QUALITATIVE TECHNIQUES FOR BIFURCATION ANALYSIS OF COMPLEX SYSTEMS*

Philip Holmes

Department of Theoretical and Applied Mechanics
Cornell University
Ithaca, New York 14853

Jerrold E. Marsden

Department of Mathematics
University of California
Berkeley, California 94720

INTRODUCTION

In this paper we consider systems whose dynamical behavior may be represented by an autonomous ordinary differential equation (ODE) with parameters,

$$\frac{dx}{dt} = A_\mu x + B(x) \equiv G_\mu(x); \quad x(0) = x_0 \tag{1}$$

Here x is an element of a finite-dimensional vector space (say $\mathbf{R}^n$) or of a suitable Banach space of functions. In the latter case, (1) represents a partial differential equation (PDE). The control parameter $\mu \in \mathbf{R}^m$ is supposed to vary slowly in comparison with the evolution rate of a typical solution $x(t)$ of (1). Thus we treat (1) as an m-parameter family of ODE's. We are primarily interested in studying the qualitative changes that occur in the vector field or (semi) flow defined by (1) as μ varies.

The techniques used in the study of (1) draw on several fields, notably those of functional analysis and differentiable topology. In this brief paper we are only able to sketch general ideas and must therefore refer the reader to texts such as Chillingworth,[5] and Marsden and McCracken[21] for background information and further details. Both texts contain a wealth of additional references.

The general problem of bifurcation of vector fields—the qualitative study of equations, such as (1)—contains as an important subproblem, the study of bifurcations of equilibria, or stationary solutions. Much of the work done so far in bifurcation theory has been addressed specifically to the latter problem. The usual definitions of a bifurcation point are couched with this in mind. Since we wish to study a more general class of problems, and, in particular, to consider the case of *global bifurcations,* we propose an alternative definition, which is a slight modification of the definitions due to Smale and Thom. First we review the usual definition.

Consider a map $F: X \times \Lambda \to Y$, where X, Λ and Y are Banach spaces, and Λ is the parameter space. Set $F(x, \lambda) = 0$ and seek the solutions. Let $x(\lambda)$ be

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a curve of known solutions; then we say that (x_0, λ_0) is a bifurcation point if, in any neighborhood of (x_0, λ_0), there is another solution $x_1(\lambda) \neq x(\lambda)$. However, consider FIGURE 1, where we have a 2-dimensional sheet Σ of solutions. According to the above definition, every point of Σ is bifurcation point, whereas we would like to distinguish the true bifurcation point (σ_0, λ_0), where a distinct curve of *new* solutions appears.

DEFINITION 1 (local bifurcation). Let $h:M \to N$ be a continuous map between topological spaces. A point $x_0 \in M$ is a bifurcation point for h if for every neighborhood U of $h(x_0)$ and V of x_0, the sets $h^{-1}(y) \cap V$ for $y \cap U$ are not all homeomorphic; i.e., $h^{-1}(y)$ changes topological type at x_0.

There is a corresponding global version, where the "whole space" $\cap h^{-1}(y)$ replaces $h^{-1}(y) \cap V$.

This can be related to the definition for an equation of the form $F(x, \lambda) = 0$, $x \in X, \lambda \in \Lambda$, by letting $\Sigma \subset X \times \Lambda$ be the set of solutions and $h: \Sigma \to \Lambda$ the

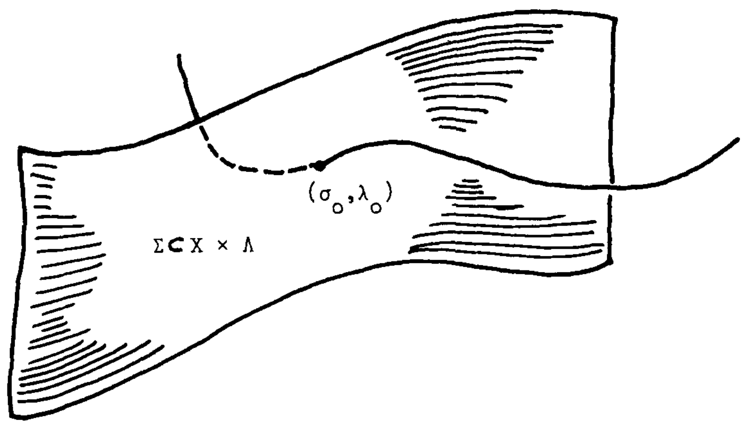


FIGURE 1. Bifurcation at a point on a sheet of solutions.

projection map onto the parameter space. "Parameter-free" DEFINITION 1 has advantages in certain situations, such as in the work of Buchner, Schecter, and Marsden[3] on scalar-curvature equations.

The condition, "same topological type," can be replaced by other relations according to the situation. For instance, if we have a set of vector fields and let the relation be, "have conjugate phase portraits," we recover Thom's definition of a bifurcation point of a family of vector fields as a member of the family which is not *structurally stable*, relative to the family cf. Ref. 5, pp. 228–9). It is noteworthy that Andronov and Pontryagin defined and discussed structural stability as early as 1937.[1]

The next two sections contain a general approach to systems represented by Equation 1. The fourth and fifth sections consider specific examples arising in engineering, which enable us to illustrate some of the relevant concepts.

Existence and Uniqueness of Solutions and Bifurcations of Fixed Points

In case Equation 1 is an ODE on $\mathbf{R}^n$, the establishment of local existence and uniqueness theorems is frequently a trivial matter, provided the nonlinear function $B(x)$ has reasonable properties. The proof of *global* existence is generally not so simple, and may require the use of Liapunov functions or related energy methods. However, a proof of local existence suffices for application of the center manifold theorem (THEOREM 2). When (1) is defined on a function space, existence-uniqueness results are often considerably more difficult to obtain, but in many cases may be obtained by use of this theorem originally due to Segal[26] (see Holmes and Marsden[18] for further details and proof).

THEOREM 1. Let X be a Banach space and let A, with domain $D(A)$ be the generator of a C^0 (linear) semigroup $U_t = e^{tA}$ on X. Let $B:X \to X$ be of class C^k, $k \geq 1$, and let $G = A + B$, $D(G) = D(A)$. Then there is a unique local semiflow $F_t(x) = F(t, x)$ defined on an open set of $[0, \infty) \times X$ containing $\{0\} \times X$ such that for $x_0 \in D(A)$, $F_t(x_0) \in D(A)$ and $F_t(x_0)$ is the unique solution of,

$$\frac{dx}{dt} = G(x)$$

$$x(0) = x_0 \in D(G)$$

REMARK. In the terminology of Marsden and McCracken, G generates a *smooth semiflow*.[21] The proof shows that if G_μ depends continuously on a parameter μ (with domain fixed), then so does its semiflow $F_t^\mu:X \to X$. From Chernoff and Marsden we note that separate continuity implies joint continuity.[4] As we shall see, a certain amount of smoothness (say $C^k, k \geq 3$) is required for application of the center manifold theorem and for subsequent bifurcation analysis.

Checking the hypotheses of THEOREM 1 in specific cases is generally a lengthy process (for equations of panel and pipe flutter, see Holmes and Marsden.[17,18] We obtain global existence results with a modification of Liapunov's second method:[18]

PROPOSITION 1. Suppose the conditions of THEOREM 1 hold, and there is a C^1 function $H:X \to \mathbf{R}$ such that:

(i) There is a monotone increasing function, $\phi:[x, \infty) \to [0, \infty)$, where $[a, \infty) \supset$ range of H satisfying $\|x\| < \phi[H(x)]$;

(ii) There is a constant $K \geq 0$ such that if $x(t)$ satisfies Equation 1, then

$$\frac{d}{dt} H[x(t)] \leq KH[x(t)]$$

Then $F_t(x_0)$ is defined for all $t \geq 0$ and $x_0 \in X$. If, in addition, H is bounded on bounded sets and,

(iii) $\dfrac{d}{dt} H[x(t)] \leq 0, \quad \text{if} \quad \|x(t)\| \geq B$

then any solution of (1) remains uniformly bounded in X for all time (i.e., given x_0), there is a constant $C = C(x_0)$ such that $\|x(t)\| \leq C$ for all $t \geq 0$.

If H decreases along solution curves of (1) on all of X and $H(0) = 0$ is the global minimum of H, then we can conclude that $x = 0$ is globally stable and that no bifurcations are possible while this condition holds.

In the succeeding discussion we will assume that the bifurcations to be studied occur from a curve of known solutions which, prior to bifurcation, are hyperbolic sinks and thus *locally* stable. We will further assume that the eigenvalues of (1), linearized at such a sink, can be calculated directly or at least estimated by numerical computations based on a finite-dimensional model.[14,17] Specifically, letting $\overline{x}(\mu)$ denote the sink, we consider the system,

$$\frac{dx}{dt} = DG_\mu(\overline{x}(\mu))x \tag{2}$$

and compute the spectrum $\sigma\{DG_\mu[\overline{x}(\mu)]\}$ of the (Fréchet) derivative of G_μ at $\overline{x}(\mu)$. If the eigenvalues all lie strictly in the left-hand half plane, then $\overline{x}(\mu)$ is a hyperbolic sink. Bifurcation from $\overline{x}(\mu)$ occurs at some parameter value μ_0 when $\sigma\{DG_{\mu_0}[x(\mu_0)]\}$ has at least one eigenvalue on the imaginary axis. If this eigenvalue passes into the right-hand half plane as μ changes, then a bifurcation occurs in which $\overline{x}(\mu)$ becomes a repelling fixed point (either a source or a saddle). In this situation the classical bifurcation theorems, such as that of Hopf[21] can be applied to determine the nature of the secondary solutions bifurcation from $\overline{x}(\mu)$ at $\overline{x}(\mu_0)$. For example, closed orbits and/or additional fixed points may appear near $\overline{x}(\mu)$ for $\mu > \mu_0$.

However, in general, only a relatively small number of eigenvalues cross the imaginary axis simultaneously at $\mu = \mu_0$, thus $\overline{x}(\mu)$ "loses stability" in relatively few directions. In the case of an ODE on a Banach space of functions, for example, suppose a finite number d of eigenvalues crosses the imaginary axis as μ passes through μ_0. For $\mu > \mu_0$, then (say $\mu = \mu_0 + \epsilon$, ϵ small) the *unstable manifold* $W^u[\overline{x}(\mu)]$ of $\overline{x}(\mu)$ is of dimension d and the *stable manifold* $W^s[\overline{x}(\mu)]$, of codimension d. At least in some neighborhood of $\overline{x}(\mu_0)$ and μ_0 in $X \times \Lambda$ the new solutions can be studied by restricting our attention to a d-dimensional submanifold of X. If d is small (say $d = 1, 2$) then this dramatically reduces the complexity of the problem, and, in specific cases, enables us to obtain a complete characterization of local bifurcational behavior. To formalize this notion we use invariant manifold methods.

Invariant Manifolds and the Center Manifold Theorem

We have already mentioned the stable and unstable manifolds of a fixed point in the second section.

Definition 2. The local stable manifold of a fixed point $\overline{x} \in X$ is the set of points y in some neighborhood U of $\overline{x}$ which approach $\overline{x}$ under the flow F_t as $t \to +\infty$; thus,

$$W_{\text{loc}}^s(\overline{x}) = \{y \in U \mid F_t(y) \to \overline{x},\ t \to +\infty\}$$

The unstable manifold is obtained by reversing time in the above definition; hence,

$$W_{\text{loc}}^u(\overline{x}) = \{y \in U \mid F_t(y) \to \overline{x} \text{ as } t \to -\infty\}$$

These definitions can be globalized by taking the unions of the local manifolds over all time (cf. Ref. 5, pp. 215–221). Clearly $W_{loc}^{u}(\bar{x})$ and $W_{loc}^{s}(\bar{x})$ describe the local splitting of the "phase space" X induced by the flow F_t of (1), or by the semiflow F_t in the case of the PDE. The definitions generalize to more complex invariant sets, such as closed orbits, etc.[5,21]

When the fixed point $\bar{x}(\mu)$ is structurally unstable and $\mu = \mu_0$ is a bifurcation value then we can define a third local submanifold, the *center manifold* $M(\bar{x})$. Just as the stable and unstable manifolds are associated with (and tangent to) the eigenspaces of those eigenvalues of $DG_{\mu_0}[\bar{x}(\mu_0)]$ with negative and positive real parts, respectively, so $M(\bar{x}(\mu_0))$ is associated with those eigenvalues with zero real parts. The center manifold theorem may be stated for ODE's or for semiflows; here we give the latter version.[21] Without loss of generality we take $\bar{x}(\mu_0) = 0 \in X$.

THEOREM 2 (Center manifold theorem for flows). Let X be a Banach space admitting a C^∞ norm away from 0, and let F_t be a C^0 semiflow defined in a neighborhood of 0 for $0 \leq t \leq T$. Assume $F_t(0) = 0$ and that, for $t > 0$, $F_t(x)$ is C^{k+1} jointly in t and x. Assume also that the spectrum of the linear semigroup $DF_t(0):X \to X$ is of the form $\exp[t(\sigma_1 \cup \sigma_2)]$, where $\exp(t\sigma_1)$ lies on the unit circle—i.e., $\text{Re}(\sigma_1) = 0$—and $\exp(t\sigma_2)$ lies inside the unit circle a nonzero distance from it, for $t > 0$; i.e., $\text{Re}(\sigma_2) < 0$. Let Y be the generalized eigenspace corresponding to $\exp(t\sigma_1)$, and assume $\dim Y = d < \infty$. Then there exists a neighborhood V of 0 in X and a C^k submanifold $M \subset V$ of dimension d passing through 0 and tangent to Y at 0 such that:

(a) If $x \in M$, $t > 0$ and $F_t(x) \in V$, then $F_t(x) \in M$ (local invariance);
(b) If $t > 0$ and $F_t(x)$ remains defined and in V for all t, then $F_t(x) \to M$ as $t \to \infty$ (local attractivity).

REMARK. If F_t is C^∞ then M can be chosen so as to be C^l for any $l < \infty$. For the semigroup $F_t^{\mu}(x)$ with control parameter $\mu \in \mathbf{R}^m$, if $F_t^{\mu}(x)$ is only assumed to be C^{k+1} in x; and its x-derivatives depend continuously on t and μ, and at $\mu = \mu_0$ part of the spectrum of $DF_t^{\mu_0}(0)$ is on the unit circle, as above, then for μ near μ_0 we can choose a family of C^k invariant manifolds M_μ depending continuously on μ. *This family completely captures the bifurcational behavior locally.*

We note that Henry[13] has a version of the theorem to cover the case in which the spectrum of $DF_t(0)$ also has a component $\exp(t\sigma_3)$ comprising a *finite* number of eigenvalues *outside* the unit circle; i.e., $\text{Re}(\sigma_3) > 0$. Thus, in addition to M, we also have invariant stable and unstable manifolds W^s, W^u, the dimensions of which are determined by the number of eigenvalues within and outside the unit circle; thus $\dim W^u < \infty$. The theorem now provides a full infinite-dimensional analog of that for ODE's in $\mathbf{R}^n$.[20] However, in this case we need a further result derived from the generalized Böchner-Montgomery theorem:

PROPOSITION 2. Let F_t be a local C^k semiflow on a Banach manifold $\tilde{M}$ and suppose F_t leaves invariant a finite-dimensional submanifold $M \subset M$. Then on M, F_t is locally reversible jointly C^k in t and x, and is generated by a C^{k-1} vector field on M.[21]

THEOREM 2 and PROPOSITION 2 imply that, under their assumptions, we can find a $(d + m)$-dimensional subsystem $M \times U$, where U is a neighborhood of the critical parameter value $\mu = \mu_0$ such that $M \times U$ provides a local, finite-dimensional *essential model.* More details on the concept of essential models can be found in Holmes and Rand[19], where a general approach to the identification of nonlinear systems is suggested based on the concept of structural stability and the assumption of generic properties.

Since the new fixed points, closed orbits and other invariant sets of F_t^μ created in the bifurcation all lie in the center manifold for μ near μ_0, their structure may be considerably easier to analyze than would otherwise be the case. Of course, the global problem of relating these invariant sets to other invariant sets, perhaps created in other bifurcations, still remains, but, as we shall see in the next section, considerable progress can be made by use of the essential model concept.

An Application to Panel Flutter

The problem of panel flutter has been extensively studied by Dowell[6,7] who has used numerical time-marching methods to solve a finite-dimensional Galerkin approximation to the governing PDE. In this way a *partial* picture of the behavior was obtained, in that only the attracting invariant sets such as sinks and attracting limit cycles were found. Moreover, the convergence of a finite-dimensional Galerkin system to the full PDE is tacitly assumed. The method proposed here is, we feel, complementary to such techniques in that, while it lacks the quantitative accuracy of numerical methods, it provides a fuller description of qualitative behavior and, in particular, of the surprisingly rich bifurcational behavior of the panel. Since the work has been extensively reported elsewhere, we merely outline the main results.[14,17,18]

The equation of motion of a thin panel of length l, fixed at both ends and undergoing cylindrical bending, can be written in terms of lateral deflection $v = v(z, t)$ as,

$$\alpha \dot{v}'''' + v'''' - \{\Gamma + k\,|\,v'\,|^2 + \sigma(v', \dot{v}')^2\}v'' + \rho v' + \sqrt{\rho}\,\delta\dot{v} + \ddot{v} = 0 \qquad (3)$$

Here $(\dot{\ }) = \partial/\partial t$, $(') = \partial/\partial z$ and $|\cdot|$ and $(\cdot,\cdot)$ denote the L_2 norm and inner product, respectively; $\alpha, k, \sigma, \delta > 0$ are fixed mechanical parameters and only the axial load; Γ and the dynamic pressure of the fluid flowing over the panel ρ vary. We collect these in a control parameter $\mu = (\rho, \Gamma)$. Note the strong symmetry of (3) due to the absence of even-order nonlinear terms. We prove the existence, uniqueness, and smoothness of the semiflow generated by (3) in Reference 18.

Working with fixed points and eigenvalues calculated from a finite-dimensional (four mode) approximation,[14] we obtain the bifurcation set of FIGURE 2, in which the bifurcation curves for fixed points only are shown. On B_s, a pair of sinks bifurcate off the trivial solution $x = \{v, \dot{v}\} = \{0, 0\} = \{0\} \in X$ and a Hopf bifurcation occurs on B_{h_1} in which an *attracting* closed orbit is created. The stability and "direction" of the family of orbits created on B_{h_1} can be checked by use of the "$V'''(\cdot)$ algorithm" described by Marsden and McCracken.†[21] The two non-

†B. Hassard (SUNY Buffalo) reports that he has carried out computations for 4, 6, 8, and 10 mode models using a version of the stability formula due to himself and Y.-H. Wan. He detected a strong convergence as the order of the approximation increases. The results reported and summarized here are qualitatively correct, although possibly in error by $\simeq 5\%$ in quantitative accuracy.[14,17]

trivial fixed points $\bar{x} = \{v, \dot{v}\} = \{\pm\bar{v}, 0\}$ undergo simultaneous Hopf bifurcations on B_{h_2}. Here the bifurcations are *subcritical* so that *repelling* orbits encircle the sinks for parameter values "below" B_{h_2}. We thus have a partial picture of bifurcational behavior near the point $O = \mu_0$, where B_{h_1}, B_{h_2}, and B_s meet.

To complete the picture we note that at O, $\mu_0 = (\Gamma_0, \rho_0)$, $A_\mu : X \to X$ has a zero eigenvalue with multiplicity two, and that near $\{0\} \in X$ and the point $\mu_0 = (\Gamma_0, \rho_0) \in \mathbf{R}^2$ we can define a 2-dimensional center manifold M and a 4-dimensional essential model $M \times U$, where $U \subset \mathbf{R}^2$ is a neighborhood of μ_0. All the nontrivial invariant sets created as μ crosses $B_{h_1} \cup \mu_0 \cup B_s$ from left to right lie on $M \subset X$. Thus the problem reduces to that of completing the bifurcation picture

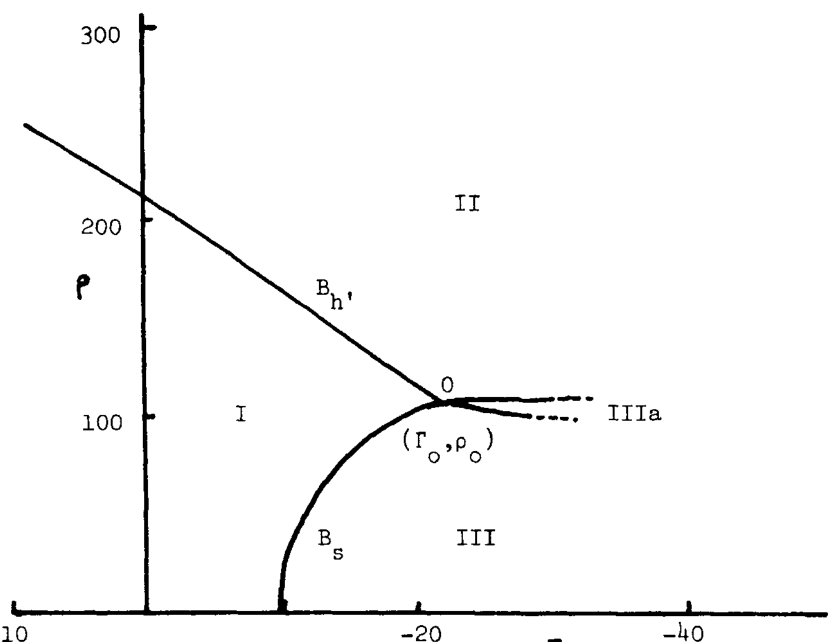


FIGURE 2. A partial bifurcation set for panel flutter.

for a two-parameter, 2-dimensional vector field. In particular, the degenerate singularity occurring at $\{0\} \in M$ for $\mu = \mu_0$ contains our information in its *versal unfolding.*[2]

We now make the key assumption that the bifurcation at $\{0\} \times \mu_0 \in M \times U$ is *generic* in the sense that, under the symmetry group acting in (3) it is the "simplest" degenerate singularity of a 2-dimensional vector field with double zero eigenvalues. We then use the classification of such *codimension-2* singularities, due to Takens,[27] and select the singularity whose unfolding contains the phase portraits we have already detected, specifically, a sink in region I of FIGURE 2, a source and limit cycle in region II, two sinks, and a saddle in region III (near B_s), etc. This is the "$m = 2$; $-$" normal form. We therefore have:

CONJECTURE 1. Flutter and divergence near $\{0\} \times (\rho_0, \Gamma_0) \in X \times \mathbf{R}^2$ can be modeled by a two-parameter vector field V_μ on a 2-manifold M, where V_μ is diffeomorphic to Takens' "$m = 2;-$" normal form (FIGURE 3). The actual vector fields portrayed in FIGURE 3a belong to the nonlinear oscillator $\ddot{x} + \nu_1 \dot{x} + \nu_2 x + x^2 \dot{x} + x^3 = 0$, which is diffeomorphic to the more complicated form of Takens.[27] It is interesting to note that the essential model is in a sense a *nonlinear normal mode.*

Note that the double *saddle connection* occurring as μ crosses the curve B_{sc} from region IIIb to IIIc is an example of a *global bifurcation* in which the phase portrait changes topological type without local bifurcations of fixed points or closed orbits occurring. The period of the two repelling closed orbits existing in region IIIb tends to infinity and the orbits reappear as a single repelling orbit in region IIIc, whose period decreases from infinity and approaches that of the attracting orbit until the two orbits coalesce and annihilate each other as μ crosses B_{lc}.[17] It is interesting to note that the use of Takens' "$m = 2;-$" normal form as an essential model for panel flutter was proposed by Holmes and Rand[19] purely on the basis of "generic" arguments, without detailed knowledge of the PDE's governing panel motions.

FORCED NONPERIODIC OSCILLATIONS

As a final example we outline some recent work of the first author on the forced oscillations of a system possessing three equilibria, two sinks, and a saddle (the actual system studied is the second-order ODE); hence,

$$\ddot{x} + \delta\dot{x} - \beta x + \alpha x^3 = f \cos \omega t; \ \alpha, \delta, \beta, \omega > 0 \text{ fixed}, \ f \geq 0, \text{ varies} \qquad (4)$$

A detailed preliminary study of Equation 4 has been completed and will appear in due course.[16] Here we merely give the main results.

Although (4) is an ODE, we believe it can be derived from the PDE for the oscillations of a buckled column under transverse sinusoidal loading by invariant manifold techniques. The PDE, in nondimensional form ($v = v(z, t)$ again represents transverse deflections), is

$$v'''' + \Gamma v'' + k\,|v'|^2 v'' + \delta\dot{v} + \ddot{v} = f(z) \cos \omega t, \qquad (5)$$

where k and δ are structural constants and $\Gamma > \pi^2$ is the fixed axial end load. Certainly a study of the behavior of (4) is necessary before the full problem (5) is tackled.

Holmes[16] proves that (4) is globally stable in the sense that after sufficient time all solution curves enter and remain within a bounded set A in the state space, thus (4) always has at least one attractor. We rewrite (4) as an autonomous system on $\mathbf{R}^2 \times S^1$, as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \beta x_1 - \delta x_2 - \alpha x_1^3 + f \cos \theta \\ \dot{\theta} &= \omega \end{aligned} \qquad (6)$$

and consider the *Poincaré map* $P_f : \Sigma \to \Sigma$ induced by the flow $\phi_t : \mathbf{R}^2 \times S_1 \to$

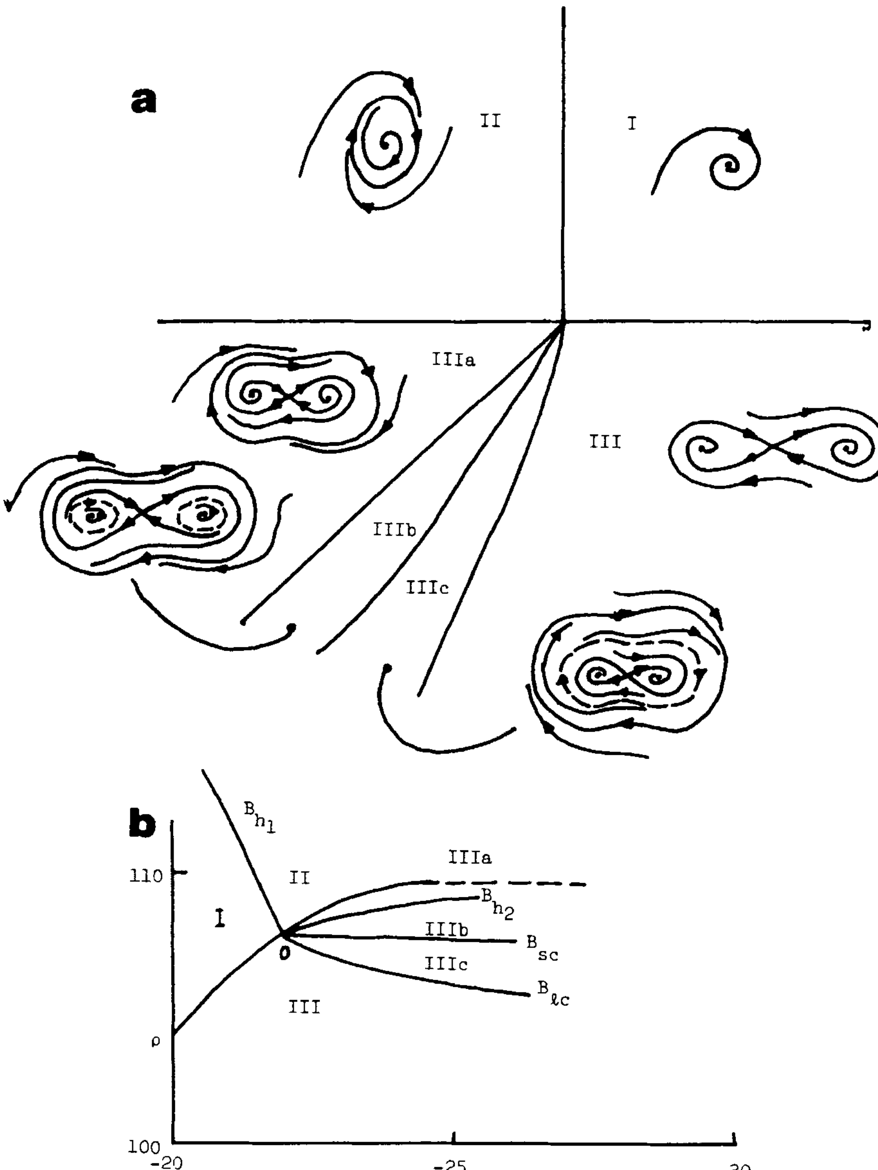


FIGURE 3. A local model for panel flutter near $\{0\} \times (\mu_0, \Lambda_0) \in X \times \mathbf{R}^2$. (a) Takens' "$m = 2;-$" normal form and associated structurally stable vector fields. (b) The completed bifurcation set.

$\mathbf{R}^2 \times S^1$ of (6), where Σ is a global cross section $\Sigma = \{(x, y, \theta) \in \mathbf{R}^2 \times S^1 \mid \theta = 0, 2\pi/\omega, \ldots\}$; P_f is the time $2\pi/\omega$ or period-1 Poincaré map.

First consider the trivial system for $f \equiv 0$. Here all cross sections of (6) are identical for all $\theta \in [0, 2\pi/\omega]$. The Poincaré map thus has a structure identical to that of the vector field of the autonomous 2-dimensional system,

$$\dot{x}_1 = x_2, \dot{x}_2 = \beta x_1 - \delta x_2 - \alpha x_1^3 \tag{7}$$

in the sense that the stable and unstable *manifolds* of the saddle (0, 0) of the Poincaré map are curves identical to the stable and unstable *seperatrices* of (0, 0) for the vector field. However, we must recall that an *orbit* of the map is a sequence of *points* and not a curve, as in the case of vector field.[5] It is easy to check that the vector field of (7), and hence the Poincaré map of (6) for $f = 0$, is (globally) *structurally stable.* We can conclude that for $f \neq 0$, small, the topological type will be identical to that for $f = 0$. This is confirmed by analog computer analysis

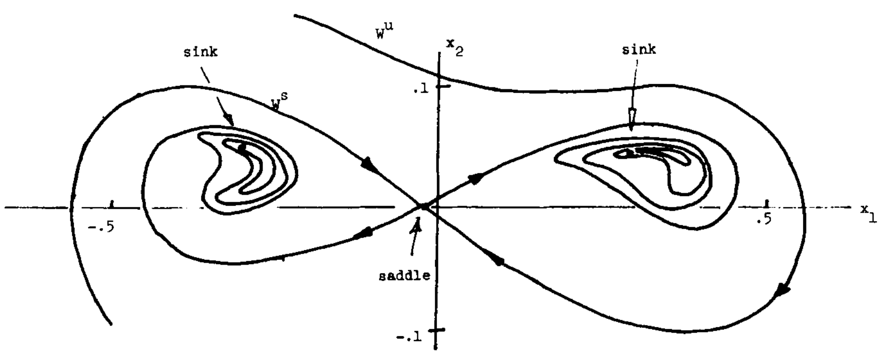


FIGURE 4. Stable and unstable manifolds of the saddle point of P_f for $f = 0.2$ ($\alpha = 1$, $\beta = 10$, $\delta = 100$, $\omega = 3.76$).

(FIGURE 4).[16] Thus, for small f, P_f has a hyperbolic saddle and two hyperbolic sinks, corresponding to the two attracting and one repelling closed orbits of (6).

As f increases it is possible to prove, using the methods of Melnikov[22], that the stable and unstable manifolds of the saddle point approach and ultimately intersect, giving rise to infinitely many homoclinic points. We note that this proof apparently has much in common with recent work of Hale[11] and others. The critical value $f = f_c(\alpha, \delta, \beta, \omega)$— ≈ 0.79 for the case studied—thus computed agrees well with that found in analog computations. FIGURE 5 shows the dispositions of stable and unstable manifolds W^s and W^u just before and after intersection takes place at $f \approx 0.76$. Note that the presence of the period-1 sinks in FIGURE 5b implies that almost all orbits converge to either one of these fixed points of P_f as $t \to \infty$. However, since P_{f_c} has tangencies of W^s and W^u, according to a theorem proved by Newhouse,[23,24] there is an open set of diffeomorphisms near P_{f_c} where each possesses an infinite number of periodic sinks. In addition to the creation of

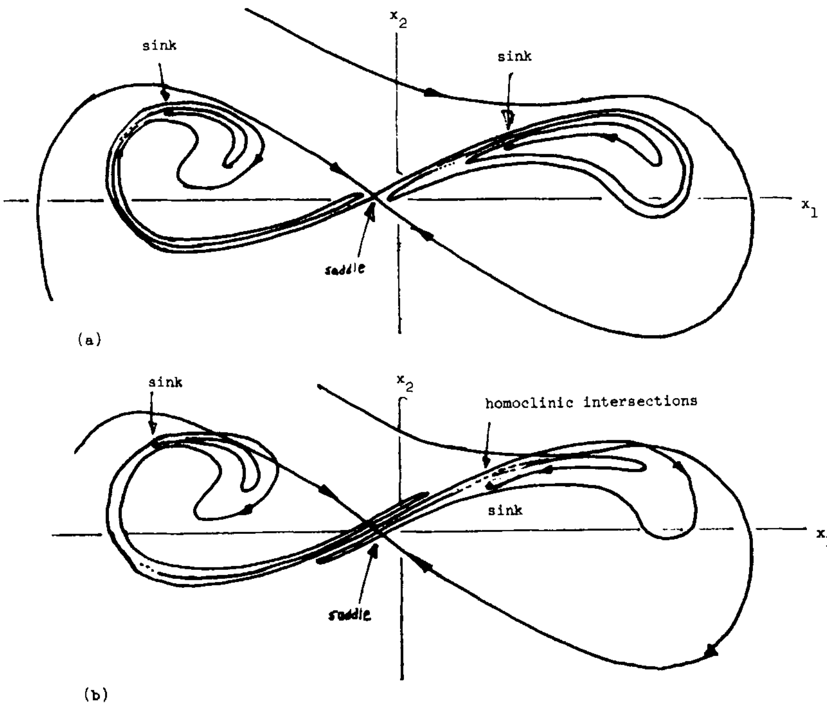


FIGURE 5. Stable and unstable manifolds of P_f for (a) $f = 0.75$ and (b) 0.90.

Smale horseshoes, then, we may have (weakly) attracting sets of periodic points of P_f for $f > f_c$.

As f continues to increase, analog computations show that the fixed points of FIGURE 5b bifurcate to sinks of period 2, and then of period 4. It is possible that further bifurcations, to periods 8, 16, 32, . . . also occur but they are difficult to detect with reliability. In any event, for $f \geq 1.08$, successive iterates of P_f are no longer attracted to a clearly periodic orbit, and they appear to wander chaotically back and forth across Σ. FIGURE 6 indicates that this wandering is in fact ordered in the sense that the orbits rapidly converge to and appear to remain on a 1-dimensional curve close to, and perhaps identified with, W^u. Computations of the power spectrum for some 50,000 samples (12,000 cycles of the forcing function) clearly suggest nonperiodic behavior.[15,16] We suggest[16] that for $f \in (1.08, 2.45)$, P_f has a *strange attractor*.[21,25]

In order to study the structure of the attractor more fully, we approximate[16] the true Poincaré map P_f by a simple cubic polynomial mapping P_d given by,

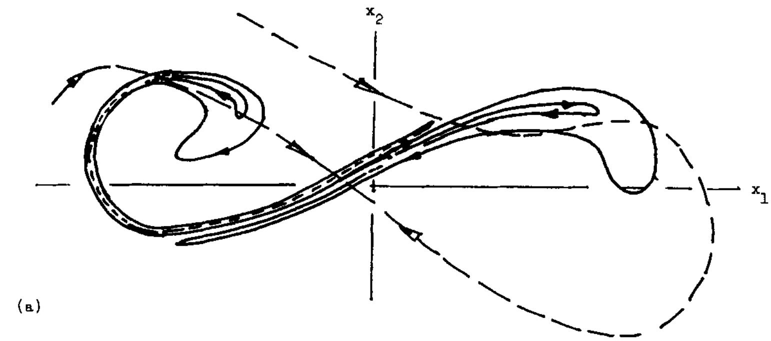
$$(x, y) \mapsto (y, -bx + dy - y^3),\ b; d > 0 \tag{8}$$

Fixing $b = 0.2$ and varying d, $d \in (1.2, 2.8)$, we were able to reproduce much of the behavior of P_f for $f \in (0, 1.2)$.

Under a suitable assumption on generic properties, and taking the symmetry of (6) into account, we might expect a Taylor series expansion of the true Poincaré map at (0, 0) to include only odd terms. We *can* explicitly calculate the linear terms by integrating (6), these terms have the form assumed in (8). We thus obtain an approximate map by ignoring all terms of order 5 and higher in the Taylor series. We assume that it is possible to choose coordinates such that the nonlinear term appears in only one component of the map.[12]

The cubic mapping (8) has much in common with the quadratic planar mapping discussed by Hénon, and digital computations strongly suggest that the invariant attracting set S_d, for a large number of values of d in the range (2.7, 2.8), has a local structure isomorphic to the product of a smooth curve and a Cantor set. Successive iterates of $P_d (d \approx 2.77)$ behave much as do successive iterates of the true Poincaré map P_f for $f \in (1.08, 2.45)$ as shown in FIGURE 7. In addition to computing successive iterates, the structure of stable and unstable manifolds of the saddle of P_f at (0, 0) was also studied. As for the true map P_f, the manifolds become tangent at a critical value $d = d_c \approx 2.60$.

Summarizing, then, we prove that (5) has homoclinic orbits for $f > f_c \cdot (\alpha, \beta, \delta, \omega) \approx 0.79$ and that (5) has at least one attracting set for all $f < \infty$.[16] Analog computer solutions of (5) indicate (but cannot prove) that the sinks of the Poincaré map P_f of (5) undergo a sequence of "flip" bifurcations for $f \approx f_0 >$



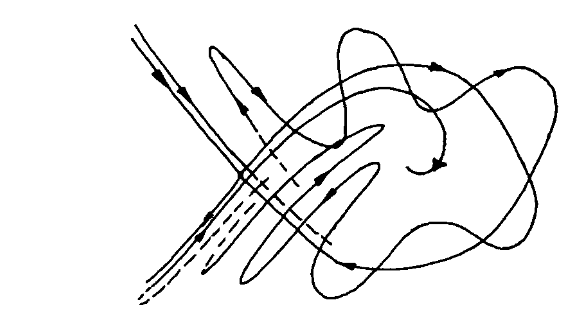


FIGURE 6. (a) The attracting invariant set of P_f for $f = 1.10$. (b) Schematic structure of one "lobe."

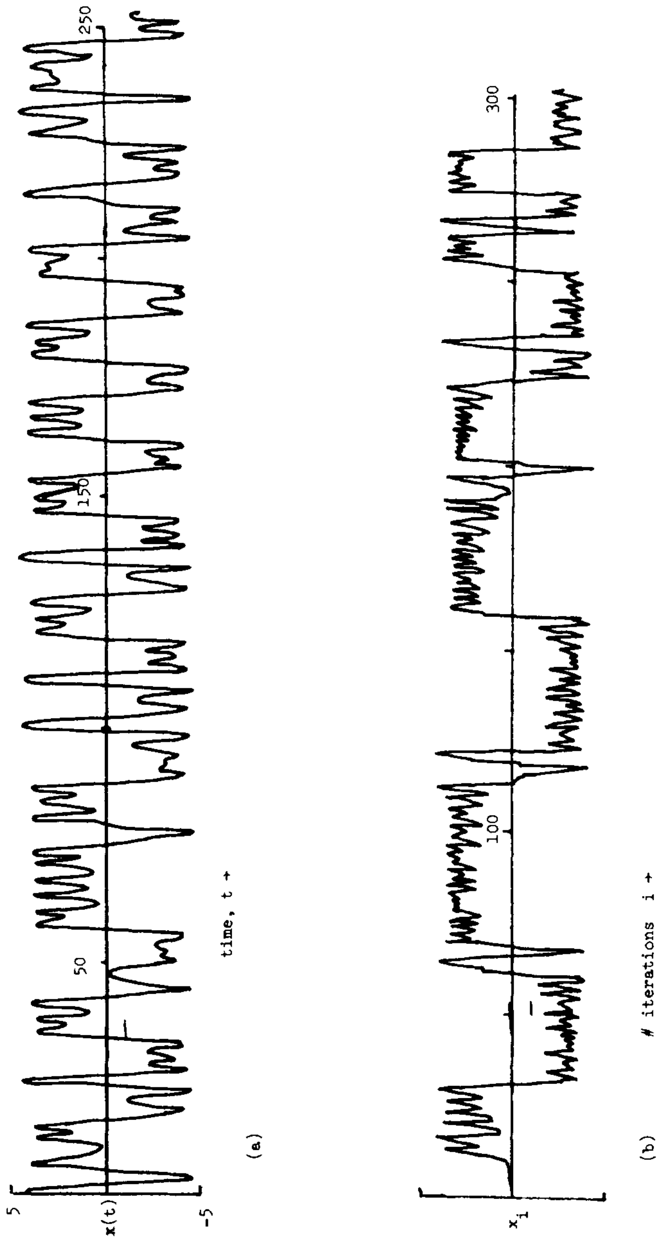


FIGURE 7. Evolutions of the ODE (6) and of the approximate Poincaré map (8).

f_c in which sinks of periods 2, 4, 8, . . . , 2^n, . . . are created. Ultimately for $f \geq f_d \approx 1.08$ ($> f_b$) all orbits apparently approach a nonperiodic attractor as $t \to +\infty$. Working with an approximate Poincaré map P_d, we can prove that P_d indeed undergoes such a succession of flip bifurcations. We then use the digital computer to study the structure of the invariant attracting set S_d of P_d for the appropriate range of values of d in which S_d appears to be topologically conjugate to the strange attractor S of P_f for $f \in (1.08, 2.45)$.

Very little is presently known about the behavior of polynomial mappings of $\mathbf{R}^2$, such as P_d (Equation 8) or the map studied by Hénon.[12] In order to obtain more information in the present case, we intend to study the cubic mapping on $\mathbf{R}$ given by,

$$y \to (dy - y^3) \qquad (9)$$

If we introduce a dummy small parameter $\epsilon < 1$ into (**8**), and rewrite the equation as

$$(x, y) \mapsto (\epsilon y, -bx + dy - y^3) \qquad (10)$$

we see that (**9**) is essentially identical to the limiting case of (**10**) when $\epsilon = 0$. In this way we hope to reduce the complexity of the problem much as Guckenheimer[8,9] and Williams[28] did in their studies of Lorenz equations. Guckenheimer has also reduced the study of the Poincaré map of the forced van der Pol oscillator to that of a map on the circle.[10]

Conclusion

We have outlined a general approach to the qualitative analysis of nonlinear dynamical problems and illustrated our methods with two examples taken from engineering science. One of the most interesting features in these examples is the detection of *global bifurcations* in which periodic orbits are created or annihilated in a manner which depends upon the global disposition of stable and unstable manifolds of some invariant set. The "figure-8" loop occurring on B_{sc} (Figure 3b) is a simple example, while the creation of infinitely many periodic points when W^u and W^s intersect homoclinically (Figure 5) is considerably more complex. In the latter case we have the creation of a countable infinity of periodic points as W^u is "pulled through" W^s, and vice versa. These examples illustrate the need for a more general definition of bifurcation, such as that proposed in the first section.

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