text{(evolution equations)} \quad & \frac{\partial \pi}{\partial \lambda} = -N \left(\text{Ric}(g) - \frac{1}{2} R(g)g \right) \mu(g) + \frac{1}{2} N g^2 \left(\pi' \cdot \pi' - \frac{1}{2} (\text{tr } \pi')^2 \right) \mu(g) - \\ & - 2N \left(\pi' \times \pi' - \frac{1}{2} (\text{tr } \pi') \pi' \right) \mu(g) + (\text{Hess } N + g \Delta N) \mu(g) - L_X \pi, \end{aligned}$$

and

$$\begin{aligned} \text{(constraint equations)} \quad & \mathcal{H}(g, \pi) = (\pi' \cdot \pi' - \frac{1}{2} (\text{tr } \pi')^2 - R(g)) \mu(g) = 0, \\ & \mathcal{J}(g, \pi) = -2(\delta_i \pi) = 2\pi_{i,j} = 0. \end{aligned}$$

Conversely, if i_λ is a slicing of $(V_4, {}^{(4)}g)$ such that the above evolution and constraint equations hold, then ${}^{(4)}g$ satisfies the (empty space) field equations.

Our notation in the theorem is as follows: $(\pi' \times \pi')^{ij} = (\pi')^{ik} (\pi')^j_k$, $\pi' \cdot \pi' = (\pi')^{ij} (\pi')_{ij}$, $\text{Hess } N = N_{(ij)}$, $\Delta N = -g^{ij} N_{(ij)}$, and $L_X \pi = (L_X \pi') \mu(g) +$

$+\pi'(\operatorname{div} X)\mu(g)$ is the Lie derivative of the tensor density $\pi = \pi' \mu_g$ (note $L_X \mu_g = (\operatorname{div} X)\mu(g)$). The Ricci tensor $R_{\mu\nu}$ of ${}^{(4)}g$ is denoted $\operatorname{Ric}({}^{(4)}g)$ and that of g by $\operatorname{Ric}(g)$. $R(g)$ is the scalar curvature. We write $\operatorname{Ein}(g) = \operatorname{Ric}(g) - \frac{1}{2}R(g)g$, the Einstein tensor of g .

A sketch of the proof of theorem 2.1 is given below.

These evolution and constraint equations are the same as those of Arnowitt Deser and Misner ([3], equations (7-3.15)); recall that our shift and divergence constraint are the negative of theirs, and that our π , \mathcal{H} and \mathcal{J} are tensor densities.

The 12 first-order evolution equations for (g, π) correspond to the six second-order equations ${}^{(4)}G^{\alpha\beta} = 0$, while four of the other Einstein equations ${}^{(4)}G^{00} = 0$ and ${}^{(4)}G^0_i = 0$ appear as the constraint equations. More explicitly, in co-ordinates determined by a slicing i_λ , ${}^{(4)}Z_\Sigma$ has components ${}^{(4)}Z_\alpha = (-N, 0)$. If we define the «perpendicular-perpendicular» and «perpendicular-parallel» projections of the Einstein tensor by

$${}^{(4)}G_{\perp\perp} = Z_\alpha Z_\beta {}^{(4)}G^{\alpha\beta} = N^2 {}^{(4)}G^{00}$$

and

$${}^{(4)}G^\perp_i = -Z_\alpha {}^{(4)}G^{\alpha}_i = N {}^{(4)}G^0_i,$$

then

$$\mathcal{H}(g, \pi) = -2 {}^{(4)}G_{\perp\perp} \mu(g)$$

and

$$\mathcal{J}(g, \pi)_i = -2 {}^{(4)}G^\perp_i \mu(g).$$

The evolution equations of this theorem are well posed. The proof of this makes use of harmonic co-ordinates, *i.e.* a special choice of lapse and shift determined implicitly. With the choice $N = 1$, $X = 0$, the equations are

$$\frac{\partial^2 g_{ij}}{\partial \lambda^2} = g^{ab} \frac{\partial^2 g_{ij}}{\partial x^a \partial x^b} + g^{ab} \frac{\partial^2 g_{ab}}{\partial x^i \partial x^j} - g^{ab} \frac{\partial^2 g_{ai}}{\partial x^b \partial x^j} - g^{ab} \frac{\partial^2 g_{aj}}{\partial x^b \partial x^i} + \text{lower-order terms}.$$

In this form the equations are not strictly hyperbolic and the known existence theorems do not apply. The use of harmonic co-ordinates makes the 4-dimensional field equations strictly hyperbolic, from which it follows that they are well posed, the result and proof of which is due to CHOQUET-BRUHAT [30]. This can also be based [31] on the strictly hyperbolic systems of Leray, and FISCHER and MARSDEN [32] treat the equations as a symmetric hyperbolic first-order system. The sharpest results, using H^s spaces with the smallest possible s , are due to HUGHES, KATO and MARSDEN [33].

In the formulation of theorem 2.1, the lapse and shift are regarded as freely specifiable. In the «thin sandwich» formulation, one regards g and \dot{g}

as Cauchy data, expresses π as a function of (g, N, X) and solves for N and X from the constraint equations

$$\mathcal{H}(g, \pi(g, N, X)) = 0,$$

$$\mathcal{J}(g, \pi(g, N, X)) = 0.$$

Upon linearizing, it is easy to see that this is not an elliptic system, so, even if it is solvable, there will be some technical problems; in particular, regularity must fail. Thus, the thin-sandwich formulation is rejected by most workers. For other difficulties with the thin-sandwich formulation, see [34].

It is important to recognize various combinations of terms in the ADM evolution equations as Lie derivatives, and we have done so in the way theorem 2.1 is written. It is also useful to write the quadratic algebraic part of $\partial\pi/\partial\lambda$ as

$$S_o(\pi, \pi) = -2\{\pi' \times \pi' - \frac{1}{2}(\text{tr } \pi')\pi'\}\mu(g) + \frac{1}{2}g^{\alpha\beta}\{\pi'_{\alpha}\pi'_{\beta} - \frac{1}{2}(\text{tr } \pi')^2\}\mu(g).$$

This is the spray of the DeWitt metric, i.e. the terms in \mathcal{H} quadratic in π' ; see below and [25]. Thus the terms in the evolution equation for π may be interpreted as follows:

$$\begin{aligned} \frac{\partial\pi}{\partial\lambda} = & N S_o(\pi, \pi) - && \text{geodesic spray of the DeWitt metric,} \\ & - N \text{Ein}(g)^{\alpha\beta}\mu(g) + && \text{force term of the scalar curvature potential,} \\ & + (\text{Hess } N + g \Delta N)^{\alpha\beta}\mu(g) - && \text{« tilt » term due to nonconstancy of } N, \\ & - L_X \pi && \text{« shift » term due to a nonzero shift.} \end{aligned}$$

The evolution equation for g may be regarded as the defining equation for π . We refer to [25, 35, 36] for more information. In this section, we shall be primarily concerned with the Hamiltonian structure of these equations.

We consider again the space \mathcal{M} of Riemannian metrics on M and the diffeomorphism group \mathcal{D} . For the compact case, we should use $\mathcal{M}^{s,p}$ with $s > n/p$, that is, Riemannian metrics of a certain Sobolev class; the diffeomorphisms and other maps and tensors we use also should belong to appropriate Sobolev classes. Similarly, in the noncompact case, the various tensors or diffeomorphisms should belong to the appropriate $M_{s,p}^{\alpha}$ spaces described in [23]. For ease of notation, however, we shall restrict ourselves to the C^∞ case.

Let $T\mathcal{M} \approx \mathcal{M} \times S_2$ denote the tangent bundle of \mathcal{M} , where S_2 is the space of C^∞ 2-covariant symmetric tensor fields on M . Let S_2^* denote the space of C^∞ 2-contravariant symmetric tensor densities on M . Define $T^*\mathcal{M} \approx \mathcal{M} \times S_2^* = \{(g, \pi) | g \in \mathcal{M}, \pi \in S_2^*\}$. We shall think of $T^*\mathcal{M}$ as the « L_x -cotangent bundle

to \mathcal{M} . For $k \in T_g \mathcal{M} \approx S_2$, $\pi \in T_g^* \mathcal{M} \approx S_2^*$, there is a natural pairing

$$(\pi, k)_{L_1} = \int_M \pi \cdot k.$$

Thus $T^* \mathcal{M}$ as defined is a subbundle of the «true» cotangent bundle. Since $T^* \mathcal{M}$ is open in $S_2 \times S_2^*$, the tangent space of $T^* \mathcal{M}$ at $(g, \pi) \in T^* \mathcal{M}$ is $T_{(g, \pi)}(T^* \mathcal{M}) \approx S_2 \times S_2^*$.

We now show that $T^* \mathcal{M}$ carries a natural symplectic structure in which the evolution equations of the theorem are Hamiltonian. In order to include the lapse function and shift vector field into this scheme, it is necessary to develop the notion of a generalized Hamiltonian system.

On $T^* \mathcal{M}$ we define the globally constant symplectic structure

$$\Omega = \Omega_{(g, \pi)}: T_{(g, \pi)}(T^* \mathcal{M}) \times T_{(g, \pi)}(T^* \mathcal{M}) \rightarrow \mathbb{R}$$

as follows: for $(h_1, \omega_1), (h_2, \omega_2) \in T_{(g, \pi)}(T^* \mathcal{M}) = S_2 \times S_2^*$,

$$\Omega_{(g, \pi)}((h_1, \omega_1), (h_2, \omega_2)) = \int_M \omega_2 \cdot h_1 - \omega_1 \cdot h_2.$$

Let

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}: S_2^* \times S_2 \rightarrow S_2 \times S_2^*$$

be defined by

$$\begin{pmatrix} \omega \\ h \end{pmatrix} \mapsto J \begin{pmatrix} \omega \\ h \end{pmatrix} = \begin{pmatrix} h \\ -\omega \end{pmatrix}$$

so that

$$J^{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}: S_2 \times S_2^* \rightarrow S_2^* \times S_2, \quad \begin{pmatrix} h \\ \omega \end{pmatrix} \mapsto J^{-1} \begin{pmatrix} h \\ \omega \end{pmatrix} = \begin{pmatrix} -\omega \\ h \end{pmatrix}.$$

Then

$$\Omega((h_1, \omega_1), (h_2, \omega_2)) = - \int_M (h_1, \omega_1) \cdot J^{-1} \begin{pmatrix} h_2 \\ \omega_2 \end{pmatrix}.$$

We shall return to J shortly.

Let

$C^\infty = C^\infty(M; \mathbb{R})$ denote the smooth real-valued functions on M ,

C_d^∞ = smooth scalar densities on M ,

\mathcal{X} = smooth vector fields on M

and

Λ_d^1 = smooth 1-form densities on M .

Consider the functions

$$\mathcal{H}: T^*\mathcal{M} \rightarrow C_d^\infty; (g, \pi) \mapsto \mathcal{H}(g, \pi) = (\pi' \cdot \pi' - \frac{1}{2}(\text{tr } \pi')^2 - R(g))\mu(g),$$

$$\mathcal{J} = -2\delta: T^*\mathcal{M} \rightarrow \Lambda_d^1; (g, \pi) \mapsto 2(\delta, \pi) = 2\pi_{i,j},$$

and

$$\Phi = (\mathcal{H}, \mathcal{J}): T^*\mathcal{M} \rightarrow C_d^\infty \times \Lambda_d^1; (g, \pi) \mapsto (\mathcal{H}(g, \pi), \mathcal{J}(g, \pi)).$$

At this point it is necessary to compute the derivatives of \mathcal{H} , \mathcal{J} and Φ and their L_2 adjoints. The results are collected in the following

2.2. Proposition. - If we let $(g, \pi) \in T^*\mathcal{M}$, $(h, \omega) \in T_{(g, \pi)}(T^*\mathcal{M}) = S_2 \times S_d^2$ and $(N, X) \in C^\infty \times \mathcal{X}$, the derivatives of \mathcal{H} , \mathcal{J} , Φ (as defined above)

$$D\mathcal{H}(g, \pi): S_2 \times S_d^2 \rightarrow C_d^\infty,$$

$$D\mathcal{J}(g, \pi): S_2 \times S_d^2 \rightarrow \Lambda_d^1,$$

$$D\Phi(g, \pi): S_2 \times S_d^2 \rightarrow C_d \times \Lambda_d^1,$$

and their natural adjoints

$$D\mathcal{H}(g, \pi)^*: C^\infty \rightarrow S_d^2 \times S_2,$$

$$D\mathcal{J}(g, \pi)^*: \mathcal{X} \rightarrow S_d^2 \times S_2,$$

$$D\Phi(g, \pi)^*: C^\infty \times \mathcal{X} \rightarrow S_d^2 \times S_2$$

are given as follows:

$$\begin{aligned} D\mathcal{H}(g, \pi) \cdot (h, \omega) &= -S_g(\pi, \pi) \cdot h + (\text{Ein } g) \cdot h - (\delta\delta h + \Delta \text{tr } h)\mu(g) + \\ &\quad + 2((\pi')^b - \frac{1}{2}(\text{tr } \pi')g) \cdot \omega, \end{aligned}$$

$$\begin{aligned} D\mathcal{H}(g, \pi)^* N &= (-NS_g(\pi, \pi) + [N \text{Ein } g] - (\text{Hess } N + g\Delta N))^* \mu(g), \\ &\quad 2N((\pi')^b - \frac{1}{2}(\text{tr } \pi')g), \end{aligned}$$

$$D\mathcal{J}(g, \pi) \cdot (h, \omega) = 2(\omega_{i,j} + h_{ik}\pi^{k,j} + \pi^{ik}(h_{i,j} - \frac{1}{2}h_{j,i})),$$

$$D\mathcal{J}(g, \pi)^* \cdot X = (L_X \pi, -L_X g),$$

$$D\Phi(g, \pi) \cdot (h, \omega) = (D\mathcal{H}(g, \pi) \cdot (h, \omega), D\mathcal{J}(g, \pi) \cdot (h, \omega))$$

and

$$\begin{aligned} D\Phi(g, \pi)^* \cdot (N, X) &= D\mathcal{H}(g, \pi)^* \cdot N + D\mathcal{J}(g, \pi)^* \cdot X = \\ &= \left(-N S_g(\pi, \pi) + [N \operatorname{Ein}(g) - (\operatorname{Hess} N + g \Delta N)]^g \mu(g) + \right. \\ &\quad \left. + L_X \pi, 2N((\pi')^g - \tfrac{1}{2}(\operatorname{tr} \pi')g) - L_X g \right). \end{aligned}$$

Remark. We are of course indulging in some abuse of notation, mixing invariant and index notations. We shall often use indices when it saves explanation and is easier to see.

Proof. To compute the derivative of \mathcal{H} ,

$$D\mathcal{H}(g, \pi): T_{(g, \pi)}(T^*\mathcal{M}) \approx S_2 \times S_d^2 \rightarrow T_{\mathcal{H}(g, \pi)} C_d^\infty \approx C_d^\infty,$$

we use

$$D\mathcal{H}(g, \pi) \cdot (h, \omega) = D_g \mathcal{H}(g, \pi) \cdot h + D_\pi \mathcal{H}(g, \pi) \cdot \omega.$$

One must be cautious here and take the partial derivatives of \mathcal{H} as a function of π and not π' . We do this by writing

$$\mathcal{H}(g, \pi) = \left(\frac{1}{\sqrt{\det g}} (\sqrt{\det g} \pi' \cdot \sqrt{\det g} \pi') - \frac{1}{2} (\sqrt{\det g} \operatorname{tr} \pi')^2 \right) d^2x - R(g) \mu(g).$$

Then the partial derivatives are given by

$$\begin{aligned} D_g \mathcal{H}(g, \pi) \cdot h &= \frac{1}{\sqrt{\det g}} \left(2 \left(\pi \times \pi - \frac{1}{2} (\operatorname{tr} \pi) \pi \right) - \frac{1}{2} \left(\pi \cdot \pi - \frac{1}{2} (\operatorname{tr} \pi)^2 \right) g \right) \cdot h - \\ &\quad - \sqrt{\det g} \left(\delta \delta h + \Delta \operatorname{tr} h - \operatorname{Ric}(g) + \frac{1}{2} g R(g) \right)^g \cdot h, \end{aligned}$$

where $\pi \times \pi = \pi' \times \pi' \sqrt{\det g} \cdot \sqrt{\det g} d^2x$ and $\pi \cdot \pi = \pi' \cdot \pi' \sqrt{\det g} \cdot \sqrt{\det g} d^2x$. In addition,

$$D_\pi \mathcal{H}(g, \pi) \cdot \omega = \frac{2}{\sqrt{\det g}} \left(\pi \cdot \omega - \frac{1}{2} (\operatorname{tr} \pi) (\operatorname{tr} \omega) \right) = 2 \left((\pi')^g - \frac{1}{2} (\operatorname{tr} \pi') g \right) \cdot \omega,$$

so that

$$\begin{aligned} D\mathcal{H}(g, \pi) \cdot (h, \omega) &= -S_g(\pi, \pi) \cdot h + (\operatorname{Ein}(g) \cdot h) \mu(g) - \\ &\quad - (\delta \delta h + \Delta \operatorname{tr} h) \mu(g) + 2((\pi')^g - \tfrac{1}{2}(\operatorname{tr} \pi')g) \cdot \omega. \end{aligned}$$

In computing the partial derivatives we have used the following expressions:

$$D_g \mu(g) \cdot h = \frac{1}{2} (\text{tr } h) \mu(g),$$

$$D_g \left(\frac{1}{\sqrt{\det g}} \right) \cdot h = -\frac{1}{2} \frac{1}{\sqrt{\det g}} \text{tr } h,$$

$$D_g(\pi \cdot \pi) \cdot h = 2(\pi \times \pi) \cdot h,$$

$$D_g(\text{tr } \pi) \cdot h = D_g(g \cdot \pi) \cdot h = h \cdot \pi$$

and

$$D_g R(g) \cdot h = \Delta \text{tr } h + \delta \delta h - h \cdot \text{Ric}(g).$$

The last equation is the classical variation formula for the scalar curvature; a convenient reference for these variation formulae is [20]. See also appendix I.

As usual, the L_2 -adjoint of $D\mathcal{H}(g, \pi)^* : (C_d^\infty)^* \approx C^\infty \rightarrow (S_2 \times S_d^2)^* = S_2^* \times S_2$ is defined by

$$\int N D\mathcal{H}(g, \pi) \cdot (h, \omega) = \int \langle D\mathcal{H}(g, \pi)^* \cdot N, (h, \omega) \rangle,$$

where the last inner product is the natural pairing between $S_d^2 \times S_2$ and $S_2 \times S_d^2$.

A straightforward integration by parts of the $-(\delta \delta h + \Delta \text{tr } h) \mu(g)$ term shows that

$$D\mathcal{H}(g, \pi)^* \cdot N =$$

$$= (-NS_g(\pi, \pi) + (N \text{Ein}(g) - \text{Hess } N - g \Delta N)^* \mu(g), 2N((\pi')^\flat - \frac{1}{2}(\text{tr } \pi')g)).$$

(We may integrate freely by parts since M is compact without boundary.)

We now compute the derivative of

$$\mathcal{J}(g, \pi) = 2\pi'_{ij} = 2g_{ik}(\pi'^k_{,j} + \pi'^l \Gamma^k_{jl}),$$

i.e. the map

$$D\mathcal{J}(g, \pi) : S_2 \times S_d^2 \rightarrow A_d^1.$$

For $(h, \omega) \in S_2 \times S_d^2$, we write, as above,

$$D\mathcal{J}(g, \pi) \cdot (h, \omega) = D_g \mathcal{J}(g, \pi) \cdot h + D_\pi \mathcal{J}(g, \pi) \cdot \omega.$$

Since $\mathcal{J}(g, \pi)$ is linear in π ,

$$D_\pi \mathcal{J}(g, \pi) \cdot \omega = 2\omega^i_{,j}.$$

The partial derivative $D_g \mathcal{J}(g, \pi) \cdot h$ is computed as follows:

$$\begin{aligned} D_g(\pi_{i,j}) \cdot h &= D_g(g_{ik} \pi^{kj}) \cdot h = h_{ik} \pi^{kj} + g_{ik} D_g(\pi^{kj} + \pi^j \Gamma_{ji}^k) \cdot h = \\ &= h_{ik} \pi^{kj} + g_{ik} \pi^j D_g(\Gamma_{ji}^k) \cdot h = \\ &= h_{ik} \pi^{kj} + g_{ik} \pi^{jj} \frac{1}{2} (h_{ji}^k + h_{ji}^k - h_{ji}^k) = \\ &= h_{ik} \pi^{kj} + \pi^{jj} h_{jji} - \frac{1}{2} \pi^{jj} h_{jji} = -h_{ik} (\delta \pi)^k + \pi^{jj} (h_{jji} - \frac{1}{2} h_{jji}), \end{aligned}$$

where we have used the variational formula for the Christoffel symbols

$$D_g(\Gamma_{ji}^k) \cdot h = \frac{1}{2} (h_{ji}^k + h_{ji}^k - h_{ji}^k).$$

Thus

$$\begin{aligned} D\mathcal{J}(g, \pi) \cdot (h, \omega) &= 2(w_{i,j} + h_{ik} \pi^{kj} + \pi^{jj} (h_{jji} - \frac{1}{2} h_{jji})) = \\ &= 2(-(\delta \omega)^j - h_{ik} (\delta \pi)^k + \pi^{jj} (h_{jji} - \frac{1}{2} h_{jji})). \end{aligned}$$

The adjoint map

$$D\mathcal{J}(g, \pi)^*: \mathcal{X} \rightarrow \mathcal{S}_g^2 \times \mathcal{S}_2$$

can be computed by integrating by parts, but it can be more easily computed as follows: note that for all vector fields $X \in \mathcal{X}$,

$$\int_M X \cdot \mathcal{J}(g, \pi) = 2 \int_M X^i \pi_{i,j} = - \int_M \pi^{jj} (X_{jji} + X_{jji}) = - \int_M \langle \pi, L_X g \rangle.$$

Since the contraction $X \cdot \mathcal{J}(g, \pi)$ is natural, i.e. does not depend on the metric,

$$D\left(\int \langle X, \mathcal{J}(g, \pi) \rangle\right) \cdot (h, \omega) = \int \langle X, D\mathcal{J}(g, \pi) \cdot (h, \omega) \rangle = \int \langle D\mathcal{J}(g, \pi)^* \cdot X, (h, \omega) \rangle,$$

and so

$$\begin{aligned} D\int \langle X, \mathcal{J}(g, \pi) \rangle \cdot (h, \omega) &= - \int D(\langle \pi, L_X g \rangle) \cdot (h, \omega) = - \int \langle \omega, L_X g \rangle - \int \langle \pi, L_X h \rangle = \\ &= - \int \langle \omega, L_X g \rangle + \int \langle L_X \pi, h \rangle = \int \langle (L_X \pi, -L_X g), (h, \omega) \rangle. \end{aligned}$$

Thus

$$D\mathcal{J}(g, \pi)^* \cdot X = (L_X \pi, -L_X g) \in \mathcal{S}_g^2 \times \mathcal{S}_2.$$

For the map $\Phi = (\mathcal{H}, \mathcal{J}): T^* \mathcal{M} \rightarrow C_a^\infty \times \Lambda_a^1$, the derivative is clearly given by $D\Phi(g, \pi) = (D\mathcal{H}(g, \pi), D\mathcal{J}(g, \pi)): \mathcal{S}_2 \times \mathcal{S}_g^2 \rightarrow C_a^\infty \times \Lambda_a^1$.

To compute the adjoint

$$D\Phi(g, \pi)^*: C^\infty \times \mathcal{X} \rightarrow \mathcal{S}_g^2 \times \mathcal{S}_2,$$

note that, for all $(h, \omega) \in S_2 \times S_2^*$,

$$\begin{aligned} \int \langle (h, \omega), D\Phi(g, \pi)^*(N, X) \rangle &= \int \langle D\Phi(g, \pi) \cdot (h, \omega), (N, X) \rangle = \\ &= \int \langle D\mathcal{H}(g, \pi) \cdot (h, \omega), N \rangle + \int \langle D\mathcal{J}(g, \pi) \cdot (h, \omega), X \rangle = \\ &= \int \langle (h, \omega), D\mathcal{H}(g, \pi)^* \cdot N \rangle + \int \langle (h, \omega), D\mathcal{J}(g, \pi)^* \cdot X \rangle. \end{aligned}$$

So $D\Phi(g, \pi)^*(N, X) = D\mathcal{H}(g, \pi)^* \cdot N + D\mathcal{J}(g, \pi)^* \cdot X$. Substitution of the expressions obtained for $D\mathcal{H}(g, \pi)^*$ and $D\mathcal{J}(g, \pi)^*$ completes the proof. \square

It is important to note that the L_2 adjoints we have been computing are just the physicist's functional derivatives. To see this, suppose

$$\mathcal{F}: T^*\mathcal{M} \rightarrow C_0^\infty$$

is a scalar density. Let $F: T^*\mathcal{M} \rightarrow \mathbb{R}$,

$$F(g, \pi) = \int_{\mathcal{M}} \mathcal{F}(g, \pi).$$

Then, for $(h, \omega) \in T_{(g, \pi)}(T^*\mathcal{M})$,

$$\begin{aligned} dF(g, \pi) \cdot (h, \omega) &= \int D\mathcal{F}(g, \pi) \cdot (h, \omega) = \int \langle D\mathcal{F}(g, \pi)^* \cdot 1, (h, \omega) \rangle = \\ &= \int \langle D_g \mathcal{F}(g, \pi)^* \cdot 1, h \rangle + \int \langle D_\pi \mathcal{F}(g, \pi)^* \cdot 1, \omega \rangle. \end{aligned}$$

In physicist's notation, $h = \delta g$, $\omega = \delta \pi$, $dF = \delta F$,

$$D_g \mathcal{F}(g, \pi)^* \cdot 1 = \frac{\delta \mathcal{F}}{\delta g} \quad \text{and} \quad D_\pi \mathcal{F}(g, \pi)^* \cdot 1 = \frac{\delta \mathcal{F}}{\delta \pi},$$

so that

$$\delta \mathcal{F} = \left(\frac{\delta \mathcal{F}}{\delta g}, \frac{\delta \mathcal{F}}{\delta \pi} \right) = (D_g \mathcal{F}(g, \pi)^* \cdot 1, D_\pi \mathcal{F}(g, \pi)^* \cdot 1) = D\mathcal{F}(g, \pi)^* \cdot 1.$$

If, for example, $\mathcal{F}(g, \pi)$ depends on at most second derivatives of the metric, then

$$\begin{aligned} \frac{\delta \mathcal{F}}{\delta g} \cdot \delta g &= \int \langle D_g \mathcal{F}(g, \pi)^* \cdot 1, h \rangle = \int D_g \mathcal{F}(g, \pi) \cdot h = \\ &= \int (\partial_i \mathcal{F} \cdot h + \partial_{\varepsilon_i} \mathcal{F} \cdot \partial_i h + \partial_{\varepsilon_i \varepsilon_j} \mathcal{F} \cdot \partial_i \partial_j h) = \\ &= \int (\partial_i \mathcal{F} - \partial_i (\partial_{\varepsilon_i} \mathcal{F}) + \partial_i \partial_j (\partial_{\varepsilon_i \varepsilon_j} \mathcal{F})) \cdot h, \end{aligned}$$

which gives the usual expression for the functional derivative, namely

$$\frac{\delta \mathcal{F}}{\delta g} = \partial_\sigma \mathcal{F} - \partial_i (\partial_{\varepsilon_{i\sigma}} \mathcal{F}) + \partial_i \partial_j (\partial_{\varepsilon_{ij\sigma}} \mathcal{F}).$$

Thus the adjoint is a convenient way to write the functional derivative. For a scalar function $f: M \rightarrow \mathbb{R}$, we also have

$$\frac{\delta}{\delta g} (f \mathcal{F}(g, \pi)) = D_\sigma (f \mathcal{F}(g, \pi))^* \cdot 1 = (D_\sigma \mathcal{F}(g, \pi))^* \cdot f,$$

as can easily be checked.

In terms of the maps $\mathcal{H}(g, \pi)$, $\mathcal{J}(g, \pi)$ and $\Phi(g, \pi)$, we have the following correspondences with the physicist's notation and our adjoint notation:

$$\begin{aligned} \left(\frac{\delta \mathcal{H}(g, \pi)}{\delta g}, \frac{\delta \mathcal{H}(g, \pi)}{\delta \pi} \right) &= (D_\sigma \mathcal{H}(g, \pi)^* \cdot 1, D_\pi \mathcal{H}(g, \pi)^* \cdot 1) = D\mathcal{H}(g, \pi)^* \cdot 1 \in S_d^2 \times S_2, \\ \left(\frac{\delta (N\mathcal{H}(g, \pi))}{\delta g}, \frac{\delta (N\mathcal{H}(g, \pi))}{\delta \pi} \right) &= D\mathcal{H}(g, \pi)^* \cdot N, \\ \left(\frac{\delta (X \cdot \mathcal{J}(g, \pi))}{\delta g}, \frac{\delta (X \cdot \mathcal{J}(g, \pi))}{\delta \pi} \right) &= (D_\sigma \mathcal{J}(g, \pi)^* \cdot X, D_\pi \mathcal{J}(g, \pi)^* \cdot X) = \\ &= D\mathcal{J}(g, \pi)^* \cdot X \in S_d^2 \times S_2 \end{aligned}$$

and

$$\begin{aligned} \left(\frac{\delta}{\delta g} (N\mathcal{H} + X \cdot \mathcal{J}), \frac{\delta}{\delta \pi} (N\mathcal{H} + X \cdot \mathcal{J}) \right) &= \\ &= (D_\sigma (N\mathcal{H} + X \cdot \mathcal{J})^* \cdot 1, D_\pi (N\mathcal{H} + X \cdot \mathcal{J})^* \cdot 1) = \\ &= D(N\mathcal{H} + X \cdot \mathcal{J})^* \cdot 1 = D \left((\mathcal{H}, \mathcal{J}) \cdot \begin{pmatrix} N \\ X \end{pmatrix} \right)^* \cdot 1 = (D\Phi(g, \pi))^* \cdot \begin{pmatrix} N \\ X \end{pmatrix} \in S_d^2 \times S_2, \end{aligned}$$

where $\Phi = (\mathcal{H}, \mathcal{J})$.

As is shown in [3], the evolution equations of theorem 2.1 are Hamilton's equations with Hamiltonian $N\mathcal{H} + X \cdot \mathcal{J}$, i.e.

$$\begin{aligned} \frac{\partial g}{\partial \lambda} &= \frac{\delta}{\delta \pi} (N\mathcal{H} + X \cdot \mathcal{J}), \\ \frac{\partial \pi}{\partial \lambda} &= -\frac{\delta}{\delta g} (N\mathcal{H} + X \cdot \mathcal{J}). \end{aligned}$$

By using the symplectic structure on $T^*\mathcal{M}$ defined by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} : S_d^2 \times S_2 \rightarrow S_d^2 \times S_2, \quad \begin{pmatrix} \omega \\ h \end{pmatrix} \mapsto J \begin{pmatrix} \omega \\ h \end{pmatrix} = \begin{pmatrix} h \\ -\omega \end{pmatrix}$$

and the correspondence

$$\left(\frac{\delta}{\delta g} (N\mathcal{H} + X \cdot \mathcal{J}), \frac{\delta}{\delta \pi} (N\mathcal{H} + X \cdot \mathcal{J}) \right) = (D\Phi(g, \pi))^* \cdot \begin{pmatrix} N \\ X \end{pmatrix},$$

the Hamiltonian equations above, i.e. those in theorem 2.1, can be written in a very compact form.

2.3. Theorem. — *The Einstein system, defined by the evolution equations and constraint equations of theorem 2.1 can be written as*

$$\begin{aligned} (\text{evolution equations}) \quad \frac{\partial}{\partial \lambda} \begin{pmatrix} g \\ \pi \end{pmatrix} &= J \circ (D\Phi(g, \pi))^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}, \\ (\text{constraint equations}) \quad \Phi(g, \pi) &= (\mathcal{H}(g, \pi), \mathcal{J}(g, \pi)) = 0, \end{aligned}$$

where (N, X) are the lapse function and shift vector field associated with the slicing, and where $D\Phi(g, \pi)^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}$ is given by proposition 2.2.

Remark. In this form of the evolution equations, the symplectic structure of the cotangent bundle enters explicitly. The principal interpretation of this theorem is that the evolution equations are generated by the adjoint $D\Phi(g, \pi)^*$ to the linearized constraints. We shall explore the consequences of writing the Einstein equations in this form in the sections to follow.

Sketch of proof of theorems 2.1 and 2.3. The Lagrangian density which generates the empty-space Einstein equations is

$$\mathcal{L}_{\text{geom}}({}^{(4)}g) = \frac{1}{16\pi} R({}^{(4)}g) \mu({}^{(4)}g),$$

where

$$\mu({}^{(4)}g) = \sqrt{-\det {}^{(4)}g} d^4x = N \sqrt{\det g} d^3x d\lambda = N\mu(g) d\lambda.$$

A computational part of the proof, which we shall not do, is to show that $\mathcal{L}_{\text{geom}}$ can be written in the $(3 + 1)$ -dimensional form as (see [3], eq. (7-3.13), and [21], eq. (21-90))

$$\begin{aligned} 16\pi \mathcal{L}_{\text{geom}}({}^{(4)}g) &= R({}^{(4)}g) \mu({}^{(4)}g) = NR({}^{(4)}g) \mu(g) d\lambda = \\ &= \left(\pi^i \frac{\partial g_{ij}}{\partial \lambda} - N\mathcal{H}(g, \pi) - X^j \mathcal{J}(g, \pi) \right) d\lambda + \\ &+ 2 \left(\left(\pi^i X^j - \frac{1}{2} X^i \text{tr} \pi - (\text{grad } N)^i \mu(g) \right)_{,i} - \frac{\partial}{\partial \lambda} \text{tr} \pi \right) d\lambda. \end{aligned}$$

Here i_λ is a slicing of V_4 , so that V_4 can be identified with $I \times M$. Note that our $\pi = \pi' \mu(g) = \pi' \sqrt{\det g} d^3x = \pi_{\text{ADM}} d^3x$ contains the d^3x term to complete $\sqrt{\det g}$ to a volume element, whereas the π_{ADM} does not. Similarly, the volume element $\mu^{(4)}(g)$ contains $d^4x = d^3x d\lambda$, explaining the overall multiplicative factor $d\lambda$. Also note that, although our shift $X' = -N'$, our $\mathcal{J}(g, \pi) = 2\pi_{i'}^{i'}$ is also the negative of the ADM $\mathcal{K}_i = -2\pi_{i'}^{i'}$, so that the overall term $X \cdot \mathcal{J}$ has the same sign.

Set $\beta = \beta' = 2(\pi', X' - \frac{1}{2} X' \text{tr} \pi - (\text{grad } N)' \mu(g))$, a vector density on M ; note that $\beta'_{i'} = \beta'^{i'} = \text{div } \beta$. The action for gravity can be written as

$$\begin{aligned} 16\pi S_{\text{geom}}^{(4)}(g) &= 16\pi \int_{V_4} \mathcal{L}_{\text{geom}}^{(4)}(g) = \\ &= \int_I \int_M \left(\pi \cdot \frac{\partial g}{\partial \lambda} - N \mathcal{H}(g, \pi) - X \cdot \mathcal{J}(g, \pi) \right) d\lambda + \int_I \int_M \left(\text{div } \beta - \frac{\partial}{\partial \lambda} \text{tr } \pi \right) d\lambda. \end{aligned}$$

Integrating the $\text{div } \beta$ term to zero on M , and dropping the total time derivative term (after changing the order of integration)

$$\int_{I=[a,b]} \int_M \left(\frac{\partial}{\partial \lambda} \text{tr } \pi \right) d\lambda = \int_M \int_I d\lambda \left(\frac{\partial}{\partial \lambda} \text{tr } \pi \right) = \int_M (\text{tr } \pi)_{\lambda=b} - \int_M (\text{tr } \pi)_{\lambda=a}$$

as a constant that will not enter into the variation of S_{geom} , we have

$$16\pi S_{\text{geom}}^{(4)}(g) = \int_I \int_M \left(\pi \cdot \frac{\partial g}{\partial \lambda} - N \mathcal{H} - X \cdot \mathcal{J} \right) d\lambda = \int_I \int_M \left(\pi \cdot \frac{\partial g}{\partial \lambda} - \Phi(g, \pi) \cdot \begin{pmatrix} N \\ X \end{pmatrix} \right) d\lambda.$$

Varying the action with respect to $^{(4)}g$ in the direction $^{(4)}h$ which vanishes on $\{a\} \times M$ and $\{b\} \times M$ induces a variation of (g, π) in the direction (h, ω) , and (h, ω) also vanishes on $\{a\} \times M$ and $\{b\} \times M$.

Thus, taking the extremum of the action for an arbitrary variation (h, ω) vanishing on the end manifolds $\{a\} \times M$ and $\{b\} \times M$ gives

$$\begin{aligned} 0 &= 16\pi dS_{\text{geom}}^{(4)}(g) \cdot ^{(4)}h = \int_I \int_M \left(\omega \cdot \frac{\partial g}{\partial \lambda} + \pi \cdot \frac{\partial h}{\partial \lambda} \right) d\lambda - \\ &- \int_I \int_M \left\langle D\Phi(g, \pi) \cdot (h, \omega), \begin{pmatrix} N \\ X \end{pmatrix} \right\rangle d\lambda = \int_I \int_M \left(\omega \cdot \frac{\partial g}{\partial \lambda} - \frac{\partial \pi}{\partial \lambda} \cdot h \right) d\lambda + \\ &+ \left(\int_M (\pi \cdot h)_{\lambda=b} - \int_M (\pi \cdot h)_{\lambda=a} \right) - \int_I \int_M \left\langle (h, \omega), D\Phi(g, \pi)^* \cdot \begin{pmatrix} N \\ X \end{pmatrix} \right\rangle d\lambda = \\ &= \int_I \int_M \left\langle (h, \omega), \left(\left(-\frac{\partial \pi}{\partial \lambda}, \frac{\partial g}{\partial \lambda} \right) - D\Phi(g, \pi)^* \cdot \begin{pmatrix} N \\ X \end{pmatrix} \right) \right\rangle d\lambda, \end{aligned}$$

where the term involving the total time derivative

$$\int_M \int \left(\frac{\partial}{\partial \lambda} (\pi \cdot h) \right) d\lambda = \int (\pi \cdot h)_{\lambda=0} - \int (\pi \cdot h)_{\lambda=1}$$

integrates to zero (in the λ -variable) by virtue of the vanishing of h on the end manifolds. Since the variation (h, ω) was arbitrary, we conclude that

$$\left(-\frac{\partial \pi}{\partial \lambda}, \frac{\partial g}{\partial \lambda} \right) = (D\Phi(g, \pi))^* \cdot \begin{pmatrix} N \\ X \end{pmatrix},$$

so that

$$J \begin{pmatrix} -\frac{\partial \pi}{\partial \lambda} \\ \frac{\partial g}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial \lambda} \\ \frac{\partial \pi}{\partial \lambda} \end{pmatrix} = J \circ (D\Phi(g, \pi))^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}. \quad \square$$

Actually, the form of the Einstein equations as they appear in theorem 2.3 can be extended to include field theories coupled to gravity. This extended form is at the basis of a covariant formulation of Hamiltonian systems [28, 36]. For example, the canonical formulation of the covariant scalar wave equation $\square \varphi = m^2 \varphi + F'(\varphi)$ on a space-time $V_4 = I \times M$, in terms of a general lapse and shift is as follows:

Consider the Hamiltonian

$$\mathcal{H}(\varphi, \pi_\varphi) = \left\{ \frac{1}{2} ((\pi'_\varphi)^2 + |\nabla \varphi|^2 + m^2 \varphi^2) + F(\varphi) \right\} \mu(g)$$

for the scalar field (the background metric is considered as implicitly given for this example). We can construct a 2-covariant symmetric tensor density \mathcal{F} obtained by varying $\mathcal{H}(\varphi, \pi_\varphi)$ with respect to g

$$\mathcal{F} = -2 D_\varphi \mathcal{H}(\varphi, \pi_\varphi)^* \cdot 1$$

and a 1-form density $\mathcal{J}(\varphi, \pi)$ from the relationship

$$\int \langle X, \mathcal{J}(\varphi, \pi_\varphi) \rangle = - \int \langle \pi, L_X \varphi \rangle,$$

so that $\mathcal{J}(\varphi, \pi_\varphi) = -\pi_\varphi \cdot d\varphi$. This condition expresses \mathcal{F} as the conserved quantity for the co-ordinate invariance group on M [25]. If we set $\Phi = (\mathcal{H}, \mathcal{J})$, then the Hamiltonian equations of motion for φ in a general slicing of the space-time with lapse N and shift X are

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} \varphi \\ \pi_\varphi \end{pmatrix} = J \circ D\Phi(\varphi, \pi_\varphi)^* \begin{pmatrix} N \\ X \end{pmatrix},$$

exactly as for general relativity. A computation shows that this system is equivalent to the covariant scalar wave equation given above.

If we couple the scalar field with gravity by regarding the scalar field as a source, the equation for the gravitational momentum $\partial\pi/\partial\lambda$ in theorems 2.1 and 2.3 is altered by the addition of the term $\frac{1}{2}N\mathcal{T}$, and the equation for $\partial g/\partial\lambda$ is unchanged. The constraint equations become

$$\mathcal{H}_{\text{geom}}(g, \pi) + \mathcal{H}_{\text{scalar}}(g, \varphi, \pi_\varphi) = 0 \quad \text{and} \quad \mathcal{J}_{\text{geom}}(g, \pi) + \mathcal{J}_{\text{scalar}}(\varphi, \pi_\varphi) = 0.$$

More generally, if one considers a total Hamiltonian $\mathcal{H}_T = \mathcal{H}_{\text{geom}} + \mathcal{H}_{\text{fields}}$ and a total universal flux tensor $\mathcal{J}_T = \mathcal{J}_{\text{geom}} + \mathcal{J}_{\text{fields}}$, and if the nongravitational fields are nonderivatively coupled to the gravitational fields, the general form of the equations

$$\frac{\partial}{\partial\lambda} \begin{pmatrix} g, \varphi_A \\ \pi, \pi^A \end{pmatrix} = J \circ (D\Phi_T(g, \varphi_A, \pi, \pi^A))^* \cdot \begin{pmatrix} N \\ X \end{pmatrix},$$

$$\Phi_T(g, \varphi_A, \pi, \pi^A) = 0$$

remains valid (see [28, 36, 37]). Here, φ_A represents all nongravitational fields, π^A their conjugate momenta, and $\Phi_T = (\mathcal{H}_T, \mathcal{J}_T)$. These results provide a unified covariant Hamiltonian formulation of general relativity coupled to other Lagrangian field theories and in fact allow the empty-space case to be extended formally to the nonderivative coupling case. KUCHAR [36], in his series of papers, gives the other side of the coin by spelling out in detail the canonical formalism for covariant field theories initiated by DIRAC (see [29] and the references therein).

Finally, we mention that the formalism of this section can be extended to the case where M is noncompact. This case has many technical problems but there is one basic difference: the fall-off rate for asymptotically flat metrics is not fast enough to allow integration by parts. This has led REGGE and TEITELBOIM [38] to conclude that the proper Hamiltonian actually generating the evolution equations contains an additional surface integral term corresponding to the mass. Thus, in the asymptotically flat case, the mass can be interpreted as the « true » generator of the evolution equations after the constraints $\Phi = 0$ are imposed. These ideas are discussed further in [23].

3. - The constraint manifold.

Let $\mathcal{C}_\mathcal{H} = \{(g, \pi) \in T^*\mathcal{M} | \mathcal{H}(g, \pi) = 0\}$ denote the set of solutions of the Hamiltonian constraint and let $\mathcal{C}_\mathcal{J} = \{(g, \pi) \in T^*\mathcal{M} | \mathcal{J}(g, \pi) = 2\pi_i{}^i, = 0\}$ denote the set of solutions of the divergence constraint. Thus $\mathcal{C} = \mathcal{C}_\mathcal{H} \cap \mathcal{C}_\mathcal{J} \subset T^*M$ is the constraint set for the Einstein system.

Two important facts about $\mathcal{C}_\mathcal{H} \cap \mathcal{C}_g$ are that the constraints are maintained by the evolution equations for any choice of lapse function and shift vector field, and that, generically, $\mathcal{C}_\mathcal{H} \cap \mathcal{C}_g$ is a smooth submanifold of $T^*\mathcal{M}$.

From the space-time point of view, maintenance of the constraints is equivalent to the contracted Bianchi identities, differential identities generated by the covariance of the field equations, as will be discussed below. Of course, this maintenance is necessary for the consistency of the evolution and constraint equations. (Otherwise, a projection, or Lagrange multiplier, would be present in the evolution equations.)

The manifold nature of $\mathcal{C}_\mathcal{H} \cap \mathcal{C}_g$, while intrinsically of interest, is the key to understanding the linearization stability of the field equations, as we shall see.

We begin by noting that the Hamiltonian and momentum functions are covariant with respect to the infinite-dimensional gauge group $\mathcal{D}(M)$ of diffeomorphisms of M . That is, for any $\eta \in \mathcal{D}(M)$ and $(g, \pi) \in T^*\mathcal{M}$,

$$\mathcal{H}(\eta^*g, \eta^*\pi) = \eta^*\mathcal{H}(g, \pi),$$

$$\mathcal{J}(\eta^*g, \eta^*\pi) = \eta^*\mathcal{J}(g, \pi),$$

and hence

$$\Phi(\eta^*g, \eta^*\pi) = \eta^*\Phi(g, \pi).$$

Here η^* denotes the usual pull-back of tensors.

If η_λ is a curve in $\mathcal{D}(M)$ with $\eta_0 = \text{identity}$, and we define the vector field X by $X = (d\eta_\lambda/d\lambda)|_{\lambda=0}$, then differentiation of the above relations in λ at $\lambda = 0$ gives the infinitesimal version of covariance:

$$D\mathcal{H}(g, \pi) \cdot (L_X g, L_X \pi) = L_X(\mathcal{H}(g, \pi)),$$

$$D\mathcal{J}(g, \pi) \cdot (L_X g, L_X \pi) = L_X(\mathcal{J}(g, \pi)),$$

and hence

$$D\Phi(g, \pi) \cdot (L_X g, L_X \pi) = L_X(\Phi(g, \pi)).$$

The next theorem computes the rate of change of \mathcal{H} and \mathcal{J} along a solution of the evolution equations for a general lapse and shift. The infinitesimal covariance accounts for the Lie derivatives in the resulting formulae.

3.1. Theorem. — *For an arbitrary lapse $N(\lambda)$ and shift $X(\lambda)$, let $(g(\lambda), \pi(\lambda))$ be a solution of the Einstein evolution equations*

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} g \\ \pi \end{pmatrix} = J \circ (D\Phi(g, \pi))^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}.$$

Then $(\mathcal{H}(\lambda), \mathcal{J}(\lambda)) = (\mathcal{H}(g(\lambda), \pi(\lambda)), \mathcal{J}(g(\lambda), \pi(\lambda)))$ satisfies the following system of equations:

$$\frac{\partial \mathcal{H}}{\partial \lambda} + L_X \mathcal{H} + \frac{1}{N} \operatorname{div} (N^2 \mathcal{J}) = 0$$

and

$$\frac{\partial \mathcal{J}}{\partial \lambda} + L_X \mathcal{J} + (dN) \mathcal{H} = 0.$$

If, for some λ_0 in the domain of existence of the solution, $(g(\lambda_0), \pi(\lambda_0)) = (g_0, \pi_0) \in \mathcal{C}_\mathcal{H} \cap \mathcal{C}_\delta$ (i.e. $\Phi(g_0, \pi_0) = 0$), then $(g(\lambda), \pi(\lambda)) \in \mathcal{C}_\mathcal{H} \cap \mathcal{C}_\delta$ for all λ for which the flow exists.

Remark. Thus, if a solution of the evolution equations intersects $\mathcal{C}_\mathcal{H} \cap \mathcal{C}_\delta$, it must lie wholly within $\mathcal{C}_\mathcal{H} \cap \mathcal{C}_\delta$.

Proof. Using the infinitesimal covariance of \mathcal{H} , we have

$$\begin{aligned} \frac{\partial \mathcal{H}(g, \pi)}{\partial \lambda} &= D\mathcal{H}(g, \pi) \cdot \left(\frac{\partial g}{\partial \lambda}, \frac{\partial \pi}{\partial \lambda} \right) = D\mathcal{H}(g, \pi) \cdot J \circ (D\Phi(g, \pi))^* \cdot \begin{pmatrix} N \\ X \end{pmatrix} = \\ &= D\mathcal{H}(g, \pi) \cdot J \circ [(D\mathcal{H}(g, \pi))^* \cdot N + D\mathcal{J}(g, \pi)^* \cdot X] = \\ &= D\mathcal{H}(g, \pi) \cdot J \circ D\mathcal{H}(g, \pi)^* \cdot N + D\mathcal{H}(g, \pi) \cdot (-L_X g, -L_X \pi) = \\ &= D\mathcal{H}(g, \pi) \cdot J \circ D\mathcal{H}(g, \pi)^* \cdot N - L_X \mathcal{H}(g, \pi) = \\ &= D\mathcal{H}(g, \pi) \cdot \begin{pmatrix} (D_\pi \mathcal{H}(g, \pi))^* \cdot N \\ - (D_g \mathcal{H}(g, \pi))^* \cdot N \end{pmatrix} - L_X \mathcal{H}(g, \pi) = \\ &= D_g \mathcal{H}(g, \pi) \cdot ((D_\pi \mathcal{H}(g, \pi))^* \cdot N) - D_\pi \mathcal{H}(g, \pi) \cdot ((D_g \mathcal{H}(g, \pi))^* \cdot N) - L_X \mathcal{H}(g, \pi). \end{aligned}$$

Since $\mathcal{H}(g, \pi)$ is algebraic in π ,

$$D_\pi \mathcal{H}(g, \pi) = \partial_\pi \mathcal{H}(g, \pi) \quad \text{and} \quad (D_\pi \mathcal{H}(g, \pi))^* \cdot N = N \partial_\pi \mathcal{H}(g, \pi).$$

From appendix I,

$$D_g \mathcal{H}(g, \pi) \cdot h = \partial_g \mathcal{H}(g, \Gamma, \pi) \cdot h - (\Delta \operatorname{tr} h + \delta \delta h) \mu(g)$$

and

$$(D_g \mathcal{H}(g, \pi))^* \cdot N = N \partial_g \mathcal{H}(g, \Gamma, \pi) - (g \Delta N + \operatorname{Hess} N) \mu(g).$$

The first two terms in the expression for $\partial \mathcal{H} / \partial \lambda$ are evaluated as follows:

$$\begin{aligned} D_g \mathcal{H} \cdot ((D_\pi \mathcal{H})^* \cdot N) - D_\pi \mathcal{H} \cdot ((D_g \mathcal{H})^* \cdot N) &= D_g \mathcal{H} \cdot (N \partial_\pi \mathcal{H}) - \partial_\pi \mathcal{H} \cdot ((D_g \mathcal{H})^* \cdot N) = \\ &= \partial_g \mathcal{H}(g, \Gamma, \pi) \cdot (N \partial_\pi \mathcal{H}) - (\Delta \operatorname{tr} (N \partial_\pi \mathcal{H}) + \delta \delta (N \partial_\pi \mathcal{H})) \mu(g) - \end{aligned}$$

$$\begin{aligned}
& -\partial_\pi \mathcal{H} \cdot (N \partial_g \mathcal{H}(g, \Gamma, \pi)) + \partial_\pi \mathcal{H} \cdot (g \Delta N + \text{Hess } N) \mu(g) = \\
& = -(\Delta \text{tr}(N \partial_\pi \mathcal{H}) + \delta \delta(N \partial_\pi \mathcal{H})) \mu(g) + \partial_\pi \mathcal{H} \cdot (g \Delta N + \text{Hess } N) \mu(g) = \\
& = -\delta \delta(N(\partial_\pi \mathcal{H} - g \text{tr} \partial_\pi \mathcal{H})) \mu(g) + \text{Hess } N \cdot (\partial_\pi \mathcal{H} - g \text{tr} \partial_\pi \mathcal{H}) \mu(g) = \\
& = -\delta \delta(2N\pi) + \text{Hess } N \cdot (2\pi) = -\frac{1}{N} \delta(N^2 \delta(2\pi)).
\end{aligned}$$

In this calculation we have used these subcalculations. Firstly,

$$\partial_\pi \mathcal{H} = 2((\pi') - \frac{1}{2}(\text{tr} \pi')g) = -2k,$$

so that

$$\partial_\pi \mathcal{H} - g \text{tr}(\partial_\pi \mathcal{H}) = -2(k - g \text{tr} k) = 2(\pi')^\flat.$$

Secondly,

$$\begin{aligned}
\delta \delta(N\pi) &= (N\pi'')_{ij} = (N_{i\ell} \pi''_{\ell j} + N\pi''_{ij})_{ij} = N_{i\ell i} \pi''_{\ell j} + N_{i\ell j} \pi''_{i\ell} + (N\pi''_{ij})_{ij} = \\
&= \text{Hess } N \cdot \pi + \frac{1}{N} N(N\pi''_{ij})_{ij} = \text{Hess } N \cdot \pi + \frac{1}{N} \delta(N^2 \delta(\pi)).
\end{aligned}$$

Thus we arrive at

$$\frac{\partial \mathcal{H}}{\partial \lambda} = -\frac{1}{N} \delta(N^2 \delta(2\pi)) - L_x \mathcal{H} = -\frac{1}{N} \text{div}(N^2 \mathcal{J}) - L_x \mathcal{H}.$$

The evolution equation for $\mathcal{J}(g, \pi)$ follows from infinitesimal covariance of $\Phi(g, \pi)$ as follows:

Let $Y \in \mathcal{X}$ be any vector field on M (independent of λ). Then

$$\begin{aligned}
\frac{d}{d\lambda} \int \langle Y, \mathcal{J}(g, \pi) \rangle &= \int \left\langle Y, \frac{d\mathcal{J}(g, \pi)}{d\lambda} \right\rangle = \int \left\langle Y, D\mathcal{J}(g, \pi) \cdot \left(\frac{\partial g}{\partial \lambda}, \frac{\partial \pi}{\partial \lambda} \right) \right\rangle = \\
&= \int \left\langle Y, D\mathcal{J}(g, \pi) \cdot J \circ (D\Phi(g, \pi))^* \cdot \begin{pmatrix} N \\ X \end{pmatrix} \right\rangle = \\
&= \int \left\langle \langle D\Phi(g, \pi) \cdot J^* \circ (D\mathcal{J}(g, \pi))^* \cdot Y, \begin{pmatrix} N \\ X \end{pmatrix} \rangle \right\rangle = \\
&= - \int \left\langle D\Phi(g, \pi) \cdot J \circ (D\mathcal{J}(g, \pi))^* \cdot Y, \begin{pmatrix} N \\ X \end{pmatrix} \right\rangle = \quad (J^* = -J) \\
&= \int \langle D\Phi(g, \pi) \cdot (L_Y g, L_Y \pi), (N, X) \rangle = \\
&= \int \langle L_Y \Phi(g, \pi), (N, X) \rangle = \quad (\text{chain rule})
\end{aligned}$$

$$\begin{aligned}
&= \int N L_Y \mathcal{H}(g, \pi) + \langle X, L_Y \mathcal{J}(g, \pi) \rangle = \\
&= - \int (L_Y N) \mathcal{H} - \int \langle L_Y X, \mathcal{J} \rangle = \quad \text{(integration by parts)} \\
&= - \int Y(dN) \mathcal{H} + \int \langle L_X Y, \mathcal{J} \rangle = - \int Y(dN) \mathcal{H} - \int \langle Y, L_X \mathcal{J} \rangle.
\end{aligned}$$

Since Y is arbitrary,

$$\frac{d\mathcal{J}}{d\lambda} + L_X \mathcal{J} + (dN) \mathcal{H} = 0.$$

Considering the evolution equations for $(\mathcal{H}(\lambda), \mathcal{J}(\lambda))$ as a linear first-order system of partial differential equations, we see that, if $(\mathcal{H}(\lambda_0), \mathcal{J}(\lambda_0))$ starts out zero, then by uniqueness for such a system it must remain zero for the entire flow. \square

Remarks.

1) An interesting feature of these equations is that the λ -derivatives of N and X do not appear.

2) In appendix II we show that these equations are equivalent to Dirac's canonical commutation relations for general relativity.

The following infinitesimal versions of 3.1 will be important in understanding and interpreting a splitting due to MONCRIEF [4] and in the construction leading to the space of gravitational degrees of freedom.

3.2. Proposition. - Let $(g, \pi) \in \mathcal{C}_\mathcal{H} \cap \mathcal{C}_\delta$. Then

$$\text{range } J \circ (D\Phi(g, \pi))^* \subset \ker D\Phi(g, \pi).$$

Proof. Let $(h, \omega) \in \text{range } J \circ (D\Phi(g, \pi))^*$ and $(N, X) \in C^\infty \times \mathcal{X}$ be such that $(h, \omega) = J \circ (D\Phi(g, \pi))^* \cdot (N, X)$. Let $(N(\lambda), X(\lambda))$ be an arbitrary lapse and shift such that $(N(0), X(0)) = (N, X)$. Let $(g(\lambda), \pi(\lambda))$ be the solution to the evolution equations with lapse and shift $(N(\lambda), X(\lambda))$ and with initial data $(g, \pi) \in \mathcal{C}_\mathcal{H} \cap \mathcal{C}_\delta$. Since $\Phi(g, \pi) = 0$, by theorem 3.1, $\Phi(g(\lambda), \pi(\lambda)) = 0$ for all λ for which the solution exists. Hence

$$\begin{aligned}
0 &= \frac{d}{d\lambda} \Phi(g(\lambda), \pi(\lambda))|_{\lambda=0} = \left(D\Phi(g(\lambda), \pi(\lambda)) \cdot \left(\frac{\partial g(\lambda)}{\partial \lambda}, \frac{\partial \pi(\lambda)}{\partial \lambda} \right) \right)|_{\lambda=0} = \\
&= \left(D\Phi(g(\lambda), \pi(\lambda)) \cdot J \circ (D\Phi(g(\lambda), \pi(\lambda)))^* \cdot \begin{pmatrix} N(\lambda) \\ X(\lambda) \end{pmatrix} \right)|_{\lambda=0} = \\
&= D\Phi(g, \pi) \cdot J \circ D\Phi(g, \pi) \cdot \begin{pmatrix} N \\ X \end{pmatrix} = D\Phi(g, \pi) \cdot (h, \omega).
\end{aligned}$$

Hence $(h, \omega) \in \ker D\Phi(g, \pi)$. \square

We now examine the manifold structure of the constraint set $\mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_g$. We introduce the following conditions on $(g, \pi) \in T^*\mathcal{M}$:

$C_{\mathcal{H}}$: if $\pi = 0$, then g is not flat;

C_g : if, for $X \in \mathcal{X}(M)$, $L_X g = 0$ and $L_X \pi = 0$, then $X = 0$;

C_{tr} : $\text{tr } \pi'$ is a constant on M .

We consider the constraints one at a time; first, the Hamiltonian constraint.

3.3. Proposition. — *Let $(g, \pi) \in \mathcal{C}_{\mathcal{H}}$ satisfy condition $C_{\mathcal{H}}$. Then $\mathcal{C}_{\mathcal{H}}$ is a C^∞ submanifold of $T^*\mathcal{M}$ in a neighborhood of (g, π) with tangent space*

$$T_{(g, \pi)} \mathcal{C}_{\mathcal{H}} = \ker D\mathcal{H}(g, \pi).$$

Proof. Consider the map $\mathcal{H}: T^*\mathcal{M} \rightarrow C_d^\infty$; $(g, \pi) \rightarrow \mathcal{H}(g, \pi)$. We shall show that, under condition $C_{\mathcal{H}}$,

$$D\mathcal{H}(g, \pi): T_{(g, \pi)}(T^*\mathcal{M}) = S_2 \times S_d^2 \rightarrow T_{\mathcal{H}(g, \pi)} C_d^\infty = C_d^\infty$$

is surjective with splitting kernel so that \mathcal{H} is a submersion at (g, π) . If we use Sobolev spaces and the implicit function theorem, and then pass to the C^∞ case via a regularity argument, it follows that $\mathcal{C}_{\mathcal{H}} = \mathcal{H}^{-1}(0)$ is a smooth submanifold in a neighborhood of (g, π) .

From the elliptic theory (sect. 1), it follows that $D\mathcal{H}(g, \pi)$ is surjective provided that its L_2 -adjoint

$$D\mathcal{H}(g, \pi)^*: C^\infty \rightarrow S_d^2 \times S_2,$$

$$(D\mathcal{H}(g, \pi))^* N =$$

$$= \left((-NS_g(\pi, \pi) + N \text{Ein}(g) - \text{Hess } N + g \Delta N)^s \mu(g), 2N((\pi')^b - \frac{1}{2}(\text{tr } \pi')g) \right)$$

is injective and has injective symbol.

The symbol of $D\mathcal{H}(g, \pi)^*$ is

$$\sigma_\xi(D\mathcal{H}(g, \pi)^*) = \left((-\xi \otimes \xi + g\|\xi\|^2)^s \mu(g), 0 \right): \\ \mathbb{R} \rightarrow \left((T_x^* M \otimes T_x^* M)_{\text{sym}} \mu(g)_x (T_x M \otimes T_x M)_{\text{sym}}^{\otimes 2} \right)$$

for $\xi \in T_x^* M$. For $s \in \mathbb{R}$, $\xi \neq 0$, $(-\xi \otimes \xi + g\|\xi\|^2)s = 0$ implies, by taking the trace, $2\|\xi\|^2 s = 0$, so $s = 0$, so that the symbol is injective. Thus from the Fredholm alternative theorem (sect. 1) we have the L_2 -orthogonal splitting

$$C_d^\infty = \text{range } D\mathcal{H}(g, \pi) \oplus \left(\ker (D\mathcal{H}(g, \pi)^*) \right) \otimes \mu(g).$$

Thus if $\ker D\mathcal{H}(g, \pi)^* = \{0\}$, $D\mathcal{H}(g, \pi)$ is surjective. Any $N \in \ker D\mathcal{H}(g, \pi)^*$ satisfies

$$a) -NS_g(\pi, \pi) + (N \operatorname{Ein}(g) - \operatorname{Hess} N - g\Delta N)^* \mu(g) = 0,$$

$$b) 2N((\pi') - \frac{1}{2}(\operatorname{tr} \pi')g) = 0.$$

Taking the trace of $b)$ gives $N(\operatorname{tr} \pi') = 0$ and so from $b)$ again $N\pi = 0$. Thus from $a)$

$$c) N \operatorname{Ein}(g) - \operatorname{Hess} N - g\Delta N = 0.$$

From the trace of $c)$

$$2\Delta N + \frac{1}{2}R(g)N = 0.$$

However, from $\mathcal{H}(g, \pi) = 0$ and $N\pi = 0$, it follows that $NR(g) = 0$. Hence

$$\Delta N = 0$$

and so $N = \text{constant}$.

If $\pi \neq 0$, then $N\pi = 0$ implies $N = 0$, since N is constant. Thus $D\mathcal{H}(g, \pi)^*$ is injective and hence $D\mathcal{H}(g, \pi)$ is surjective.

If $\pi = 0$, then, from $a)$, $N \operatorname{Ein}(g) = 0$ implies $N \operatorname{Ric}(g) = 0$. Thus, if $N \neq 0$, then $\operatorname{Ric}(g) = 0$ and hence g is flat, since $\dim M = 3$. But a flat g and $\pi = 0$ is ruled out by condition C_x . Hence $N = 0$, and again $D\mathcal{H}(g, \pi)$ is surjective. \square

Remark. If $g = g_r$ is flat and $\pi = 0$,

$$\ker D\mathcal{H}(g, \pi)^* = \{\text{constant functions on } M\} = \mathbb{R}.$$

Next we investigate the divergence constraint.

3.4. Proposition. — *If $(g, \pi) \in \mathcal{C}_\delta = \{(g, \pi) | \mathcal{J}(g, \pi) = 0\} \subset T^*\mathcal{M}$ satisfies condition C_δ , then \mathcal{C}_δ is a smooth submanifold of $T^*\mathcal{M}$ in a neighborhood of (g, π) with tangent spaces*

$$T_{(g, \pi)}\mathcal{C}_\delta = \ker D\mathcal{J}(g, \pi)^*.$$

Proof. The derivative of $\mathcal{J}(g, \pi)$ and its adjoint were computed in sect. 2:

$$D\mathcal{J}(g, \pi)^* \cdot X = (L_X \pi, -L_X g).$$

The symbol is injective (from its injectivity in the second component alone). The kernel of $D\mathcal{J}(g, \pi)^*$ is $\{X | L_X \pi = 0, L_X g = 0\}$, so that injectivity of $D\mathcal{J}(g, \pi)^*$ is exactly condition C_δ . \square

Remark. The regular points (g, π) satisfying condition C_δ are just those (g, π) having discrete isotropy group under the action of the diffeomorphism group $\mathcal{Q}(M)$ acting on $T^*\mathcal{M} \approx \mathcal{M} \times S_d^2$ by pull-back (see [6], p. 258).

To show that the intersection $\mathcal{C} = \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_\delta$ is a submanifold of $T^*\mathcal{M}$, we need additional restrictions because there may be points at which the intersection is not transversal. At this point it is necessary to assume that (g, π) satisfies the condition $\text{tr } \pi' = \text{const.}$

3.5. Theorem. — *Let $(g, \pi) \in \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_\delta$ satisfy the conditions $C_{\mathcal{H}}$, C_δ and C_{tr} . Then the constraint set $\mathcal{C} = \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_\delta$ is a C^∞ submanifold of $T^*\mathcal{M}$ in a neighborhood of (g, π) with tangent space*

$$T_{(g, \pi)}\mathcal{C} = \ker D\Phi(g, \pi),$$

where $\Phi = (\mathcal{H}, \mathcal{J})$.

Proof. We want to show $D\Phi(g, \pi) = (D\mathcal{H}(g, \pi), D\mathcal{J}(g, \pi))$ is surjective for $(g, \pi) \in \mathcal{C}$ and satisfying the given conditions. The adjoint

$$D\Phi(g, \pi)^*: C^\infty \times \mathcal{X} \rightarrow S_d^2 \times S_2,$$

$$(N, X) \mapsto D\Phi(g, \pi)^* \cdot (N, X) = D\mathcal{H}(g, \pi)^* \cdot N + D\mathcal{J}(g, \pi)^* \cdot X$$

is given in proposition 2.2. For $\xi \in T_x^*M$, $\xi \neq 0$, the symbol of this map, $\sigma_\xi(D\Phi(g, \pi)^*)$, $\xi \in T_x^*M$, may be shown to be injective, as above (see, however, remarks on various types of ellipticity in [19]). Thus it remains to show that $D\Phi(g, \pi)^*$ is injective. Let $(N, X) \in \ker D\Phi(g, \pi)^*$. Then from the formula for $D\Phi(g, \pi)^*$ we have

$$a) -NS_\delta(\pi, \pi) + (N \text{Ein}(g) - (\text{Hess } N + g \Delta N))^* \mu_g + L_X \pi = 0$$

and

$$b) 2N((\pi') - \frac{1}{2}(\text{tr } \pi')g) - L_X g = 0.$$

Taking the trace of a) and b), we get

$$c) -\frac{N}{2}\mathcal{H}(g, \pi) + 2(\Delta N)\mu_g - \text{tr } L_X \pi = 0$$

and

$$d) -N \text{tr } \pi' + 2\delta_g X = 0.$$

Now

$$\text{tr } L_X \pi = X \cdot d \text{tr } \pi - \pi \cdot L_X g + (\text{div } X)(\text{tr } \pi),$$

since

$$L_X \pi = (L_X \pi') \otimes \mu_g + \pi' \otimes (\operatorname{div} X) \mu_g$$

(in co-ordinates, $(L_X \pi)^{ij} = X^i \pi'^{j}_{|k} - \pi'^k X^j_{|k} - \pi'^k X^i_{|k} + X^k_{|k} \pi'^{ij}$).

Since $\mathcal{H}(g, \pi) = 0$, c) reduces to

$$e) \ 2(\Delta N) \mu(g) - (X \cdot d \operatorname{tr} \pi - \pi \cdot L_X g + (\operatorname{div} X)(\operatorname{tr} \pi)) = 0.$$

Using b) and d) to eliminate $L_X g$ and $\operatorname{div} X$ respectively in e) gives

$$\begin{aligned} f) \ 2(\Delta N) - X \cdot d \operatorname{tr} \pi' + \pi' \cdot L_X g - (\operatorname{div} X)(\operatorname{tr} \pi') &= \\ &= 2\Delta N + 2N\pi' \cdot \left((\pi')^\flat - \frac{1}{2} (\operatorname{tr} \pi') g \right) + \frac{N}{2} (\operatorname{tr} \pi') (\operatorname{tr} \pi') - X \cdot d \operatorname{tr} \pi' = \\ &= 2\Delta N + 2N\pi' \cdot \pi' + \frac{N}{2} (\operatorname{tr} \pi')^2 - X \cdot d \operatorname{tr} \pi' = \\ &= 2\Delta N + 2N \left(\pi' \cdot \pi' - \frac{1}{4} (\operatorname{tr} \pi')^2 \right) - X \cdot d \operatorname{tr} \pi' = 0. \end{aligned}$$

Since

$$P(\pi', \pi') \stackrel{\text{def}}{=} \pi' \cdot \pi' - \frac{1}{4} (\operatorname{tr} \pi')^2 = \left(\pi' - \frac{1}{2} (\operatorname{tr} \pi') g \right) \cdot \left(\pi' - \frac{1}{2} (\operatorname{tr} \pi') g \right),$$

we note that the coefficient of N is positive definite. Thus, if $\operatorname{tr} \pi' = \text{const}$, f) becomes

$$2\Delta N + 2P(\pi', \pi')N = 0,$$

which implies $N = 0$ unless $\pi' = 0$, in which case $N = \text{const}$. In this case, from a), $\operatorname{Ein}(g) = 0$ and so $\operatorname{Ric}(g) = 0$, i.e. g is flat since $\dim M = 3$. However, the case $(g_f, 0)$, where g_f is flat, is excluded by condition $C_{\mathcal{H}}$. Thus, $\pi' \neq 0$ and $N = 0$. Then, by a) and b), $L_X g = 0$ and $L_X \pi = 0$, which by condition C_δ implies $X = 0$. Thus $(N, X) = (0, 0)$ and so $D\Phi(g, \pi)^*$ is injective, under conditions $C_{\mathcal{H}}$, C_δ and C_{tr} . \square

Remark. That one must impose the condition $\operatorname{tr} \pi' = \text{const}$ to show that the intersection $\mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_\delta$ is a manifold is an annoying feature of the analysis. One suspects that under conditions $C_{\mathcal{H}}$ and C_δ alone the system a) and b) is injective. The difficulty is that in the system, say f) and b) for (N, X) , the $X \cdot d \operatorname{tr} \pi'$ coupling terms seems to be sufficient to prevent one from showing uniqueness for this system. The results of Moncrief, discussed in sect. 4, will shed light on this point.

In the following paper it is shown that many space-times with compact spacelike hypersurfaces that satisfy the weak energy condition $\operatorname{Ric}(\cdot \cdot g) > 0$ have $\operatorname{tr} \pi' = \text{constant}$ hypersurfaces. Thus these preferred hypersurfaces will be the place to check conditions $C_{\mathcal{H}}$ and C_δ .

In the following sections the above theorem will be the main tool in proving that generically the Einstein empty-space field equations are linearization stable.

4. - The linearized Einstein system.

Preparatory to studying linearization stability of the Einstein equations, we study the linearized Einstein system

$$D \text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h = 0,$$

where ${}^{(4)}g_0$ is a solution of the (empty space) field equations, $\text{Ein}({}^{(4)}g_0) = 0$. We shall consider these equations from both the space-time and dynamical points of view and then discuss some relationships between them.

Given a space-time $(V_4, {}^{(4)}g)$ let \square_L denote the Lichnerowicz Laplacian acting on symmetric two-tensors ${}^{(4)}h \in S_2(V_4)$; in co-ordinates $\{x^2\}$,

$$(\square_L {}^{(4)}h)_{\alpha\beta} = -g^{\mu\nu} h_{\alpha\beta;\mu;\nu} + R_\alpha{}^\mu h_{\mu\beta} + R_\beta{}^\mu h_{\mu\alpha} - 2R_\alpha{}^\nu h_{\nu\beta}.$$

Let $(\square {}^{(4)}h)_{\alpha\beta} = -g^{\mu\nu} h_{\alpha\beta;\mu;\nu}$ be the usual d'Alembertian on two-tensors (note our sign conventions).

We collect three useful variational formulae in the following

4.1. Proposition.

- a) $D \text{Ric}({}^{(4)}g) \cdot {}^{(4)}h = \frac{1}{2} \{ \square_L {}^{(4)}h - \alpha_{(4)} \delta_{(4)} {}^{(4)}h - \text{Hess tr}({}^{(4)}h) \} =$
 $= \frac{1}{2} \{ \square_L {}^{(4)}h - \alpha_{(4)} \delta_{(4)} ({}^{(4)}h - \frac{1}{2} \text{tr}({}^{(4)}h) {}^{(4)}g) \},$
- b) $DR({}^{(4)}g) \cdot {}^{(4)}h = \square \text{tr}({}^{(4)}h) + \delta_{(4)} \delta_{(4)} {}^{(4)}h - {}^{(4)}h \cdot \text{Ric}({}^{(4)}g),$
- c) $D \text{Ein}({}^{(4)}g) \cdot {}^{(4)}h = \frac{1}{2} \{ \square_L {}^{(4)}\bar{h} - \alpha_{(4)} \delta_{(4)} {}^{(4)}\bar{h} - (\delta_{(4)} \delta_{(4)} {}^{(4)}\bar{h})({}^{(4)}g) +$
 $+ \frac{1}{2} \{ ({}^{(4)}\bar{h} \cdot \text{Ric}({}^{(4)}g))({}^{(4)}g) - R({}^{(4)}g)({}^{(4)}\bar{h}) \}.$

Here, as usual,

$$\begin{aligned} \alpha_{(4)}({}^{(4)}X) &= L_{(4)x}({}^{(4)}g) = X_{\alpha;\beta} + X_{\beta;\alpha}, \\ \delta_{(4)} {}^{(4)}h &= -h_{\alpha}{}^{\beta}{}_{;\beta}, \\ \square \text{tr}({}^{(4)}h) &= -g^{\alpha\beta} h^{\gamma}{}_{\gamma;\alpha;\beta}, \end{aligned}$$

and where ${}^{(4)}\bar{h} = {}^{(4)}h - \frac{1}{2}(\text{tr}({}^{(4)}h))({}^{(4)}g)$, i.e. $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h^{\gamma}{}_{\gamma}g_{\alpha\beta}$.

These formulae in the Riemannian case have already been alluded to in sect. 2 during consideration of $D\Phi$. It is instructive to derive b) and c) from the more primitive formula a). For the method to prove a) and a direct proof of b), see appendix I.

Proof of a) \Rightarrow b). We suppress the superscript $^{(4)}$:

$$\begin{aligned} DR(g) \cdot h &= D(g^{-1} \cdot \text{Ric}(g)) \cdot h = \\ &= D(g^{-1}) \cdot h \cdot \text{Ric}(g) + g^{-1} \cdot D \text{Ric}(g) \cdot h = \\ &= -h \cdot \text{Ric}(g) + \frac{1}{2} \{g^{-1} \square_L h - \alpha_s \delta_s h - \text{Hess}(\text{tr } h)\} = \\ &= -h \cdot \text{Ric}(g) + \frac{1}{2} \{\square \text{tr } h + 2\delta_s \delta_s h + \square(\text{tr } h)\} = \\ &= \square(\text{tr } h) + \delta_s \delta_s h - h \cdot \text{Ric}(g). \end{aligned}$$

Here we have used $\text{tr} \square_L h = \square \text{tr } h$.

Proof of a) \Rightarrow c). Since $\text{Ein}(g) = \text{Ric}(g) - \frac{1}{2}R(g)g$,

$$\begin{aligned} D \text{Ein}(g) \cdot h &= D \text{Ric}(g) \cdot h - \frac{1}{2} \{DR(g) \cdot h\} g - \frac{1}{2} R(g) D(g) \cdot h = \\ &= \frac{1}{2} \{\square_L h - \alpha_s \delta_s (h - \frac{1}{2}(\text{tr } h)g)\} - \frac{1}{2} \{\square \text{tr } h + \delta_s \delta_s h - h \cdot \text{Ric}(g)\} g - \frac{1}{2} R(g) h = \\ &= \frac{1}{2} \{\square_L h - \frac{1}{2}g \square \text{tr } h - \alpha_s \delta_s h - (\delta_s \delta_s h + \frac{1}{2}g \square \text{tr } h)\} + \\ &\quad + \frac{1}{2} \{[h \cdot \text{Ric}(g) - \frac{1}{2}(\text{tr } h)R(g)]g - [R(g)h - \frac{1}{2}(\text{tr } h)R(g)g]\}. \end{aligned}$$

These are the corresponding five terms in the expression c). \square

If $\text{Ein}^{(4)}(g) = 0$, then $\text{Ric}^{(4)}(g) = 0$ and $R^{(4)}(g) = 0$, so c) becomes

$$D \text{Ein}^{(4)}(g) \cdot {}^{(4)}h = \frac{1}{2} \{\square_L {}^{(4)}h - \alpha_{(4)} \delta_{(4)} {}^{(4)}h - (\delta_{(4)} \delta_{(4)} {}^{(4)}h) {}^{(4)}g\}.$$

We shall now derive a useful second-order variational formula due to TAUB [39].

4.2. *Theorem* - Let $\text{Ein}^{(4)}(g_0) = 0$ and ${}^{(4)}h \in S_2(V_4)$. Then

$$\delta_{(4)} {}^{(4)}(D \text{Ein}^{(4)}(g_0) \cdot {}^{(4)}h) = 0.$$

If, moreover, ${}^{(4)}h$ satisfies the linearized equations, $D \text{Ein}^{(4)}(g_0) \cdot {}^{(4)}h = 0$, then

$$\delta_{(4)} {}^{(4)}(D^2 \text{Ein}^{(4)}(g_0) \cdot ({}^{(4)}h, {}^{(4)}h)) = 0.$$

Proof. Let ${}^{(4)}g(\varrho)$ be a curve of Lorentz metrics, such that ${}^{(4)}g(0) = {}^{(4)}g_0$ and $(\partial/\partial \varrho) {}^{(4)}g(\varrho)|_{\varrho=0} = {}^{(4)}h$. Now differentiate the contracted Bianchi identities

$$(1) \quad \delta_{(4)} {}^{(4)}\text{Ein}^{(4)}(g(\varrho)) = 0$$

with respect to ϱ and set $\varrho = 0$:

$$(2) \quad (D(\delta_{(a)_x}) \cdot {}^{(a)}h) \text{Ein}({}^{(a)}g_0) + \delta_{(a)_x}(D \text{Ein}({}^{(a)}g_0) \cdot {}^{(a)}h) = 0.$$

Since $\text{Ein}({}^{(a)}g_0) = 0$, this gives the first result.

For the second, differentiate (1) twice with respect to ϱ and set $\varrho = 0$:

$$\begin{aligned} D^2(\delta_{(a)_x}) \cdot {}^{(a)}h \cdot {}^{(a)}h \cdot \text{Ein}({}^{(a)}g) + 2 D(\delta_{(a)_x}) \cdot {}^{(a)}h \cdot (D \text{Ein}({}^{(a)}g_0) \cdot {}^{(a)}h) + \\ + \delta_{(a)_x}(D^2 \text{Ein}({}^{(a)}g_0)({}^{(a)}h, {}^{(a)}h) + \delta_{(a)_x}(D \text{Ein}({}^{(a)}g_0) \cdot {}^{(a)}g'(0)) = 0. \end{aligned}$$

Using the first equation in 4.2 and the conditions $\text{Ein}({}^{(a)}g) = 0$ and $D \text{Ein}({}^{(a)}g) \cdot {}^{(a)}h = 0$, this gives the second result. \square

The second-order result just obtained will be useful below in our discussions of linearization stability.

We next discuss the «canonical» decomposition

$${}^{(a)}h = {}^{(a)}\hat{h} + L_{(a)x}{}^{(a)}g,$$

where $\delta_{(a)_x}({}^{(a)}\hat{h} - \frac{1}{2} \text{tr}({}^{(a)}\hat{h})({}^{(a)}g)) = 0$, in the Lorentzian case. Such an ${}^{(a)}X$ must satisfy

$$\begin{aligned} \delta_{(a)_x}({}^{(a)}h - \frac{1}{2} \text{tr}({}^{(a)}h)({}^{(a)}g)) &= \delta_{(a)_x}(L_{(a)x}{}^{(a)}g - \frac{1}{2} \text{tr}(L_{(a)x}{}^{(a)}g)({}^{(a)}g)) = \\ &= \delta_{(a)_x}L_{(a)x}{}^{(a)}g - d\delta_{(a)_x}{}^{(a)}X = \square({}^{(a)}X - 2 \text{Ric}({}^{(a)}g) \cdot {}^{(a)}X). \end{aligned}$$

A solution to this hyperbolic equation is determined by a set of Cauchy data. Note that solutions are not unique, and the decomposition is not unique, although it does exist. In fact ${}^{(a)}\hat{h}$ is nonunique up to terms $L_{(a)x}{}^{(a)}g$, where ${}^{(a)}Y$ is a solution to

$$\square({}^{(a)}Y - 2 \text{Ric}({}^{(a)}g) \cdot {}^{(a)}Y) = 0$$

(or ${}^{(a)}X$ is nonunique up to such ${}^{(a)}Y$). The situation is to be contrasted with the Riemannian case where X is unique up to a Killing field.

To nail ${}^{(a)}X$ down we shall single out special Cauchy data, namely zero. We summarize this discussion.

4.3. Proposition. — *Let $(V_4, {}^{(a)}g)$ be a space-time with a Cauchy hypersurface Σ_0 . For ${}^{(a)}h \in S_2(V_4)$ there is a unique decomposition*

$${}^{(a)}h = {}^{(a)}\hat{h} + L_{(a)x}{}^{(a)}g$$

such that

$$i) \quad \delta_{(a)_x}({}^{(a)}\hat{h} - \frac{1}{2} \text{tr}({}^{(a)}\hat{h})({}^{(a)}g)) = 0$$

and

- ii) if $x_0 \in \Sigma_0$, ${}^{(4)}X(x_0) = 0$ and $D({}^{(4)}X)(x_0) = 0$, i.e. ${}^{(4)}X(x_0) = 0$ and $\nabla_{({}^{(4)}Z)} {}^{(4)}X(x_0) = 0$, where ${}^{(4)}Z$ is the unit normal to Σ_0 .

For any diffeomorphism $F: V_4 \rightarrow V_4$, we have the covariance property

$$\text{Ein}(F^*({}^{(4)}g_0)) = F^* \text{Ein}({}^{(4)}g_0).$$

From this it follows (as earlier) that infinitesimal covariance holds:

$$D \text{Ein}({}^{(4)}g_0) \cdot L_{({}^{(4)}X)} {}^{(4)}g = L_{({}^{(4)}X)} \text{Ein}({}^{(4)}g_0).$$

Thus, if $\text{Ein}({}^{(4)}g_0) = 0$, any gauge perturbation $L_{({}^{(4)}X)} {}^{(4)}g$ satisfies the linearized equations.

If we put this together with the above decomposition, the linearized field equations about ${}^{(4)}g_0$ for ${}^{(4)}h = {}^{(4)}\bar{h} + L_{({}^{(4)}X)} {}^{(4)}g$ become

$$\square_L {}^{(4)}\bar{h} = 0,$$

where

$${}^{(4)}\bar{h} = {}^{(4)}h - \frac{1}{2} \text{tr}({}^{(4)}h) {}^{(4)}g_0 \quad \text{and} \quad \delta_{({}^{(4)}Z)} ({}^{(4)}\bar{h}) = 0.$$

We now recall the existence and uniqueness theorems for the full nonlinear and then the linearized Einstein equations; see [40] or [41].

4.4. Theorem. — *Fix a compact M and let $(g_0, \pi_0) \in \mathcal{C}_X \cap \mathcal{C}_\delta$. Then there is a space-time $(V_4, {}^{(4)}g_0)$ and a spacelike embedding $i_0: M \rightarrow V_4$ such that*

- i) $\text{Ein}({}^{(4)}g_0) = 0$,
- ii) the metric and conjugate momentum induced on $\Sigma_0 = i_0(M)$ is (g_0, π_0) ,
- iii) $(V_4, {}^{(4)}g_0)$ is maximal (i.e. cannot be properly and isometrically embedded in another space-time with properties i) and ii)).

This space-time $(V_4, {}^{(4)}g_0)$ is unique in the sense that, if we have another $(V'_4, {}^{(4)}g'_0)$ with i)-iii) holding, there is a unique diffeomorphism $F: V_4 \rightarrow V'_4$ such that

$$\text{i) } F^*({}^{(4)}g'_0) = {}^{(4)}g_0 \text{ (i.e. } F \text{ is an isometry)}$$

and

$$\text{ii) } F \circ i_0 = i'_0.$$

Remarks. The local existence and uniqueness is found in, e.g., [30, 32, 33], as was mentioned earlier. The maximality part is due to CHOQUET-BRUHAT and GEROCH [40]. The uniqueness of F uses the fact that an isometry is determined by its action on a frame at a point.

4.5. *Theorem.* — Let $(V, {}^{(4)}g_0)$ be a vacuum space-time, i.e. $\text{Ein}({}^{(4)}g_0) = 0$ with a compact Cauchy surface $\Sigma_0 = i_0(M)$ and with induced metric and canonical momentum $(g_0, \pi_0) \in \mathcal{C}_\mathcal{X} \cap \mathcal{C}_\delta$. Let $(h_0, \omega_0) \in S_2 \times S_2^*$ satisfy the linearized constraint equations, i.e.

$$D\Phi(g_0, \pi_0) \cdot (h_0, \omega_0) = 0.$$

Then there exists an ${}^{(4)}h_0 \in S_2(V_4)$ such that

$$D \text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h_0 = 0$$

and such that the linearized Cauchy data induced by ${}^{(4)}h_0$ on Σ_0 are (h_0, ω_0) .

If ${}^{(4)}h'_0$ is another such solution, there is a unique vector field ${}^{(4)}X$ on V_4 such that

$${}^{(4)}h'_0 = {}^{(4)}h_0 + L_{{}^{(4)}X} {}^{(4)}g_0$$

and ${}^{(4)}X$ and its derivative vanish on Σ_0 .

Remark. The linearized Cauchy data are defined in the same manner as the (g, π) are defined. In fact, if ${}^{(4)}g(\varrho)$ is a curve of Lorentz metrics tangent to ${}^{(4)}h$ at ${}^{(4)}g_0$, then

$$(h_0, \omega_0) = \left(\left. \frac{\partial g(\varrho)}{\partial \varrho} \right|_{\varrho=0}, \left. \frac{\partial \pi(\varrho)}{\partial \varrho} \right|_{\varrho=0} \right),$$

where $(g(\varrho), \pi(\varrho))$ are the induced Cauchy data from ${}^{(4)}g(\varrho)$.

Theorem 4.5 is proved as follows: one begins by working in the linearized harmonic gauge $\delta_{(4)} {}^{(4)}\bar{h}_0 = 0$ for which the linearized equations are the hyperbolic equation

$$\square_L {}^{(4)}\bar{h}_0 = 0.$$

For uniqueness, one notes that, if ${}^{(4)}h_0$ and ${}^{(4)}h'_0$ are two solutions, ${}^{(4)}h_0$ and ${}^{(4)}h'_0$ both satisfy $\square_L {}^{(4)}h = 0$ and have the same Cauchy data. The result then follows by using proposition 4.3.

Now we consider the linearized equations from the dynamical point of view. Recall that, if $dx/d\lambda = A(x)$ is a nonlinear differential equation, the linearized equations about a solution $x(\lambda)$ are

$$\frac{dy}{d\lambda} = DA(x(\lambda)) \cdot y.$$

These may be obtained by supposing there is a curve $x(\lambda, \varrho)$ of exact solutions with $x(\lambda, 0) = x(\lambda)$ and $(\partial/\partial\varrho)x(\lambda, \varrho)|_{\varrho=0} = y(\lambda)$ and differentiating $\partial x/\partial\lambda = A(x)$ in ϱ and evaluating at $\varrho = 0$.

The linearized Einstein dynamical system is obtained by the same procedure. Specifically, let $\text{Ein}({}^{(4)}g_0) = 0$, ${}^{(4)}h \in S_1(\Gamma_1)$, and let ${}^{(4)}g(\varrho)$ be a curve of exact solutions, $\text{Ein}({}^{(4)}g(\varrho)) = 0$, with ${}^{(4)}g(0) = {}^{(4)}g_0$ and ${}^{(4)}h = \partial({}^{(4)}g(0))/\partial\varrho$, so that ${}^{(4)}h$ satisfies the linearized equations $D \text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h = 0$. Let $i_\lambda \in E^\infty(M, V_4, {}^{(4)}g(\varrho))$ be a fixed slicing of V_4 ; for ϱ sufficiently small, i_λ will remain spacelike in the varying metrics ${}^{(4)}g(\varrho)$. By theorem 2.3, for each ϱ , we have

$$\begin{pmatrix} \frac{\partial g(\lambda, \varrho)}{\partial \lambda} \\ \frac{\partial \pi(\lambda, \varrho)}{\partial \lambda} \end{pmatrix} = J \circ D\Phi(g(\lambda, \varrho), \pi(\lambda, \varrho))^* \cdot \begin{pmatrix} N(\lambda, \varrho) \\ X(\lambda, \varrho) \end{pmatrix}$$

and

$$\Phi(g(\lambda, \varrho), \pi(\lambda, \varrho)) = 0.$$

The lapse and shift depend on ϱ even though the slicing is fixed, because the decomposition of ${}^{(4)}X$ depends on the normal to Σ and hence on ${}^{(4)}g(\varrho)$. Also, $g(\lambda, 0) = g(\lambda)$, $\pi(\lambda, 0) = \pi(\lambda)$, $N(\lambda, 0) = N(\lambda)$ and $X(\lambda, 0) = X(\lambda)$, where $(g(\lambda), \pi(\lambda))$ are induced on Σ_λ by ${}^{(4)}g_0$ and $(N(\lambda), X(\lambda))$ are the lapse-shift decomposition of ${}^{(4)}X_{i_\lambda}$ in the metric ${}^{(4)}g_0$.

Differentiating these equations with respect to ϱ , interchanging ϱ - and λ -derivatives, letting $\varrho = 0$, and letting

$$h(\lambda) = \frac{\partial g(\lambda, 0)}{\partial \varrho}, \quad \omega(\lambda) = \frac{\partial \pi(\lambda, 0)}{\partial \varrho}$$

and

$$U(\lambda) = \frac{\partial N(\lambda, 0)}{\partial \varrho}, \quad V(\lambda) = \frac{\partial X(\lambda, 0)}{\partial \varrho}$$

gives the following:

$$\begin{aligned} & \text{(linearized evolutions equations)} \quad \begin{pmatrix} \frac{\partial h(\lambda)}{\partial \lambda} \\ \frac{\partial \omega(\lambda)}{\partial \lambda} \end{pmatrix} = \\ & = J \circ D \left[D\Phi(g(\lambda), \pi(\lambda))^* \cdot \begin{pmatrix} N(\lambda) \\ X(\lambda) \end{pmatrix} \right] \cdot \begin{pmatrix} h(\lambda) \\ \omega(\lambda) \end{pmatrix} + J \circ D\Phi(g(\lambda), \pi(\lambda))^* \cdot \begin{pmatrix} U(\lambda) \\ V(\lambda) \end{pmatrix}, \end{aligned}$$

$$\text{(linearized constraint equations)} \quad D\Phi(g(\lambda), \pi(\lambda)) \cdot (h(\lambda), \omega(\lambda)) = 0.$$

Remarks. Since J and the (natural) adjoint operator are linear and independent of the metric, they contribute no terms. (Here is a computational advantage of natural adjoints.)

As in the full nonlinear case, one regards $U(\lambda)$, $V(\lambda)$, the infinitesimal variations of the lapse and shift, as arbitrarily specifiable (possibly zero). Then $h(\lambda)$, $\omega(\lambda)$ are determined by the linearized evolution equations and their value at $\lambda = 0$.

The maintenance of the linearized constraint equations in λ is guaranteed by the linearized contracted Bianchi identities (see theorem 4.2 and what follows).

We now discuss the relationship between ${}^{(4)}h$ and (h, ω, U, V) . Let a slicing be fixed, giving co-ordinates $\{x^i\}$ and let ${}^{(4)}g(\varrho)$, $g(\lambda, \varrho)$, $\pi(\lambda, \varrho)$, etc. be as above. The following formulae are derived directly:

$$g_{ij}(\lambda, \varrho) = {}^{(4)}g_{ij}(\lambda, \varrho), \quad {}^{(4)}g_{0i}(\lambda, \varrho) = -g_{ij}(\lambda, \varrho)X^j(\lambda, \varrho),$$

$${}^{(4)}g^{00}(\lambda, \varrho) = -1/(N(\lambda, \varrho))^2,$$

$$h_{ij}(\lambda) = {}^{(4)}h_{ij}, \quad {}^{(4)}h_{0i}(\lambda) = -h_{ij}X^j - V_i,$$

$${}^{(4)}h^{00}(\lambda) = 2U(\lambda)/(N(\lambda))^2,$$

$${}^{(4)}Z_\lambda = (-N, 0), \quad {}^{(4)}Z^\lambda = \frac{1}{N}(1, X^i),$$

$${}^{(4)}h_{\perp\perp} = {}^{(4)}Z_\lambda {}^{(4)}Z^\lambda {}^{(4)}h^{\lambda\beta} = 2U/N \quad (\text{the perpendicular-perpendicular projection}),$$

$$({}^{(4)}h_{\perp})_i = -{}^{(4)}Z^\lambda {}^{(4)}h_{\lambda i} = V_i/N \quad (\text{the perpendicular-parallel projection}),$$

$$({}^{(4)}h_{\parallel})_{ij} = {}^{(4)}h_{ij} \quad (\text{the parallel-parallel projection}).$$

For gauge perturbations ${}^{(4)}h = L_{(4)r} {}^{(4)}g$, if we let ${}^{(4)}Y = Y_\perp {}^{(4)}Z - T \circ Y_{\text{shift}}$, write out the Lie derivative in co-ordinates and project, the formulae $h_{\perp\perp}$ and h_{\perp} become

$$N(L_{(4)r} {}^{(4)}g)_{\perp\perp} = 2 \left(\frac{dY_\perp}{d\lambda} + L_X Y_\perp - L_{Y_{\text{shift}}} N \right) = 2U,$$

$$N(L_{(4)r} {}^{(4)}g)_{\perp i} = \frac{\partial Y_{\text{shift}}}{\partial \lambda} + L_X Y_{\text{shift}} + N \text{grad } Y_\perp - Y_\perp \text{grad } N = V_i,$$

where $Y_{\text{shift}} = -Y_i$. Note the special case ${}^{(4)}Y = {}^{(4)}X$; then we get

$$\frac{dY_\perp}{d\lambda} = U, \quad \frac{dY_{\text{shift}}}{d\lambda} = V.$$

(KUCHAR [36, II] has general formulae for projecting the covariant derivative and hence the Lie derivative; see his equations (2.5) and (2.6).)

The next proposition details some of the relationships between the space-time and dynamical equations for the linearized system.

4.6. Theorem. — Let $\text{Ein}({}^{(4)}g) = 0$ and $\Sigma_0 = i_0(M)$ be a spacelike hypersurface with induced $(g_0, \pi_0) \in \mathcal{C}_\mathcal{P} \cap \mathcal{C}_\delta$. Let ${}^{(4)}h_0 \in S_2(V_4)$ and let (h_0, ω_0) be the deformations of (g_0, π_0) induced on Σ_0 by ${}^{(4)}h$. If ${}^{(4)}Y_{\Sigma_0}$ is a vector field on Σ_0 , ${}^{(4)}Y_{\Sigma_0} = Y_\perp {}^{(4)}Z_{\Sigma_0} + Ti_0 \circ Y_1$, then

$$\{-2(\text{D Ein}({}^{(4)}g_0)) \cdot {}^{(4)}h \cdot ({}^{(4)}Y_{\Sigma_0}, {}^{(4)}Z_{\Sigma_0})\} \mu(g_0) = \langle (Y_\perp, -Y_1), \text{D}\Phi(g_0, \pi_0) \cdot (h_0, \omega_0) \rangle.$$

If, moreover, ${}^{(4)}h_0$ satisfies the linearized equations

$$\text{D Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h_0) = 0,$$

then (h_0, ω_0) satisfies the linearized constraint equations

$$\text{D}\Phi(g_0, \pi_0) \cdot (h_0, \omega_0) = 0.$$

Proof. Let ${}^{(4)}g(\varrho)$ be a curve of Lorentz metrics tangent to ${}^{(4)}h_0$ at $\varrho = 0$. For each ϱ , we have the following identity on Σ_0 :

$$\begin{aligned} (3) \quad -2 \text{Ein}({}^{(4)}g(\varrho))({}^{(4)}Y_{\Sigma_0}, {}^{(4)}Z_{\Sigma_0}(\varrho)) \mu(g_0) &= \\ &= Y_\perp(\varrho) \mathcal{H}(g(\varrho), \pi(\varrho)) - Y_1(\varrho) \cdot \mathcal{J}(g(\varrho), \pi(\varrho)) \end{aligned}$$

(from the formulae on p. 331 above).

Differentiating this identity (3) with respect to ϱ and evaluating at $\varrho = 0$ gives the differential identity

$$\begin{aligned} (4) \quad -2(\text{D Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h)) \cdot ({}^{(4)}Y_{\Sigma_0}, {}^{(4)}Z_{\Sigma_0}) \mu(g_0) - \\ -2 \text{Ein}({}^{(4)}g_0) \cdot \left({}^{(4)}Y_{\Sigma_0}, \frac{\partial ({}^{(4)}Z_{\Sigma_0}(0))}{\partial \varrho} \right) \mu(g_0) - 2 \text{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}Y_{\Sigma_0}, {}^{(4)}Z_{\Sigma_0}) \frac{1}{2} (\text{tr } h_0) \mu(g_0) = \\ = \frac{\partial Y_\perp(0)}{\partial \varrho} \mathcal{H}(g_0, \pi_0) + Y_\perp \text{D} \mathcal{H}(g_0, \pi_0) \cdot (h_0, \omega_0) - \\ - \frac{\partial Y_1(0)}{\partial \varrho} \cdot \mathcal{J}(g_0, \pi_0) - Y_1 \text{D} \mathcal{J}(g_0, \pi_0) \cdot (h_0, \omega_0). \end{aligned}$$

Since $\text{Ein}({}^{(4)}g_0) = 0$, $\mathcal{H}(g_0, \pi_0) = 0$ and $\mathcal{J}(g_0, \pi_0) = 0$, (4) reduces to the first desired conclusion:

$$\begin{aligned} (5) \quad -2(\text{D Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h)) \cdot ({}^{(4)}Y_{\Sigma_0}, {}^{(4)}Z_{\Sigma_0}) \mu(g_0) &= \\ &= \langle (Y_\perp, -Y_1), \text{D}\Phi(g_0, \pi_0) \cdot (h_0, \omega_0) \rangle. \end{aligned}$$

If ${}^{(4)}h$ is a solution to the linearized equations, $D \text{Ein} ({}^{(4)}g_0) \cdot {}^{(4)}h_0 = 0$, then (5) becomes

$$\langle (Y_\perp, -Y_\parallel), D\Phi(g_0, \pi_0) \cdot (h_0, \omega_0) \rangle = 0.$$

Since this is true for all ${}^{(4)}Y_\Sigma$,

$$D\Phi(g_0, \pi_0) \cdot (h_0, \omega_0) = 0. \quad \square$$

Remark. Note that (3) in the proof is an identity in Lorentz geometry, i.e. it is true for all space-times whether or not they satisfy the empty-space field equations. This equation is one key for relating the space-time point of view to the dynamical point of view.

Now let us consider infinitesimal deformations of vacuum space-times ${}^{(4)}g_0$ of the special form ${}^{(4)}h = L_{({}^{(4)}\gamma)} {}^{(4)}g_0$. Deformations of this type automatically satisfy the linearized equations

$$D \text{Ein} ({}^{(4)}g_0) \cdot L_{({}^{(4)}\gamma)} {}^{(4)}g_0 = L_{({}^{(4)}\gamma)} (\text{Ein} ({}^{(4)}g_0)) = 0,$$

as we have seen.

Let Σ_0 be a compact Cauchy hypersurface in V_4 , and let (g_0, π_0) be the metric and momentum induced on Σ_0 , (h_0, ω_0) the deformations of (g_0, π_0) on Σ_0 induced by ${}^{(4)}h = L_{({}^{(4)}\gamma)} {}^{(4)}g_0$ and (Y_\perp, Y_\parallel) the tangential and normal components of ${}^{(4)}Y$ on Σ_0 . These quantities are all related together quite simply. As has been shown by MONCRIEF [7], the relationship is simple, but the original proof involved a long computation. Here we give a geometric proof based on the adjoint form of the evolution equations. The idea is to replace a deformation ${}^{(4)}g(\varrho)$ of ${}^{(4)}g_0$ by a family of embeddings and then use the evolution equations.

4.7. Theorem. — On V_4 , let $\text{Ein} ({}^{(4)}g_0) = 0$ and let $\Sigma_0 = i_0(M)$ be a compact spacelike hypersurface. Let ${}^{(4)}Y$ be a vector field on V_4 with flow F_λ and let $i_\lambda = F_\lambda \circ i_0$ (for $|\lambda|$ small, this is a one-parameter family of spacelike embeddings). Let ${}^{(4)}h = L_{({}^{(4)}\gamma)} {}^{(4)}g_0$, $(g(\lambda), \pi(\lambda))$ be the metric and momentum on $\Sigma_\lambda = i_\lambda(M)$ and let $(h(\lambda), \omega(\lambda))$ be the infinitesimal deformation of (g, π) induced on Σ_λ by ${}^{(4)}h$. Then

$$\begin{pmatrix} h(\lambda) \\ \omega(\lambda) \end{pmatrix} = J \circ D\Phi(g(\lambda), \pi(\lambda))^* \cdot \begin{pmatrix} Y_\perp(\lambda) \\ Y_{\text{shift}}(\lambda) \end{pmatrix}.$$

Proof. Consider the curve ${}^{(4)}g(\varrho) = F_\varrho^* {}^{(4)}g_0$ through ${}^{(4)}g_0$ with tangent ${}^{(4)}h = L_{({}^{(4)}\gamma)} {}^{(4)}g_0$. Let $g(\lambda, \varrho)$ and $\pi(\lambda, \varrho)$ denote the metric and canonical momentum induced on Σ_λ by ${}^{(4)}g(\varrho)$, so that $h(\lambda) = (\partial g / \partial \varrho)(\lambda, 0)$ and $\omega(\lambda) = (\partial \pi / \partial \varrho)(\lambda, 0)$.

Now $(g(\lambda, \varrho), \pi(\lambda, \varrho))$ induced on Σ_λ by ${}^{(4)}g(\varrho)$ is equal to the metric and canonical momentum induced on $\Sigma_{\varrho, \lambda} = F_\varrho \circ F_\lambda \circ i_0(M)$ by the space-time ${}^{(4)}g_0$. For example, for the induced metrics,

$$g(\lambda, \varrho) = i_\lambda^* {}^{(4)}g(\varrho) = (F_\lambda \circ i_0)^* F_\varrho^* ({}^{(4)}g_0) = (F_\varrho \circ F_\lambda \circ i_0)^* {}^{(4)}g_0,$$

which is the metric induced on $\Sigma_{\varrho, \lambda}$ in the space-time ${}^{(4)}g_0$.

Now fix λ and consider the curve of embeddings $i(\varrho, \lambda) = F_\varrho \circ F_\lambda \circ i_0$, and note that its generator is

$$\frac{\partial i}{\partial \varrho}(\varrho, \lambda) = \frac{\partial F_\varrho}{\partial \varrho} \circ F_\lambda \circ i_0 = {}^{(4)}Y \circ F_\varrho \circ F_\lambda \circ i_0 = {}^{(4)}Y \circ i(\varrho, \lambda).$$

Since this is now a 1-parameter family of spacelike embeddings in the parameter ϱ (for ϱ sufficiently small), by the evolution equations in adjoint form we get

$$\begin{pmatrix} \frac{\partial g(\lambda, \varrho)}{\partial \varrho} \\ \frac{\partial \pi(\lambda, \varrho)}{\partial \varrho} \end{pmatrix} = J \circ D\Phi(g(\lambda, \varrho), \pi(\lambda, \varrho))^* \cdot \begin{pmatrix} Y_\perp(\lambda, \varrho) \\ -Y_\parallel(\lambda, \varrho) \end{pmatrix},$$

where $Y_\perp(\lambda, \varrho)$ and $Y_\parallel(\lambda, \varrho)$ are the normal and tangential components of ${}^{(4)}Y \circ i(\varrho, \lambda)$, respectively.

Evaluating at $\varrho = 0$ gives

$$\begin{pmatrix} h(\lambda) \\ \omega(\lambda) \end{pmatrix} = J \circ D\Phi(g(\lambda), \pi(\lambda))^* \cdot \begin{pmatrix} Y_\perp(\lambda) \\ -Y_\parallel(\lambda) \end{pmatrix}. \quad \square$$

Remark. This result amounts to an integration of the linearized evolution equations in the special case that ${}^{(4)}h = L_{({}^{(4)}\gamma)} {}^{(4)}g_0$ and ${}^{(4)}Y \circ i_\lambda$ is chosen as the generator of the 1-parameter family of spacelike embeddings $i_\lambda = F_\lambda \circ i_0$, where F_λ is the flow of ${}^{(4)}Y$. The lapse and shift for this family is $(N, X) = (Y_\perp - Y_\parallel)$, so that

$$\begin{pmatrix} h(\lambda) \\ \omega(\lambda) \end{pmatrix} = J \circ D\Phi(g(\lambda), \pi(\lambda))^* \begin{pmatrix} Y_\perp(\lambda) \\ -Y_\parallel(\lambda) \end{pmatrix} = J \circ D\Phi(g(\lambda), \pi(\lambda))^* \begin{pmatrix} N(\lambda) \\ X(\lambda) \end{pmatrix} = \begin{pmatrix} \frac{\partial g(\lambda)}{\partial \lambda} \\ \frac{\partial \pi(\lambda)}{\partial \lambda} \end{pmatrix}.$$

4.8. Proposition. — $(h(\lambda), \omega(\lambda))$ in theorem 4.7 satisfy the linearized Einstein evolution equations.

Proof. Recall the equations $dY_{\perp}/d\lambda = U$ and $dY_{\text{shift}}/d\lambda = V$ derived above. Hence differentiating

$$\begin{pmatrix} h(\lambda) \\ \omega(\lambda) \end{pmatrix} = J \circ D\Phi(g(\lambda), \pi(\lambda))^* \begin{pmatrix} Y_{\perp}(\lambda) \\ Y_{\text{shift}}(\lambda) \end{pmatrix}$$

with respect to λ gives

$$\begin{aligned} \begin{pmatrix} \frac{\partial h(\lambda)}{\partial \lambda} \\ \frac{\partial \omega(\lambda)}{\partial \lambda} \end{pmatrix} &= J \circ D \left[D\Phi(g(\lambda), \pi(\lambda))^* \begin{pmatrix} Y_{\perp}(\lambda) \\ Y_{\text{shift}}(\lambda) \end{pmatrix} \right] \cdot \begin{pmatrix} \frac{\partial g(\lambda)}{\partial \lambda} \\ \frac{\partial \pi(\lambda)}{\partial \lambda} \end{pmatrix} + \\ &+ J \circ D\Phi(g(\lambda), \pi(\lambda))^* \begin{pmatrix} \frac{\partial Y_{\perp}(\lambda)}{\partial \lambda} \\ \frac{\partial Y_{\text{shift}}(\lambda)}{\partial \lambda} \end{pmatrix} = \\ &= J \circ D \left[D\Phi(g(\lambda), \pi(\lambda))^* \begin{pmatrix} Y_{\perp}(\lambda) \\ Y_{\text{shift}}(\lambda) \end{pmatrix} \right] \cdot \begin{pmatrix} h(\lambda) \\ \omega(\lambda) \end{pmatrix} + J \circ D\Phi(g(\lambda), \pi(\lambda))^* \begin{pmatrix} U(\lambda) \\ V(\lambda) \end{pmatrix}, \end{aligned}$$

showing that $(h(\lambda), \omega(\lambda))$ indeed does satisfy the linearized equations. \square

Using the results of this section, we now go on to study linearization stability of the empty-space field equations.

5. - Linearization stability of the vacuum Einstein equations.

Linearization stability concerns the validity of first-order perturbation theory. The idea is the following. Suppose we have a differentiable function F and points x_0 and y_0 such that $F(x_0) = y_0$. A standard procedure for finding other solutions to the equation $F(x) = y_0$ near x_0 is to solve the linearized equation $DF(x_0) \cdot h = 0$ and assert that $x = x_0 + \rho h$ is, for small ρ , an approximate solution to $F(x) = y_0$. Technically this assertion may be stated as follows: there exists a curve of exact solutions $x(\rho)$ for small ρ such that $F(x(\rho)) = y_0$, $x(0) = x_0$ and $x'(0) = h$. If this assertion is valid, we say that F is *linearization stable* at x_0 . It is easy to give examples in which the assertion is false. For instance, in 2 dimensions $F(x_1, x_2) = x_1^2 + x_2^2 = 0$ has no solutions except $(0, 0)$, although the linearized equation $DF(0, 0) \cdot h = 0 \cdot h = 0$ has many solutions. Thus it is a nonvacuous question whether an equation is linearization stable at some given solution, or not. Intuitively, linearization stability means that first-order perturbation theory is valid near x_0 and there are no spurious directions of perturbation.

The question of linearization stability is important for relativity. In the literature it was often assumed that solutions to the linearized equations do in fact approximate solutions to the exact equations. However, in [42], BRILL and DESER asserted that for the flat three-torus, with zero extrinsic curvature, there are solutions to the linearized constraint equations which are not approximated by a curve of exact solutions. They gave a second-order perturbation argument to show that, subject to the condition $\text{tr } \pi = 0$, there are no other nearby solutions to the constraint equations, except essentially trivial modifications, even though there are many nontrivial solutions to the linearized equations. Since then, FISCHER and MARSDEN [6] have given a rigorous proof. It is analogous to and is proved by techniques used in the following *isolation theorem* in geometry.

5.1. Theorem. (FISCHER and MARSDEN [19].) — *If M is compact and g_F is a flat metric on M , then there is a neighborhood U_{ϵ_F} of g_F in the space of metrics \mathcal{M} such that any metric g in the neighborhood U_{ϵ_F} with $R(g) > 0$ is flat.*

The proof amounts to a version of the Morse lemma adapted to infinite-dimensional space with special attention needed because of the co-ordinate invariance of the scalar curvature map.

The results on linearization stability are due, independently, to CHOQUET-BRUHAT and DESER [43] for flat space and to FISCHER and MARSDEN [5] for the general case of empty space-times with a compact hypersurface. The methods used are rather different. Later, O'MURCHADHA and YORK [44] generalized the Choquet-Bruhat and Deser method to the case of space-times with a compact hypersurface. We comment on this method later. The flat-space result is:

5.2. Theorem. — *Near Minkowski space, the Einstein empty-space equations $\text{Ein}({}^{(4)}g) = 0$ are linearization stable.*

In this theorem, one must use suitable function spaces with asymptotic conditions and asymptotically flat space-times. We will consider the non-compact case in the following paper where we will prove that time-symmetric asymptotically flat empty-space solutions are linearization stable. Actually, the original Choquet-Bruhat and Deser paper dealt with asymptotic conditions of a special form. We shall deal with asymptotic conditions in the general case. Here we shall consider only those space-times which are developed from Cauchy data on a compact hypersurface.

We begin by defining linearization stability for the empty-space Einstein equations.

Let $\text{Ein}({}^{(4)}g_0) = 0$. An infinitesimal deformation of ${}^{(4)}g_0$ is a solution ${}^{(4)}h \in S_2(V_4)$ of the linearized equations

$$D \text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h = 0.$$

The Einstein equations are *linearization stable* at ${}^{(4)}g_0$ (or ${}^{(4)}g_0$ is *linearization stable*) if, for every infinitesimal deformation ${}^{(4)}h$ of ${}^{(4)}g_0$, there exists a C^1 curve ${}^{(4)}g(\varrho)$ of exact solutions to the empty-space field equations (on the same V_4)

$$\text{Ein}({}^{(4)}g(\varrho)) = 0$$

such that ${}^{(4)}g(0) = {}^{(4)}g_0$ and $\partial {}^{(4)}g(0)/\partial \varrho = {}^{(4)}h_0$.

This definition has to be qualified slightly to be strictly accurate. Namely, for any compact set $D \subset V_4$, we only require ${}^{(4)}g(\varrho)$ to be defined for $|\varrho| < \varepsilon$, where ε may depend on D . The reason for this is that ${}^{(4)}g(\varrho)$ will be developed from a curve of Cauchy data $(g(\varrho), \pi(\varrho))$ and so ${}^{(4)}g(\varrho)$ will be uniformly close to ${}^{(4)}g_0$ on compact sets for $|\varrho| < \varepsilon$, but not on all of V_4 in general.

Since we are fixing our hypersurface topology M , all Cauchy developments lead to topologically equivalent space-times $V_4 = R \times M$, so fixing V_4 is not a serious restriction. (Topological perturbations are, of course, another story.)

If one uses the linearized dynamical Einstein system, linearization stability of the Einstein equations is equivalent to linearization stability of the constraint equations, as we shall see below. In fact, linearization stability of a well-posed hyperbolic system of partial differential equations is equivalent to linearization stability of any nonlinear constraints present.

In terms of the linearized map $D\Phi(g, \pi)$, we can give necessary and sufficient conditions for the constraint equations

$$\Phi(g, \pi) = 0$$

to be linearization stable at (g_0, π_0) ; that is, if $(h, \omega) \in S_2 \times S_2^2$ satisfies the linearized equations

$$D\Phi(g_0, \pi_0) \cdot (h, \omega) = 0,$$

then there exists a differentiable curve $(g(\varrho), \pi(\varrho)) \in T^*\mathcal{M}$ of exact solutions to the constraint equations

$$\Phi(g(\varrho), \pi(\varrho)) = 0$$

such that $(g(0), \pi(0)) = (g_0, \pi_0)$ and

$$\left(\frac{\partial g(0)}{\partial \varrho}, \frac{\partial \pi(0)}{\partial \varrho} \right) = (h, \omega).$$

The main result follows:

5.3. Theorem. — Let $\Phi = (\mathcal{H}, \mathcal{J}): T^*\mathcal{M} \rightarrow C_2^\infty \times A_2^1$ be defined as in sect. 2 so $\mathcal{C}_\mathcal{H} \cap \mathcal{C}_\mathcal{J} = \Phi^{-1}(0)$. Let $(g_0, \pi_0) \in \mathcal{C}_\mathcal{H} \cap \mathcal{C}_\mathcal{J}$. The following conditions are equivalent:

i) the constraint equations

$$\Phi(g, \pi) = 0$$

are linearization stable at (g_0, π_0) ,

ii) $D\Phi(g_0, \pi_0): S_2 \times S_d^2 \rightarrow C_d^\infty \times A_d^1$ is surjective,

iii) $D\Phi(g_0, \pi_0)^*: C^\infty \times \mathcal{X} \rightarrow S_d^2 \times S_2$ is injective.

Remark. In sect. 3 we listed sufficient conditions in order for ii) to be valid, namely the conditions $C_{\mathcal{X}}$, C_d and C_w .

Proof of 5.3. In sect. 3 we showed that $D\Phi(g_0, \pi_0)^*$ is elliptic. Thus, the equivalence of ii) and iii) is an immediate consequence of the Fredholm alternative.

ii) implies i). The kernel of $D\Phi(g_0, \pi_0)$ splits by the Fredholm alternative. Thus the implicit function theorem implies that, near (g_0, π_0) , $\Phi^{-1}(0)$ is a smooth manifold. (Here one must use the Sobolev spaces and pass to C^∞ by a regularity argument as in [19].) Since any tangent vector to a smooth manifold is tangent to a curve in the manifold, i) results.

i) implies iii). This is less elementary and will just be sketched. Assume i) and that $D\Phi(g_0, \pi_0)^* \cdot (N, X) = 0$, but $(N, X) \neq 0$. We will derive a contradiction by showing that there is a necessary second-order condition on first-order deformations (h, ω) that must be satisfied in order for the deformation to be tangent to a curve of exact solutions to the constraints. Thus let (h, ω) be a solution to the linearized equations, and let $(g(\varrho), \pi(\varrho))$ be a curve of exact solutions of

$$(6) \quad \Phi(g(\varrho), \pi(\varrho)) = 0$$

through (g_0, π_0) and tangent to (h, ω) . Differentiating (6) twice and evaluating at $\varrho = 0$ gives

$$(7) \quad D\Phi(g_0, \pi_0) \cdot (g'(0), \pi'(0)) + D^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega)) = 0,$$

where $g'(0) = \partial^2 g(0) / \partial \varrho^2$ and $\pi'(0) = \partial^2 \pi(0) / \partial \varrho^2$. If we contract (7) with (N, X) and integrate over M , the first term of (7) gives

$$\int \langle (N, X), D\Phi(g_0, \pi_0) \cdot (g'(0), \pi'(0)) \rangle = \int \langle D\Phi(g_0, \pi_0)^* \cdot (N, X), (g'(0), \pi'(0)) \rangle = 0,$$

since $(N, X) \in \ker D\Phi(g_0, \pi_0)^*$.

Thus the first term of (7) drops out, leaving the necessary condition

$$(8) \quad \int \langle (N, X), D^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega)) \rangle = 0,$$

which must hold for all $(h, \omega) \in \ker D\Phi(g_0, \pi_0)$. As we shall see below, (8) is « hypersurface invariant » (cf. theorems 4.2 and 4.6), so we can assume $N \neq 0$, say $N > 0$, in a neighborhood U of M . A (long) calculation as in [6, 19] shows that, if h is transverse traceless, ω is transverse and some additional algebraic conditions making $(h, \omega) \in \ker D\Phi(g_0, \pi_0)$ hold, their (8) becomes

$$(9) \quad \int N(\nabla h)^2 + \text{lower-order terms} = 0.$$

On the other hand, by [45], the space of h 's satisfying these conditions $\cap \{(h, \omega) \text{ with support in } U\}$ is infinite dimensional. But a relation like (9) cannot hold in this infinite-dimensional space by Rellich's theorem (see sect. 1) (see [46] for details). \square

Remarks. a) The procedure for finding a second-order condition when linearization stability fails is quite general. See [6, 19] for other applications.

b) The implication i) \Rightarrow ii) uses $\dim M > 3$. For the equation $R(g) = g$, i) \Rightarrow ii) (replacing Φ by R) is not true on two manifolds [19].

From the linearization stability of the constraint equations we can deduce linearization stability of the space-time and *vice versa*, as follows:

5.4. Theorem. — *Let $(V_*, {}^{(4)}g_0)$ be a vacuum space-time which is the maximal development of Cauchy data (g_0, π_0) on a compact hypersurface $\Sigma_0 = i_0(M)$. Then the Einstein equations*

$$\text{Ein}({}^{(4)}g) = 0 \quad \text{on } V_*$$

are linearization stable at ${}^{(4)}g_0$ if and only if the constraint equations

$$\Phi(g, \pi) = 0$$

are linearization stable at (g_0, π_0) .

In particular, if conditions $C_{\mathcal{H}}$, C_δ and C_π hold for (g_0, π_0) , then the Einstein equations are linearization stable.

Proof. Assume first that the constraint equations are linearization stable. Let ${}^{(4)}h_0$ be a solution to the linearized equations at ${}^{(4)}g_0$ and let (h_0, ω_0) be the induced deformation of (g, π) on Σ_0 . By proposition 4.6, (h_0, ω_0) satisfies the linearized constraint equations. By assumption, there is a curve $(g(\varrho), \pi(\varrho)) \in \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_\delta$ tangent to (h_0, ω_0) at (g_0, π_0) .

By the existence theory for the Cauchy problem, there is a curve ${}^{(4)}g(\varrho)$ of maximal solutions on $V_* \cong \mathbb{R} \times M$ of $\text{Ein}({}^{(4)}g(\varrho)) = 0$ and with Cauchy data $(g(\varrho), \pi(\varrho))$. (As earlier, for any compact set $D \subset V_*$ and $\varepsilon > 0$, there is a $\delta > 0$ such that ${}^{(4)}g(\varrho)$ is within ε of ${}^{(4)}g_0$ (using any convenient topology) on D .)

We now must show that we can transform the curve ${}^{(4)}g(0)$ by diffeomorphisms so that ${}^{(4)}h_0$ is its tangent at $\varrho = 0$. It is here that we use the uniqueness results for the linearized and full Einstein system (theorems 4.4 and 4.5).

Since ${}^{(4)}g(0)$ and ${}^{(4)}g_0$ have the same Cauchy data, there exists an $F \in \mathcal{D}(V_4)$ such that $F|_{\Sigma_0} = id_{r_1}$, $DF|_{\Sigma_0} = id$ (on $TV_4|_{\Sigma_0}$) and ${}^{(4)}g_0 = F^*({}^{(4)}g(0))$. (The symbol $|$ means restriction.) Thus the curve ${}^{(4)}\bar{g}(\varrho) = F^*({}^{(4)}g(\varrho))$ satisfies ${}^{(4)}\bar{g}(0) = {}^{(4)}g_0$.

By the conditions on F , ${}^{(4)}\bar{g}(\varrho)$ and ${}^{(4)}g(\varrho)$ induce the same Cauchy data on Σ_0 , viz. $(g(\varrho), \pi(\varrho))$. Thus if

$${}^{(4)}\bar{h} = \frac{\partial {}^{(4)}\bar{g}(0)}{\partial \varrho},$$

${}^{(4)}\bar{h}$ induces the same linearized Cauchy data

$$(\bar{h}_0, \bar{\omega}_0) = \left(\frac{\partial g(0)}{\partial \varrho}, \frac{\partial \pi(0)}{\partial \varrho} \right) = (h_0, \omega_0)$$

as ${}^{(4)}h$. Moreover, since ${}^{(4)}\bar{h}$ is tangent to a curve of exact solutions ${}^{(4)}\bar{g}(\varrho)$, ${}^{(4)}\bar{h}$ is a solution to the linearized equations $D \text{Ein}({}^{(4)}\bar{g}(0)) \cdot {}^{(4)}\bar{h} = D \text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}\bar{h} = 0$.

Therefore, by uniqueness of solutions to the linearized equations, there exists a unique ${}^{(4)}X$ such that ${}^{(4)}X|_{\Sigma_0} = 0$, $D({}^{(4)}X)|_{\Sigma_0} = 0$, and such that

$${}^{(4)}\bar{h} = {}^{(4)}h_0 + L_{({}^{(4)}X)}({}^{(4)}g_0).$$

Let F_ϱ be the flow of ${}^{(4)}X$, $F_0 = id_{r_1}$. Then $F_\varrho|_{\Sigma_0} = id$ and $DF_\varrho|_{\Sigma_0} = id$. Let

$${}^{(4)}\bar{g}(\varrho) = (F_\varrho^{-1})^*({}^{(4)}\bar{g}(\varrho)) = (F_\varrho^{-1})^*(F^*({}^{(4)}g(\varrho))).$$

Then ${}^{(4)}\bar{g}(\varrho)$ is a curve of exact solutions with ${}^{(4)}\bar{g}(0) = {}^{(4)}g_0$ and tangent

$$\frac{\partial {}^{(4)}\bar{g}(0)}{\partial \varrho} = \frac{\partial {}^{(4)}\bar{g}(0)}{\partial \varrho} - L_{({}^{(4)}X)}({}^{(4)}\bar{g}(0)) = {}^{(4)}\bar{h} - L_{({}^{(4)}X)}({}^{(4)}g_0) = {}^{(4)}h_0.$$

This is the curve we have been looking for (note that ${}^{(4)}\bar{g}(\varrho)$ has the same Cauchy data $(g(\varrho), \pi(\varrho))$ as ${}^{(4)}g(\varrho)$).

Secondly, assume that the equations $\text{Ein}({}^{(4)}g) = 0$ are linearization stable. To prove that the constraint equations are linearization stable, let $(g_0, \pi_0) \in \mathcal{C}_\mathcal{K} \cap \mathcal{C}_\delta$, ${}^{(4)}g_0$ be its maximal development, so Σ_0 is a Cauchy surface in V_4 . Let $(h_0, \omega_0) \in \ker D\Phi(g_0, \pi_0)$ and let, by the linearized existence theorem 4.5, ${}^{(4)}h \in \ker D \text{Ein}({}^{(4)}g_0)$ be such that it induces the data (h_0, ω_0) . By assumption, ${}^{(4)}h$ is tangent to a curve ${}^{(4)}g(\varrho)$ of exact solutions (in the sense explained earlier). Let $(g(\varrho), \pi(\varrho)) \in \mathcal{C}_\mathcal{K} \cap \mathcal{C}_\delta$ be the Cauchy data induced by ${}^{(4)}g(\varrho)$ on Σ_0 . By

definition of induced Cauchy data, the curve $(g(\varrho), \pi(\varrho))$ is tangent to (h_0, ω_0) at (g_0, π_0) . \square

EHLERS has emphasized the importance of obtaining explicit estimates on how far away the perturbed solution ${}^{(4)}g_0 + \varrho^{(4)}g$ is from an associated exact solution ${}^{(4)}g(\varrho)$. Such an estimate can be made from the analysis given here. Indeed, in terms of the Cauchy data (g, π) , if $(g(\varrho), \pi(\varrho))$ is the curve of exact solutions corresponding to a perturbation (h, ω) , we can say that

$$\|(g(\varrho), \pi(\varrho)) - (g_0 + \varrho h, \pi_0 + \varrho \omega)\|_{-2} \leq \frac{1}{2} \|D^2 \Phi(g_0, \pi_0)((h, \omega), (h, \omega))\|_{-2} \varrho^2 + O(\varrho^3),$$

where $\|\cdot\|_{-2}$ refers to the $H^1 \times H^{-1}$ norm on (g, π) .

In any particular example, the coefficient of ϱ^2 could presumably be worked out explicitly. The coefficient of ϱ^3 , if needed, is the corresponding third derivative, and so forth.

In [47, 48], O'MURCHADHA and YORK provide an approach to the linearization stability of the constraint equations which is rather different from ours. Building on the conformal techniques developed by LICHNEROWICZ and CHOQUET-BRUHAT (with the crucial additional step of allowing $\text{tr } k$ to be a constant depending on λ , the slicing parameter) to analyze the constraint equations, they generalize the Choquet-Bruhat and Deser [43] approach to the constraint equations to nonflat initial data. As in our «smooth submanifold approach» [5], they also attempt to find those $(g, \pi) \in \mathcal{C}_\mathcal{K} \cap \mathcal{C}_\delta$ near which $\mathcal{C}_\mathcal{K} \cap \mathcal{C}_\delta$ is a submanifold. Their method of proof uses immersions, in contrast to ours which used submersions, but the final results are nearly the same.

Their main idea is as follows: Consider the set of triples $(g, \tilde{\pi}^{TT}, \tau)$, where $g \in \mathcal{M}$, $\pi \in S_2^{TT} = \{\pi \in S_2^2 | \delta_\pi \pi = 0, \text{tr}_\pi \pi = 0\}$ and $\tau \in C^\infty(M; \mathbb{R})$, and also the set of $(\tilde{g}, \tilde{\pi}^{TT}, \tau)$, where $\tilde{g} = g \otimes \mu_g^{-1}$ and $\tilde{\pi}^{TT} = \pi^{TT} \otimes \mu_g^{\frac{1}{2}}$ are the «conformal parts» of g and π , respectively, in the sense that $(\tilde{g}, \tilde{\pi}^{TT})$ are invariant under any conformal transformation $g \mapsto \varphi^* g$. The idea of the conformal method is to use a conformal transformation to map any such triple $(\tilde{g}, \tilde{\pi}^{TT}, \tau)$ to a solution $(g, \pi) \in \mathcal{C}_\mathcal{K} \cap \mathcal{C}_\delta$ of the constraint equations such that the conformal parts of (g, π^{TT}) and $\text{tr } \pi$ are the same as the original triple $(\tilde{g}, \tilde{\pi}^{TT}, \tau)$. This is accomplished by rewriting the constraint equations as four nonlinear elliptic equations for a conformal factor $\varphi > 0$ and a vector field X , generalizing $\Delta \varphi + 8R(g)\varphi = 0$ for the time-symmetric case (see [49]). Under suitable conditions, there are solutions (φ, X) which describe the conformal transformation that takes $(\tilde{g}, \tilde{\pi}^{TT}, \tau)$ to a solution (g, π) of the constraint equations. Moreover, by linearizing this set of four elliptic equations, O'MURCHADHA and YORK show that, if (φ, X) is such a solution and if X is not a conformal Killing vector field for the conformal class of matrices \tilde{g} , then solutions (φ, Y) exist nearby and in fact define a local immersion of the set $(\tilde{g}, \tilde{\pi}^{TT}, \tau)$ into the constraint set. Again one concludes that $\mathcal{C}_\mathcal{K} \cap \mathcal{C}_\delta$ is a manifold near such a (g, π) and so linearization stability holds. This method yields slightly weaker results in as much

as it does not prove linearization stability for (g, π) which admit vector fields X which are conformal Killing vector fields but not «double» Killing vector fields.

Recently, MONCRIEF [7] has proven that for $(g, \pi) \in \mathcal{C}_\pi \cap \mathcal{C}_g$, the map $D\Phi(g, \pi)^*$ is injective if and only if a space-time ${}^{(4)}g$ generated by (g, π) has no (nontrivial) Killing vector fields ${}^{(4)}Y$ (i.e. $L_{{}^{(4)}Y}{}^{(4)}g = 0$ implies ${}^{(4)}Y = 0$); together with theorems 5.3 and 5.4, Moncrief's result then gives necessary and sufficient conditions for a space-time with compact Cauchy spacelike hypersurfaces to be linearization stable.

Moncrief's result still does not give necessary and sufficient conditions for $D\Phi(g, \pi)^*$ to be injective in terms of the (g, π) (the conditions C_π , C_g and C_π are sufficient but not necessary), but bypasses the $\text{tr } \pi' = \text{const}$ problem completely, rendering it much less important.

Moncrief's theorem is an important improvement over theorem 5.4, since the condition for linearization stability can now be expressed in terms of the space-time metric ${}^{(4)}g$ rather than in terms of the (g, π) of some arbitrarily embedded hypersurface.

The use of the adjoint form of the equations of motion helps to understand and give an easily digested proof of the result.

5.5. Theorem. (MONCRIEF [7].) — *Let ${}^{(4)}g$ be a solution to the empty-space field equations $\text{Ein}({}^{(4)}g) = 0$. Let $\Sigma_0 = i_0(M)$ be a compact Cauchy hypersurface with induced metric g_0 and canonical momentum π_0 . Then $\ker D\Phi(g_0, \pi_0)^*$ (a finite-dimensional vector space) is isomorphic to the space of Killing vector fields of ${}^{(4)}g$. In fact,*

$$(Y_\perp, -Y_\parallel) \in \ker D\Phi(g_0, \pi_0)^*$$

if and only if there exists a Killing vector field ${}^{(4)}Y$ of ${}^{(4)}g$ whose normal and tangential components to Σ_0 are Y_\perp and Y_\parallel .

Remark. Related references to this result are [50, 51]. Note that there are no assumptions that ${}^{(4)}Y$ be timelike or spacelike.

Proof. It is straightforward to prove, and well known, that the space of Killing vector fields is isomorphic to its space of normal and tangential components on any spacelike Cauchy hypersurface. Thus we need only prove the last remark.

The necessity follows immediately from proposition 4.7, for if ${}^{(4)}Y$ is a Killing vector field, ${}^{(4)}h = L_{{}^{(4)}Y}{}^{(4)}g_0 = 0$, and so

$$0 = \begin{pmatrix} h_0 \\ \omega_0 \end{pmatrix} = J \circ D\Phi(g_0, \pi_0)^* \cdot \begin{pmatrix} Y_\perp \\ -Y_\parallel \end{pmatrix}.$$

Another instructive argument is as follows. Let F_λ be the flow of ${}^{(4)}Y$.

For λ in a neighborhood of 0, $i_\lambda = F_\lambda \circ i_0$ is a well-defined one-parameter family of spacelike embeddings with generator ${}^{(0)}Y_\lambda = {}^{(0)}Y \circ i_\lambda$ as in sect. 1. Let $(Y_\perp(\lambda), Y_\parallel(\lambda))$ be the normal and tangential components of ${}^{(0)}Y_\lambda$. Let $(g(\lambda), \pi(\lambda))$ be the metrics and momenta induced on Σ_λ by ${}^{(0)}g$. In general, for a family of embeddings given by $i_\lambda = F_\lambda \circ i_0$, this will be the same as the metrics and momenta induced on Σ_0 by the space-time metrics ${}^{(0)}g(\lambda) = F_\lambda^* {}^{(0)}g$ (for example, for the metrics $g(\lambda)$, we have $g(\lambda) = i_\lambda^* {}^{(0)}g = (F_\lambda \circ i_0)^* {}^{(0)}g = i_0^* (F_\lambda^* {}^{(0)}g)$, which is the metric induced on Σ_0 by $F_\lambda^* {}^{(0)}g$). Since ${}^{(0)}Y$ is a Killing vector field, $F_\lambda^* {}^{(0)}g = {}^{(0)}g$ and so $g(\lambda) = g_0$, $\pi(\lambda) = \pi_0$ for all λ . Thus, by the adjoint form of the evolution equations,

$$0 = \begin{pmatrix} \frac{\partial g}{\partial \lambda} \\ \frac{\partial \pi}{\partial \lambda} \end{pmatrix} = J \circ D\Phi(g(\lambda), \pi(\lambda))^* \begin{pmatrix} Y_\perp(\lambda) \\ -Y_\parallel(\lambda) \end{pmatrix}.$$

Evaluating at $\lambda = 0$, $(Y_\perp, -Y_\parallel) \in \ker D\Phi(g_0, \pi_0)^*$.

Second, we prove sufficiency. Let $(Y_\perp, -Y_\parallel) \in \ker D\Phi(g_0, \pi_0)^*$. We wish to extend (Y_\perp, Y_\parallel) to a Killing field ${}^{(0)}Y$. Choose a slicing i_λ and let N_λ, X_λ be its lapse and shift. To define $Y_\perp(\lambda), Y_\parallel(\lambda)$, take the perpendicular-perpendicular ($\perp\perp$) and perpendicular-parallel ($\perp\parallel$) projections of Killing's equations $L_{i_\lambda} {}^{(0)}g = 0$. As on p. 358, this yields

$$\begin{aligned} \frac{\partial Y_\perp}{\partial \lambda} + L_X Y_\perp + L_{Y_\parallel} N &= 0, \\ -\frac{\partial Y_\parallel}{\partial \lambda} - L_X Y_\parallel + N \operatorname{grad} Y_\perp - Y_\perp \operatorname{grad} N &= 0. \end{aligned}$$

For given $N(\lambda, x), X(\lambda, x)$ and initial conditions (Y_\perp, Y_\parallel) , these equations define a unique Y_\perp, Y_\parallel on V_λ with the given initial conditions. (The proof of existence and uniqueness is easiest to see in Gaussian normal co-ordinates.) Thus we get a vector field ${}^{(0)}Y$ on V_λ with these normal and tangential components on each hypersurface. Let ${}^{(0)}h = L_{i_\lambda} {}^{(0)}g$ and $(h(\lambda), \omega(\lambda), U(\lambda), V(\lambda))$ be the induced deformation of (g, π, N, X) as described in sect. 4. By construction, ${}^{(0)}h_{\perp\perp} = 0$, ${}^{(0)}h_{\perp\parallel} = 0$, so from p. 358,

$$U(\lambda) = \frac{1}{2} N(\lambda) {}^{(0)}h_{\perp\perp}(\lambda) = 0 \quad \text{and} \quad V(\lambda) = N(\lambda) {}^{(0)}h_{\perp\parallel}(\lambda) = 0.$$

Thus $(h(\lambda), \omega(\lambda))$ satisfy the linear system obtained in sect. 4:

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} h \\ \omega \end{pmatrix} = J \circ D \left\{ D\Phi(g, \pi)^* \begin{pmatrix} N \\ X \end{pmatrix} \right\} \begin{pmatrix} h \\ \omega \end{pmatrix}.$$

From proposition 4.7, we get on Σ_0

$$\begin{pmatrix} h(0) \\ \omega(0) \end{pmatrix} = J \circ D\Phi(g(0), \pi(0))^* \begin{pmatrix} Y_{\perp}(0) \\ -Y_1(0) \end{pmatrix} = 0.$$

Thus $(h(\lambda), \omega(\lambda)) = (0, 0)$ for all λ . Therefore, since $h(\lambda) = 0$, $h_{\perp\perp}(\lambda) = 0$ and $h_{\perp i}(\lambda) = 0$, ${}^{(4)}h = 0$ and so ${}^{(4)}Y$ is a Killing field. \square

Remark. If ${}^{(4)}Y_{\perp} > 0$, we can take a slicing i_{λ} generated by ${}^{(4)}Y$ which has lapse-shift decomposition $(Y_{\perp}, -Y_1)$, independent of λ . By the evolution equations in such a slicing

$$\begin{pmatrix} \frac{\partial g(\lambda)}{\partial \lambda} \\ \frac{\partial \pi(\lambda)}{\partial \lambda} \end{pmatrix} = J \circ D\Phi(g(\lambda), \pi(\lambda))^* \begin{pmatrix} Y_{\perp} \\ -Y_1 \end{pmatrix}$$

for all λ . This has the unique solution $g(\lambda) = g_0$, $\pi(\lambda) = \pi_0$, since $(Y_{\perp}, -Y_1) \in \ker D\Phi(g_0, \pi_0)^*$. Hence $(g, \pi, Y_{\perp}, -Y_1)$ are all independent of λ . Hence ${}^{(4)}Y$ is a Killing vector field for ${}^{(4)}g_0$. Thus the proof of theorem 5.5 is very easy in this case.

The problem with this procedure when Y_{\perp} is not > 0 is that we can no longer generate a slicing by $(Y_{\perp}, -Y_1)$.

As an important corollary of this result, we observe that the condition $\ker D\Phi(g_0, \pi_0)^* = \{0\}$ is hypersurface independent (since it is equivalent to the absence of Killing vector fields, which is hypersurface independent). The condition is also obviously unchanged if we pass to an isometric space-time.

Putting all this together yields the main linearization stability theorem.

5.6. Theorem. — Let ${}^{(4)}g_0$ be a solution of the vacuum field equations $\text{Ein}({}^{(4)}g_0) = 0$ on V_4 . Assume that $(V_4, {}^{(4)}g_0)$ has a compact Cauchy surface Σ_0 and that $(V_4, {}^{(4)}g_0)$ is the maximal development.

Then the Einstein equations on V_4 ,

$$\text{Ein}({}^{(4)}g) = 0,$$

are linearization stable at ${}^{(4)}g_0$ if and only if ${}^{(4)}g_0$ has no Killing vector fields.

We conclude this section by examining the case in which ${}^{(4)}g_0$ is not linearization stable. The goal is to find necessary and sufficient conditions on a solution ${}^{(4)}h$ of the linearized equations so that ${}^{(4)}h$ is tangent to a curve of exact solutions through ${}^{(4)}g_0$. The necessary conditions will be derived; for sufficiency, see [52].

In theorem 5.3 we showed that, if ${}^{(4)}h$ is tangent to a curve of exact solutions and $(N, X) \in \ker D\Phi(g_0, \pi_0)^*$, then

$$\int_{\Sigma_0} \langle (N, X), D^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega)) \rangle = 0.$$

Following MONCRIEF [53], we can re-express this second-order condition in terms of the space-time, just as the condition $\ker D\Phi(g_0, \pi_0) = \{0\}$ was so expressed.

5.7. *Theorem.* (MONCRIEF [53].) — Let $\text{Ein}({}^{(4)}g_0) = 0$, and let ${}^{(4)}h \in S_2(V_4)$ satisfy the linearized equations

$$D \text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h = 0.$$

Let ${}^{(4)}Y$ be a Killing vector field of ${}^{(4)}g_0$ (so that ${}^{(4)}g_0$ is linearization unstable). Let Σ_0 be a compact Cauchy hypersurface and let (Y_\perp, Y_\parallel) be the normal and tangential components of ${}^{(4)}Y$ on Σ_0 . Then a necessary second-order condition for ${}^{(4)}h$ to be tangent to a curve of exact solutions is

$$\begin{aligned} (10) \quad \int_{\Sigma_0} (D^2 \text{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)) \cdot ({}^{(4)}Y_{\Sigma_0}, {}^{(4)}Z_{\Sigma_0}) \mu(g_0) = \\ = \int_{\Sigma_0} \langle (Y_\perp, -Y_\parallel), D^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega)) \rangle = 0. \end{aligned}$$

Proof. Suppose ${}^{(4)}g(\varrho)$ is a curve of exact solutions

$$(11) \quad \text{Ein}({}^{(4)}g(\varrho)) = 0$$

with

$${}^{(4)}g(0) = {}^{(4)}g_0 \quad \text{and} \quad \frac{\partial {}^{(4)}g(0)}{\partial \varrho} = {}^{(4)}h.$$

Differentiating (11) twice and evaluating at $\varrho = 0$ gives

$$(12) \quad D \text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}g''(0) + D^2 \text{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h) = 0,$$

where

$${}^{(4)}g''(0) = \frac{\partial^2 {}^{(4)}g(0)}{\partial \varrho^2}$$

is the « acceleration » of the curve ${}^{(4)}g(\varrho)$ at $\varrho = 0$. Note that since $\text{Ein}({}^{(4)}g_0) = 0$ and $D \text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h = 0$, by Taub's theorem 4.2, the divergence of each term of (12) is zero.

Now let ${}^{(4)}Y$ be a Killing vector field of ${}^{(4)}g_0$, let Σ_0 be a compact Cauchy hypersurface, and let (Y_\perp, Y_\parallel) be the normal and tangential components of ${}^{(4)}Y$ and Σ_0 . If we contract (12) with ${}^{(4)}Y_{x_i} = {}^{(4)}Y \circ i_0$ and ${}^{(4)}Z_{x_i}$ and use proposition 4.6, the first term gives

$$\begin{aligned} (D \operatorname{Ein} ({}^{(4)}g_0) \cdot {}^{(4)}g''(0)) \cdot ({}^{(4)}Y_{x_i}, {}^{(4)}Z_{x_i}) \mu(g_0) = \\ = -\frac{1}{2} \langle (Y_\perp, -Y_\parallel), D\Phi(g_0, \pi_0) \cdot (g''(0), \pi''(0)) \rangle, \end{aligned}$$

where $g''(0)$ and $\pi''(0)$ are the deformation of (g_0, π_0) on Σ_0 induced by ${}^{(4)}g''(0) \in S_2(V_4)$. If we integrate (12) over Σ_0 , this first term integrates to 0, since ${}^{(4)}Y$ is a Killing vector field. Thus,

$$\begin{aligned} \int_{\Sigma_0} (D \operatorname{Ein} ({}^{(4)}g_0) \cdot {}^{(4)}g''(0)) \cdot ({}^{(4)}Y_{x_i}, {}^{(4)}Z_{x_i}) \mu(g_0) = \\ = -\frac{1}{2} \int_{\Sigma_0} \langle (Y_\perp, -Y_\parallel), D\Phi(g_0, \pi_0) \cdot (g''(0), \pi''(0)) \rangle = \\ = -\frac{1}{2} \int_{\Sigma_0} \langle D\Phi(g_0, \pi_0)^* \cdot (Y_\perp, -Y_\parallel), (g''(0), \pi''(0)) \rangle = 0, \end{aligned}$$

thus giving the second-order condition

$$0 = \int_{\Sigma_0} [D^2 \operatorname{Ein} ({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)] \cdot (Y_{x_i}, Z_{x_i}) \mu(g_0).$$

The first equality in (10) comes from the following result which is proved exactly as in proposition 4.6.

5.8. Lemma. — *If $\operatorname{Ein} ({}^{(4)}g_0) = 0$, $D \operatorname{Ein} ({}^{(4)}g_0) \cdot ({}^{(4)}h) = 0$, ${}^{(4)}g(\rho)$ is any curve (not necessarily exact solutions) through ${}^{(4)}g_0$ tangent to ${}^{(4)}h$ and ${}^{(4)}Y_{x_i}$ is a vector field on a spacelike hypersurface Σ_0 (with normal ${}^{(4)}Z_{x_i}$), then*

$$\begin{aligned} (13) \quad -2 (D \operatorname{Ein} ({}^{(4)}g_0) \cdot {}^{(4)}g''(0)) \cdot ({}^{(4)}Y_{x_i}, {}^{(4)}Z_{x_i}) \mu(g_0) - \\ -2 [D^2 \operatorname{Ein} ({}^{(4)}g_0) ({}^{(4)}h_0, {}^{(4)}h_0)] \cdot ({}^{(4)}Y_{x_i}, {}^{(4)}Z_{x_i}) \mu(g_0) = \\ = \langle (Y_\perp, -Y_\parallel), D\Phi(g_0, \pi_0) \cdot (g''(0), \pi''(0)) \rangle + \\ + \langle (Y_\perp, -Y_\parallel), D^2\Phi(g_0, \pi_0) \cdot ((h_0, \omega_0), (h_0, \omega_0)) \rangle. \end{aligned}$$

If (13) is integrated over Σ_0 , in the context of theorem 5.7, (10) results. \square

Remarks. 1) By Taub's theorem 4.2, if $\operatorname{Ein} ({}^{(4)}g_0) = 0 = D \operatorname{Ein} ({}^{(4)}g_0) \cdot ({}^{(4)}h_0)$, then $D^2 \operatorname{Ein} ({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)$ has zero divergence. Thus, if ${}^{(4)}Y$ is a Killing vector field, then the vector field

$${}^{(4)}W = {}^{(4)}Y \cdot (D^2 \operatorname{Ein} ({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h))$$

also has zero divergence. Thus the necessary second-order condition

$$(14) \quad \int_{\Sigma_0} \langle {}^{(4)}W_{x_i}, {}^{(4)}Z_{x_i} \rangle \mu(g_0) = 0$$

on first-order deformations is independent of the Cauchy hypersurface on which it is evaluated.

2) The construction of the divergence-free vector field ${}^{(4)}W$ is due to TAUB [54]. The integral of ${}^{(4)}W$ over a Cauchy hypersurface then represents a conserved quantity for the gravitational field, constructed from a solution ${}^{(4)}h$ of the linearized equations and from a Killing vector field ${}^{(4)}Y$. The interesting feature of this conserved quantity, as shown by theorem 5.7, is that, unless it is zero, the first-order solution ${}^{(4)}h$ from which ${}^{(4)}W$ was constructed is not tangent to any curve of exact solutions.

In summary, the second-order condition « works » as follows. Let $\text{Ein}({}^{(4)}g_0) = 0$, $D \text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h = 0$, and let ${}^{(4)}Y$ be a Killing vector field of ${}^{(4)}g_0$. If on any compact Cauchy hypersurface

$$\int_{\Sigma_0} (D^2 \text{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)) \cdot ({}^{(4)}Y_{x_i}, {}^{(4)}Z_{x_i}) \mu(g_0) \neq 0,$$

then ${}^{(4)}h$ is not tangent to any curve of exact solutions of the empty-space field equations, i.e. ${}^{(4)}h$ is a spurious direction of perturbation.

Recent work [52] has shown that this second-order condition is sufficient as well. Thus the basic questions concerning linearization stability of the vacuum equations for space-times with compact Cauchy hypersurfaces have been answered. (See sect. 8 for further remarks.)

6. - Decomposition of tensors.

We continue to restrict our attention to the case of compact spacelike hypersurfaces M . While the decompositions undoubtedly do work in the noncompact case, weighted Sobolev or Hölder spaces are sufficiently tricky that rigorous proofs are less routine than in the compact case. For example, as is well recognized, decomposing tensors which only fall off as $1/r$ is usually impossible; $1/r^2$ fall-off is generally required. A discussion of the noncompact case is contained in [23].

Recall the canonical *splitting* that was discussed in sect. 1 as an application of the Fredholm alternative theorem:

$h = \hat{h} + L_x g$, where $L_x g$ is the part of h that is tangent to the orbit of g under the action of the diffeomorphism group $= \mathcal{D}(M)$ and where $\delta \hat{h} = 0$.

In this splitting we use the usual L_x -adjoint $2\delta_x$ of $\alpha_x(X) = L_x g$. However, for relativity, as we have seen, it is convenient to use natural adjoints acting on spaces of dual tensor objects. Since α_x maps vector fields to symmetric two-covariant tensors, its natural L_x -adjoint, α_x^* , maps symmetric two-contravariant tensor densities to one-form densities

$$(\alpha_x)^*: S_2^* \rightarrow A_1^1, \quad \pi \mapsto (\alpha_x)^* \cdot \pi = 2(\delta_x \pi).$$

Using the metric g , we obtain an isomorphism of S_2 with S_2^* by $h \mapsto h^{\sharp} \mu(g)$. Its inverse is $\pi \mapsto (\pi')^{\flat}$. Note that $\ker \alpha_x^* \subset S_2^*$. We write $(\ker \alpha_x^*)^*$ for the corresponding set of dual objects in S_2 (i.e. the image of $\ker \alpha_x^*$ under the inverse of the above isomorphism). Then the splitting of sect. 2 is written in terms of natural adjoints and dual tensor objects as

$$S_2 = \text{range } \alpha_x \oplus (\ker \alpha_x^*)^*.$$

York's splitting. YORK [55] gives a decomposition which arises in a similar way to the canonical splitting, but with the conformal group replacing the diffeomorphism group. The conformal group, the set of all possible conformal transformations, is the semi-direct product of the set \mathcal{P} of positive functions and the diffeomorphism group \mathcal{D} . $\mathcal{P} \cdot \mathcal{D}$ acts on g by pull-back under a diffeomorphism followed by multiplication by a positive function:

$$\begin{aligned} \mathcal{P} \cdot \mathcal{D} \times \mathcal{M} &\rightarrow \mathcal{M}, \\ ((p, \eta), g) &\mapsto p(\eta^* g). \end{aligned}$$

For g fixed, let $\Psi_g: \mathcal{P} \cdot \mathcal{D} \rightarrow \mathcal{M}$, $(p, \eta) \mapsto p(\eta^* g)$ be the orbit map, and let

$$\tau_g = T\Psi_g: T_{id}(\mathcal{P} \cdot \mathcal{D}) \approx C^\infty(M; \mathbb{R}) \times \mathcal{X} \rightarrow T_g \mathcal{M} \cong S_2, \quad (f, X) \mapsto fg + L_X g,$$

be its tangent. The range of τ_g is then the tangent space to the orbit under the action of $\mathcal{P} \cdot \mathcal{D}$. The L_x -adjoint of τ_g is

$$\tau_g^*: S_2 \rightarrow C^\infty \times \mathcal{X}: h \mapsto (\text{tr } h, 2(\delta h)^{\sharp}).$$

The Fredholm theory does not apply directly here because τ_g is a first-order operator on X and is a zeroth-order operator on f . However, use of a stronger concept of ellipticity due to DOUGLIS and NIRENBERG (cf. [56]) enables us to still apply a modified Fredholm alternative theorem to obtain the splitting

$$S_2 = (\ker \tau_g^*)^* \oplus \text{range } \tau_g,$$

so that, for $h \in S_2$,

$$(15) \quad h = h^{TT} + fg + L_x g,$$

where $h^{TT} \in \ker \tau_g^*$ is a transverse traceless tensor, i.e. $\delta h^{TT} = 0$ and $\text{tr } h^{TT} = 0$. We get essentially the same geometric picture as before (fig. 3).

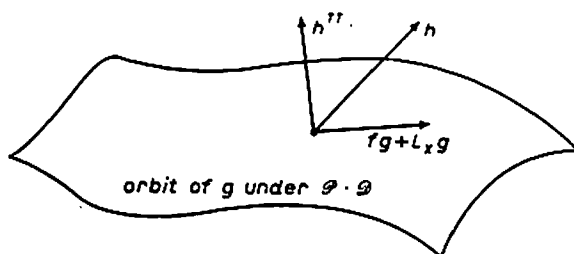


Fig. 3.

From (15), $\text{tr } h = nf - 2\delta X$, so that

$$f = \frac{1}{n} \text{tr } h + \frac{2}{n} \delta X.$$

Thus (15) can be rewritten as the finer splitting

$$(16) \quad h = h^{TT} + L_x g + \frac{2}{n} (\delta X) g + \frac{(\text{tr } h)}{n} g.$$

One calls $L_x g + (2/n)X = LX$ the «longitudinal part» or the «conformal Killing form of X ». Note that the trace of LX is zero, so that the third term is pointwise orthogonal to each of the first two.

This splitting can also be obtained by working in the space \mathcal{W} of «conformally invariant metrics», $g \otimes \mu(g)^{-2/n}$; LX is tangent to the \mathcal{D} orbit of $g \otimes \mu(g)^{-2/n}$ in this space. The h^{TT} part can be regarded as in the direction of a slice for the action of $\mathcal{P} \cdot \mathcal{D}$ on \mathcal{M} or \mathcal{D} on \mathcal{W} ; see fig. 4. (For further ge-

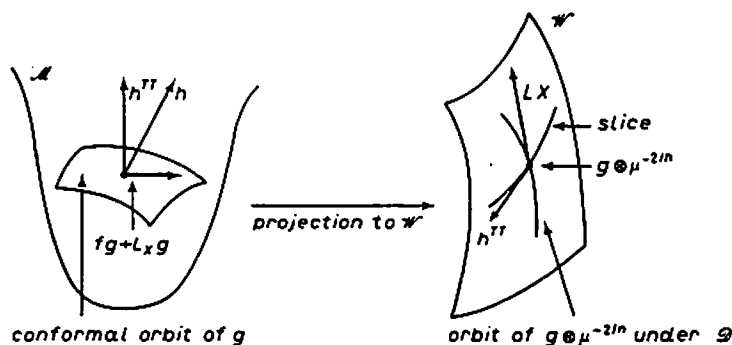


Fig. 4.

ometry of this splitting, see [55, 57].) Finally we note that the infinite dimensionality of the space of TT tensors is not obvious; see [45].

Barbance-Deser splitting. The Barbance [58]-Deser [8]-Berger-Ebin [18] splittings come from splitting the divergence-free part \tilde{h} in the canonical splitting $h = \tilde{h} + L_x g$. This decomposition works for (M, g) compact with constant scalar curvature. More can be said when g is Einstein; that is, $\text{Ric}(g) = \lambda g$. The differential operator used in these splittings is the derivative of the scalar curvature, $\gamma_* = DR(g)$. The kernel of γ_* consists of tensors tangent to the space of metrics with a specified scalar curvature $R(g) = \varrho$; the range of the adjoint is the L_2 -orthogonal space to this kernel. Thus for any $g \in \mathcal{M}$, $T_g \mathcal{M}$ splits as $T_g \mathcal{M} = \ker \gamma_* \oplus (\text{range } \gamma_*)^\perp$.

If $R(g) = \text{const}$, this decomposition and the canonical decomposition are compatible; that is, $\text{range } \alpha_* \subset \ker \gamma_*$. Indeed, if η_t is the flow of X ,

$$R(\eta_t^* g) = R(g), \quad \text{so } 0 = \frac{d}{dt} R(\eta_t^* g)|_{t=0} = DR(g) \cdot L_x g = \gamma_*(\alpha_*(X)).$$

Thus intersecting the two splittings, we get the finer L_2 -orthogonal splitting

$$(17) \quad T_g \mathcal{M} = (\ker \gamma_* \cap (\ker \alpha_*^\perp)^\perp) \oplus \text{range } \alpha_* \oplus (\text{range } \gamma_*^\perp)^*.$$

Now suppose g is Einstein, $\text{Ric}(g) = \lambda g$, and let $\tilde{h} \in \ker \gamma_* \cap (\ker \alpha_*^\perp)^\perp$, so $\delta \tilde{h} = 0$ and $\gamma_*(\tilde{h}) = \Delta \text{tr } \tilde{h} + \delta \delta \tilde{h} - \tilde{h} \cdot \text{Ric}(g) = 0$. Thus $\Delta(\text{tr } \tilde{h}) = \lambda \tilde{h} \cdot g = \lambda \text{tr } \tilde{h}$. If $\lambda < 0$, this implies $\text{tr } \tilde{h} = 0$; if $\lambda = 0$, $\text{tr } \tilde{h}$ is constant. If $\lambda > 0$, then by [59] we know that the first eigenvalue of Δ is

$$\lambda_1 > \frac{n}{n-1} \lambda > \lambda,$$

so $\text{tr } \tilde{h} = 0$. Thus, if $\tilde{h} \in \ker \gamma_* \cap (\ker \alpha_*^\perp)^\perp$, $\text{tr } \tilde{h}$ is either zero or constant.

Thus for the case in which (M, g) is an Einstein manifold, (17) above becomes the « Barbance-Deser-Berger-Ebin splitting »:

$$(18) \quad \begin{cases} \lambda \neq 0: h = h^{rr} + L_x g + (g \Delta f + \text{Hess } f - \lambda f g), \\ \lambda = 0: h = \left(h^{rr} + \frac{c}{n} g \right) + L_x g + (g \Delta f + \text{Hess } f), \end{cases}$$

where

$$c = \frac{1}{\text{vol } M} \int_M \text{tr } h \mu_g.$$

This splitting for $\lambda = 0$ was also used by BRILL and DESER [42].

Moncrief's splitting. Recent work by MONCRIEF [4] generalizes the above decomposition of $T_g\mathcal{M}$ to splittings of $T_{(g,\pi)}(T^*\mathcal{M})$. MONCRIEF observed from the various decompositions used in perturbation theory that one should really decompose the components of the tangent vector $(h, \omega) \in T_{(g,\pi)}(T^*\mathcal{M})$ simultaneously. Moncrief's decomposition can be derived by considering the operator $D\Phi(g, \pi)$ of sect. 2, and recalling our form of the evolution equations

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} g \\ \pi \end{pmatrix} = J \circ D\Phi(g, \pi)^* \cdot \begin{pmatrix} N \\ X \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

We have shown previously in sect. 2 that the operator $D\Phi(g, \pi)^*$ is elliptic, and so, therefore, is $J \circ D\Phi(g, \pi)^*$. Since $(J \circ D\Phi(g, \pi)^*)^* = -D\Phi(g, \pi) \circ J$, we have immediately the two splittings

$$T_{(g,\pi)}(T^*\mathcal{M}) = (\text{range } (D\Phi(g, \pi)^*))^* \oplus \ker (D\Phi(g, \pi))$$

and

$$T_{(g,\pi)}(T^*\mathcal{M}) = \text{range } (J \circ D\Phi(g, \pi)^*) \oplus (\ker (D\Phi(g, \pi) \circ J))^*,$$

where $(\)^*$ means, as above, the space of dual tensor objects. Thus,

$$\text{range } D\Phi(g, \pi)^* \subset S_d^2 \times S_2,$$

$$\ker (D\Phi(g, \pi) \circ J) \subset S_d^2 \times S_2,$$

$$(\text{range } (D\Phi(g, \pi)^*))^* \subset S_2 \times S_d^2$$

and

$$(\ker (D\Phi(g, \pi) \circ J))^* \subset S_2 \times S_d^2.$$

The summand $\ker D\Phi(g, \pi)$ represents the infinitesimal deformations (h, ω) of (g, π) that maintain $\Phi(g, \pi)$, and $(\text{range } (D\Phi(g, \pi)^*))^*$ represents the infinitesimal deformations which change $\Phi(g, \pi)$. Thus, if $\Phi(g, \pi) = 0$, $\ker D\Phi(g, \pi)$ represents those infinitesimal deformations that conserve the constraints.

From infinitesimal conservation of Φ , proposition 3.2, we know that, for $(g, \pi) \in \mathcal{C}_\mathcal{M} \cap \mathcal{C}_\delta$,

$$\text{range } J \circ D\Phi(g, \pi)^* \subset \ker D\Phi(g, \pi).$$

Thus these two splittings can be intersected to give Moncrief's splitting:

6.1. *Theorem.* (MONCRIEF [4].) - For $(g, \pi) \in \mathcal{C}_X \cap C_\delta$, the tangent space $T_{(g, \pi)}(T^* \mathcal{M}) \approx S_2 \times S_d^2$ splits L_2 -orthogonally as

$$T_{(g, \pi)}(T^* \mathcal{M}) = (\text{range } D\Phi(g, \pi)^*)^* \oplus \text{range } (J \circ D\Phi(g, \pi)^*) \oplus \\ \oplus (\ker (D\Phi(g, \pi) \circ J))^* \cap \ker (D\Phi(g, \pi))$$

For the purposes of the figure below, we number the summands as ① \oplus ② \oplus ③.

The two summands ② and ③ in the splitting

$$\ker D\Phi(g, \pi) = \text{range } (J \circ D\Phi(g, \pi)^*) \oplus (\ker (D\Phi(g, \pi) \circ J))^* \cap \ker (D\Phi(g, \pi))$$

can be interpreted as follows. Elements of the summand ② infinitesimally deform (g, π) to Cauchy data that generate isometric space-times, and elements of the summand ③ infinitesimally deform (g, π) in the direction of new Cauchy data that generate nonisometric solutions to the empty-space field equations (see fig. 5).

Below we shall see the geometrical significance of this second summand more clearly. For now, we note that Moncrief's splittings can be regarded as a decomposition of 12 functions of 3 variables $(h_{ij}(x^k), \omega^{ij}(x^k))$ into 3 sets of 4 functions of 3 variables.

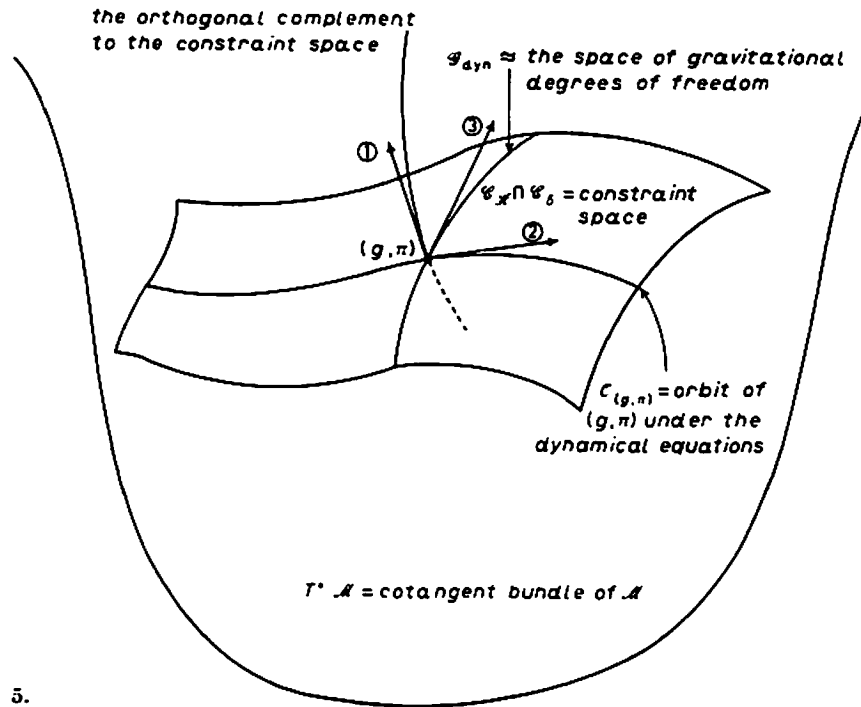


Fig. 5.

Finally we remark that, if $\pi = 0$ and $R(g) = 0$, then Moncrief's decomposition reduces to two copies of the Berger-Ebin splitting, given by eq. (17) above. If, moreover, $\text{Ric}(g) = 0$ (so that g is flat), then Moncrief's decomposition gives the Barbançe-Deser-Berger-Ebin splitting with $\lambda = 0$, as given by eq. (18) above.

7. - Reduction of phase space and the symplectic space of gravitational degrees of freedom.

We now review some results of symplectic geometry that provide a basis for a more unified description of the various splittings [9]. These results are based on a general reduction of phase spaces for which there is an invariant Hamiltonian system under some group action [14]. A further application of these results leads to the construction of the symplectic space of gravitational degrees of freedom [10].

A background reference for the material in this section is [60, 61].

Let P be a manifold and Ω a symplectic form on P ; that is, Ω is a closed (weakly) nondegenerate two-form. For relativity, P will be $T^*\mathcal{M}$ and Ω will be the canonical symplectic form $-J^{-1}$ as described above.

Let G be a topological group which acts canonically on P ; that is, for each $g \in G$, the action of g on P , $\Phi_g: p \mapsto g \cdot p$, preserves Ω . Assume there is a moment Ψ for the action, as defined by Souriau [62]. This means the following: Ψ is a map from P to \mathfrak{G}^* , the dual to the Lie algebra $\mathfrak{G} = T_e G$ of G , such that

$$\Omega(\xi_p(p), v_p) = \langle d\Psi(p) \cdot v_p, \xi \rangle$$

for all $\xi \in \mathfrak{G}$, where ξ_p is the corresponding infinitesimal generator (Killing form) on P , and $v_p \in T_p P$. Another way to define Ψ is to require that, for each ξ , $p \mapsto \langle \Psi(p), \xi \rangle$ be an energy function for the Hamiltonian vector field ξ_p . This concept of a moment is an important geometrization of the various conservation theorems of classical mechanics and field theory, including Noether's theorem.

It is easy to prove that, if H is a Hamiltonian function on P with corresponding Hamiltonian vector field X_H (i.e. $dH(p) \cdot v = \Omega_p(X_H(p), v)$), or equivalently $i_{X_H} \Omega = dH$, and if H is invariant under G , then Ψ is a constant of the motion for X_H ; i.e., if F_t is the flow of X_H , then $\Psi \circ F_t = \Psi$.

As an example, consider a group G acting on a configuration space Q . This action lifts to a canonical action on the phase space T^*Q . The moment in this case is given by

$$\langle \Psi(\alpha_q), \xi \rangle = \langle \xi_q(q), \alpha_q \rangle,$$

where α_q belongs to T^*Q . If G is the set of translations or rotations, Ψ is

linear or angular momentum, respectively. As expected, Ψ is a vector, and the transformation property required of this vector is equivariance of the moment under the co-adjoint action of G on \mathfrak{G} ; that is, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\Phi_g} & P \\ \downarrow & & \downarrow \\ \mathfrak{G}^* & \xrightarrow{Ad_g^*} & \mathfrak{G}^* \end{array}$$

must commute. We shall consider only equivariant moments.

There are several classical theorems concerning reduction of phase spaces. In celestial mechanics there is Jacobi's elimination of the node which states that in a rotationally invariant system we can eliminate four of the variables and still have a Hamiltonian system in the new variables. Another classical theorem of Hamiltonian mechanics states that the existence of k first integrals in involution allows a reduction of $2k$ variables in the phase space. Both of these theorems follows from a theorem of Marsden and Weinstein [14] on the reduction of phase space.

To construct this reduced space, let G_μ be the isotropy group of μ :

$$G_\mu = \{g \in G \mid Ad_g^* \mu = \mu\}.$$

Consider $\Psi^{-1}(\mu) = \{p \mid \Psi(p) = \mu\}$. The equivariance condition implies that G_μ preserves $\Psi^{-1}(\mu)$, so we can consider $P_\mu = \Psi^{-1}(\mu)/G_\mu$. In case $\Psi^{-1}(\mu)$ is a manifold (e.g. if μ is a regular value) and G acts freely and properly on this manifold, we have

7.1. Theorem. (MARSDEN and WEINSTEIN [14].) — P_μ inherits a natural symplectic structure from P , and a Hamiltonian system on P which was invariant under the canonical action of G projects naturally to a Hamiltonian system on P_μ .

In Jacobi's elimination of the node, G is SO_3 , so \mathfrak{G} is \mathbb{R}^3 and the co-adjoint action is the usual one. Thus the isotropy subgroup G_μ of a point μ in \mathbb{R}^3 is S^1 . If n is the dimension of P , then $\Psi^{-1}(\mu)$ is the solution set for three equations, so the dimension of $\Psi^{-1}(\mu)/G_\mu$ is $n - 3 - 1 = n - 4$. For k first integrals in involution, G is a k -dimensional Abelian group, so the co-adjoint action is trivial and $G_\mu = G$. Thus the dimension of $\Psi^{-1}(\mu)/G$ is $n - 2k$. Another known theorem that follows from theorem 7.1 is the Kostant-Kirilov theorem which states that the orbit of a point μ in \mathfrak{G}^* under the adjoint action is a symplectic manifold.

Now we shall show how to obtain a general splitting theorem for symplectic manifolds, one piece of which is tangent to the reduced space P_μ [9]. This includes the splitting theorems for symmetric tensors as a special case.

A splitting theorem for a symplectic manifold P requires a positive definite but possibly only weakly nondegenerate metric, or other such structure to give a dualization. This is so that orthogonal complements may be defined. Otherwise quotient spaces are needed. Suppose we know, say from the Fredholm theorem, that

$$(19) \quad T_p P = \text{range } (T_p \Psi)^* \oplus \ker T_p \Psi$$

(here $(T_p \Psi)^*$ is the usual L_2 -adjoint). Of course, in finite dimensions this is automatic. Define

$$\alpha_p: \mathfrak{G}_\mu \rightarrow T_p P: \xi \mapsto \xi_p(p),$$

where \mathfrak{G}_μ is the Lie algebra of G_μ . Suppose we also have the splitting

$$(20) \quad T_p P = \text{range } \alpha_p \oplus \ker \alpha_p^*.$$

There is a general compatibility condition between these two splittings, namely $\text{range } \alpha_p \subset \ker T_p \Psi$, which follows readily from equivariance. In fact,

$$\text{range } \alpha_p = T_p(G \cdot p) \cap \ker T_p \Psi.$$

This compatibility condition implies the finer splitting

$$(21) \quad T_p P = \text{range } (T_p \Psi)^* \oplus \text{range } \alpha_p \oplus (\ker T_p \Psi \cap \ker \alpha_p^*),$$

i.e.

$$(22) \quad T_p P \approx \text{range } (T_p \Psi)^* \oplus T_p(\text{orbit under } G_\mu) \oplus \ker T_p \Psi / G_\mu.$$

Note that the third summand is the tangent space to P_μ . The geometric picture is the following (fig. 6):

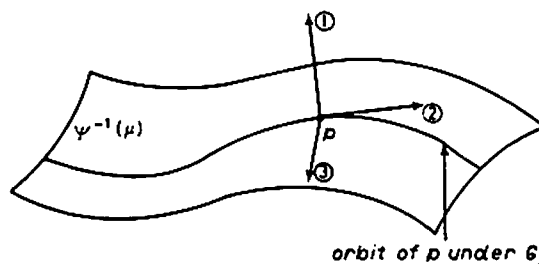


Fig. 6.

① belongs to $\text{range } T_p \Psi^*$, the orthogonal complement of the tangent space to the level set $\Psi^{-1}(\mu)$;

② belongs to $\text{range } \alpha_p$, the tangent space to the orbit of p under G_μ ;

③ is in $(\ker T_p \Psi \cap \ker \alpha_p^*)$, and is the part of the decomposition which is tangent to the reduced symplectic manifold.

② and ③ together are $\ker T_p \Psi$, the tangent space to $\Psi^{-1}(\mu)$.

Both Moncrief's and York's splittings are applications of this result.

For the case of Moncrief's splitting, P is $T^*\mathcal{M}$ and the «group» is $G = E^\infty(M, V_4, {}^{(4)}g)$, the spacelike embeddings of M to Cauchy hypersurfaces in $(V_4, {}^{(4)}g)$, a maximal space-time. Although this is not a group, it is enough like $\mathcal{T} \cdot \mathcal{D}$, the semi-direct product of functions τ (time translations) with diffeomorphisms η of M for the analysis to work. G «acts» on (g, π) as follows. Let ${}^{(4)}g$, $\text{Ein}({}^{(4)}g) = 0$, be a space-time which has (g_0, π_0) as Cauchy data on an embedded Cauchy hypersurface

$$\Sigma_0 = i_0(M), \quad i_0: M \rightarrow V_4.$$

Then $i \in E^\infty(M, V_4, {}^{(4)}g)$ maps (g_0, π_0) to the (g, π) induced on the hypersurface $\Sigma = i(M)$. The set of all such (g, π) define the orbit of (g_0, π_0) . These orbits are disjoint, so define an equivalence relation \sim .

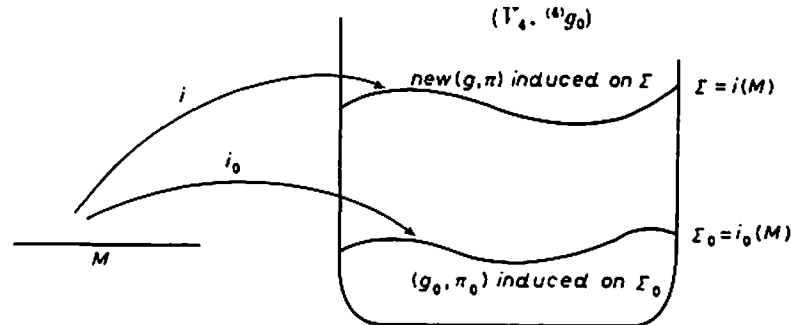


Fig. 7.

Although this is not an action (since E^∞ is not a group), it has well-defined orbits and the above symplectic analysis applies [10]. If we use the adjoint form of the Einstein evolution system, the moment of «this action» on a tangent vector ${}^{(4)}X_{\Sigma_0} \in T_{\Sigma_0} E^\infty(M, V_4, {}^{(4)}g)$ with lapse N and shift X is computed to be

$$\Psi_{(g, \pi)}({}^{(4)}X) = \int N \mathcal{H}(g, \pi) + X \cdot \mathcal{J}(g, \pi).$$

Here the ${}^{(4)}X_{\Sigma_0}$ or the (N, X) can be thought of as belonging to the «Lie algebra» of E^∞ . (See appendix II.)

Since $\Psi^{-1}(0)$ is precisely the constraint set $\mathcal{C}_\mathcal{H} \cap \mathcal{C}_\mathcal{S}$, we choose $\mu = 0$, so

$G_\mu = G$. From the equations of motion,

$$\alpha_{(g,\pi)}: \mathcal{G} \rightarrow T_{(g,\pi)}(T^*\mathcal{M}), \quad (N, X) \mapsto J \circ D\Phi(g, \pi)^* \begin{pmatrix} N \\ X \end{pmatrix}.$$

Thus the symplectic decomposition (21) becomes

$$T_{(g,\pi)}(T^*\mathcal{M}) = (\text{range } D\Phi(g, \pi)^*)^* \oplus \text{range } (J \circ D(g, \pi)^*) \oplus \\ \oplus \ker D\Phi(g, \pi) \cap (\ker D\Phi(g, \pi) \circ J),$$

which is Moncrief's splitting. Here the third summand represents the tangent space to the reduced space $P_\mu \approx \mathcal{G}_\mu \cap \mathcal{G}_\delta / \sim$. This quotient, by the equivalence relation described above, is naturally isomorphic to the space of gravitational degrees of freedom, namely $\mathcal{E}(V_4)/\mathcal{D}(V_4)$, the set of solutions to the vacuum Einstein equations $\mathcal{E}(V_4) = \{^4g | \text{Ein}(^4g) = 0\}$ modulo the space-time diffeomorphism group $\mathcal{D}(V_4) = \text{Diff}(V_4)$. This is the space of isometry classes of empty-space solutions of the Einstein equations. This is the space of gravitational degrees of freedom; the co-ordinate gauge group has been factored out.

For the noncompact case, one must be much more careful about the definition of the space of gravitational degrees of freedom (e.g., one does not want to identify all solutions; this case is qualitatively different because of the presence of a mass function and gravitational radiation, as is discussed in [23]).

For York's decomposition, the manifold is the same, namely $T^*\mathcal{M}$, but the group is the conformal group $\mathcal{P} \cdot \mathcal{D}$ acting on $T^*\mathcal{M}$, as described before. The infinitesimal generators of the action of the conformal group are

$$\alpha_{(g,\pi)}(p, X) = pg + L_X g$$

and the moment is computed to be

$$(23) \quad \Psi_{(g,\pi)}(p, X) = \int p \, \text{tr } \pi + \int X \cdot \mathcal{J}(g, \pi)$$

so

$$\Psi^{-1}(0) = \{(g, \pi) | \delta\pi = 0 \text{ and } \text{tr } \pi = 0\} = \mathcal{G}_\mu \cap \mathcal{G}_\delta,$$

the intersection of the sets where the divergence of π is zero and where the trace of π is zero.

One can show, as for $\mathcal{G}_\mu \cap \mathcal{G}_\delta$, that $\mathcal{G}_\mu \cap \mathcal{G}_\delta$ is a manifold in a neighborhood of those $(g, \pi) \in \mathcal{G}_\mu \cap \mathcal{G}_\delta$ such that (g, π) has no simultaneous conformal Killing vector fields, i.e., if $L_X g = f_1 g$, $L_X \pi = f_2 g$, then $X = 0$ (see [57]).

The universal decomposition (21) splits an element (h, ω) of $T_{(g,\pi)}(T^*\mathcal{M})$ into two copies of York's decomposition described in sect. 6. In this case, the

reduced phase space is

$$\mathcal{C}_\delta \cap \mathcal{C}_u / \mathcal{P} \cdot \mathcal{Q}.$$

From [47], we see that this space is isomorphic to the space $\mathcal{C}_\mathcal{K} \cap \mathcal{C}_\delta / \sim$.

As we have emphasized, in the case of compact hypersurfaces, one identifies all (g, π) 's which occur on slicings in one space-time. In the noncompact case one does *not* do this, as is explained in [23]. This point and the general definition of « true degrees of freedom » is consequently confusing at first.

In the present compact case, however, we find it useful to write

$$\mathcal{G}_{dyn} = \mathcal{G}_{dynamical} = \mathcal{C}_\mathcal{K} \cap \mathcal{C}_\delta / \sim$$

and

$$\mathcal{G}_{conf} = \mathcal{G}_{conformal} = \mathcal{C}_\delta \cap \mathcal{C}_u / \mathcal{P} \cdot \mathcal{Q}.$$

Both are representations of $\mathcal{G} = \mathcal{E}(V_\delta) / \mathcal{D}(V_\delta)$. The natural symplectic structure on $T^*\mathcal{M}$ associated with the dynamics induces naturally the symplectic structure on \mathcal{G}_{dyn} . We do not know if the isomorphism between \mathcal{G}_{dyn} and \mathcal{G}_{conf} is a canonical transformation, i.e. if the symplectic structure on \mathcal{G}_{conf} associated with the dynamics is the natural symplectic structure on \mathcal{G}_{conf} . However, it seems unlikely.

The symplectic structure on \mathcal{G} described above may be important for the problem of quantizing gravity. This would be of physical interest in the noncompact case in connection with gravitational waves.

The symplectic structure presented here is implicit in the work of Bergmann [63], Dirac [2] and DeWitt [35, 64]. The present formulation, however, allows one to be rather precise and geometrical. First of all, it may allow one to use the Segal (cf. [65, 66]) or Kostant-Souriau [62] formalism to carry out a full quantization or a semi-classical quantization. Secondly, the approach presented here enables one to show that near metrics ${}^{(0)}g$ in $\mathcal{E}(V_\delta)$ with no isometries (and hence no space-time Killing vector fields) $\mathcal{E}(V_\delta) / \mathcal{D}(V_\delta)$ is a smooth manifold and is locally isomorphic in a natural way to $\mathcal{C}_\mathcal{K} \cap \mathcal{C}_\delta / \sim$, and thus carries a canonical symplectic structure. Thus in a neighborhood of Einstein flat space-times without Killing vector fields, the space $\mathcal{G} = \mathcal{E}(V_\delta) / \mathcal{D}(V_\delta)$ of gravitational degrees of freedom is itself a symplectic manifold, or, if you prefer, a gravitational phase space without singularities, each element of which represents an empty-space geometry.

8. - Current work and open problems.

Many of the areas that we have discussed in this paper are currently under investigation.

Linearization stability for nongravitational fields coupled to gravity is an active area of research. For example, the case of the coupled Einstein-Maxwell system has been solved by ARMS [67]. The conditions for stability in this case are the absence of a simultaneous Killing vector field for ${}^{(4)}g$ and the electromagnetic-field tensor F . The methods have been extended to general Yang-Mills fields coupled to gravity as well.

D'EATH [68, 69] has examined the case of linearization stability of Robertson-Walker universes and finds them to be linearization stable. In the case $k = 0$, he considers perturbations which die away at spatial infinity.

The question of linearization stability for asymptotically flat space-times is not fully settled. A main difficulty is that, in general, the splitting theorems of sect. 1 are quite delicate and often break down. We consider the time-symmetric asymptotically flat case in [23].

The situation for several important cases of interest for black-hole research, namely the Schwarzschild and Kerr solutions, remains open, but should be settled in the near future.

The sufficiency of the second-order conditions presented in sect. 5 has been proven in current investigations [52]. We have been able to show that, if there are k linearly independent Killing vector fields for ${}^{(4)}g_0$, then these k extra second-order conditions are sufficient for linearization stability. The (g_0, π_0) induced on any Cauchy hypersurface is a singular point of $\mathcal{C}_\pi \cap \mathcal{C}_g$, and locally $\mathcal{C}_\pi \cap \mathcal{C}_g$ looks like a (manifold) \times (an intersection of k cones). The proof depends on the slice theorem for relativity [10] and on some techniques from bifurcation theory at multiple eigenvalues.

The authors are currently engaged in the general question of the Hamiltonian structure of tensor field theories coupled to gravity [28]. A fundamental work in this area is due to KUCHAR [36].

Our approach is to develop a Hamiltonian formalism, modeled on the adjoint form of the Einstein equations, for any covariant field theory coupled to gravity. In the case of Lagrangians which do not depend on derivatives of the gravitation field, our results are similar to the pure gravitational case. Briefly, if ${}^{(4)}\varphi_A$ is a space-time field whose dynamics are described by a Lagrangian density $\mathcal{L}({}^{(4)}\varphi_A, {}^{(4)}g)$ which does not depend on derivatives of ${}^{(4)}g$, then the projections of ${}^{(4)}\varphi_A$ on spacelike hypersurfaces gives rise to $\varphi_{A\text{dynamical}}$ and $\varphi_{A\text{degenerate}}$, now tensor quantities on the hypersurface. One also has

$$H_{\text{total}} = H_{\text{geom}} + H_{\text{fields}},$$

$$J_{\text{total}} = J_{\text{geom}} + J_{\text{fields}},$$

and functions \mathcal{C}^{deg} which correspond to the constraints of the theory due to degenerate fields.

If we let $\Phi_r = (H_r, J_r, \mathcal{C}^{\text{deg}})$, the Hamiltonian picture is fully described

by the system

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} g \\ q^{A_{d\gamma n}} \\ \pi \\ \pi^{A_{d\gamma n}} \end{pmatrix} = J \circ D\Phi_r(g, q^{A_{d\gamma n}}; \pi, \pi^{A_{d\gamma n}})^* \cdot \begin{pmatrix} N \\ X \\ q^{A_{deg}} \end{pmatrix},$$

$$\Phi_r(g, q^{A_{d\gamma n}}; \pi, \pi^{A_{d\gamma n}}) = 0.$$

Thus the expression $N\Phi'_r + X\mathcal{J}'_r + \varphi_{A_{deg}}\varphi^{A_{deg}}$ acts as the generator of translations.

The formal similarities to the pure gravitational case allow one to take over at little extra effort the splitting theorems and reduction procedures of sect. 7. Thus, in particular, we are able to construct a symplectic manifold which represents the total space of degrees of freedom due to gravity and to the external fields. This approach may be tied in with the Dirac theory of constraints using [70].

For alternative approaches to the space of gravitational degrees of freedom and its symplectic structure, see [71] and [72].

Finally we mention that many of the topics presented here may be extended to noncompact cases. Linearization stability results are completely different in this case (see [23]). Moreover, in the dynamical formulation, the mass function acts as the generator of time translations and indeed appears to be the proper Hamiltonian (see [38] and sect. 10 of [23]).

APPENDIX I

Variational derivatives of the scalar curvature.

In computing the variational derivatives of a tensor that depends on the metric and its derivatives, the partial derivatives that arise are not tensor quantities. Here we consider only the scalar curvature map, but the general procedure is useful in computing the stress-energy tensor of a Lagrangian density which may depend on the derivatives of the background metric tensor (see [28, 36 III]).

For the map $R(g)$ we have

$$D_\alpha(R(g)) \cdot h = \partial_\alpha R(g) \cdot h + \partial_{\partial_\alpha} R(g) \cdot \partial h + \partial_{\partial^2_\alpha} R(g) \cdot \partial^2 h.$$

The three partial-derivative terms do not correspond to the three tensor terms in the expression

$$D_\alpha R(g) \cdot h = -h \cdot \text{Ric}(g) + \Delta \text{tr } h + \delta \delta h.$$

For example, the term $\partial_a R(g) \cdot h$ is not a tensor; it involves variations of the Christoffel symbols with respect to the metric g alone, and not its derivatives. In fact,

$$\partial_a(\Gamma_{jk}^i) \cdot h = -\frac{1}{2} h'^a (g_{aj,k} + g_{ak,j} - g_{jk,a}),$$

whereas the total variation of the Christoffel symbols is a tensor,

$$D_a \Gamma(g) \cdot h = \partial_a \Gamma(g) \cdot h + \partial_{\epsilon_a} \Gamma(g) \cdot \partial h = \frac{1}{2} g'^a (h_{aj,k} + h_{ak,j} - h_{jk,a}).$$

To get a variational method that yields partial derivatives that are tensors, we consider the scalar curvature as a function of the undifferentiated metric coefficients that do *not* appear in the Christoffel symbols, and of the Christoffel symbols, and we write $R(g, \Gamma)$ to represent the functional dependence.

Since the Ricci tensor depends only on the connection Γ , the undifferentiated metric coefficients do not appear except in the definition of the Christoffel symbols. Hence we write $\text{Ric}(\Gamma)$, and

$$R(g, \Gamma) = g^{-1} \cdot \text{Ric}(\Gamma).$$

If we use the chain rule for functional derivatives, the derivative of $R(g, \Gamma)$ is given by

$$\begin{aligned} DR(g) \cdot h &= D_{(g, \Gamma)} R(g, \Gamma) \cdot (h, D_a \Gamma(g) \cdot h) = D_g R(g, \Gamma) \cdot h + D_\Gamma R(g, \Gamma) \cdot (D_a \Gamma(g) \cdot h) = \\ &= \partial_a R(g, \Gamma) \cdot h + D_\Gamma R(g, \Gamma) \cdot (D_a \Gamma(g) \cdot h), \end{aligned}$$

where now each term is a tensor,

$$\partial_a R(g, \Gamma) \cdot h = -h \cdot \text{Ric}(\Gamma) = -h \cdot \text{Ric}(g),$$

and the second term is evaluated as

$$\begin{aligned} D_\Gamma(R(g, \Gamma)) \cdot (D_a \Gamma(g) \cdot h) &= g^{-1} \cdot (D_\Gamma \text{Ric}(\Gamma) \cdot (D_a \Gamma(g) \cdot h)) = \\ &= g^{-1} \cdot (D \text{Ric}(g) \cdot h) = \Delta \text{tr } h + \delta \delta h, \end{aligned}$$

where the equality $D \text{Ric}(g) \cdot h = D_\Gamma \text{Ric}(\Gamma) \cdot (D_a \Gamma(g) \cdot h)$ follows because $\text{Ric}(g)$ depends only on the connection.

Applying this procedure to the Hamiltonian density $\mathcal{H}(g, \pi)$ for general relativity, we write

$$\mathcal{H}(g, \Gamma, \pi) = \mathcal{K}(g, \pi) - R(g, \Gamma) \mu(g),$$

where $\mathcal{K}(g, \pi)$ are the kinetic terms of $\mathcal{H}(g, \pi)$ and are algebraic in (g, π) . Thus

$$\begin{aligned} D_a \mathcal{H}(g, \pi) \cdot h &= D_{(g, \Gamma)} \mathcal{H}(g, \Gamma, \pi) \cdot (h, D \Gamma(g) \cdot h) = \\ &= D_g \mathcal{H}(g, \Gamma, \pi) \cdot h + D_\Gamma \mathcal{H}(g, \Gamma, \pi) \cdot (D \Gamma(g) \cdot h) = \\ &= \partial_a \mathcal{H}(g, \Gamma, \pi) \cdot h - (D_\Gamma(R(g, \Gamma)) \cdot (D \Gamma(g) \cdot h)) \mu(g) = \\ &= \partial_a \mathcal{H}(g, \Gamma, \pi) \cdot h - (\Delta \text{tr } h + \delta \delta h) \mu(g), \end{aligned}$$

where

$$\hat{c}_s \mathcal{H}(g, \Gamma, \pi) = \hat{c}_s \mathcal{H}(g, \pi) \cdot h + (\text{Ein}(g) \cdot h) \mu(g) = -S_s(\pi, \pi) \cdot h + (\text{Ein}(g) \cdot h) \mu(g) .$$

Integrating by parts, we then find

$$D_s \mathcal{H}(g, \pi)^* \cdot N = N \hat{c}_s \mathcal{H}(g, \Gamma, \pi) - (g \Delta N + \text{Hess } N) \mu(g) .$$

In the above expressions, each of the partial derivatives is a tensor.

APPENDIX II

Poisson brackets and the Dirac canonical commutation relations.

This appendix gives a few complements to the results of sect. 2.

Let $F: T^* \mathcal{M} \rightarrow \mathbb{R}$ be a real-valued function of $T^* \mathcal{M}$ that comes from a density $\mathcal{F}: T^* \mathcal{M} \rightarrow C_d^\infty$,

$$F(g, \pi) = \int_{\mathcal{M}} \mathcal{F}(g, \pi) .$$

Then the *Hamiltonian vector field* of F

$$X_F: T^* \mathcal{M} \rightarrow T(T^* \mathcal{M})$$

is defined by

$$dF(g, \pi) \cdot (h, \omega) = \Omega(X_F(g, \pi), (h, \omega)) ,$$

where Ω is the symplectic structure on $T^* \mathcal{M}$.

II.1. *Proposition.* - The Hamiltonian vector field X_F is given by

$$X_F(g, \pi) = J \circ (D\mathcal{F}(g, \pi))^* \cdot 1 .$$

Proof. Recall that

$$\Omega(X_F(g, \pi), (h, \omega)) = - \int \langle X_F(g, \pi), J^{-1}(h, \omega) \rangle ,$$

and so

$$\begin{aligned} dF(g, \pi) \cdot (h, \omega) &= \int D\mathcal{F}(g, \pi) \cdot (h, \omega) = \int \langle D\mathcal{F}(g, \pi)^* \cdot 1, (h, \omega) \rangle = \\ &= - \int \langle J \circ D\mathcal{F}(g, \pi)^* \cdot 1, J^{-1}(h, \omega) \rangle = \Omega(J \circ D\mathcal{F}(g, \pi)^* \cdot 1, (h, \omega)) . \end{aligned}$$

Here we used the identity $J^* = -J$.

In particular, if $F = \int N\mathcal{H} + X \cdot \mathcal{J}$, then

$$X_F(g, \pi) = J \circ D(N\mathcal{H} + X \cdot \mathcal{J})^* \cdot 1 = J \circ D\Phi(g, \pi)^* \cdot \begin{pmatrix} N \\ X \end{pmatrix},$$

again showing that the Einstein evolution equations are Hamilton's equations on the symplectic manifold $T^*\mathcal{M}$ with Hamiltonian density $N\mathcal{H} + X \cdot \mathcal{J}$.

Now suppose $F_1, F_2: T^*\mathcal{M} \rightarrow \mathbb{R}$ are real-valued functions on $T^*\mathcal{M}$ that arise from densities \mathcal{F}_1 and \mathcal{F}_2 , respectively. Then their Poisson bracket,

$$\{F_1, F_2\}: T^*\mathcal{M} \rightarrow \mathbb{R},$$

is defined by

$$\{F_1, F_2\}(g, \pi) = \Omega(X_{F_1}(g, \pi), X_{F_2}(g, \pi)),$$

where X_F is the Hamiltonian vector field for F .

II.2. *Proposition.* — The Poisson bracket $\{F_1, F_2\}$ defined above is given by

$$\begin{aligned} \{F_1, F_2\}(g, \pi) = & \int \langle D_g \mathcal{F}_1(g, \pi)^* \cdot 1, D_\pi \mathcal{F}_2(g, \pi)^* \cdot 1 \rangle - \\ & - \int \langle D_\pi \mathcal{F}_1(g, \pi)^* \cdot 1, D_g \mathcal{F}_2(g, \pi)^* \cdot 1 \rangle. \end{aligned}$$

Proof.

$$\begin{aligned} \{F_1, F_2\}(g, \pi) &= \Omega(X_{F_1}(g, \pi), X_{F_2}(g, \pi)) = \\ &= - \int \langle X_{F_1}(g, \pi), J^{-1} \circ X_{F_2}(g, \pi) \rangle = \\ &= - \int \langle J \circ D\mathcal{F}_1(g, \pi)^* \cdot 1, J^{-1} \circ J \circ D\mathcal{F}_2(g, \pi)^* \cdot 1 \rangle = \\ &= - \int \langle D\mathcal{F}_1(g, \pi)^* \cdot 1, J^* \circ D\mathcal{F}_2(g, \pi)^* \cdot 1 \rangle = \\ &= \int \langle D\mathcal{F}_1(g, \pi)^* \cdot 1, J \circ D\mathcal{F}_2(g, \pi)^* \cdot 1 \rangle = \\ &= \int \left\langle (D_g \mathcal{F}_1(g, \pi)^* \cdot 1, D_\pi \mathcal{F}_2(g, \pi)^* \cdot 1), \begin{pmatrix} D_\pi \mathcal{F}_2(g, \pi)^* \cdot 1 \\ -D_g \mathcal{F}_2(g, \pi)^* \cdot 1 \end{pmatrix} \right\rangle = \\ &= \int \langle D_g \mathcal{F}_1(g, \pi)^* \cdot 1, D_\pi \mathcal{F}_2(g, \pi)^* \cdot 1 \rangle - \int \langle D_\pi \mathcal{F}_1(g, \pi)^* \cdot 1, D_g \mathcal{F}_2(g, \pi)^* \cdot 1 \rangle. \end{aligned}$$

Remark. According to the correspondences in sect. 2, this may be written in physics notation as

$$\{F_1, F_2\} = \int \left(\frac{\delta \mathcal{F}_1}{\delta g} \frac{\delta \mathcal{F}_2}{\delta \pi} - \frac{\delta \mathcal{F}_1}{\delta \pi} \frac{\delta \mathcal{F}_2}{\delta g} \right).$$

Now consider the case when $F_2 = \int N\mathcal{H} + X \cdot \mathcal{J}$. Then, from the above proof,

$$\begin{aligned} \{F, N\mathcal{H} + X \cdot \mathcal{J}\}(g, \pi) &= \int \langle D\mathcal{F}(g, \pi)^* \cdot 1, J \circ (DN\mathcal{H} + X \cdot \mathcal{J})^* \cdot 1 \rangle = \\ &= \int \left\langle D\mathcal{F}(g, \pi)^* \cdot 1, J \circ D\Phi(g, \pi)^* \cdot \begin{pmatrix} N \\ X \end{pmatrix} \right\rangle = \int D\mathcal{F}(g, \pi) \cdot \begin{pmatrix} \frac{\partial g}{\partial \lambda} \\ \frac{\partial \pi}{\partial \lambda} \end{pmatrix} = \\ &= \int \frac{d}{d\lambda} \mathcal{F}(g, \pi) = \frac{d}{d\lambda} F(g, \pi). \end{aligned}$$

What this means is the following. Given (g, π) and (N, X) , let $(N(\lambda), X(\lambda))$ be an arbitrary lapse and shift such that $N(0) = N, X(0) = X$. Let $(g(\lambda), \pi(\lambda))$ be the solution of the Einstein evolution equations with lapse and shift $(N(\lambda), X(\lambda))$ and initial data (g, π) . Let $F(\lambda) = F(g(\lambda), \pi(\lambda))$. Then

$$\frac{dF}{d\lambda}(g, \pi) = \frac{dF}{d\lambda}(\lambda).$$

Thus, as we expected, a Poisson bracket with the Hamiltonian $\int N\mathcal{H} + X \cdot \mathcal{J}$ generates λ -derivatives of $F(g(\lambda), \pi(\lambda))$, where $(g(\lambda), \pi(\lambda))$ is the flow with initial data (g, π) and lapse and shift $(N(\lambda), X(\lambda))$ such that $N(0) = N, X(0) = X$.

Now we consider the case when $F_1 = \int N_1\mathcal{H} + X_1 \cdot \mathcal{J}$, $F_2 = \int N_2\mathcal{H} + X_2 \cdot \mathcal{J}$. The next theorem computes Dirac's [2] canonical commutation relationships for general relativity. (See also [13, 35].)

II.3. *Theorem.* - Given $N_1, N_2 : M \rightarrow \mathbb{R}$, $X_1, X_2 : M \rightarrow TM$, and

$$F_1 = \int (N_1\mathcal{H} + X_1 \cdot \mathcal{J}) : T^*M \rightarrow \mathbb{R},$$

$$F_2 = \int (N_2\mathcal{H} + X_2 \cdot \mathcal{J}) : T^*M \rightarrow \mathbb{R},$$

then

$$\{F_1, F_2\} = \int (L_{X_1}N_2 - L_{X_2}N_1)\mathcal{H} + \int \langle (N_2 \operatorname{grad} N_1 - N_1 \operatorname{grad} N_2), \mathcal{J} \rangle + \int \langle L_{X_1}X_2, \mathcal{J} \rangle,$$

and, in particular,

$$\left\{ \int N_1\mathcal{H}, \int N_2\mathcal{H} \right\} = \int \langle N_2 \operatorname{grad} N_1 - N_1 \operatorname{grad} N_2, \mathcal{J} \rangle,$$

$$\left\{ \int N\mathcal{H}, \int X \cdot \mathcal{J} \right\} = \int (L_X N)\mathcal{H},$$

$$\left\{ \int X_1 \cdot \mathcal{J}, \int X_2 \cdot \mathcal{J} \right\} = \int \langle L_{X_1}X_2, \mathcal{J} \rangle.$$

Proof. By the remarks preceding the theorem,

$$\left\{ \int N_1 \mathcal{H} + X_1 \cdot \mathcal{J}, \int N_2 \mathcal{H} + X_2 \cdot \mathcal{J} \right\} = \frac{\partial}{\partial \lambda} \int (N_1 \mathcal{H} + X_1 \cdot \mathcal{J}) = \int \left(N_1 \frac{\partial \mathcal{H}}{\partial \lambda} + X_1 \cdot \frac{\partial \mathcal{J}}{\partial \lambda} \right),$$

where the λ -derivatives of \mathcal{H} and \mathcal{J} are computed with respect to a flow generated by (g, π) and (N_2, X_2) . Thus, using theorem 3.1, we have

$$\begin{aligned} \int N_1 \frac{\partial \mathcal{H}}{\partial \lambda} + X_1 \cdot \frac{\partial \mathcal{J}}{\partial \lambda} &= \\ &= \int N_1 \left(-L_{X_2} \mathcal{H} - \frac{1}{N_2} \operatorname{div} ((N_2)^2 \mathcal{J}) \right) + \int X_1 \cdot (-L_{X_2} \mathcal{J} - (dN_2) \mathcal{H}) = \\ &= \int (L_{X_1} N_1) \mathcal{H} + d \left(\frac{N_1}{N_2} \right) (N_2)^2 + \int \langle L_{X_1} X_1, \mathcal{J} \rangle - (L_{X_1} N_2) \mathcal{H} = \\ &= \int (L_{X_1} N_1 - L_{X_1} N_2) \mathcal{H} + \int \langle N_2 \operatorname{grad} N_1 - N_1 \operatorname{grad} N_2, \mathcal{J} \rangle + \int \langle L_{X_1} X_1, \mathcal{J} \rangle. \end{aligned}$$

By bilinearity and antisymmetry of the Poisson bracket,

$$\begin{aligned} \left\{ \int N_1 \mathcal{H} + X_1 \cdot \mathcal{J}, \int N_2 \mathcal{H} + X_2 \cdot \mathcal{J} \right\} &= \\ &= \left\{ \int N_1 \mathcal{H}, \int N_2 \mathcal{H} \right\} + \left\{ \int N_1 \mathcal{H}, \int X_2 \cdot \mathcal{J} \right\} - \left\{ \int N_2 \mathcal{H}, \int X_1 \cdot \mathcal{J} \right\} + \left\{ \int X_1 \cdot \mathcal{J}, \int X_2 \cdot \mathcal{J} \right\}. \end{aligned}$$

Comparing with the above, we identify

$$\begin{aligned} \left\{ \int N_1 \mathcal{H}, \int N_2 \mathcal{H} \right\} &= \int \langle N_2 \operatorname{grad} N_1 - N_1 \operatorname{grad} N_2, \mathcal{J} \rangle, \\ \left\{ \int N \mathcal{H}, \int X \cdot \mathcal{J} \right\} &= \int (L_X N) \mathcal{H}, \\ \left\{ \int X_1 \cdot \mathcal{J}, \int X_2 \cdot \mathcal{J} \right\} &= \int \langle L_{X_1} X_1, \mathcal{J} \rangle. \quad \square \end{aligned}$$

These later relations, Dirac's canonical commutation relations for general relativity, are thus equivalent to the evolution equations for \mathcal{H} and \mathcal{J} with a general lapse and shift function.

Let $i_0 \in E^\infty(M; V_*, {}^{(4)}g)$. Then using the normal ${}^{(4)}Z_{\Sigma_0}$ to the embedded hypersurface $\Sigma_0 = i_0(M)$, the lapse-shift decomposition gives a decomposition of the tangent space

$$T_{i_0} E^\infty(M; V_*, {}^{(4)}g) \approx C^\infty(M; \mathbb{R}) \times \mathcal{X}(M);$$

${}^{(4)}X_{\Sigma_0} \mapsto (N, X)$, the lapse and shift of ${}^{(4)}X_{\Sigma_0} = N {}^{(4)}Z_{\Sigma_0} - T_{i_0} \cdot X$.

Define a bracket structure on $C^\infty(M; \mathbb{R}) \times \mathcal{X}(M)$ as follows:

$$[(N_1, X_1), (N_2, X_2)] = (L_{X_1} N_2 - L_{X_2} N_1, N_2 \operatorname{grad} N_1 - N_1 \operatorname{grad} N_2 + L_{X_1} X_2).$$

With this bracket, $C^\infty(M; R) \times \mathcal{X}(M)$ is given the structure of an algebra, but it is not a Lie algebra. Through the gradient terms, this bracket depends on the metric g induced on the hypersurface, but it does not depend on the momentum π . Thus at different i_0 's the algebra structure changes.

KUCHAR [36] takes the point of view that this bracket structure on the space of embeddings can be understood in its own right, and that the canonical commutation relationships of general relativity are a representation (in fact, the unique representation when no external fields are present) of the «group» of embeddings $E^\infty(M; V_4, {}^{(4)}g)$.

KUCHAR has recently suggested enlarging the dynamical phase space to $T^*(\mathcal{M} \times E^\infty(M; V_4, {}^{(4)}g))$. This has the pleasant feature that now we have a group $\mathcal{Q}(V_4)$ which acts on this space (cf. [25], sect. 4, 7).

REFERENCES

- [1] P. A. M. DIRAC: *Can. Journ. Math.*, **2**, 129 (1950).
- [2] P. A. M. DIRAC: *Phys. Rev.*, **114**, 924 (1959).
- [3] R. ARNOWITT, S. DESER and C. W. MISNER: *The dynamics of general relativity*, in *Gravitation, and Introduction to Current Research*, edited by L. WITTEN (New York, N. Y., 1962), p. 227.
- [4] V. MONCRIEF: *Journ. Math. Phys.*, **16**, 1556 (1975).
- [5] A. FISCHER and J. MARSDEN: *Bull. Amer. Math. Soc.*, **79**, 995 (1973).
- [6] A. FISCHER and J. MARSDEN: *Linearization stability of nonlinear partial differential equations*, in *Proceedings of the Symposium on Pure Mathematics AMS*, Vol. **27**, Part 2 (1975), p. 219.
- [7] V. MONCRIEF: *Journ. Math. Phys.*, **16**, 493 (1975).
- [8] S. DESER: *Ann. Inst. H. Poincaré*, **7**, 149 (1967).
- [9] J. ARNS, A. FISCHER and J. MARSDEN: *Compt. Rend.*, **281**, 517 (1975).
- [10] A. FISCHER and J. MARSDEN: *The space of gravitational degrees of freedom* (in preparation).
- [11] J. W. YORK: *Phys. Rev. Lett.*, **28**, 1082 (1972).
- [12] J. W. YORK: *Journ. Math. Phys.*, **14**, 456 (1973).
- [13] L. D. FADEEV: *Symplectic structure and quantization of the Einstein gravitation theory*, in *Actes du Congrès International de Mathématique*, Vol. **3** (1970), p. 35.
- [14] J. MARSDEN and A. WEINSTEIN: *Rep. Math. Phys.*, **5**, 121 (1974).
- [15] R. PALAIS: *Seminar on the Atiyah-Singer Index Theorem* (Princeton, N. J., 1965).
- [16] R. PALAIS: *Foundations of Global Nonlinear Analysis* (New York, N. Y., 1968).
- [17] A. FRIEDMAN: *Partial Differential Equations* (New York, N. Y., 1969).
- [18] M. BERGER and D. EBIN: *Journ. Diff. Geom.*, **3**, 379 (1969).
- [19] A. FISCHER and J. MARSDEN: *Duke Math. Journ.*, **42**, 519 (1975).
- [20] A. LICHNEROWICZ: *Propagateurs et commutateurs en relativité générale*, in *Publ. Scientifiques, IHES*, Vol. **10** (1961), p. 293.
- [21] C. W. MISNER, K. THORNE and J. A. WHEELER: *Gravitation* (San Francisco, Cal., 1974).
- [22] D. EBIN: *On the space of Riemannian metrics*, in *Symposium on Pure Mathematics, Amer. Math. Soc.*, **15** (1970), p. 11.
- [23] Y. CHOQUET-BRUHAT, A. E. FISCHER and J. E. MARSDEN: this volume, p. 396.
- [24] D. EBIN and J. MARSDEN: *Ann. of Math.*, **92**, 102 (1970).

- [25] A. FISCHER and J. MARSDEN: *Journ. Math. Phys.*, **13**, 546 (1972).
- [26] J. A. WHEELER: *Geometrodynamics and the issue of the finite state*, in *Relativity, Groups and Topology*, edited by B. DEWITT and C. DEWITT (New York, N. Y., 1964).
- [27] K. KUČAŘ: *Journ. Math. Phys.*, **13**, 768 (1972); **15**, 708 (1974).
- [28] A. FISCHER and J. MARSDEN: *Gen. Rel. Grav.*, **7**, 915 (1976).
- [29] P. A. M. DIRAC: *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Monograph Series, No. 2, Yeshiva University, N. Y. (1964).
- [30] Y. CHOQUET-BRUHAT: *Acta Math.*, **88**, 141 (1952).
- [31] Y. CHOQUET-BRUHAT: *Cauchy problem*, in *Gravitation, an Introduction to Current Research*, edited by L. WITTEN (New York, N. Y., 1962).
- [32] A. FISCHER and J. MARSDEN: *Comm. Math. Phys.*, **28**, 1 (1972).
- [33] T. HUGHES, T. KATO and J. MARSDEN: *Arch. Rat. Mech. Anal.*, **63**, 273 (1977).
- [34] D. CHRISTODOULOU and F. FRANCAVIGLIA: this volume, p. 480.
- [35] B. DEWITT: *Phys. Rev.*, **160**, 1113 (1967).
- [36] K. KUČAŘ: *Journ. Math. Phys.*, **17**, 777 (I), 792 (II), 801 (III) (1976); **18**, 1589 (IV) (1976).
- [37] A. E. FISCHER and J. E. MARSDEN: *Hamiltonian field theories on space-time* (in preparation).
- [38] T. REGGE and C. TEITELBOIM: *Ann. of Phys.*, **88**, 286 (1974).
- [39] A. TAUB: *Variational Principles in General Relativity* (Bressanone, 1970), p. 206.
- [40] Y. CHOQUET-BRUHAT and R. GEROCH: *Comm. Math. Phys.*, **14**, 329 (1969).
- [41] S. W. HAWKING and G. F. R. ELLIS: *The Large Scale Structure of Space-Time* (Cambridge, 1973).
- [42] D. BRILL and S. DESER: *Comm. Math. Phys.*, **32**, 291 (1973).
- [43] Y. CHOQUET-BRUHAT and S. DESER: *Ann. of Phys.*, **81**, 165 (1973); *Compt. Rend.*, **275**, 1019 (1972).
- [44] N. O'MURCHADHA and J. W. YORK: *Phys. Rev. D*, **10**, 437 (1974).
- [45] J. P. BOURGUIGNON, D. EBIN and J. MARSDEN: *Compt. Rend.*, **282**, 867 (1976).
- [46] J. M. ARMS and J. E. MARSDEN: *The absence of Killing fields is necessary for linearization stability of Einstein's equations* (preprint).
- [47] N. O'MURCHADHA and J. W. YORK: *Journ. Math. Phys.*, **14**, 1551 (1973).
- [48] N. O'MURCHADHA and J. W. YORK: *Phys. Rev. D*, **10**, 428 (1974).
- [49] H. ARAKI: *Ann. of Phys.*, **7**, 456 (1959).
- [50] J. BEREZDEVIN: *Journ. Math. Phys.*, **15**, 1963 (1974).
- [51] B. COLL: *Compt. Rend.*, **282**, 247 (1976); *Ann. Inst. H. Poincaré*, **25**, 393 (1976); *Journ. Math. Phys.*, **18**, 1918 (1977).
- [52] A. FISCHER, J. MARSDEN and V. MONCRIEF: *Linearization stability of the Einstein equations* (in preparation).
- [53] V. MONCRIEF: *Journ. Math. Phys.*, **17**, 1893 (1976).
- [54] A. TAUB: *Comm. Math. Phys.*, **15**, 235 (1969).
- [55] J. W. YORK: *Ann. Inst. H. Poincaré*, **21**, 319 (1974); *Phys. Rev. Lett.*, **26**, 1656 (1971); *Journ. Math. Phys.*, **14**, 456 (1973).
- [56] L. HORMANDER: *Ann. of Math.*, **83**, 129 (1966).
- [57] A. FISCHER and J. MARSDEN: *Can. Journ. Math.*, **29**, 193 (1977).
- [58] C. BARBANCE: *Compt. Rend.*, **258**, 5336 (1964) (also **264**, 515 (1967)).
- [59] A. LICHNEROWICZ: *Géométrie des groupes de transformation* (Paris, 1958).
- [60] P. CHERNOFF and J. MARSDEN: *Properties of Infinite Dimensional Hamiltonian Systems*, Springer Lecture Notes Series, No. 425 (1974).
- [61] J. MARSDEN: *Application of Global Analysis in Mathematical Physics* (Boston, Mass., 1974).
- [62] J. M. SOURIAU: *Structure des systèmes dynamiques* (Paris, 1970).

- [63] P. G. BERGMANN: *Phase integrals and Poincaré invariants*, in *Magic Without Magic*, edited by J. R. KLAUDER (San Francisco, Cal., 1972).
- [64] B. DEWITT: *Covariant quantum geometrodynamics*, in *Magic Without Magic*, edited by J. R. KLAUDER (San Francisco, Cal., 1972).
- [65] I. SEGAL: *Journ. Math. Pures Appl.*, **44**, 71 (1965).
- [66] I. SEGAL: *Symplectic structures and the quantization problem for wave equations*, in *Symposia Mathematics*, Vol. 14 (New York, N. Y., 1974), p. 99.
- [67] J. ARNS: *Journ. Math. Phys.*, **18**, 830 (1977).
- [68] P. D'EATH: *Three Perturbation Problems in General Relativity* (Cambridge, 1974).
- [69] P. D'EATH: *Ann. of Phys.*, **98**, 237 (1976).
- [70] R. ABRAHAM and J. MARSDEN: *Foundations of Mechanics*, second edition (Reading, Mass., 1978).
- [71] J. KIJOWSKI and W. SZCZYRBA: *Comm. Math. Phys.*, **46**, 183 (1976).
- [72] W. SZCZYRBA: *On the geometric structure of the set of solutions of Einstein equations. Dissertation Math. (Rozprawy Mat.)*, **150** (1977).