COUNTABLE AND NET CONVERGENCE

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It is well known that Lebesgue's dominated convergence theorem does not hold for nets; that is, having a countable sequence is essential (see [1], p. 95). On the other hand, for a real valued function on an interval, sequences do suffice; that is, $\lim_{x\to y} f(x)=a$ iff $\lim_{n\to\infty} f(x_n)=a$ for every sequence $x_n\to y$. The purpose of this note is to isolate the basic reasons for these phenomena.

DEFINITION. *A directed set A is called countably accessible iff there is a countable sequence a_n in A such that $a_n\to\infty$, that is, for any $b\in A$ there is an N such that $a_n\geqq b$ if $n\geqq N$.*

THEOREM. *Let X be a topological space and A a countably accessible directed set. Suppose $f: A\to X$ is a net and for every countable sequence $b_n\to\infty$ in A, $f(b_n)$ converges to $x\in X$. Then f converges to x.*

Proof. If f did not converge to x, there would be a neighborhood U of x such that for any $b\in A$ there is a $b'\geqq b$ with $f(b')\notin U$. However, if $a_n\to\infty$ then $f(a_n')$ does not converge to x even though $a_n'\to\infty$, a contradiction.

References

1. S. K. Berberian, Measure and Integration, Macmillan, New York, 1965.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1950.

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