# Optimal Control of Nonlinear Systems with Temporal Logic Specifications

**Eric M. Wolff**[1]

Ufuk Topcu[2] and Richard M. Murray[1]

[1]Caltech and [2]UPenn

University of Michigan

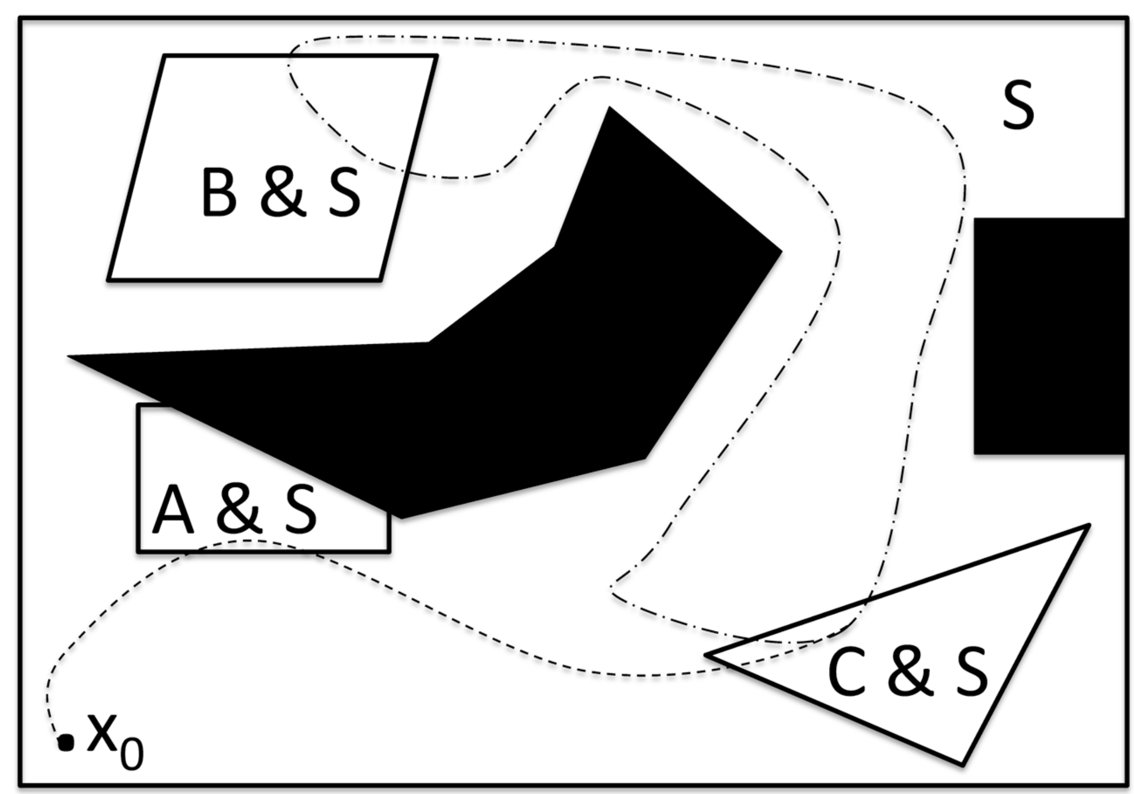
October 1, 2013

# Autonomous Systems in the Field


# Motivation

- How do we specify tasks for autonomous systems?
- How do we compute optimal solutions?
- How do we handle high-dimensional continuous dynamics?

# An Example Problem




**Spec**: Avoid obstacles, pick-up supplies at region A and then do surveillance on regions B and C.

# Main Contributions

- Trajectory generation techniques for **high-dimensional (10+ dim)** and **nonlinear** systems with temporal logic specifications
  - Automata-guided temporal logic planning
    - Wolff and Murray [ISRR 2013]
  - Mixed-integer linear encoding of LTL
    - Wolff, Topcu, Murray [IROS 2013, ICRA 2014-sub]

- Improve on discrete abstraction techniques

# Related Work

- Discrete abstractions (Alur00, Belta06, Habets06, Kloetzer08, Pappas06, Tabuada06, Wongpiromsarn10)

**Low dimensional systems (<= 6)**

- Mathematical programming:
  - Constrained trajectory generation (Bemporad99, Earl06, Richards02)
  - Finite-horizon LTL properties (Karaman08, Kwon08)

**Simple tasks**

# These Problems are Hard!

- Dynamic constraints -> **undecidable**

- Task specification -> **PSPACE**

# Outline

- Preliminaries
  - System model
  - Linear temporal logic (LTL)
- Automata-guided temporal logic planning
  - Constrained reachability
  - Examples
- Mixed-integer linear encoding of LTL
  - A finite-dimensional encoding
  - Examples

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# System Model

- Discrete-time nonlinear system

  $x_{t+1} = f(x_t, u_t)$

  $x \in X \subseteq R^n$

  $u \in U \subseteq R^m$

- Labels

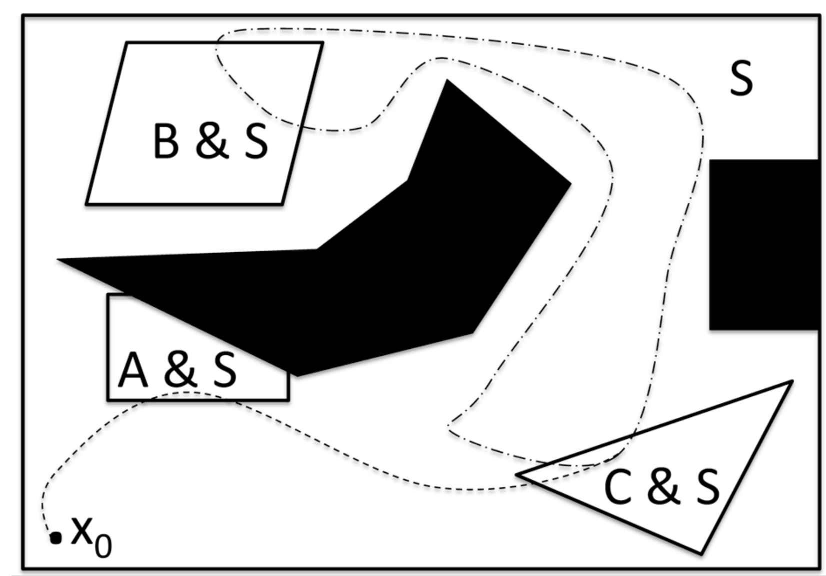
  $L : X \rightarrow 2^{AP}$

- Trajectory:

  $\mathbf{x} = x(x_0,u) = x_0x_1x_2 \ldots$

  $x_{i+1} = f(x_i,u)$ for some $u \in U$ for $i = 0,1,\ldots$

- Word: $L(\mathbf{x}) = L(x_0)L(x_1)L(x_2)\ldots$



# Linear Temporal Logic (LTL)

Want to specify properties such as:

- Response: always SIGNAL after a REQUEST arrives
- Liveness: always eventually PICKUP
- Safety: always remain SAFE
- Priority: do JOB1 until JOB2
- Guarantee: eventually reach GOAL

**Linear temporal logic (LTL):**

- A logic for reasoning about how properties change over time
- Reason about infinite sequences $\sigma = s_0 s_1 s_2 \ldots$ of states
- Propositional logic: $\wedge$ (and), $\vee$ (or), $\implies$ (implies), $\neg$ (not)
- Temporal operators: $\mathcal{U}$ (until), $\bigcirc$ (next), $\square$ (always), $\diamond$ (eventually)

# Linear Temporal Logic (LTL)

Want to specify properties such as:

- Response: $\square($ REQUEST $\implies$ SIGNAL $)$
- Liveness: $\square\diamond$ PICKUP
- Safety: $\square$ SAFE
- Priority: JOB1 $\mathcal{U}$ JOB2
- Guarantee: $\diamond$ GOAL

**Linear temporal logic (LTL):**

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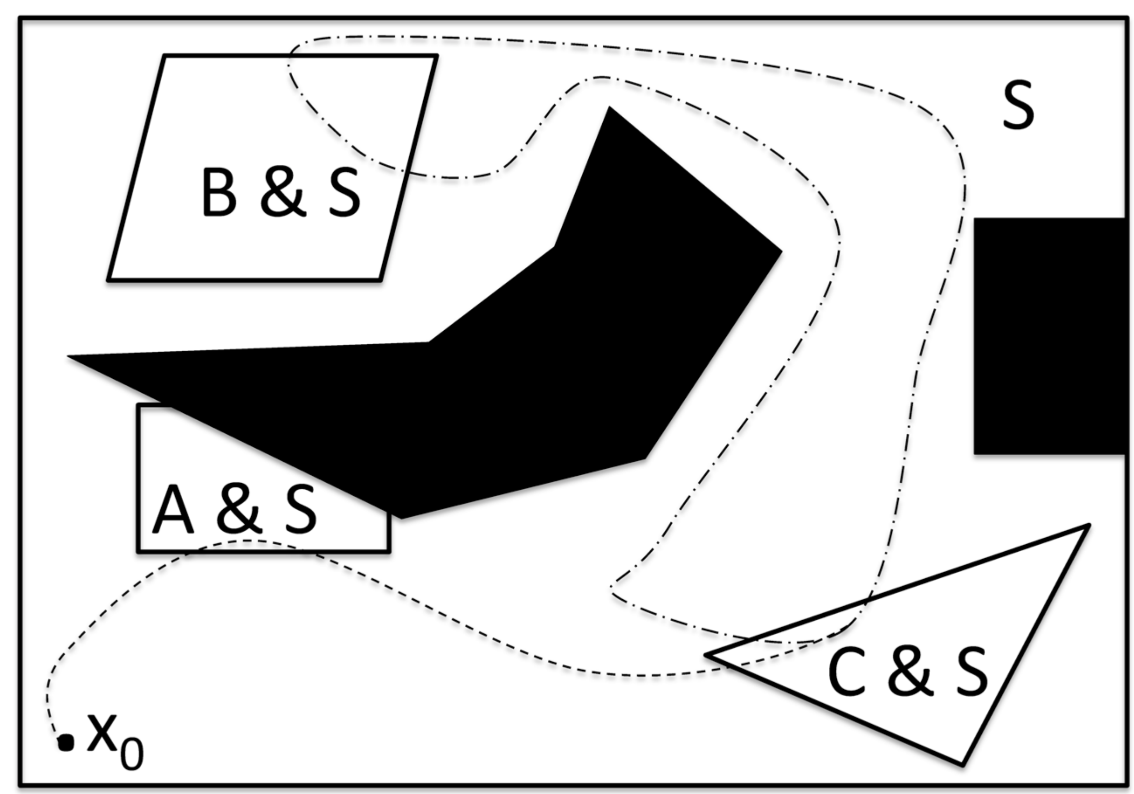
# Problem Overview




**Spec**: Avoid obstacles, pick-up supplies at region A and then do surveillance on regions B and C.

# From Logic to Automaton

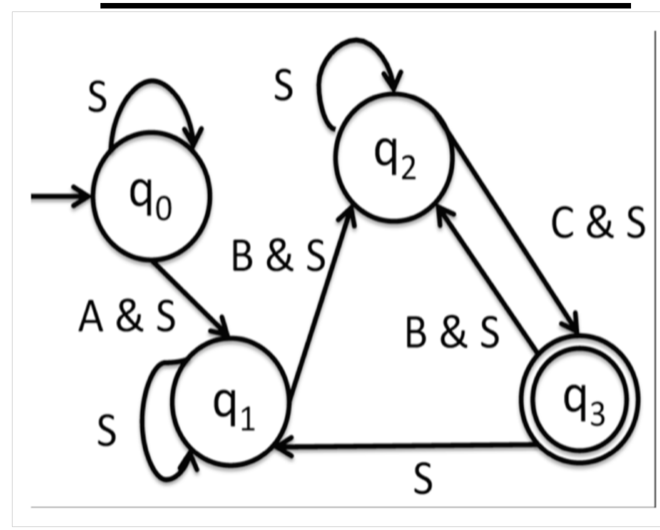
**Informal spec**: Avoid obstacles, pick-up supplies at region A and then do surveillance on regions B and C.

LTL Specification

φ = <> A & []<> B & []<> C & [] S

Automatic translation from **logic** to **automaton**!

Büchi Automaton




**Gastin,Oddoux**: http://www.lsv.ens-cachan.fr/~gastin/ltl2ba/

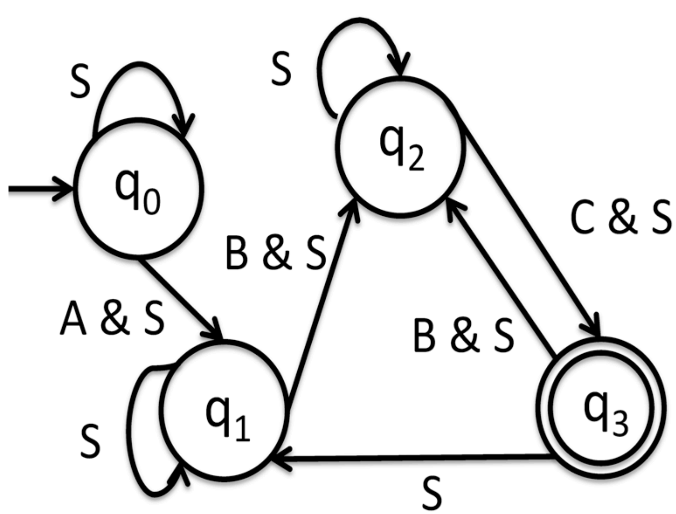
# Problem Statement

- **Given**:
  - a deterministic nonlinear dynamical system,
  - initial state $x_0$,
  - a Büchi automaton A (specification)

- **Goal**: Find a control input sequence **u** such that the word $L(\mathbf{x}(x_0,\mathbf{u}))$ is accepted by A

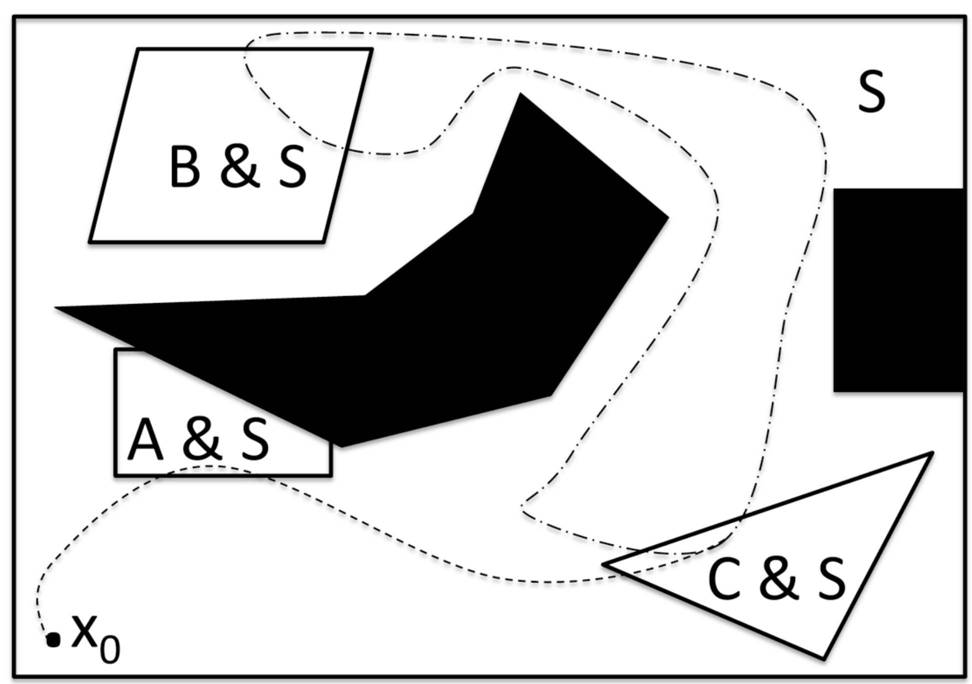
# Solution

- **Main idea**: Use specification automaton to guide constrained reachability computations
- Compute a word (from the trajectory) that is accepted by the automaton

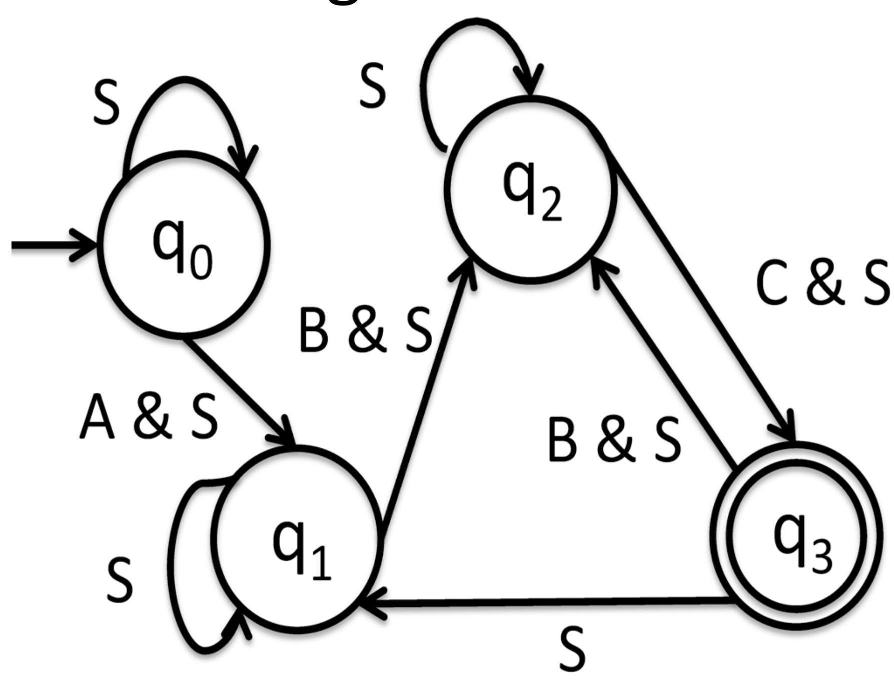


# Solution

Environment



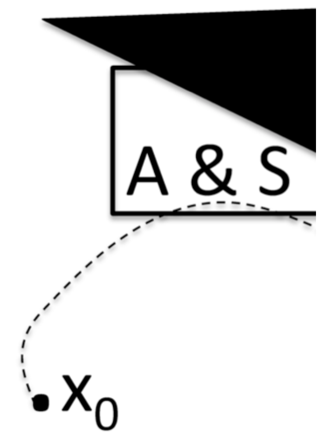

Logic Automaton




**Spec**: φ = <> A & []<> B & []<> C & [] S

# Constrained Reachability

- **Given**:
  - sets $\mathbf{X_1}$ , $\mathbf{X_2} \subseteq X$ ,
  - horizon length N




- **Goal**: Find a control input sequence u and a horizon length N such $x_1,\ldots,x_{N-1} \in \mathbf{X_1}$, $x_N \in \mathbf{X_2}$ such that $x_{t+1} = f(x_t,u_t)$ for $t = 1,\ldots,N-1$

- CstReach($\mathbf{X_1}$,$\mathbf{X_2}$)

# Solving Constrained Reachability

- CstReach($\mathbf{X_1}$,$\mathbf{X_2}$) can be encoded as a mixed-integer linear program using the big-M formulation

- Big-M
  - Enforce that state is in union of polyhedra
  - $H_i\, x <= K_i + M(1-z_i)$, $z_i \in \{0,1\}$, $sum(z) = 1$

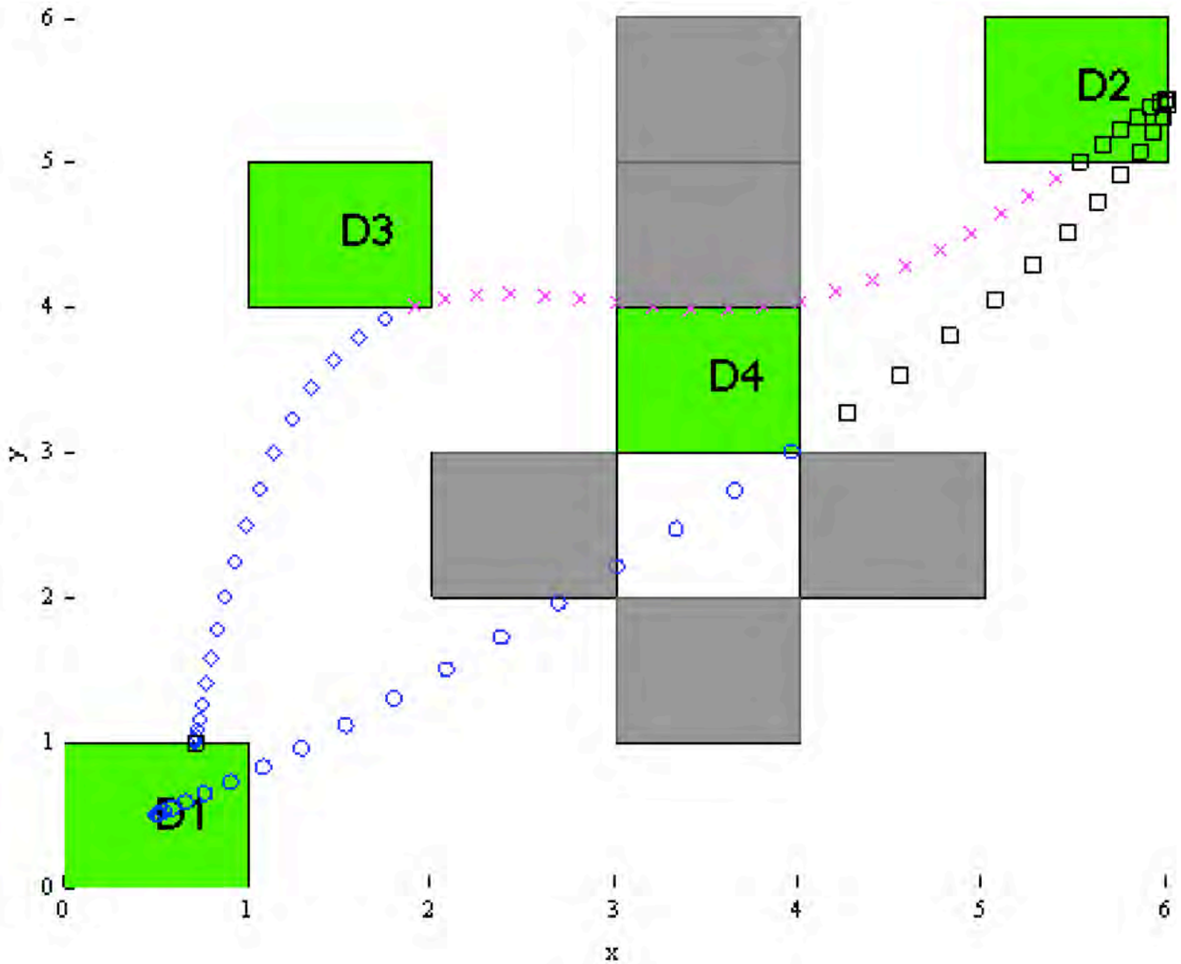
- Independent of dynamics

# Examples

- Systems
  - Quadrotor (10 dim)
  - Chained integrators (4, 12, 20 dim)
  - Car-like robot (nonlinear + drift)

- Specifications
  - Visit *n* goals
  - Repeatedly visit *n* goals

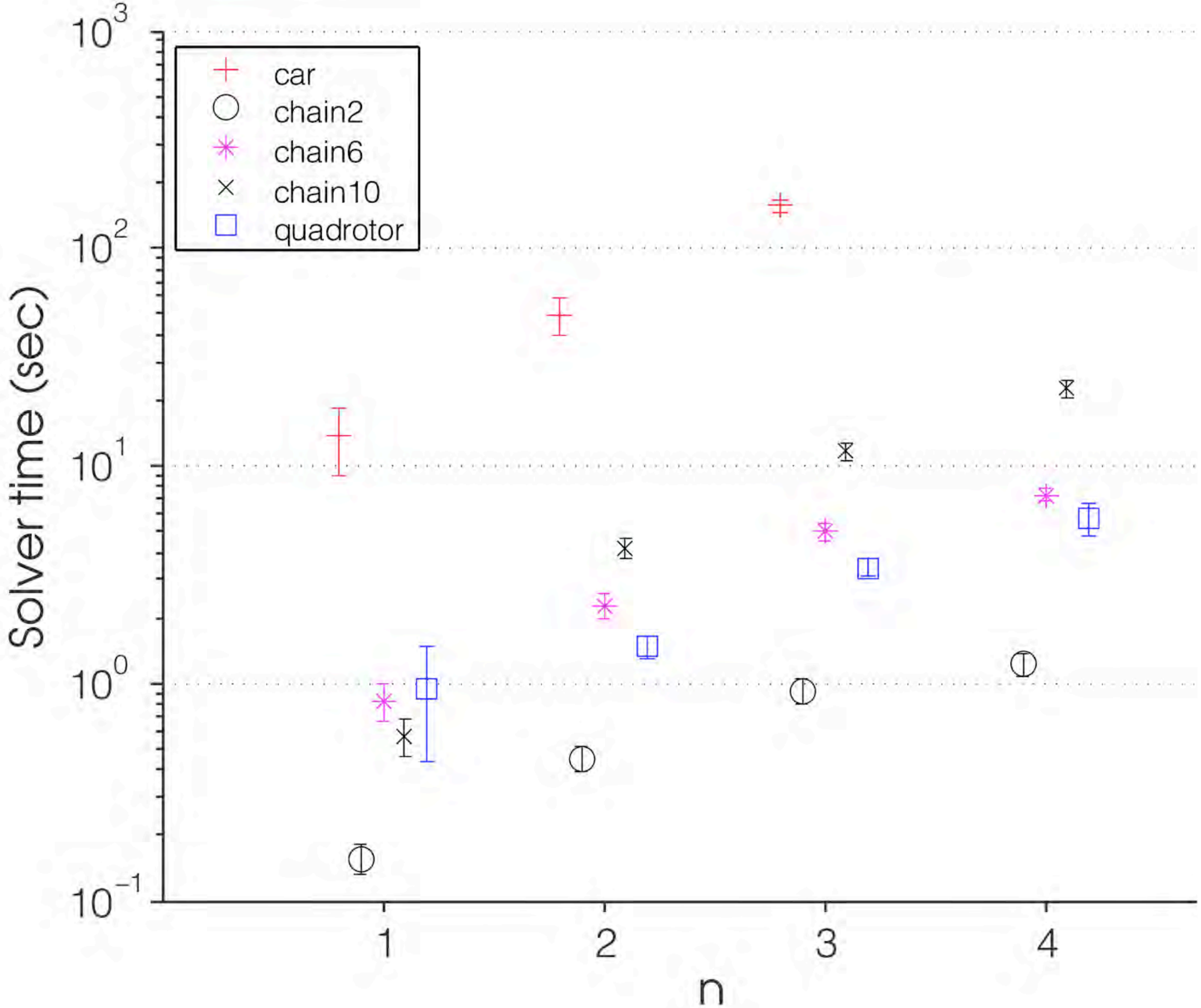
# Examples



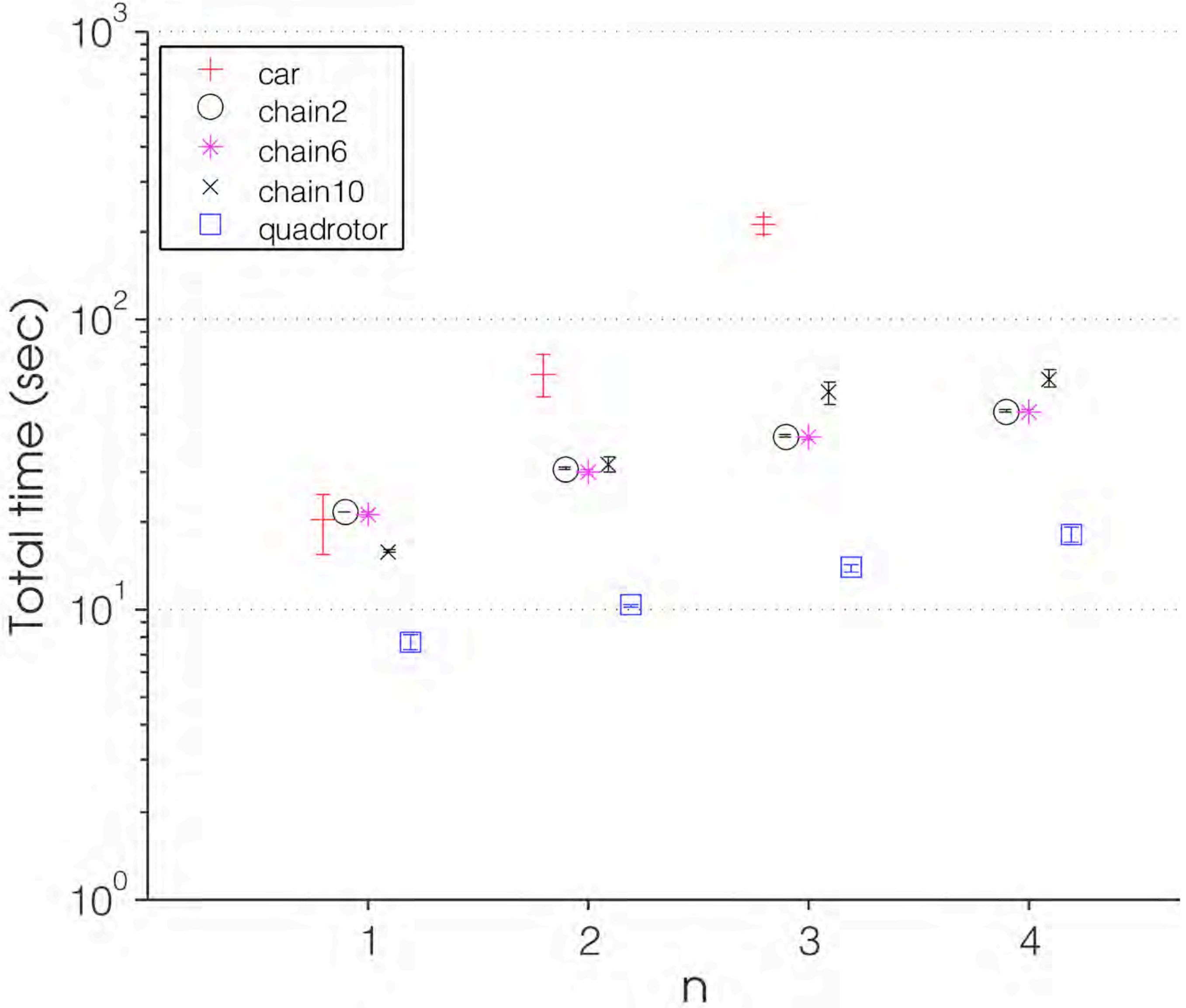

**Model**: 10-dim quadrotor

**Spec**: φ = <> D1 & <> D2 & <> D3 & <> D4 & [] safe

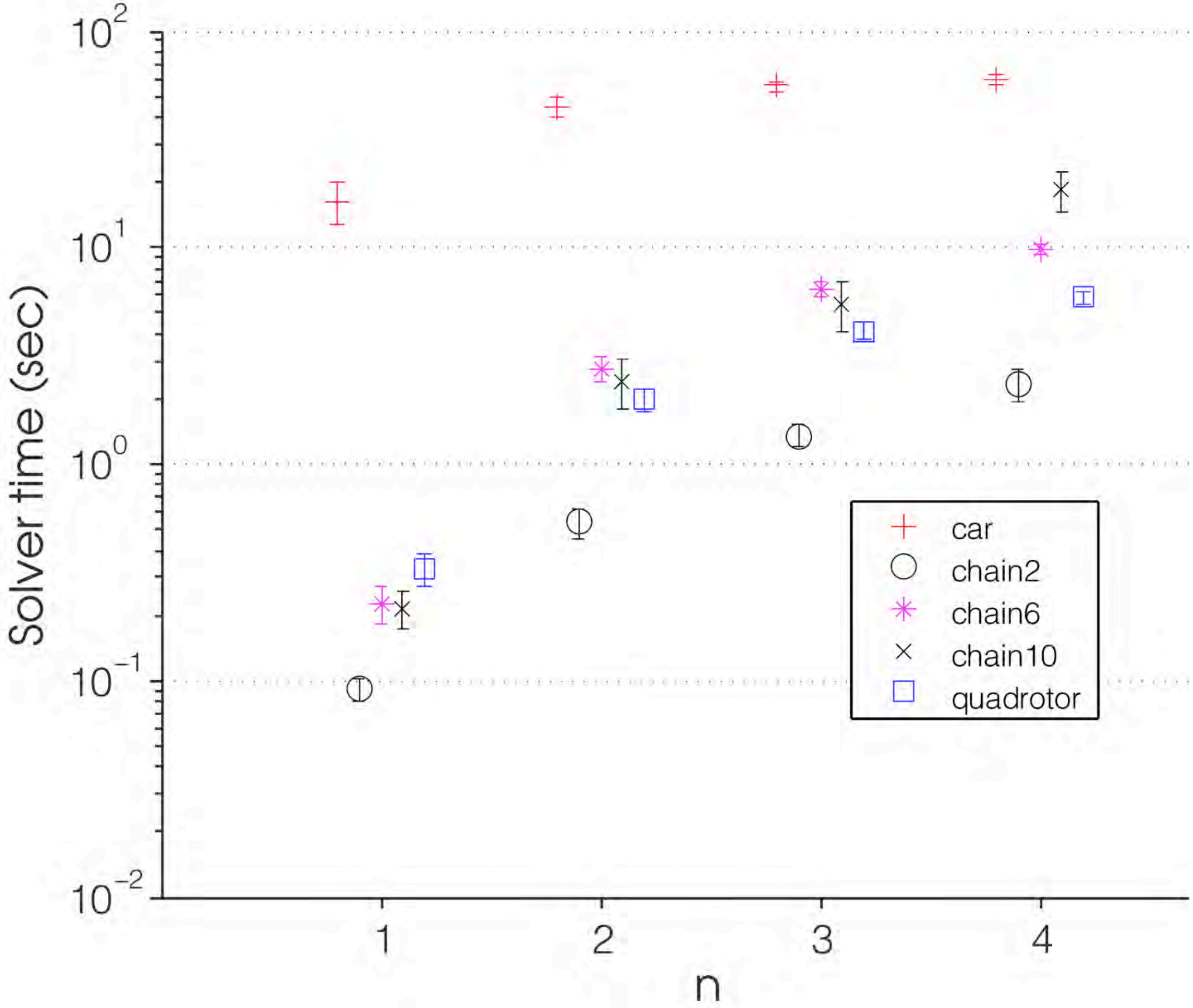
# Eventually (solver)



# Eventually (total)

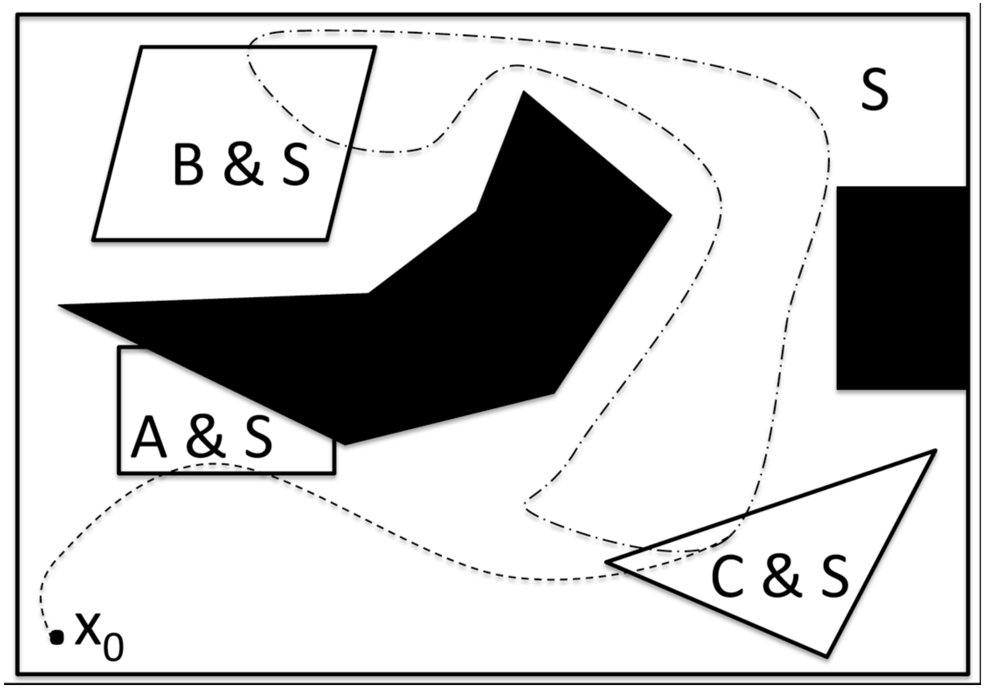


# Repeatedly (solver)

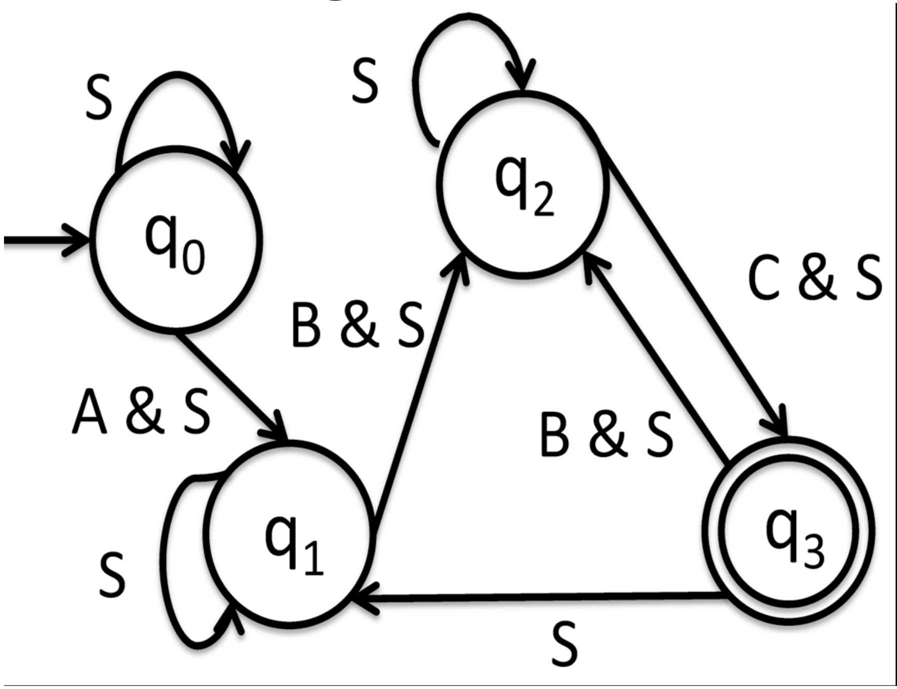


# Review (1/2)

Environment




Logic Automaton




**Spec**: φ = <> A & []<> B & []<> C & [] S

# Future Work (1/2)

- Stochastic constrained reachability
  - [Horowitz, Wolff, Murray ACC14-sub]

- Improved composition of subproblems

- Improved heuristics

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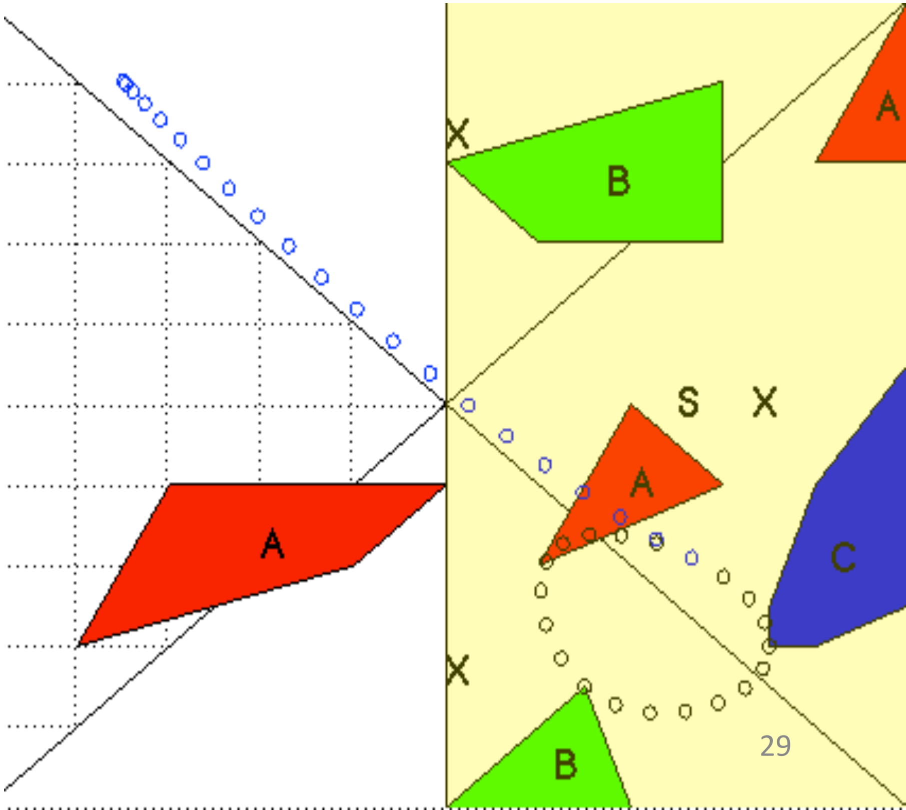
# Problem Description

- Discrete-time nonlinear systems
  - Piecewise affine
  - Differentially flat
  - Small-time locally controllable

- Cost function J
- Task specification φ

- min $J(\mathbf{x}(x_0,\mathbf{u}))$
  s.t. $\mathbf{x}(x_0,\mathbf{u}) \models \varphi$



# A Fragment of LTL

## Core

$$\varphi_{safe} := \Box\psi$$

$$\varphi_{goal} := \Diamond\psi$$

$$\varphi_{per} := \Diamond\Box\psi$$

$$\varphi_{live} := \Box\Diamond\psi$$

# A Fragment of LTL

## Core

$\varphi_{safe} := \square\psi$

$\varphi_{goal} := \diamond\psi$

$\varphi_{per} := \diamond\,\square\,\psi$

$\varphi_{live} := \square\,\diamond\,\psi$

## Response

$\varphi^1_{resp} := \square(\psi \implies \bigcirc\phi)$

$\varphi^2_{resp} := \square(\psi \implies \diamond\phi)$

$\varphi^3_{resp} := \diamond\,\square\,(\psi \implies \bigcirc\phi)$

$\varphi^4_{resp} := \diamond\,\square\,(\psi \implies \diamond\phi)$

# A Fragment of LTL

## Core

$$\varphi_{safe} := \Box\psi$$

$$\varphi_{goal} := \Diamond\psi$$

$$\varphi_{per} := \Diamond \Box \psi$$

$$\varphi_{live} := \Box \Diamond \psi$$

## Response

$$\varphi^1_{resp} := \Box(\psi \implies \bigcirc\phi)$$

$$\varphi^2_{resp} := \Box(\psi \implies \Diamond\phi)$$

$$\varphi^3_{resp} := \Diamond \Box (\psi \implies \bigcirc\phi)$$

$$\varphi^4_{resp} := \Diamond \Box (\psi \implies \Diamond\phi)$$

## Fairness

$$\varphi^1_{fair} := \Diamond\psi \implies \bigwedge_{j=1}^{m} \Diamond\phi_j$$

$$\varphi^2_{fair} := \Diamond\psi \implies \bigwedge_{j=1}^{m} \Box \Diamond \phi_j$$

$$\varphi^3_{fair} := \Box \Diamond \psi \implies \bigwedge_{j=1}^{m} \Box \Diamond \phi_j$$

# A Fragment of LTL

**Core**

$$\varphi_{safe} := \Box\psi$$

$$\varphi_{goal} := \Diamond\psi$$

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$$\varphi^3_{fair} := \Box\Diamond\psi \implies \bigwedge_{j=1}^{m} \Box\Diamond\phi_j$$

- What is missing?
  - Nested temporal operators
  - Negations

# A Fragment of LTL

## Core

$$\varphi_{safe} := \Box\psi$$

$$\varphi_{goal} := \Diamond\psi$$

$$\varphi_{per} := \Diamond\Box\psi$$

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$$\varphi^3_{fair} := \Box\Diamond\psi \implies \bigwedge_{j=1}^{m} \Box\Diamond\phi_j$$

**Recently extended to all of LTL**

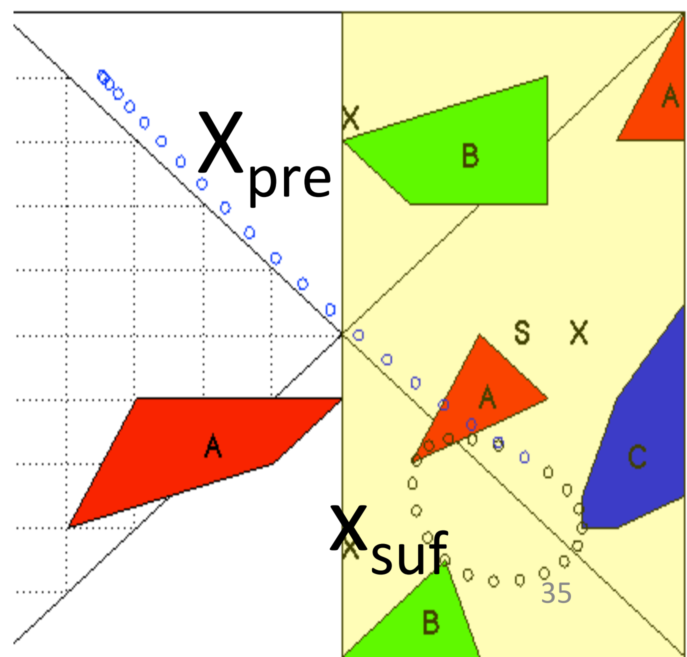
[Wolff,Topcu,Murray ICRA14-sub]

# Finite Parameterization of Trajectory

- Let $x = x_{pre} (x_{suf})^{\omega}$
  - $x_{pre}$ and $x_{suf}$ are finite
  - $x_{suf}$ is a cycle

- Labels are disjunctions of polytopes

- Use a binary variable for each polyhedron every stage
  - Big M: $H x <= K + M(1-z), \; z \in \{0,1\}$
  - Convex hull

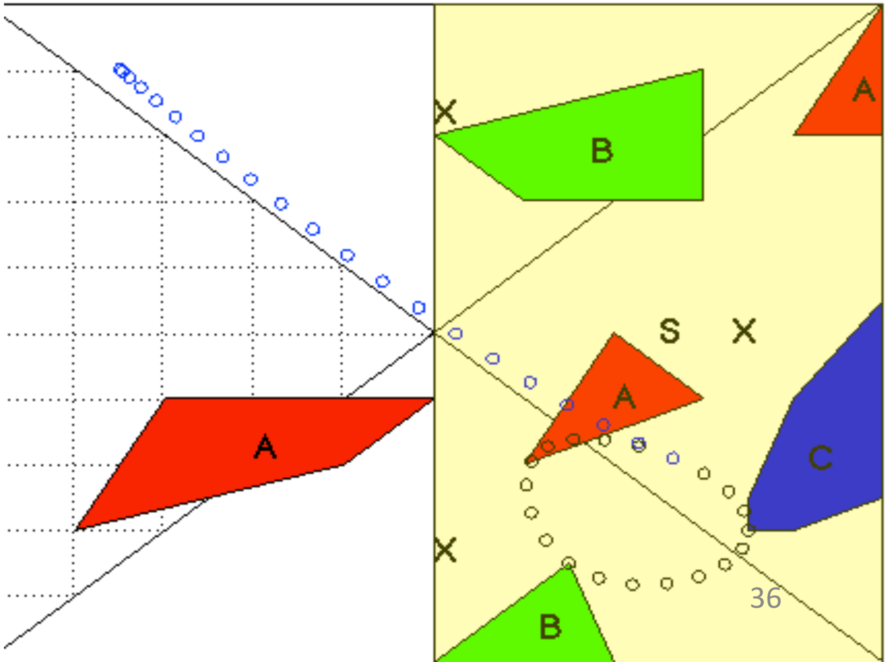


# Linking the System and Logic

- Use a binary variable for each polyhedron every stage
  - Big M: $H x <= K + M(1-z), \; z \in \{0,1\}$
  - Convex hull

- Does system satisfy label ψ at time t?

$$P_t^{\psi} := \sum_{i \in \mathcal{I}_t^{\psi}} z_t^{\psi_i}$$



$P_t^{\psi} := \sum_{i \in \mathcal{I}_t^{\psi}} z_t^{\psi_i}$

# Encoding LTL Constraints

$$\varphi_{safe} := \Box \psi$$

$$P_t^{\psi} \geq 1 \quad \forall t \in \mathcal{T}_{pre},$$
$$P_t^{\psi} \geq 1 \quad \forall t \in \mathcal{T}_{suf}.$$

$$\varphi_{goal} := \Diamond \psi$$

$$\sum_{t \in \mathcal{T}_{pre}} P_t^{\psi} + \sum_{t \in \mathcal{T}_{suf}} P_t^{\psi} \geq 1.$$

$$\varphi_{per} := \Diamond \Box \psi$$

$$P_t^{\psi} \geq 1 \quad \forall t \in \mathcal{T}_{suf}$$

$$\varphi_{live} := \Box \Diamond \psi$$

$$\sum_{t \in \mathcal{T}_{suf}} P_t^{\psi} \geq 1.$$

# Complexity

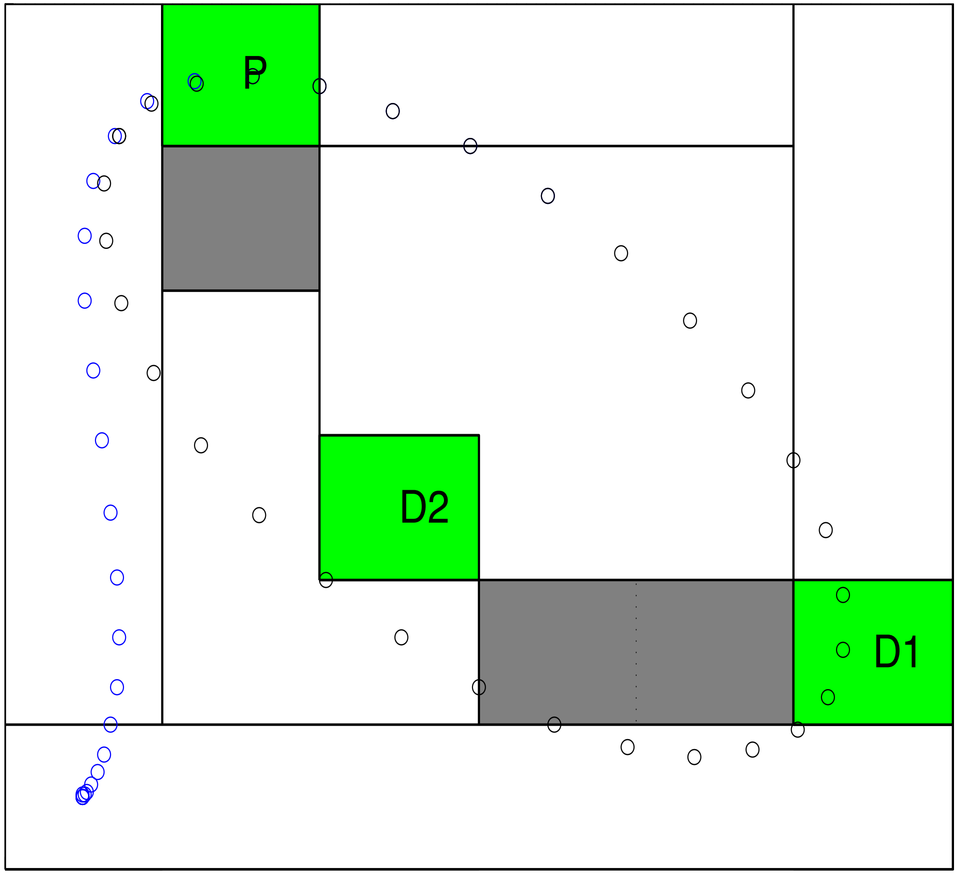
- Add all constraints (dynamics + LTL) and solve MILP using off-the-shelf software

- NP-complete
  - Branch + bound
  - Solvers work well in practice

- With H polyhedrons and T time steps
  - Safety = $H^T$ $\varphi_{safe} := \Box\psi$
  - Goal = HT $\varphi_{goal} := \Diamond\psi$

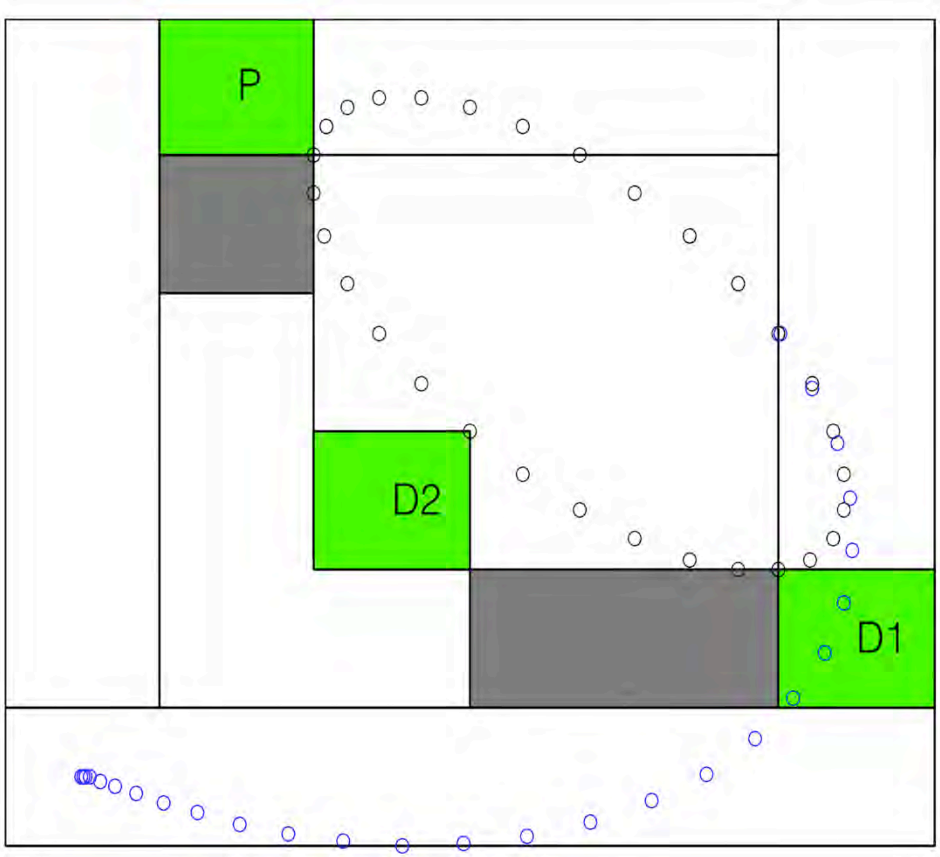
# Examples

- Systems
  - Quadrotor (10-dim)
  - Chained integrators
  - Car-like robot
- Solution
  - Simultaneous (Sim)
  - Sequential (Seq)

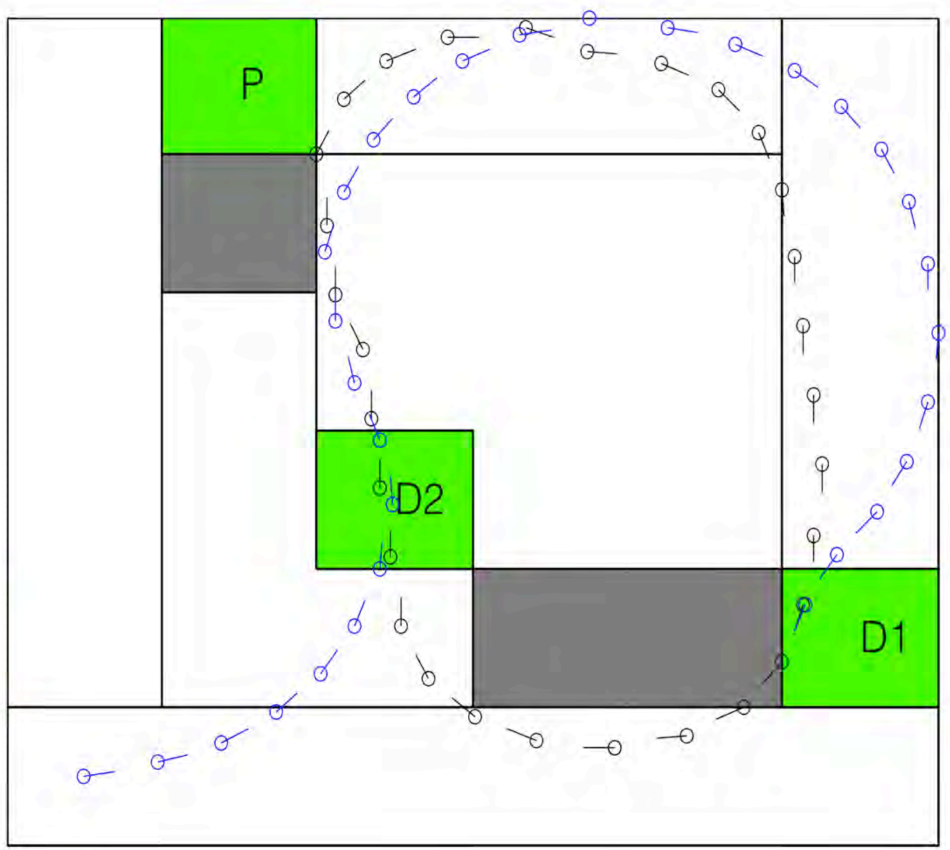



**Integrator (20 dim)**
**Solution in 6 sec.**

# Examples




**Quadrotor**



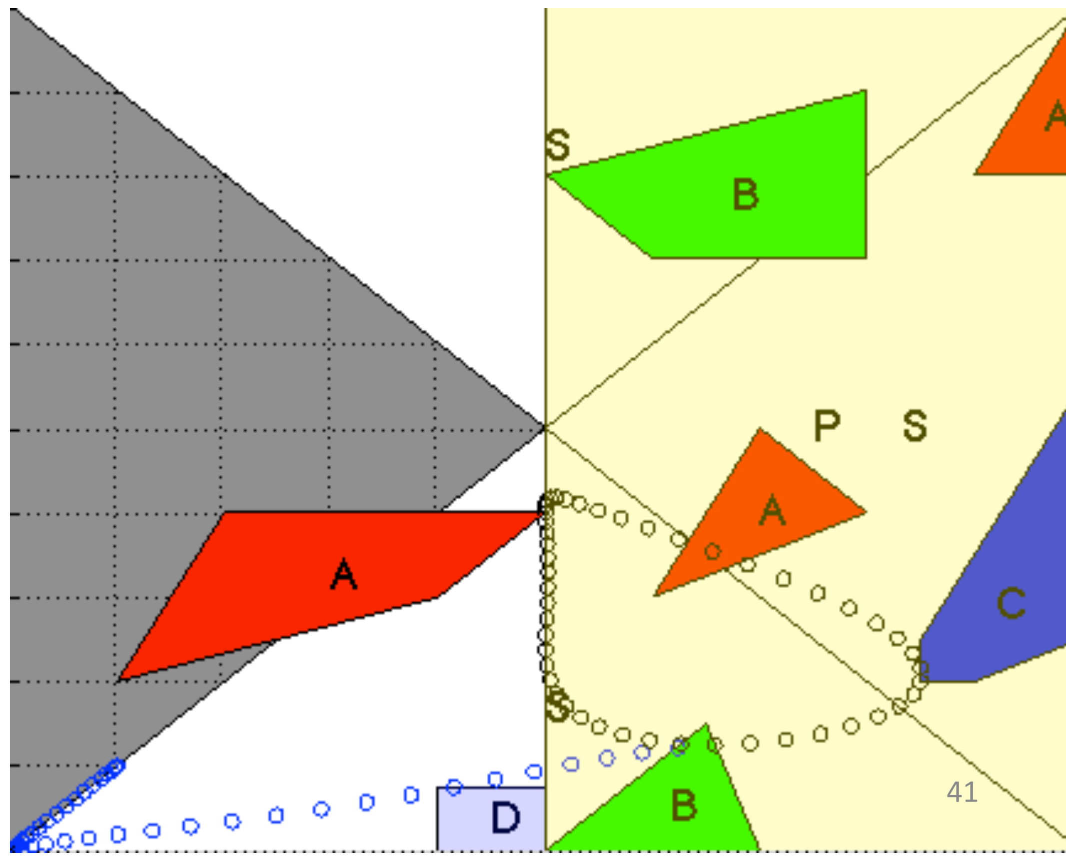

**Car-like robot**

# Quadrotor Example

- 10 dimensional quadrotor
- Linearized about hover (with fixed yaw)

- Feasible solutions found in seconds

- Cost function rewards staying left



# Results

| | | Feasible soln. (sec) | | Num. solved | |
|---|---|---|---|---|---|
| Model | Dim. | Sim. | Seq. | Sim. | Seq. |
| chain-2 | 4 | 1.10 ±.09 | 0.64 ±.06 | 20 | 20 |
| chain-6 | 12 | 4.70 ±.48 | 2.23 ±.15 | 20 | 20 |
| chain-10 | 20 | 9.38 ±1.6 | 3.74 ±.29 | 20 | 19 |

# Results

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| chain-6 | 12 | 4.70 ±.48 | 2.23 ±.15 | 20 | 20 |
| chain-10 | 20 | 9.38 ±1.6 | 3.74 ±.29 | 20 | 19 |
| quadrotor | 10 | 4.20 ±.66 | 1.80 ±.15 | 20 | 20 |
| quadrotor-flat | 10 | 2.26 ±.36 | 1.99 ±1.0 | 20 | 20 |

# Results

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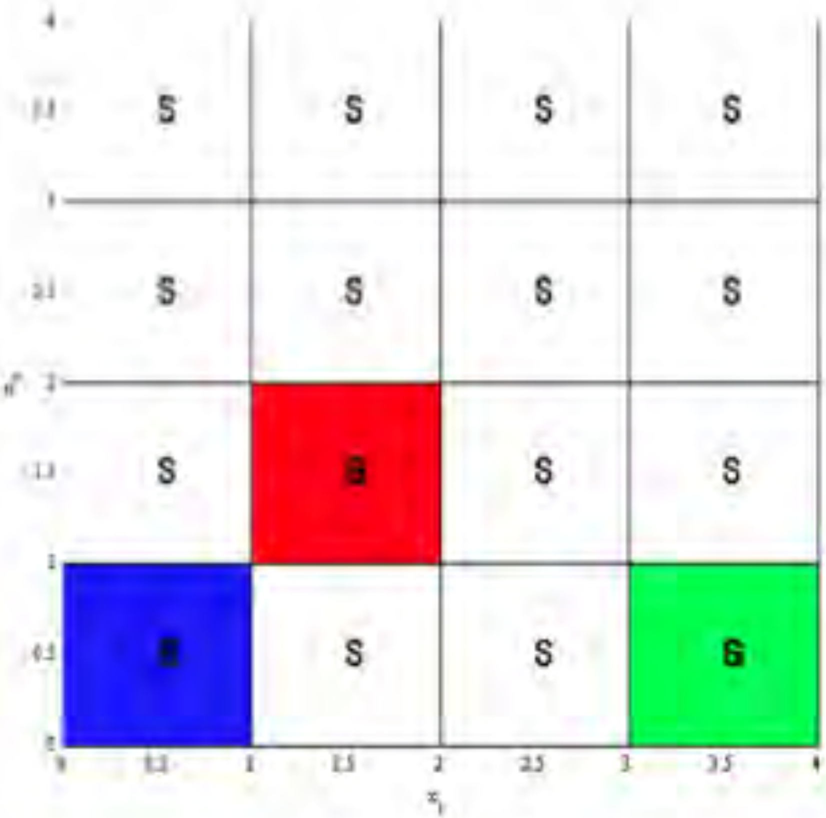
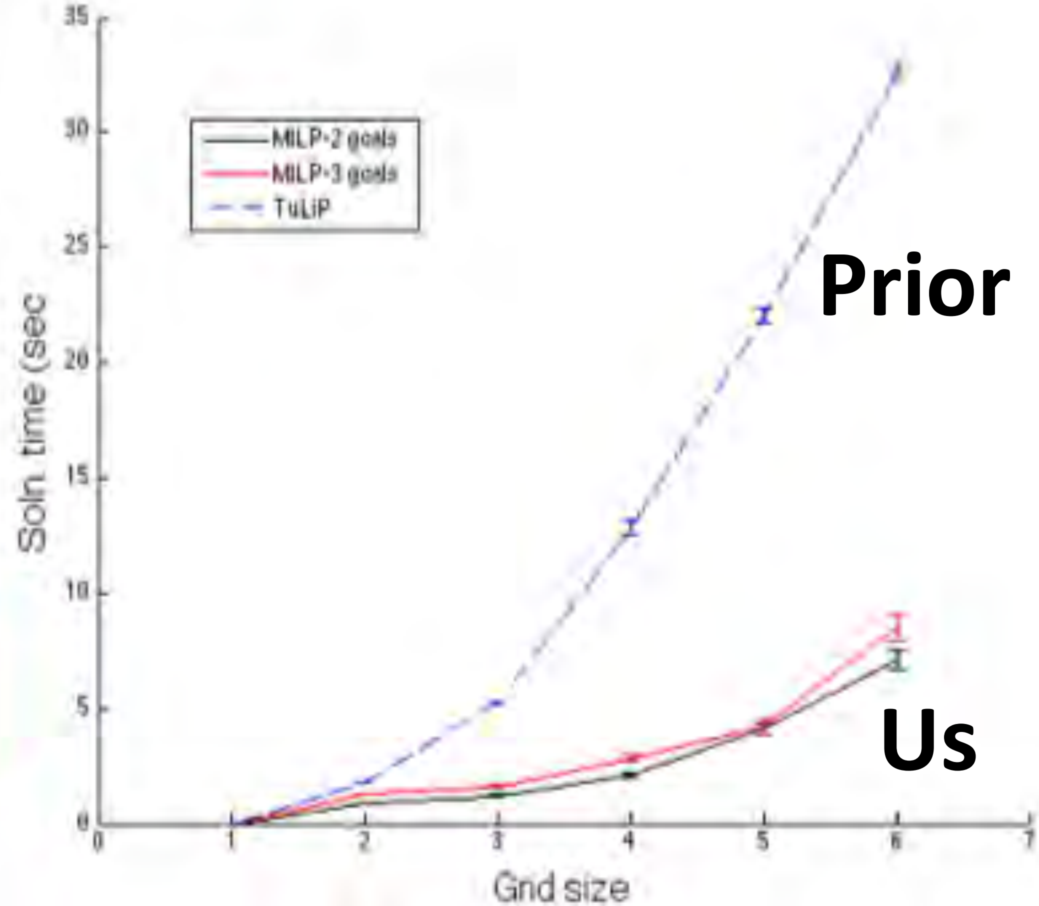
| car-3 | 3 | 43.9 ±.77 | 10.7 ±2.0 | 4 | 20 |
| car-4 | 3 | 42.4 ±1.7 | 18.7 ±3.1 | 2 | 18 |
| car-flat | 3 | 15.8 ±3.8 | 14.0 ±4.4 | 12 | 14 |

# Improvement Over Abstractions



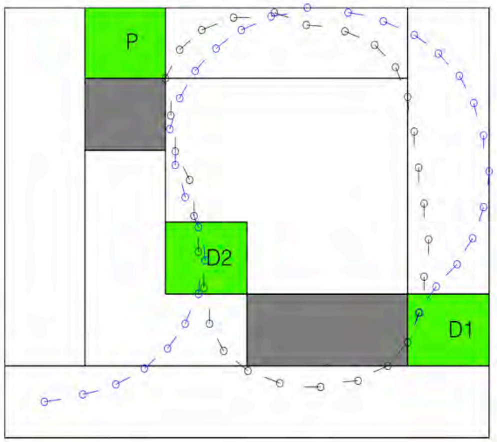



- Randomly generated `gridworld' problems
- Better performance than finite abstractions

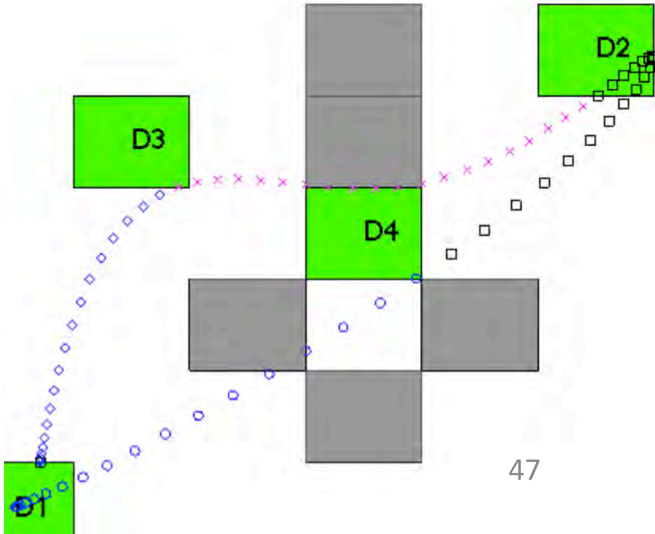
# Future Work (2/2)

- Disturbances and reactivity (RHC)
- Timed specifications
- Using discrete solutions to simplify optimization

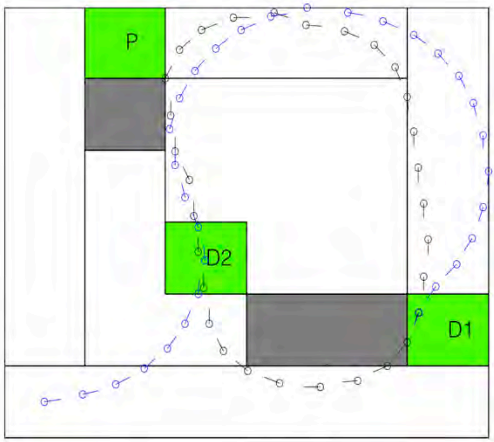
# Conclusions




- Two new LTL planning methods
  - Automata-guided temporal logic planning [ISRR2013]
  - Mixed-integer encoding of LTL [IROS 2013, ICRA 2014-sub]
- High dimensional and nonlinear systems
- Optimal control



# Conclusions




- Two new LTL planning methods
  - Automata-guided temporal logic planning [ISRR2013]
  - Mixed-integer encoding of LTL [IROS 2013, ICRA 2014-sub]
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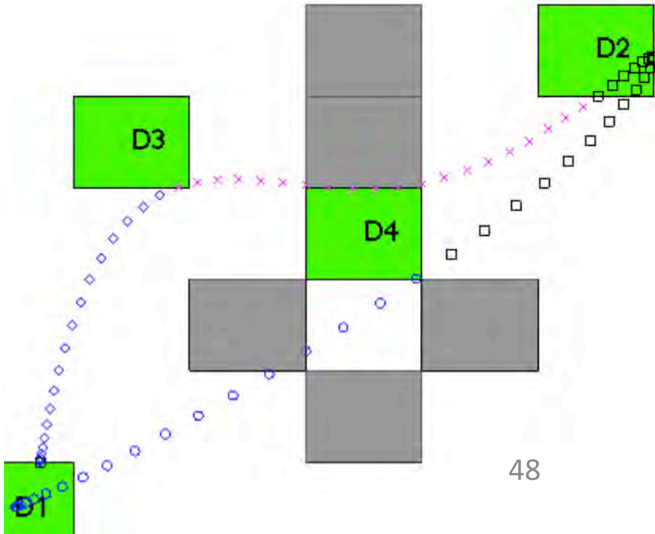
We can solve **high dimensional**, **nonlinear** LTL motion planning problems that were previously impossible.



# Thank you!


- Contact: Eric M. Wolff
  - Email: ewolff@caltech.edu
  - Web: www.cds.caltech.edu/~ewolff/



- Funding: NDSEG fellowship, Boeing, AFOSR